Cake Eating, Chattering, and Jumps: Existence Results for Variational Problems

Romer, Paul M.

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ABSTRACT

This paper establishes a simple existence result for solutions to variational problems of the form $\int_0^\infty G(x,\dot{x},t)\,dt$ or $\int_0^\infty G(x,\dot{x},\ddot{x},t)\,dt$. The key assumptions are that G have an integrable upper bound, that it satisfy a growth condition and that it be concave as a function of the highest order derivative in the problem, other arguments held constant. The discussion illustrates why three well known types of problems fail to have solutions. For two of these--chattering and cake eating--extended solution concepts are contrasted with simple modifications that restore the existence of a conventional solution. In a third case--state variables with jumps--the source of the difficulty is fundamental. For these problems a natural extended solution, analogous to the extension from probability density functions to general distribution functions is suggested.

1. INTRODUCTION

Many problems in economics can be stated as variational problems of the form "maximize $\int_0^\infty G(x,\dot{x},t)\,dt$." When the integrand G is concave as a function of x and $\dot{x}$, t held constant, solutions to this problem can be analyzed by means of a well-known set of sufficient conditions. When G does not possess this kind of concavity, a theorist must rely on necessary conditions and an existence theorem. Without such a theorem there is no guarantee that there exists a path which satisfies the necessary conditions or that one of the paths which do satisfy the necessary conditions will be a solution. In finite dimensional problems, economists often omit explicit reference to an existence result because it is obvious that the problem under consideration does indeed have a solution; however, experience with variational problems suggests caution because apparently reasonable problems can fail to have a solution. This paper presents a general existence result and uses it to discuss what goes wrong in each of the three kinds of problems referred to in the title and how a economic theorist faced with such a problem should proceed.

Because it is useful for understanding problems which exhibit "chattering," the main theorem actually considers a more general variational problem of the form "maximize $\int_0^\infty G(x,\dot{x},\ddot{x},t)\,dt$." Stated in this form, the key assumptions are that G have an integrable upper bound, that it satisfy a growth condition and that it be concave as a function of the highest order derivative in the problem, other arguments held constant. Because of the form of the growth condition, this extension to integrands stated in terms of higher order derivatives cannot be derived from the usual statement of the problem by stacking a higher order system to form a first order system. This extension (and an easy generalization to n'th order derivatives) also makes explicit the role played by concavity. In contrast to problems in finite dimensional spaces, some concavity assumption is crucial for establishing the existence of a solution; but this assumption is much weaker than joint concavity of the integrand in all arguments except t. In particular, it is weak enough to allow consideration of most economic problems with some form of non-convexity.

Problems which fail to have a solution because of chattering do not satisfy this convexity assumption. An extended solution concept which formally restores the existence of a solution in these cases has been proposed by L. C. Young [26]. The discussion here argues that in economic problems, a solution in the usual sense fails to exist because the problem is not completely specified. The correct response is not to extend the notion of what is a solution but rather to consider cost terms associated with higher order derivatives which are usually neglected. This modified problem will satisfy the convexity assumption. The "cake eating" problem of Gale [12] fails to have a solution because it does not have an integrable upper bound. For this problem an extended solution concept has been proposed by Artstein [2]. Once again, economic arguments suggest that it is more natural to modify the statement of the problem so that it will have such a bound. In the third kind of problem considered here, the obvious "solution" is a path for the state variable $x(t)$ which is discontinuous and therefore lies outside the space of functions allowed as possible solutions. These "jumps" arise because of the absence of a growth condition. The previous two kinds of problems can be modified to recover solutions in the usual sense, but the absence of a growth condition is fundamental in many economic problems. In particular, this condition cannot be present in the equilibrium problem of a price-taking agent. In this case a natural and persuasive extended solution concept is available from the work of Rockafellar [20]. Paths with derivatives in the usual sense can be extended to paths which have Radon-Nykodym derivatives. Formally, this is analogous to the generalization from densities to general measures used in the commodity differentiation literature. (See Jones [13] and the references cited therein.)

The organization of the paper is as follows. Section 2 gives a precise statement of the general problem in terms of a functional defined over a function space. The existence theorem is stated and its relation to other work is discussed. Section 3 considers the three problems described above, illustrating in each case the role played by the assumption which is violated and showing how solutions can be found either by looking in a larger function space or by modifying the statement of the problem. Section 4 gives the proof of the theorem. It amounts to an application of three basic results from analysis--Fatou's lemma, Mazur's lemma and a compactness result of de la Vallee-Poussin--which require respectively the integrable upper bound, the convexity assumption and the growth condition.

2. EXISTENCE THEOREM

2.1 *FORMALISM AND STATEMENT*

The classical problem of Lagrange, from the calculus of variations, is to calculate an extreme point of an integral functional which depends on a (possibly vector valued) state variable, $x(t)$, and its first derivative, $\dot{x}(t)$, both defined on some interval $I \subset \mathbb{R}$. To accommodate the extension here to higher order derivatives, we choose to write this functional in the form $\int_0^\infty F(x(t),\dot{x}(t),\ddot{x}(t),t)\,\delta(t)\,dt$. The choice of the unbounded interval of integration is harmless since $\delta(t)$ can be chosen to be the indicator function for some interval. For many economic problems, $\delta(t)$ will represent a discount function. When it arises naturally in a problem, it is useful to separate it from any other possible time dependence in the integrand F.

To state the maximization problem precisely, let F be an extended real valued function $F\colon \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^M \times [0,\infty) \to \mathbb{R} \cup \{-\infty\}$ so all constraints can be stated implicitly. Let LI^M denote the space of locally (Lebesgue) integrable functions $y\colon [0,\infty) \to \mathbb{R}^M$. Given any value $d \in \mathbb{R}^M$, we can define an integration mapping which sends y to $I_d y$ where $I_d y(t) = \int_0^t y(s)\,ds + d$. Let b and a be the prespecified initial values for $x(0)$ and $\dot{x}(0)$ respectively. Then under suitable measurability and boundedness assumptions on F and δ, we can define a functional $W\colon LI^M \to \mathbb{R} \cup \{-\infty\}$ by $W(y) = \int_0^\infty F(I_b I_a y(t), I_a y(t), y(t), t)\,\delta(t)\,dt$. One could always stack $(x,\dot{x})$ and $(\dot{x},\ddot{x})$ and write such a problem using only one derivative of a higher dimensional state variable, but the statement of the theorem in this extended form is weaker in an economically interesting way. We can always assume that the dependence of F on its first argument is trivial so this formulation will contain the classical problem as a special case.

For simplicity, the main theorem is stated and proved only for the case where $\delta(t)$ is a strictly positive discount function over $[0,\infty)$. By a trivial modification of lemma 3 in Section 4, it can be extended to the case where δ is the indicator of some finite interval. To state the theorem, we need one additional definition. An extended real valued function G defined on a topological space D is *upper-semi-continuous* (u.s.c.) if the subgraph of G, sub $G = \{(z,b) \in D \times \mathbb{R}: b \leq G(z)\}$, is closed in $D \times \mathbb{R}$.

Theorem: *Assume that* $\delta: [0,\infty) \to \mathbb{R}$ *is non-increasing, strictly positive and (Lebesgue) integrable. Assume that* $F: \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^M \times [0,\infty) \to \mathbb{R} \cup \{-\infty\}$ *is u.s.c. and satisfies the following conditions:*

(i) *For all* $u,v \in \mathbb{R}^M \times \mathbb{R}^M$ *and almost all* t, $F(u,v,\cdot,t): \mathbb{R}^M \to \mathbb{R} \cup \{-\infty\}$ *is concave.*

(ii) *There exists a constant* $p > 1$ *and a measurable function* $m(t)$, *with* $m(t)\delta(t)$ *integrable, such that for all* $(u,v,z) \in \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^M$ *and almost all* t,

$$F(u,v,z,t) \leq m(t) - ||z||^p.$$

Then $W: LI \to \mathbb{R} \cup \{-\infty\}$ *is well defined and there exists an element* $y^* \in LI$ *such that* $\infty > W(y^*) \geq W(y)$ *for all* $y \in LI$.

The proof of the theorem is deferred until Section 4.

2.2 *RELATION TO EXISTING RESULTS*

The proof of the theorem follows an approach which dates back to the work of McShane [16]. As always, it relies on compactness and some form of continuity. In this case it is convenient to use the weak topology on an L^1 space because it allows a simple characterization of compact sets. In economic problems, this topology is often avoided because preferences stated in the variational form used here are not continuous. (See for example Bewley [6].) This objection is irrelevant here because the functional is upper-semi-continuous in the weak L^1 topology. Combined with compactness, upper-semi-continuity is all that is necessary for the existence of a maximum. The functional W is defined over all of L1, but it will be shown that the upper-contour set, $\{y \in L1: W(y) \geq b\}$ for any $b \in \mathbb{R}$, is contained in the specified subset of L^1 functions.

The treatment here follows most closely the existence result in Ekeland and Temam [10]. Relative to that result or the results of Bates [3] and Baum [4], which explicitly consider infinite horizon control problems, the innovation here is to observe that the growth condition and concavity assumption are required only on the "highest order derivative" in any problem stated in terms of multiple derivatives. As is clear from the proof in Section 4, the theorem can be immediately extended to problems with n'th order derivatives of the state variable (i.e. n'th order integrals of function y(t)) for any integer n. The results of Chichilnisky [7] [8], while technically very different in approach, contain an observation similar to the one here; in non-concave maximization problems which do not have solutions, any perturbation of the problem which acts to limit the rate of change of the highest order derivative in the problem will produce a problem which does have a solution. The existence result proved by Magill [17] follows a similar approach to the one followed here for the special case where F depends only on x and $\dot{x}$ and is jointly concave in these arguments. The assumption that F depends upper-semi-continuously on t is stronger than is strictly necessary; essentially all that is required is measurable dependence. Nothing in the proof is changed if F is assumed to be a normal integrand as defined in [19]. For a description of how a control problem can be stated formally as a variational problem, see [21].

3. PROBLEMS WITHOUT SOLUTIONS

This section considers three problems known not to have solutions when posed as variational problems in the form stated above. Each illustrates the role of one of the key assumptions in the theorem. In each case we compare extended solution concepts proposed for these problems with the alternative of slightly modifying the statement of the problem so it has a conventional solution.

3.1 *CHATTERING*

Consider an optimizing growth model with a bounded, strictly increasing instantaneous utility function U: $\mathcal{R} \rightarrow \mathcal{R} \cup \{-\infty\}$ that is discounted at a constant exponential rate δ. Assume that output as a function of a single capital good is described by a conventional production function f: $\mathcal{R} \rightarrow \mathcal{R} \cup \{-\infty\}$. If consumption goods and capital goods can be exchanged one for one and if capital depreciates at a rate

ρ, the integrand for this economy can be written as $F(k,\dot{k},t) = U(f(k)-\rho k-\dot{k})$. Assume that there is an optimal steady state value $\bar{k}$ for the capital stock and that the initial stock for this economy equals this value. Then the solution to the problem $\max \int_0^\infty F(k,\dot{k},t)e^{-\delta t}dt$ has $\dot{k}$ identically zero and constant gross investment $\rho\bar{k}$.

Now suppose that the problem is modified so that there is a fixed minimum level $m > \rho\bar{k}$ for positive gross investment. Forgone consumption of less than this amount causes no increase in $\dot{k}$, so gross investment equal to $\rho\bar{k}$ and $\dot{k}$ identically zero cannot be a solution. A sequence of functions $\{\dot{k}^n\}$ that converges to the supremum of this modified problem can be constructed by forcing gross investment to alternate back and forth between 0 and m with resting times at the two points such that on average $\dot{k}$ is close to zero and k is close to $\bar{k}$. As the period of the alternation decreases with n, the path for $\dot{k}^n$ "chatters" increasingly rapidly between its two values. The limit of the objective functional along this sequence equals the supremum for the problem, but the sequence does not converge to any well-defined function $\dot{k}(t)$ on $[0,\infty)$. The problem as posed has no solution. Because of the minimum level of gross investment, the integrand for this problem is not concave in $\dot{k}$, but all the other assumptions of the theorem can be verified.

The behavioral prediction which emerges from this model is nonsensical. Every path for $\dot{k}(t)$ in L1 is dominated by some other path, so no path can be chosen. As suggested in the introduction, one obvious way to remedy this situation is to expand the class of mathematical objects of choice. For non-convex problems there is a well developed theory of generalized curves based on the work of L. C. Young [26]. In this extension, $\dot{k}(t)$ is allowed to be a measure over the real line at each point in time. This includes the classical case where $\dot{k}(t)$ is a measure with a single atom for all t. For the problem here, the generalized solution is to choose a measure $\dot{k}(t)$ which assigns positive probability to each of the two values for $\dot{k}(t)$ corresponding to the values of 0 and m for gross investment. For the correct relative probabilities, k(t) will be constant at the initial value $\bar{k}$. This will be the solution to the maximization problem stated in terms of an extended version of the objective functional which integrates the expectation of the integrand with respect to the choosen measure. Because of the strict concavity of U, the value of this objective will differ from the value of the solution to the original problem. Observationally, the solutions to the two problems are equivalent except for the restriction that in the second case, $\dot{k}(t)$ is not directly observable. Only the average value of $\dot{k}(t)$ over arbitrarily small time intervals is observed.

This extension does provide a formal solution to the problem as posed, but it is artificial and methodologically unattractive. According to this model, there is an important non-convexity in investment, but it has no observable consequence. Presumably, the reason an economist adds a non-convexity to the statement of a problem is to capture observable features of behavior. An alternative approach to this problem is to exploit the economic intuition that there will be some (possibly small) cost associated with rapidly changing rates of investment. Ultimately, the incremental cost should outweigh the advantage of higher frequency oscillations. Specifically suppose that quadratic costs $-\alpha\ddot{k}^2$ in the rate of change of net investment must also be subtracted from gross output. Then the modified problem stated in terms of $\ddot{k}$ and its first two integrals will have a solution by the theorem.[1] This solution will have the property that $\dot{k}$ will vary continuously between the values implied by gross investment equal to 0 or m. The frequency of this oscilation will decrease as α gets larger and the costs in $\ddot{k}$ become relatively more important. Other than producing an economically sensible solution to this problem, this kind of analysis of existence yields a general insight about such problems. If the costs associated with $\ddot{k}$ are small (i.e. α is small) then little error is introduced by dropping them altogether in problems which are concave in $\dot{k}$ since in these cases $|\ddot{k}|$ will be small. For problems which are not concave in $\dot{k}$, the term involving $\ddot{k}$ is decisive for the qualitative features of the solution. In general, approximations of this form must include derivatives up to and including one in which the integrand is concave.

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[1] The form of the theorem stated above is necessary for this result. A theorem dealing with a state variable $(k,\dot{k})$ and its derivative $(\dot{k},\ddot{k})$ would not apply because the problem will satisfy a growth condition only with respect to $\ddot{k}$, not with respect to the vector $(\dot{k},\ddot{k})$.

3.2 *CAKE EATING*

The bound in condition (ii) from the theorem contains two parts, the integrable upper bound $m(t)$ (integrable here means with respect to $\delta(t)dt$) and the growth condition involving the term $||y||^p$. One problem which can arise when there is no integrable upper bound is well known. The improper integral defining the functional W can take on the value $+\infty$. There is nothing per se wrong with allowing an objective function to achieve the other endpoint of the extended real line, but in most models this implies an unrealistic form of satiation. Typically, if you can achieve $+\infty$ starting from $k(0)$, you can also achieve it from $1/2\ k(0)$. Various catching up criteria have been proposed for discriminating amongst such paths, most of which are equivalent to some renormalization of the instantaneous utility function U. But the absence of an integrable upper bound implies a possible continuity problem that is far more troublesome than satiation. Consider the undiscounted problem max $\int_0^\infty U\ (f(k(t))-\dot{k}(t))dt$ where $U(c) = \ln c$ and where f takes the form $f(k) = k$ for $k \in [0, 1]$, $f(k) = 1$ for $k>1$. That is, f is a productive technology for values of k up to 1; beyond that it is a pure storage technology. Since U and f were chosen so that $U(f(1)) = 0$, we can show by elementary means that this problem has an optimum, hence an overtaking optimum, for any $k(0) \leq 1$. With any translation of the function U, the problem will still have an overtaking optimum. But for any value of $k(0) > 1$, the problem cannot have an optimum of any form. In this case the difficulty concerns how best to allocate the consumption of the amount $k(0) - 1$ over the infinite time horizon. This amount contributes nothing to production and can only be stored. How best to consume this amount is the cake eating problem described by Gale [12]. Because of diminishing marginal utility, a sequence of paths for consumption that consume the excess at successively slower rates over longer intervals will converge to the supremum of the problem. But this sequence of consumption paths converges to one where none of the excess is consumed. In the natural topology on consumption paths, the objective functional fails to be upper-semi-continuous.

Artstein [2] has described an extended solution concept for this problem designed to remove this difficulty, but it has the unattractive feature that a given consumption path can be assigned different utility values. As in the last example, the alternative to extending the functional and the set of possible solutions is to restrict the statement of the problem. In the usual growth theory context, restricting attention to problems satisfying an integral upper bound requires joint restrictions on the possible sets of preferences and technologies. For example, with constant relative risk aversion utility, $U(c) = c^{\gamma}/\gamma$, $\gamma \leq 1$, linear technology $f(k) = rk$, and exponential discounting, $e^{-\delta t}$, the bound is possible only if $\delta > \gamma r$. In principle, one should be able to specify preferences and technologies independently. That this is not possible in general for fixed discount rate preferences suggests that these are only an analytically convenient approximation to a more complicated set of preferences like those studied by Lucas and Stokey [15] or Epstein and Hynes [11] which have endogenously determined discount rates. The existence theorem above then gives conditions on the possible values of δ so that a constant discount rate approximation to the true preferences will have the correct continuity properties given the specified technology. Inferring that some approximate specifications of preferences are inadmissible given certain technologies seems preferable to introducing preferences which are not uniquely defined in terms of consumption.

3.3 *JUMPS*

A third class of problems which fail to have solutions in the sense defined above arises when the growth condition involving $||y||$ raised to a power p fails to hold. What this term in the bound does is ensure that high values of $||y||$ are sufficiently costly that $||y||$ is effectively bounded. Any problem with restrictions like $||y|| \le M$ will automatically satisfy this condition by setting the integrand equal to $-\infty$ for $||y|| > M$. If there is no cost or constraint which limits $||y||$, then it may be possible to construct a sequence of paths converging to the supremum of a problem such that $||y||$ goes to ∞ and $I_a y(t)$ converges to a path which has a discrete jump. As an example, consider the profit maximizing problem of a competitive firm facing a constant interest rate r, max $\int_0^\infty (f(k)-\dot{k})e^{-rt}dt$. Let k(0) be an arbitrary initial value. It is easy to see that the limit of any maximizing sequence for this problem has a path for k(t) which jumps instantaneously at t=0 to a value $\bar{k}$ such that $Df(\bar{k}) = r$. In general these jumps can occur at any time t. One could try to re-establish existence in the sense of the theorem by adding a cost term $-\ddot{k}^2$, but this arbitrarily prohibits the perfectly reasonable economic concept of a discrete sale or purchase of a finite amount of capital between two firms at a point in time. This kind of restriction is even more artificial in the context of a financial investment problem for an individual consumer. Here portfolio shares are the state variables. There is no economic reason to force trades to take place at finite rates and prohibit discrete portfolio shifts.

There is an important economic reason why this kind of problem arises so naturally in equilibrium problems where agents maximize subject to prices which are taken as given. In problems with enough convexity, economists and mathematicians have long been aware of the duality between state variables and co-state variables and the possible interpretation of co-state variables as equilibrium prices. It is also widely recognized that in problems with state constraints the price or co-state variables, which are generally absolutely continuous, can jump when a state variable hits a constraint. (See [1] for a recent discussion.) The problem of a competitive agent is the dual to such a problem. By the definition of competitive behavior, there are no restrictions on the quantities any agent can trade but there are implicit constraints on the shadow prices. Quantities can jump when the price constraints are binding. More precisely, the solution for the path of quantities in the agent's primal problem is also the path of co-state variables for his dual problem. Shadow prices will be the state variables in the derived dual problem, and it will contain state variable (i.e. price) constraints.

In contrast to the last two examples, this is a case with a natural and easily interpreted extension of the basic solution concept. It is developed in detail by Rockafellar [20] for finite horizon problems with integrands which are concave (or convex) in all arguments except time, t. Romer [22] has a partial extension to infinite horizon problems. Now, instead of choosing paths y(t) which are locally Lebesgue integrable functions, one selects regular Borel measures dz(t). Let $\dot{z}(t)dt$ and $\frac{dz}{d\theta}(t)dt$ represent the absolutely continuous and singular parts of dz respectively, where $d\theta(t)$ is some non-negative singular measure and $dz/d\theta$ is the Radon-Nykodym derivative of dz with respect to $d\theta$. Assuming for simplicity that F depends only on a state variable and its first derivative, we can define

$$J(dz) = \int_0^\infty F(z(t), \dot{z}(t), t)\,\delta(t)dt + \int_0^\infty r_F\left(\frac{dz}{d\theta}(t), t\right)\delta(t)d\theta(t).$$

Here r_F is the recession function of $F(z, \cdot, t)$ and z(t) is defined by $z(t) = \int_{[0,t)} dz(s) + z^0$. See [20] for details. Extending the statement from measures with densities y(t) to Borel measures dz(t) is analogous to the extension in probability theory from probability distributions with densities to general distributions. In problems like that of a competitive agent where the integrand is concave in all arguments other than t, standard sufficient conditions can be extended to include problems of this form with general measures ([20], [22]). For some forms of this problem with concave F, existence results have also been established ([14], [25]).

4. PROOF OF THE THEOREM

The proof of the theorem is simplified by establishing two preliminary lemmas. Given a function which is concave in some arguments but not in others (e.g. a saddle function), the first lemma gives conditions under which it is possible to separate the graph of the function from a point lying above it using a "strip" of a hyperplane.

LEMMA 1: *Let* $F\colon \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R} \cup \{-\infty\}$ *be u.s.c. Assume that for all* $z \in \mathbb{R}^N$, $F(z,\cdot)$ *is concave. Assume also that F is bounded by the estimate* $F(z,y) \le m - ||y||^p$ *where* $p>1$. *Given* (z^*,y^*), *suppose that* $\gamma > F(z^*,y^*)$. *Then there exists* $r \in \mathbb{R}^M$ *and* $\sigma>0$ *such that* $||z-z^*|| < \sigma$ *implies* $F(z,y) \le (y^*-y)\cdot r + \gamma$.

PROOF: [10, p.241].

Let $\delta(t)$ be as specified in the statement of the theorem so $\delta(t) > 0$ everywhere and $\int_0^\infty \delta(t)dt$ is finite. Let $\mathcal{M}$ be the Lebesgue completion of the Borel sigma algebra on $[0,\infty)$ and let Δ be the measure defined by δ, $\Delta(A) = \int_A \delta(t)dt$. Let $L^M_{1,\delta}$ denote the Lebesgue integrable functions from $([0,\infty),\mathcal{M},\Delta)$ into $\mathbb{R}^M$. Recall that a sequence $\{y_j\}$ in $L^M_{1,\delta}$ converges weakly to y^* if

$$\langle g,y_j\rangle \to \langle g,y^*\rangle$$

for all $g \in (L^M_{1,\delta})^*$, the space of continuous linear functionals on $L^M_{1,\delta}$. Since Δ and Lebesgue measure have the same sets of measure zero, we can represent $(L^M_{1,\delta})^*$ as L^M_∞, the usual set of essentially bounded functions on $[0,\infty)$. Then $g \in L^M_\infty$ implies $\langle g,y\rangle = \int_0^\infty y\cdot g d\Delta$.

LEMMA 2: *If* $\{y_j\}$ *converges weakly to* y^* *in* $L^M_{1,\delta}$, *then* $I_a y_j$ *and* $I_b I_a y_j$ *converge pointwise to* $I_a y^*$ *and* $I_b I_a y^*$ *for all* $a,b \in \mathbb{R}^M$.

PROOF: Pick any t in $[0,\infty)$ and assume for notational simplicity that $M=1$. Let $g\colon [0,\infty) \to \mathbb{R}$ be defined by $g(s) = 1/\delta(s)$ for $s \in [0,t]$, $g(s) = 0$ for $s>t$. Since $\delta(t)$ is non-increasing and strictly positive, $g(s) \le 1/\delta(t) < \infty$ so $g \in L_\infty$. Then by the weak convergence of y_j,

$$I_a y_j(t) - a = \int_0^t y_j(s)ds = \int_0^\infty y_j(s)g(s)d\Delta(s)$$
$$\to \int_0^\infty y^*(s)g(s)d\Delta(s) = I_a y^*(t) - a.$$

Thus $I_a y_j$ converges pointwise to $I_a y^*$; in particular $I_a y_j(s)$ converges to $I_a y^*(s)$ for all $s \in [0,t]$. Then $I_b I_a y_j(t)$ will converge to $I_b I_a y^*(t)$ by the Dominated Convergence Theorem provided $I_a y_j(r)$ is bounded for all j and all $r \in [0,t]$. By the Uniform Boundedness Principle [9, II.3.27, p.68], we know that a weakly convergent sequence is bounded in the norm. Thus for all $r \in [0,t]$, for all j and for some $B \in \mathcal{R}$, $B \geq \int_0^\infty |y_j(s)| d\Delta(s) \geq \delta(t) |\int_0^r y_j(s) ds|$ so $|I_a y_j(r)| \leq B/\delta(t) + |a|$.

Using this argument componentwise, the result also holds for M>1.

Q.E.D.

This result can easily be extended to give pointwise convergence of integrals of any order. The next lemma is one way to use the fact that in normed linear spaces the strong (i.e. norm) closure and weak closure of a convex set coincide.

MAZUR'S LEMMA: *Let V be a normed linear space and $\{u_j\}$ a sequence converging weakly to u^*. Then there exists a sequence of convex combinations of elements in the tail of the original sequence,*

$$v_j = \sum_{k=j}^{N(j)} \lambda_{jk} u_k \text{ where } \sum_{k=j}^{N(j)} \lambda_{jk} = 1$$

for all j, such that v_j converges to u^* in the norm topology.

PROOF: [10, p.6].

With these preliminaries we can prove the theorem stated in Section 2.

PROOF OF THE THEOREM: Because F is u.s.c., it is Borel measurable, so $F(I_b I_a y(t), I_a y(t), y(t), t)$ is Lebesgue-measurable [24, p.70]. Then $W: LI^M \to \mathcal{R} \cup \{-\infty\}$ is well defined because

$$\int_0^\infty F\delta(t)dt \leq \int_0^\infty m(t)\delta(t)dt < \infty.$$

Let $s = \sup_{y \in L_1^M} W(y)$. We can assume $s > -\infty$; otherwise the theorem is trivial. Let $\{y_j\}$ be a maximizing sequence for W so $\lim_{j\to\infty} W(y_j) = s$. Without loss of generality we can assume that there exists some $B \in \mathcal{R}$ such that $B \le W(y_j)$ for all j. Using the bound on F, this implies $\int_0^\infty ||y_j||^p \delta(t)dt \le \int_0^\infty m(t)\delta(t)dt - B$, so $\{y_j\}$ is uniformly bounded in $L^M_{p,\delta}$. By a result of de la Vallee-Poussin for finite measure spaces [10, p.239], this implies that $\{y_j\}$ is weakly relatively compact in $L^M_{1,\delta}$. Then reindexing if necessary, $\{y_j\}$ converges weakly to some $y^* \in L^M_{1,\delta}$.

It remains to show that W is weakly u.s.c. in $L^M_{1,\delta}$. For notational convenience, let $z_j(t) = (I_b I_a y_j(t), I_a y_j(t)) \in \mathcal{R}^{2M}$. By Lemma 2, $z_j(t)$ converges to $z^*(t) = (I_b I_a y^*(t), I_a y^*(t))$ for all t in $[0,\infty)$. By Mazur's lemma there exists an array $\{\lambda_{jk}\}$ such that

$$v_j(t) = \sum_{k=j}^{N(j)} \lambda_{jk} y_k(t)$$

converges almost everywhere to $y^*(t)$. Define $\ell(t)$ by

$$\ell(t) = \limsup_{j\to\infty} \sum_{k=j}^{N(j)} \lambda_{jk} F(z_k(t), y_k(t), t),$$

and fix a t such that $v_j(t) \to y^*(t)$. We want to show that $\ell(t) \le F(z^*(t), y^*(t), t)$. Suppose not. Let γ be such that $F(z^*(t), y^*(t), t) < \gamma < \ell(t)$. By Lemma 2 there exists $\sigma > 0$ and $r \in \mathcal{R}^M$ such that $||z_j(t) - z^*(t)|| < \sigma$ implies $F(z_j(t), y_j(t), t) \le (y^*(t) - y_j(t)) \cdot r + \gamma$. Since $z_j(t)$ converges to $z^*(t)$,

$$\limsup_{j\to\infty} \sum_{k=j}^{N(j)} \lambda_{jk} F(z_k(t) y_k(t), t)$$

$$\le \limsup_{j\to\infty} \sum_{k=j}^{N(j)} \lambda_{jk} (y^*(t) - y_k(t)) \cdot r + \gamma = \gamma.$$

Then $\ell(t) \leq \gamma$, but by assumption $\gamma < \ell(t)$. By contradiction, we have $\ell(t) \leq F(z^*(t), y^*(t), t)$. Since this expression holds for almost all t, integrate with respect to $\delta(t)dt$ to get

$$\int_0^\infty \limsup_{j\to\infty} \sum_{k=j}^{N(j)} \lambda_{jk} F(z_k(t), y_k(t), t)\,\delta(t)\,dt \leq W(y^*).$$

Because $F\delta$ has the integrable upper bound $m\delta$, we can use Fatou's lemma to conclude that

$$\limsup_{j\to\infty} \sum_{k=k}^{N(j)} \lambda_{jk} W(y_k) \leq W(y^*).$$

But since $W(y_j)$ converges to s, this implies $s \leq W(y^*)$.

Q.E.D.

It is useful to recapitulate where the three basic assumptions corresponding to the three problems in Section 3 enter the proof. The concavity of the integrand in the highest derivative is necessary to apply Mazur's Lemma. The integrable upper bound is required for Fatou's lemma. The growth condition is essentially equivalent to the characterization of weak relative compactness in L_1 due to de la Vallee Poussin. It is also used in Lemma 1 to get the local form of a separating hyperplane argument necessary for the convex combinations taken in the proof, but weaker conditions would have been sufficient for this purpose. In brief, Mazur stops chattering, Fatou solves cake eating and de la Vallee Poussin rules out jumps.

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