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REPEATED GAMES, FINITE AUTOMATA, AND COMPLEXITY*

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Abstract

We study the structure of Nash equilibria in 2-player repeated games played with finite automata, when complexity considerations enter into the players' decision calculus. We employ a more general measure of complexity than that used by previous authors, viz., the complexity of a machine depends not only on the number of states but also on the complexity of the transition mechanism in the machine. We show that, in this set-up, the resulting Nash equilibria are trivial: the machines recommend actions each period which are stage game best responses to one another.

*This is an updated version of an earlier paper with the same title. In particular, we consider more general measures of complexity. We are grateful to Professor Ariel Rubinstein for comments and suggestions. We are also grateful to seminar participants at the Universities of Rochester and Arizona, particularly to John Boyd and Mahmoud El-Gamal.
Repeated Games, Finite Automata, and Complexity

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1. Introduction

In this paper we study the structure of Nash equilibria in 2–player repeated games played with finite automata. Such an approach was first suggested by Aumann (1981) for the study of decision making with bounded rationality; recent applications include Neyman (1985) in the context of a finitely repeated prisoners' dilemma game using machines of fixed sizes (see also Zemel (1986)); Kalai and Stanford (1988), who show that every strategy of an infinitely repeated game can be uniquely described by an automaton of minimal size; and Ben–Porath and Peleg (1987), who show that any payoff of an infinitely repeated game can be approximated using automata with a finite set of states (for a more complete list of references, the reader should consult the excellent survey in Kalai (1987)). In the current paper we are concerned with the effects of complexity and implementation costs on the set of equilibria in an infinitely repeated game, issues that are analyzed in Rubinstein (1986) and Abreu and Rubinstein (1988). In particular, we examine the set of equilibria when a more general measure of complexity is employed by the players.

We build on the model of Abreu and Rubinstein (1988) where the strategic interaction between the two players occurs over the simultaneous selection of finite automata, or (Moore) machines, with which to implement their repeated game strategies. Such machines consist of (i) a finite set of states, one of which is identified as the initial state, (ii) an output function determining the action taken at the $t^{th}$ stage of the game as a function of the $t^{th}$ state, and (iii) a transition function describing the $(t+1)^{st}$ state as a function of the $t^{th}$ state as well as the action by the other player at the $t^{th}$ stage of the game. Any pair of machines for the two players...
then determines a sequence of action pairs and ultimately a repeated game payoff.

In Abreu and Rubinstein (1988) preferences for player i defined over machine pairs \((M_i, M_j)\) are increasing in the repeated game payoffs induced by the machines, and decreasing in the complexity of \(M_i\). Complexity is measured by the number of states a machine possesses; thus the lower the number of states, the less costly is the implementation of the strategy. Within this framework, Abreu and Rubinstein (1988) show that the set of Nash equilibrium payoffs in the repeated game is dramatically reduced from the standard Folk Theorem result (cf. Fudenberg and Maskin (1985)). In particular, since the player's preferences are allowed to be lexicographic in repeated game payoffs and complexity, this result demonstrates a non-robustness of the Folk Theorem to "infinitesimal" considerations of decision costs.

Abreu and Rubinstein (1988) note that their measure of implementation cost captures the following complexity issues: (i) if monitoring an opponent's behavior is costly, players will attempt to economize on states held to keep track of the other's actions, and (ii) if "punishment states", that is, states which lead (through the output function) to actions designed to punish deviant behavior by the opponent, are costly to maintain, players will eliminate such states if they are never used. In particular, this shows that "empty" threats to take certain actions cannot exist in equilibrium. However they also mention that their measure is quite special, and ignores other possible dimensions with which to measure a machine's complexity such as the complexity of the transition mechanism. (This point is also raised in Rubinstein (1986).) Indeed, we believe that their measure does not allow for a sufficiently fine distinction between machines that appear quite different from the perspective of implementation costs.

A simple example will serve to illustrate this point. Consider the repeated prisoner's dilemma, where in each period a player has two options: Cooperate (C) or Defect (D). One possible strategy for player 1 is to employ a machine that, regardless
of 2's choice of action, plays action C initially and then switches to D forever. Diagrammatically, this machine can be represented as

\[ \text{start} \rightarrow q_C \overset{C,D}{\rightarrow} q_D \overset{C,D}{\leftarrow} \]

(Here, as in the next two figures, player 1's machine recommends C (resp. D) when the state is \( q_C \) (resp. \( q_D \)). The letters on the arrows represent 2's choice of action, and the arrows themselves represent the transitions.)

A second possible machine for player 1 that appears slightly more complicated is one which adopts the "grim" trigger strategy (play C until 2 plays D, then play D forever). This also involves two states:

\[ \text{start} \rightarrow q_C \overset{C,D}{\rightarrow} q_D \overset{C,D}{\leftarrow} \]

Finally, a tit-for-tat strategy can also be implemented using only two states:

\[ \text{start} \rightarrow q_C \overset{C,D}{\rightarrow} q_D \overset{C,D}{\leftarrow} \]

All three of these machines employ two states, and hence are of equal complexity as defined by Abreu and Rubinstein (1988). Yet we would argue that the first machine is inherently less "complex" than the second, and the second less "complex" than the third, where clearly this augmented definition of complexity is a function of the number of transitions or distinct routes between states. The rationale for this measure is related to those noted by Abreu and Rubinstein (1988): if monitoring the opponents' behavior is costly, then at each state players will attempt to economize on the number of events which require distinct transitions. Thus the grim strategy is less complex under this measure than tit-for-tat, since at state \( q_D \) player 1 no longer needs to
monitor player 2's behavior; similarly the first machine is less complex than the second. In general, if two machines differ only in that one has more transitions emanating from each state, then such a machine requires a higher degree of precision in the monitoring of the opponent's behavior. Second, it may simply be that the transitions, like states, are costly to maintain; the lower the number of contingencies the machine has to identify to proceed, the less costly is the implementation of the strategy.

In what follows, the preference structure of Abreu–Rubinstein (1988) is extended to capture this notion of transitional complexity as one aspect of the complexity of a machine. In section 2, we present 3 different complexity measures, of varying degrees of restrictiveness. Given a machine $M_i$ for player $i$, a machine $M_j$ is at least as good as another machine $M_j'$ if (i) $M_j$ yields at least as high a repeated game payoff as $M_j'$, and (ii) $M_j$ is at most as complex as $M_j'$. Such preferences as we consider include the case where transitional complexity enters lexicographically and last, thus giving an "infinitesimal" change over Abreu–Rubinstein (1988), but also includes numerous other cases. In particular, we allow for the case where players use different complexity measures from each other, reflecting possible differential advantages.

The main result of the paper (Theorem 2 below) is simply stated: under any of these measures, any Nash equilibrium of the machine game is such that at each stage of the repeated game the players adopt one-shot Nash equilibrium actions. In payoff space this implies that, if preferences are lexicographic with repeated game payoffs evaluated according to the limit of means, the only equilibrium payoffs to the players are those that are (rational) convex combinations of stage-game Nash equilibrium payoffs. So, for example, when finite automata play the repeated prisoners' dilemma, the only Nash equilibrium payoff is that resulting when both players play the suboptimal strategy "defect" forever. Interestingly, for some other games—such as the "Battle of the Sexes," the set of equilibrium payoffs resulting from our formulation of preferences, is the same as that in Abreu–Rubinstein (1988). A comparison of our
results with theirs, together with an intuitive explanation of our result is provided after the proof of our main result. In the course of proving our main theorem, we also prove that in the limit–average payoffs case, the machines in a N.E. must start cycling between states right from the beginning. This is in contrast to the result in Abreu–Rubinstein (1988), where the machines eventually settle down into a cycle. We also show (Theorem 1) that all the qualitative results of Abreu–Ruvinstein (1988) regarding the lengths of cycles, etc., continue to hold in our framework.

The paper is organized as follows: Section 2 gives the description of the model; Section 3 contains the statement and proof of the main theorem; Section 4 includes a discussion of possible extensions of the model; and the Appendix shows that the relevant results in Abreu and Rubinstein (1988) continue to hold in the current model.

2. The model

2.1 Notation and Definitions

Let $G = (S_1, S_2, u_1, u_2)$ denote a two–person game in normal form, where $S_i$ is a finite set of actions for player $i$ and $u_i: S_1 \times S_2 \to \mathbb{R}$ is $i$'s payoff function. Let $N(G)$ denote the set of Nash equilibria of the game $G$; i.e. $(s_1^*, s_2^*) \in N(G)$ iff $\forall (s_1, s_2) \in S_1 \times S_2$, $u_1(s_1^*, s_2^*) \geq u_1(s_1, s_2^*)$ and $u_2(s_1^*, s_2^*) \geq u_2(s_1, s_2^*)$.

The supergame of $G$, denoted $G^\infty$, consists of an infinite sequence of repetitions of $G$ at times $t = 1, 2, 3, \ldots$. At period $t$ the players simultaneously select an element of $S_i$, denoted $s_i^t$, which become common knowledge.

The players determine a sequence of actions in the supergame $G^\infty$ by simultaneously selecting finite automata, or (Moore) machines [see Hopcroft and Ullman (1979)]. A machine for player $i$, denoted $M_i$, is a four–tuple $(Q_i, q_i^1, \lambda_i, \mu_i)$, where $Q_i$ is a finite set of states, $q_i^1 \in Q_i$ is the initial state, $\lambda_i: Q_i \to S_i$ is the output function, and $\mu_i : Q_i \times S_j \to Q_i$ is the transition function. A pair of machines $(M_1, M_2)$ induces a sequence of state pairs $(q_i^t)$ and action pairs $(s_i^t)$ in the obvious manner:
\[ q^1 = (q^1_1, q^1_2), \]
\[ s^t = (\lambda_1(q^t_1), \lambda_2(q^t_2)), \]
\[ q^{t+1} = (\mu_1(q^t s^t_1), \mu_2(q^t s^t_1)). \]

Since the machines are finite both \( q^1 \) and \( q^t \) eventually cycle. Let \( T_i \) denote the length of the cycle of \( q_i^t \), and \( T \) that of \( q^t \); further let the first cycle in \( q^t \) begin and end at \( t_1 \) and \( t_2 \), respectively. We refer to the time period \( t < t_1 \) as the pre-cycle phase. By the stationarity of the output and transition functions the continuation of \( q^t \) after \( t_2+1 \) is just like after \( t_1 \).

\( \pi_1(M_1, M_2) \) denotes the repeated game payoff induced by the machines \( (M_1, M_2) \), where this is evaluated either according to the limit of means or with discounting:

(i) limit of means
\[
\pi_1(M_1, M_2) = \lim_{T' \to \infty} \frac{1}{T'} \sum_{t=1}^{T'} u_i(s^t) = \frac{1}{t_2 - t_1 + 1} \sum_{t=t_1}^{t_2} u_i(s^t);
\]

(ii) discounting
\[
\pi_1(M_1, M_2) = \frac{1 - \delta}{\delta} \sum_{t=1}^{\infty} \delta^t u_i(s^t).
\]

2.2 Measures of Complexity

As mentioned in the Introduction, we consider 3 different measures of complexity designed to capture the notion of monitoring costs. The following notation will be used: \( |Q_i| \) will denote the number of elements in \( Q_i \). \( R(q_i) \) will denote the number of distinct transitions from \( q_i \in Q_i \). That is, \( R(q_i) \) is the number of equivalence classes at \( q_i \), where \( s_j \) and \( s_{j'} \) are defined to be equivalent at \( q_i \) if \( \mu_i(q_i, s_j) = \mu_i(q_i, s_{j'}). \) Finally, \( R(M_i) \) will denote the number of distinct transitions in
$M_i$, i.e., $R(M_i) = \sum_{q_i \in Q_i} R(q_i)$. Where $M_i$ is understood we shall use the more compact notation $R_i$ for $R(M_i)$.

**Measure 1** The simplest measure of state–transitional complexity that we consider is obtained by counting the number of state–action pairs $(q_i, s_j)$ that require distinct transitions. This is easily seen to be the same as $R(M_i)$. Thus, under this measure of complexity, $M_i$ is more complex than $M_i'$ if and only if $R(M_i) > R(M_i')$. Note that: (i) this measure completely orders all machines with respect to their complexity, and (ii) under this measure the examples of the previous section are increasing in complexity, although they all employ only 2 states.

**Measure 2** Our second complexity measure extends the first. Rather than just count the number of transitions in a machine, we define a 2–dimensional measure of complexity $(|Q_i|, R(M_i))$ for a machine $M_i$. Under this measure, $M_i$ is more complex than $M_i'$ if $|Q_i| \geq |Q_i'|$ and $R(M_i) > R(M_i')$ with at least one strict inequality. As defined, this measure does not completely order machines according to their degree of complexity. We note, however, that any extension making it a complete ordering would not change our results. In particular, the measure could be completed lexicographically in either way: $M_i$ is more complex than $M_i'$ iff $|Q_i| > |Q_i'|$ or $|Q_i| = |Q_i'|$ and $R(M_i) > R(M_i')$; or $M_i$ is more complex than $M_i'$ iff $R(M_i) > R(M_i')$ or $R(M_i) = R(M_i')$ and $|Q_i| > |Q_i'|$. As a special case, therefore, is the one where the number of transitions are appended on to the Abreu–Rubinstein (1988) preferences lexicographically and last.

We also note that measures 1 and 2 could yield different rankings of complexity, i.e., neither need nest the other. Consider for example the machines described below which implement strategies in the repeated–prisoner's dilemma:
Then, the lexicographic completion of measure 2 with states first would pick machine 1 as more complex, while measure 1 would always pick machine 2 as the more complex.

Measure 3¹ By considering a 2–dimensional measure of complexity, measure 2 does capture both potential size and maintenance costs (number of states), and monitoring costs (number of transitions). However, measure 2 also does not permit a sufficiently fine distinction between machines. To illustrate, consider the following 2 machines implementing strategies in the Prisoner's dilemma:

Both use 3 transitions; machine 3 uses 3 states while machine 4 uses only 2. Measure 2 would therefore unambiguously select machine 4 as the less complex. This, however, does not always seem reasonable, especially when states are "cheap" but transitions are

¹John Boyd independently suggested this as a measure of complexity. Mahmoud El–Gamal suggested to us the possibility of incorporating differential transition costs such that not all transitions in the machine have the same cost. For example, if more transitions emanate from one state than another, transitions from the first state would have a greater weight than those from the second. We believe this would have no impact on our results.
not: the transitions in machine 3 involve no monitoring at all, while in machine 4 this is not true for those emanating from \( q_c \). The measure appears in this case to pay insufficient attention to monitoring costs.

The dilemma is easily resolved when we note that every state must have at least one transition emanating from it. Monitoring costs at a state exist only if there is more than one transition leading from it. Thus, we define \( M_i \) to be more complex than \( M_i' \) if \( (R(M_i) - |Q_i|) \geq (R(M_i') - |Q_i'|) \) and \(|Q_i| \geq |Q_i'|\) with at least one strict inequality.

As with measure 2, this measure is not complete, but this is irrelevant for our purpose. The reader is free to complete these measures in any way he or she finds desirable. Some aspects of completing these measures are however interesting. It was observed earlier that measures 1 and 2 could yield different rankings of complexity. The same is true of measures 1 and 3, and of measures 2 and 3. Consider, for example, the case where measure 3 is completed lexicographically with net transitions \( (R(M_i) - |Q_i|) \) first. (We write M1 for machine 1, etc.) Then, between M3 and M4, measure 3 picks M3 as the less complex,\(^2\) while M3 would be the more complex under any completion of measure 2. Similarly, between M1 and M2, measure 3 under lexicographic completion with states first, would pick M2 as the less complex, while measure 1 would pick M1 for this spot. It is interesting to observe, on the other hand, that measure 1 is a special completion of the 2-dimensional measure 3, obtained by summing the coordinates \( (R(M_i) - |Q_i|, |Q_i|) \) of the latter. In general, however, none of the measures need nest any of the others.\(^3\)

\(^2\)Measure 3 is very flexible. If made lexicographic with states first it would rank M3 as more complex than M4.

\(^3\)We are currently engaged in studying a variation on measures 2 and 3, where the only transitions that have a cost associated with them are those between distinct states. Preliminary results show that actions other than stage game Nash equilibrium actions can form part of a machine game equilibrium.
2.3 Equilibrium in the machine game

It is irrelevant for the results of this paper which of the measures of complexity is considered in the sequel. Therefore, in what follows, we will use the terms "more complex," or "at least as complex as." Any of the 3 measures may be used to make the definitions precise. In particular, we allow players to use different measures from each other to reflect possible differential advantages: states may matter more to player 1 than transitions, while for player 2 the situation may be reversed, with indeed, states being irrelevant.

Player 1's preferences over machine pairs $(M_1, M_2)$, denoted by $\succeq_1$ are assumed to satisfy the following: if (a) $M_1'$ is at most as complex as $M_1$ and (b) $\pi_1(M_1', M_2) \geq \pi_1(M_1, M_2)$, then $(M_1', M_2) \succeq_1 (M_1, M_2)$. If (a) holds, and (b) holds strictly, $(M_1', M_2) >_1 (M_1, M_2)$. Similarly, if (b) holds while $M_1'$ is strictly less complex than $M_1$.

Player 2's preferences over pairs $(M_1, M_2)$, denoted by $\succeq_2$, are defined analogously.

**Definition.** A Nash equilibrium of the machine game is a pair $(M_1^*, M_2^*)$ such that for all $M_1, M_2$, the following relations hold:

$(M_1^*, M_2^*) \succeq_1 (M_1, M_2)$,

$(M_1^*, M_2^*) \succeq_2 (M_1, M_2)$.

Let $N(G_m)$ denote the set of Nash equilibria of the machine game.
3. Results

We begin this section by first proving that all the qualitative results of Abreu–Rubinstein (1988) regarding the structure of the machines and equilibrium play continue to hold in our framework. These results will be used below to prove our main result that all Nash equilibria of the machine game are now trivial.

A point to be noted in the sequel is the following: when a machine is physically unaltered except for the dropping of one or more transitions, it becomes unambiguously less complex under all 3 measures. The same thing happens when a state is dropped, for now at least one transition – leading from the dropped state – also gets eliminated.

**Theorem 1:** For all \((M_1, M_2) \in N(G_m)\),

(i) \(T_1 = T_2 = T\);

(ii) no state is repeated within a cycle by either player;

(iii) no non-cycle state is repeated within the pre-cycle phase by either player;

(iv) the set of pre-cycle states and cycle states are distinct.

**Proof.** Abreu and Rubinstein (1988) prove (i)–(iv) for discounting when only the number of states matters; as is shown in the Appendix their proof goes through in the current model as well. They further note that with the limit of means, while (i)–(iii) continue to hold, there may exist a "Phase II" in which cycle states are employed, yet not in the correct (according to the cycle) order; thus (iv) is not necessarily satisfied. If Phase II is empty then the proof for the discounting case goes through with the limit of means; hence what remains to be shown is that, when transitions matter, Phase II is always empty.

Suppose to the contrary that \(q_i\) were both a pre-cycle and cycle state for player i.
Let $t, t'$ denote respectively the first time $q_i$ occurs and the first time it occurs in the cycle phase, and let $q_j^t = q_j^{t_1}$ and $q_j^* = q_j^{t_2}$; clearly $q_j^t \neq q_j^*$. Let $\tilde{Q}_1, \tilde{Q}_2$ be the set of cycle states for 1 and 2, respectively. It is clear that $\forall \ t \in [t, t') q_j^t \in \tilde{Q}_j$, else player j could save a state and at least 1 transition by replacing $q_j^t$ with $q_j^*$ at $t$ (recall that the cycle has begun at $(q_i^t, q_j^t)$). It follows that $\forall \ t \in [t, t') q_i^t \in Q_i$ as well.

We claim that for at least one player, say $j$, $\exists \ q_j \in \tilde{Q}_j$ and $s_i, s_j^* \in S_j$ such that

\[(*) \quad \mu_j(q_j, s_i) \neq \mu_j(q_j, s_j^*).
\]

If $(*)$ were not true, then there would exist a unique path through the sets $\tilde{Q}_i$ and $\tilde{Q}_j$, implying that the cycle would have begun at $t$, contradicting our hypothesis that $(q_i^t, q_j^t)$ is not part of the cycle.

Now in the cycle the states in $\tilde{Q}_j$ occur in a particular order, say $q_j^1, \ldots, q_j^C$. But then if $(*)$ holds, player $j$ can adopt the following strategy and save on the number of transitions employed: play $q_j^*$ at $t$, and define

\[
\mu_j^*(q_j^k, s_i) = q_j^{k+1} \quad \forall \ s_i \in S_i, \ k < C,
\]

\[
\mu_j^*(q_j^C, s_i) = q_j^1 \quad \forall \ s_i \in S_i.
\]

Of course, $q_j^* = q_j^k$ for some $k$, so $j$ employs strictly less transitions than previously, with the same repeated game payoffs and the same number of states.

Similarly, if it were $i$ for whom $(*)$ held, $i$ could adopt a different machine and do strictly better. QED

Abreu and Rubinstein (1988) show that if the players' preferences are weakly monotonic in repeated game payoffs and states, the set of potential equilibrium payoffs is quite limited relative to the Folk Theorem. The following result shows that if in addition players' preferences take transitional complexity into account, the potential equilibrium paths themselves become trivial.
Theorem. For all $(M_1, M_2) \in N(G_m)$,

(i) $(\lambda_1(q^t_1), \lambda_2(q^t_2)) \in N(G) \quad \forall \ t$; i.e., any Nash equilibrium of the machine game induces one-shot Nash equilibrium actions at each stage of the repeated game.

(ii) In the limit-average case the cycle begins in the first period, i.e., the set of pre-cycle states is empty.

Proof: By Theorem 1, we claim that $\forall q_i \in Q_i$ and $s_j, s'_j \in S_j$:

\[ (** ) \quad \mu_i(q_i, s_j) = \mu_i(q_i, s'_j). \]

To see this, suppose that $q_i$ is such that (**) does not hold, where $q_i = q^t_i$; let $q_j = q^t_j$. Then there exists $s_j \in S_j$ such that

\[ \mu_i(q_i, s_j) = a_i \neq \mu_i(q_i, \lambda_j(q_j)), \]

where the state $\mu_i(q_i, \lambda_j(q_j))$ is the unique (by Theorem 1) successor to $q_i$ along the path of states induced by the machines $M_1, M_2$. But then by defining a new machine $M'_i$ which differs from $M_i$ only in that $\forall s_j \in S_j$: $\mu'_i(q_i, s_j) = \mu_i(q_i, \lambda_j(q_j))$ player $i$ receives the same repeated game payoff with the same number of states but with at least one less transition. This contradicts the assumption that $(M_1, M_2) \in N(G_m)$.

Suppose now that, per absurdum, the set of pre-cycle states in the limit-average case were not empty. Let $N_1, N_2$ denote respectively the number of pre-cycle and cycle states in $M_1$. We will construct an alternative machine $M_1^*$ for player 1 such that (i) $M_1^*$ makes use of only the $N_2$ cycle states in both the pre-cycle and cycle phases, and (ii) $(M_1^*, M_2) >_1 (M_1, M_2)$. Note that the existence of such an $M_1^*$ does not contradict (iv) in Theorem 1 above since $(M_1^*, M_2)$ may not be in $N(G_m)$; but $(M_1, M_2) \in N(G_m)$ is contradicted.

Since neither players repeats pre-cycle states in the pre-cycle phase, the pre-cycle
phase has length $N_1$ periods. Let $n$ be the remainder from dividing $N_1$ by $N_2$, and let
\{q_1^1, \ldots, q_1^{N_2}\}$ be 1's cycle states in $M_1$ in the order in which they occur in the cycle.
Define $M_1^*$ by $M_1^* = \{Q_1^*, q_1^1, \lambda_1^*, \mu_1^*\}$ where

\[
\begin{align*}
Q_1^* &= \{q_1^1, \ldots, q_1^{N_2}\} \\
q_1^* &= q_1^{N_2-n+1} \\
\lambda_1^*(q_1^k) &= \lambda_1(q_1^k) \\
\mu_1^*(q_1^k, s_j) &= \mu_1(q_1^k, s_j).
\end{align*}
\]

That is, $M_1^*$ starts at $q_1^{N_2-n+1}$ and uses the same output and transition functions as
$M_1$. By (**), this has no impact on player 2's transitions, and by construction the
cycle begins at $(N_1 + 1)$, and yields player 1 exactly the same repeated game payoff.
but $M_1^*$ uses strictly fewer states and transitions than $M_1$, so certainly $(M_1^*, M_2) >_1
(M_1, M_2)$, or $(M_1, M_2) \notin N(G_m)$. a contradiction. This proves the second part of the
Theorem.

Since pre-cycle states are empty in the limit–average case, equation (**), of course
proves the first part of the Theorem: if i's transitions are independent of what j does
then j will prefer to play stage–game best response strategies at each point in time.

QED

Why is this result true? The intuition underlying the formal proof is as
straiighforward as the following verbal argument suggests: if transitions have a cost
associated with them, players' machines in equilibrium must use all the available
transitions. Thus, whereas in Abreu and Rubinstein (1988) a state will be discarded if
it is never employed, (so that punishment states must occur at least once), in addition
a transition will now be discarded if it is never employed (by making the transition
equivalent to some transition which is employed). This implies that "punishment
contingencies", i.e. those state–action pairs which lead (through the transition function) to a punishment state, must occur at least once. But then these contingencies are now part of the equilibrium path of play, implying that there cannot be any threats in equilibrium, empty or otherwise.

To illustrate the difference between the Abreu and Rubinstein (1988) model and the current model, consider again the repeated prisoners' dilemma, where preferences are lexicographic and repeated game payoffs are evaluated according to the limit of means. If both players adopt the machine implementing the "grim" strategy, the resulting equilibrium path is (C,C) forever. Yet this is not an equilibrium when the number of states matters, since both have an incentive to drop the "punishment state" q_D (clearly this conclusion holds in the current model as well). Alternatively, Rubinstein (1986) shows that (C,C) can be sustained for all but the initial period if the players both adopt the following "pretty grim" strategy:

![Diagram](image)

This constitutes equilibrium behavior in the model of Abreu and Rubinstein (1988) since now the punishment state q_D occurs along the equilibrium path, if only once. However when transitions matter this will no longer be an equilibrium, since in particular the transition from q_C to q_D, or rather the "punishment contingency" (q_C,D), is never pursued. In fact, an implication of the above result is that the only equilibrium in the repeated prisoners' dilemma is where both players play D forever. On the other hand, Abreu and Rubinstein (1988) show that in the repeated battle of the sexes game all Nash equilibria are such that players at each stage are at one of the one-shot Nash equilibria. Hence in this game the additional consideration of transition costs will have no effect on the potential equilibrium payoffs.
4. Conclusion

In this paper we have explored the sensitivity of the results of Abreu and Rubinstein (1988) to their measure of a machine's complexity. We have shown that if, in addition to the number of states, the complexity of the transition mechanism enters into this measure the set of possible Nash equilibrium payoffs is dramatically reduced in certain games (e.g. prisoners' dilemma) and is untouched in others (e.g. battle of the sexes). However we concur with Abreu and Rubinstein (1988) that the story on implementation costs in repeated games is nowhere near complete, and hope to explore further such issues in future research.

In general the motivation of Rubinstein (1986), Abreu and Rubinstein (1988) and ourselves concerns limiting the extent of the Folk Theorem in infinitely repeated games by explicitly including complexity considerations in the players' preferences, where complexity relates to the cost of implementing a repeated game strategy. These considerations are facilitated by the use of finite automata, which provide simple measures of such complexity. Another use of finite automata, as noted in the Introduction, is in the study of bounded rationality in finitely repeated games, in particular the finitely repeated prisoners' dilemma (Neyman (1985), Zemel (1986)), where now the maximum number of states a player can used is fixed at the outset. In these papers the motivation is to generate cooperation as an equilibrium, in part to justify the empirical regularity of such behavior (cf. Axelrod (1980)). Given the results of the current paper, then, it appears that these two strands of the literature have actually passed each other by, in that Neyman (1985) gets cooperation in finite time while we get only non-cooperation in infinite time! An implication of this fact appears to be the necessity of developing a more general theory of bounded rationality and decision costs in repeated games which would address the issue of complexity both in terms of the computation of optimal strategies as well as the actual implementation of such strategies.
Appendix

Abreu and Rubinstein (1988) prove Theorem 1 above for discounting using the following three results. We modify their proofs to show that the results hold in the current model.

Let \((M_1, M_2) \in \mathcal{N}(G_m)\). Define \(A_1(k_1, k_2)\) as the average (discounted or undiscounted) payoff to i between the periods \(k_1\) and \(k_2\) inclusive, and let \(\pi^*_i(k)\) denote the average discounted payoff to i from period \(k\) on.

**Lemma A1.** Suppose \(q_1^{k_1} = q_1^{k_2+1}\), \(k_2 \geq k_1\). Then \(A_2(k_1, k_2) = \pi^*_2(k_1)\).

**Proof.** Suppose it were the case that \(A_2(k_1, k_2) > \pi^*_2(k_1)\). Note that this implies that \(q_2^{k_1} \neq q_2^{k_2+1}\). Define a new machine \(M_2^*\) for player 2 identical to \(M_2\) but with the following change:

\[
\mu^*_2(q_2^{k_2}, s) = q_2^{k_1} \quad \forall s \text{ such that } \mu_2(q_2^{k_2}, s) = q_2^{k_2+1}.
\]

Clearly, \(M_2^*\) is no more complex than \(M_2\) but \(\pi_2(M_1, M_2^*) > \pi_2(M_1, M_2)\), contradicting \((M_1, M_2) \in \mathcal{N}(G_m)\).

Similarly, if \(A_2(k_1, k_2) < \pi^*_2(k_1)\), player 2 could do better with a machine \(M_2^*\) identical to \(M_2\) but with:

\[
\mu^*_2(q_2^{k_1-1}, s) = q_2^{k_2+1} \quad \forall s \text{ suc that } \mu_2(q_2^{k_1-1}, s) = q_2^{k_1-1}.
\]

Q.E.D.
Let \( m_1 \) denote the minimal \( t \) such that \( q^{t}_1 \) is repeated.

**Lemma A2.** \( m_1 = m_2 \).

**Proof.** Suppose \( m_2 > m_1 \); let \( \overline{m}_1 > m_1 \) satisfy \( \overline{m}_1 = q^{m_1}_1 \). Define \( M_2' \) by

\[
Q_2' = Q_2 \backslash \{ q^{m_1}_2 \},
\]

\[
\lambda'_2(q_2) = \lambda_2(q_2) \forall q_2 \in Q_2',
\]

\[
\mu'_2(q_2, \cdot) = \mu_2(q_2, \cdot) \forall q_2 \neq q^{m_1-1}_2,
\]

\[
\mu'_2(q^{m_1-1}_2, s) = q^{\overline{m}_1}_2 \forall s \in S_1.
\]

Since \( q^{\overline{m}_1-1}_2 \) and \( q^{m_1}_2 \) are not repeated in \((M_1, M_2)\), the sequence of states in \((M_1, M_2')\) is \( q^1, \ldots, q^{\overline{m}_1-1}_2, q^{\overline{m}_1}_2, \ldots \). By Lemma A1, \( \pi'_2(M_1, M_2') = \pi_2(M_1, M_2) \), but \( M_2' \) is less complex than \( M_2 \) (since \( q^{m_1}_2 \) and the transitions leading to and from it are absent in \( M_2' \), whereas only 1 transition has been added); a contradiction.

Q.E.D.

Recall that \( t_1 \) is the beginning of the initial cycle.

**Lemma A3.** \( t_1 = m_1 \).

**Proof.** Let \( \overline{m}_1 \) be the minimal \( t > m_i \) such that \( q^{t}_1 = q^{m_i}_1 \), and suppose that \( \overline{m}_1 > \overline{m}_2 \). If \( q^{m_1}_2 \neq q^{t}_2 \) for any \( t < \overline{m}_1 \), define \( M_2' \) by

\[
Q_2' = Q_2 \backslash \{ q^{\overline{m}_2}_2 \},
\]

\[
\lambda'_2(q_2) = \lambda(q_2) \forall q_2 \in Q_2,
\]

\[
\mu'_2(q^{m_1}_2, s) = q^{\overline{m}_1}_2 \forall s \in S_1.
\]
\[ \mu'_2(\cdot, \cdot) = \mu_2(\cdot, \cdot) \forall (q'_2, s) \neq (q_2, s_1, \overline{s}_1), \]
\[ \mu'_2(q'_2, s_1) = q'_2. \]

Then by Lemma A1, \( \pi_2(M_1, M'_2) = \pi_2(M_1, M_2) \), but (for the same reason as in lemma A2), \( M'_2 \) is less complex than \( M_2 \), a contradiction. If \( q^{m_1}_2 = q^k_2 \) (where \( k \neq m_1 \)), define \( M'_1 \) by
\[ Q'_1 = Q_1 \setminus \{q^1_1\}, \]
\[ \lambda'_1(\cdot) = \lambda_1(\cdot), \]
\[ \mu'_1(q'_1, s'_2) = \mu_1(q'_1, s'_2) \text{ if } t \neq m_1 - 1, \overline{m}_1 - 1, \]
\[ \mu'_1(q'_1, s) = q^1_1 \forall s \in S_2, t = m_1 - 1, \]
\[ \mu'_1(q'_1, \overline{s}_2) = q^k_1, \]
\[ \mu'_1(q'_1, s) = \mu_1(q'_1, s) \forall s \neq \overline{s}_2. \]

Then the sequence of states is \( q^1, q^{m_1 - 1}_1, q^m, q^{m_2 + 1}_1, q^{m_1 - 1}_1 \) followed by the cycle \( (q^k, q^{k+1}, ..., q^{m_1 - 1}_1) \). By Lemma A1, \( \pi_1(M_1, M_2) = \pi_1(M'_1, M_2) \), but now \( M'_1 \) is less complex than \( M_1 \). Contradiction. QED

Lemma A3 shows that all pre-cycle states are distinct and Lemma A2 shows that the cycles in \( q^t_1 \) and \( q^t_2 \) begin at the same time. To see that all cycle states are distinct, and therefore that \( T_1 = T_2 = T \), a construction similar to that in Lemma A2 is used (see Abreu and Rubinstein (1988), p. 1270).
References


