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TRANSVERSALITY CONDITION AND OPTIMALITY IN A CLASS OF INFINITE HORIZON CONTINUOUS TIME ECONOMIC MODELS

By

Siu Fai Leung

ABSTRACT: This paper studies a class of widely used infinite horizon continuous time economic models first formulated by Kamien and Schwartz (1971). We show that the method of solution for this class of models offered in the literature is unwarranted because it assumes that the transversality condition is a necessary optimality condition. We show that the existing theorems in the literature do not provide a satisfactory justification for the necessity of the transversality condition. Two different approaches are provided to solve the problem. The first approach requires some restrictive assumptions and the second approach provides a complete solution to the problem without making any additional assumption.

KEYWORDS: Transversality condition, optimal control, infinite horizon models, limit pricing

1. INTRODUCTION

In this paper, we will study a class of infinite horizon continuous time optimal control problems that have been widely used in economics. This class of models first appears in Kamien and Schwartz (1971) in an attempt to bring uncertainty into the Bain-Sylos-Labini-Modigliani theory of limit pricing. The model that Kamien and Schwartz formulate turns out to have a profound impact not only on the subsequent development of limit pricing theory but also on some other areas in economics as well. In the area of limit pricing, refinements and generalizations of the Kamien and Schwartz model have been made, for example, in Baron (1972, 1973), Kamien and Schwartz (1975), De Bondt (1976), Deshmukh and Chikte (1976), Lippman (1980), and Phlips (1980).[1] Other than limit pricing, the same model has been applied to study, for example, new product pricing (Kamien and Schwartz, 1972a), optimal plant size (Kamien and Schwartz, 1972b), market structure and innovation (Loury, 1979), and most recently, crime and punishment (Davis, 1988). Although the economic issues addressed in these papers may be very different and hence there may be some variations in the models, the mathematical structures of these models are analytically the same, because they are all based on the Kamien and Schwartz (1971) model. For brevity, let us call these models the KS class of models. It will be shown later that this class of models is a broad one, for example, it also covers Lucas' (1971) model of research and development.

Although there are many variations in the KS class of models, the solutions to the different optimal control problems in this class of models are all based on the solution that Kamien and Schwartz (1971) established for their optimal control problem (hereafter the KS solution). All the comparative statics results derived from the KS class of models are built on the KS solution, which is the only solution available and is implicitly considered to be the solution in the literature. In this paper, we show that the proof that Kamien and Schwartz (1971) (hereafter the KS proof) use to establish the KS solution is unfounded because it suffers from an important problem that has never been noticed in the literature. Therefore, whether the widely endorsed KS solution is valid remains to be shown. Despite the widespread use of the KS class of models in economics, we demonstrate that a complete solution to the problem has not yet been established. The problem of the KS proof lies in the use of an unverified transversality condition for an infinite horizon optimal control problem.

For a finite horizon optimal control problem with no constraint on the terminal state, the multiplier (auxiliary variable, costate variable) must be equal to zero at the terminal time. This condition, known as the transversality condition or the terminal condition, is one of the necessary optimality conditions obtained from the Pontryagin maximum principle.[2] If there are inequality constraints on the terminal state, the transversality condition can be modified accordingly. The transversality condition plays an important role in optimal control theory because it is useful in eliminating nonoptimal solutions, proving sufficient conditions for optimality and performing stability analyses. Short of a rigorous proof, it was once believed that the transversality condition is also a necessary optimality condition for an infinite horizon optimal control problem. This conjecture was shown to be false when Shell (1969) provided the first counter-example to show that the transversality condition is not a necessary optimality condition. At about the same time, Arrow and Kurz (1970) reported another counter-example constructed by Halkin (see also Halkin, 1974). For the last twenty years, the counter-examples of Shell and Halkin have generated a number of studies to provide sufficient conditions under which the transversality condition is a necessary optimality condition for an infinite horizon optimal control problem, see for example Bensoussan, Hurst and Naslund (1974), Aubin and Clarke (1979), Michel (1982), Benveniste and Scheinkman (1982), Araujo and Scheinkman (1983).[3] The conditions supplied by these studies are quite general and they have been used to justify the transversality conditions found in many different economic models. We show, however, that none of the existing theorems in the literature can satisfactorily justify the transversality condition of the KS class of models. It is shown that this class of models does not satisfy many of the conditions (such as concavity) required by the existing theorems. In some cases, the conditions are satisfied only if some assumptions with little economic appeal are imposed on the model. The KS class of models possess several nonstandard features that cause the justification of the transversality condition a technically difficult problem. Therefore the KS proof, which relies on an unverified transversality condition, remains unfounded.

In this paper, we provide two different approaches to solve the problem. The first approach presents a straightforward way to justify the transversality condition by making some boundedness assumptions. We argue that this approach is not satisfactory because the assumptions are too restrictive in many economic situations. The second approach presents an entirely different way to tackle the problem. The approach does not involve the transversality condition and no extra assumptions have to be imposed on the models. The approach yields the transversality condition as a byproduct. It also reveals clearly the role of the assumptions in the models. We further illustrate the usefulness of the approach by applying it to solve Lucas' (1971) model of research and development.

2. THE KS CLASS OF MODELS

For expository reasons, throughout this paper, we will couch our discussion of the KS class of models in terms of the limit pricing model in Kamien and Schwartz (1971), but we emphasize that the discussion will be essentially the same for other models as well. Consider the following model of limit pricing first formulated by Kamien and Schwartz (1971). A monopoly firm or a cartel faces potential entry of rival firms. The incumbent firm has to choose a price policy to maximize its expected profit and to deter the entry of rival firms. Assuming the product price $p(t) \in [0,\bar{p}]$ for some finite positive number $\bar{p}$, the firm's decision problem is to choose p(t) to maximize (1) subject to (2) and (3). For convenience, we call this control problem *[P1]*:

$$(1) \qquad \max_{p(\cdot)} \int_0^\infty e^{-(r-g)t}[\pi_1(p(t))(1-F(t)) + \pi_2(g)F(t)]dt$$

subject to *[P1]*

$$(2) \qquad F'(t) = h(p(t),g)(1-F(t))$$

$$(3) \qquad F(0) = 0,$$

where

r = discount rate

g = market growth rate

p(t) = product price at time t

$\pi_1(p(t))$ = profit at time t, before entry of rival firms

$\pi_2(g)$ = profit after entry of rival firms

$F(t)$ = distribution function of the time of entry of rival firms

$F'(t)$ = density function corresponding to $F(t)$

$h(p(t),g)$ = hazard rate of entry of rival firms at time t

We briefly describe how the integral in (1) is derived. Without potential entry of rival firms, the incumbent firm can just choose $p(t)$ to maximize profit $e^{gt}\pi_1(p(t))$ at each t. When there is potential entry, a higher price $p(t)$ will induce a higher probability of entry so that the incumbent firm cannot just maximize $e^{gt}\pi_1(p(t))$. Before any rival firm enters into the market, the incumbent firm earns $e^{gt}\pi_1(p(t))$ at each time t. If a rival firm enters into the market at time s, then the incumbent firm will earn $e^{gt}\pi_2(g)$ for all $t \geq s$. Therefore, the present value of the incumbent firm's expected profit is given by

$$\int_0^\infty e^{-rt}[e^{gt}\pi_1(p(t))(1-F(t))]dt + \int_0^\infty e^{-rs}[\int_s^\infty e^{-r(t-s)}e^{gt}\pi_2(g)dt]F'(s)ds.$$

The integrand $e^{-rt}[e^{gt}\pi_1(p(t))(1-F(t))]$ is the present value of the incumbent firm's profit at time t when entry of rival firms has not yet occured by time t, multiplied by the probability of that event. The integrand $e^{-rs}[\int_s^\infty e^{-r(t-s)}e^{gt}\pi_2(g)dt]F'(s)$ is the present value of the incumbent firm's profit when entry occurs at time s, multiplied by the density $F'(s)$ at time s. Assume $r-g > 0$, then $\int_0^\infty e^{-rs}[\int_s^\infty e^{-r(t-s)}e^{gt}\pi_2(g)dt]F'(s)ds$ can be simplified to $\int_0^\infty e^{-(r-g)t}\pi_2(g)F(t)dt$ (a finite integral), by using Fubini's Theorem to interchange the double integrals. Hence, (1) is obtained.

Facing potential entry of rival firms, the incumbent firm sets the limit price to maximize expected profits and to deter entry of rivals. Although the incumbent firm cannot preclude entry altogether, it can affect the probability of entry of rival firms. The hazard rate of entry of rival firms is assumed to be a function of the market growth rate and the price set by the firm, i.e., $h(p(t),g)$. Since the hazard rate is defined by $F'(t)/[1-F(t)]$, the incumbent firm faces the law of motion (2). The initial condition (3) indicates that the probability of entry at time zero is 0.

Problem *[P1]* is an infinite horizon continuous time optimal control problem, with state variable $F(t)$ and control variable $p(t)$. This kind of formulation has attracted considerable interests among economists. By renaming the symbols $p(t)$, $\pi_1(p(t))$ and $\pi_2(g)$, many economists have essentially used the same mathematical model (1)-(3) to study a variety of different economic issues other than limit pricing. Two examples are provided below to illustrate the usefulness and wide applicability of *[P1]*. These examples show that there can be many variations in the KS class of models.

EXAMPLE I: Crime and Punishment (Davis, 1988)

In Davis' intertemporal model of crime and punishment, the problem is to

$$\max_{p(.)} \int_0^\infty e^{-rt}[\pi_1(p(t))(1-F(t)) - \theta F'(t) + \pi_2 F(t)]dt$$

subject to

$$F'(t)=h(p(t))(1-F(t))$$

$$F(0)=0,$$

where $\pi_1(p(t))$ is the income or utility received from some illegal activity and $p(t)$ is the rate at which offenses are committed. Using Davis' examples,

$\pi_1(p(t))$ might represent the profit function of a firm subject to pollution control laws or price controls and p(t) might represent the rate of pollution or the price charged in excess of the ceiling.[4] F(t) is the distribution function of the time of detection of the illegal activity. If the illegal activity is detected, the firm has to pay a fine θ at the time of detection. The firm will earn income π_2 from some legal activity thereafter. One can further complicate Davis' model by including imprisonment as a punishment, and making the fine θ and the income π_2 depending on the offense rate.

EXAMPLE II: Cournot Oligopoly with Uncertain Entry (Kamien and Schwartz, 1975; Deshmukh and Chikte, 1976; and Lippman, 1980)

Suppose at time t, there are n identical firms in an industry, each with profits q(t)D(Q(t)+q(t))-C(q(t)), where q(t) is the quantity of output produced by each firm, D(.) is the industry inverse demand function, Q(t) is the quantity produced by the other (n-1) firms, and C(.) is the cost function. Let V(n+1) be the value function when there are n+1 firms in the industry, then each firm faces the following recursive problem:

$$V(n) = \max_{q(.)} \int_0^\infty e^{-rt}\{[q(t)D(Q(t)+q(t))-C(q(t))](1-F(t)) + rV(n+1)F(t)\}dt$$

subject to

$$F'(t) = h(D(Q(t)+q(t)))(1-F(t))$$

$$F(0) = 0,$$

where h(.) is the hazard function of entry of the (n+1)st firm. Although this control problem is more complicated than [*P1*] (e.g., the properties of the value function have to be derived), the structures of the two problems are basically the same. This example illustrates that the KS class of models is not only limited to two period models. In the limit pricing example in [*P1*], there are only two periods, the first period is the time before the entry of rival firms and the second period is the time after the entry. Similarly, there are only two periods in example I, the period before the illegal activity is detected and the period after. In example II, however, multiple periods are allowed because of the recursive structure such that there can be sequential entry of firms. It is easy to see that both the limit pricing and the crime and punishment examples can be extended to multiple period models.

3. SOLVING THE KS CLASS OF MODELS

The Hamiltonian for problem [*P1*] is given by

$$H = \lambda_0 e^{-(r-g)t}[\pi_1(p(t))(1-F(t)) + \pi_2(g)F(t)] + \lambda(t)h(p(t),g)(1-F(t)).$$

Since our focus is on the justification of the transversality condition and not on the existence of an optimal solution, the multiplier λ_0 is assumed to be 1. In order to solve the optimal control problem, Kamien and Schwartz (1971) make the following assumptions:

(A1) π_1 is twice continuously differentiable and strictly concave in p.

(A2) For any $k \in [0,\pi_2]$, the equation $\pi_1(p) = k$ has two real, distinct and positive solutions of p.

(A3) $g < r$, $0 \le \pi_2 < \max_p \pi_1(p)$.

(A4) $\partial\pi_2/\partial g \le 0$, $h(p(t),g) \ge 0$, $h(0,g) = 0$, $\partial h/\partial p \ge 0$, $\partial^2 h/\partial p^2 \ge 0$, $\partial h/\partial g \ge 0$.

Assumptions (A1) and (A2) imply that we can take $\bar{p}$ to be the nonzero root of the equation $\pi_1(p) = 0$. With assumptions (A1)-(A4), Kamien and Schwartz (1971, p.447-448) show that an optimal policy is to choose a constant price p^* for all $t \in [0,\infty)$, defined by the following equation:

(4) $$\pi_1'(p^*)(r-g+h(p^*,g)) = h_1(p^*,g)(\pi_1(p^*)-\pi_2(g)).$$

Their proof involves three steps:

Step 1. To show that if the policy $p(t) = p^*$ exists, then it is admissible, i.e., there exist functions $F^*(t)$ and $\lambda^*(t)$ such that $(p^*,F^*(t),\lambda^*(t))$ satisfy the following three conditions (5),(6) and (7).

(5) $$\partial H/\partial p = [e^{-(r-g)t}\pi_1'(p(t)) + \lambda(t)h_1(p(t),g)](1-F(t)) = 0$$

(6) $$\lambda'(t) = -\partial H/\partial F = e^{-(r-g)t}[\pi_1(p(t))-\pi_2(g)] + \lambda(t)h(p(t),g)$$

(7) $$\lim_{t\to\infty} \lambda(t)F(t) = 0.$$

Step 2. To show that p^* exists.

Step 3. To prove that p^* is optimal.

It is the first step that is problematic. In the process of finding the function $\lambda^*(t)$, they assume that the transversality condition (7) is a necessary optimality condition. Although the work of Shell and Halkin show that the transversality condition may not be a necessary optimality condition for infinite horizon problems, Kamien and Schwartz do not provide any justification for (7). Despite the wide applicability of the formulation of problem [*P1*], the necessity of (7) has not yet been demonstrated in the literature.[5] The conclusion that the constant price p^* is an optimal solution, however, has been accepted without any reservation in the literature.

4. JUSTIFYING THE TRANSVERSALITY CONDITION

In this section, we will investigate whether the existing results in the literature can be used to justify the transversality condition (7).

Specifically, we will examine the theorems in Bensoussan, Hurst and Naslund (1974), Aubin and Clarke (1979), Michel (1982), Benveniste and Scheinkman (1982), Araujo and Scheinkman (1983) because these are the only available results in the literature.[6] We first begin with Michel (1982). Michel's results appear to be the most general one because only a few conditions are required to be satisfied. The key condition, for the transversality condition to be a necessary optimality condition, is that the set of the laws of motion has to contain a neighborhood of zero for all large t. This condition is, however, violated in *[P1]* because $h(p(t))(1-F(t)) \geq 0$, as the hazard function is always nonnegative and the distribution function F(t) is bounded between 0 and 1, so that the set of the laws of motion does not contain a neighborhood of zero for any $t \in [0,\infty)$. One can see that it is impossible to transform or reparametrize *[P1]* into one such that the set of the laws of motion will contain a neighborhood of zero, thus Michel's result is not applicable. To examine whether the other results in the literature are applicable, we have to transform the original problem *[P1]* into a suitable formulation required by these theorems. This is because Benveniste and Scheinkman (1982), and Araujo and Scheinkman (1983) only deal with calculus of variations and not optimal control problems, while Bensoussan, Hurst and Naslund (1974), and Aubin and Clarke (1979) require that the law of motion has to satisfy a certain linear relation.

Let $\alpha = r-g$ be the effective discount rate. Since g is assumed to be a constant, the dependence of the functions h(p,g) and $\pi_2(g)$ on g can be suppressed for convenience. Equations (2) and (3) imply that F(t) = $1-\exp[-\int_0^t h(p(s))ds]$. Let $z(t) = \int_0^t h(p(s))ds$, then $z'(t) = h(p(t))$. Assume $h'(p(t)) \neq 0$ for $p(t) \in (0,\bar{p})$, then there exists a function $\pi(.)$ such that

$\pi_1(p(t)) = \pi(h(p(t))$ for $p(t) \in [0,\bar{p}]$. In addition, the dependence of $h(p(t))$ on $p(t)$ can be suppressed and can simply be written as $h(t)$. With this reformulation, problem [*P1*] will then be equivalent to the following calculus of variations problem [*P2*]:

$$\max_{h(.)} \int_0^\infty e^{-\alpha t}[\pi(h(t))e^{-z(t)} + \pi_2(1-e^{-z(t)})]dt \qquad [P2]$$

subject to

$$z'(t) = h(t)$$

$$z(0) = 0.$$

The control variable is the hazard rate of entry $h(t)$, instead of the price $p(t)$. Let $\mu(t)$ be the multiplier, then the transversality condition for [*P2*] is

(8) $$\lim_{t\to\infty} \mu(t)z(t) = 0.$$

It is easy to see that the Kamien and Schwartz (1971) method of solving [*P1*] is also applicable to solving [*P2*], therefore the solution p^* given by (4) is optimal if we can show that (8) is a necessary optimality condition. Let $G(z(t),h(t)) = G(z(t),z'(t)) = \pi(h(t))e^{-z(t)} + \pi_2(1-e^{-z(t)})$. Clearly, $G(z(t),z'(t))$ is not concave in $(z(t),z'(t))$, therefore the theorems of Benveniste and Scheinkman (1982) and Araujo and Scheinkman (1983) are not applicable since they require the objective functions to be concave. The law of motion $(z'(t) = h(t))$ is linear in the control variable $h(t)$, which satisfies one of the conditions of Bensoussan, Hurst, and Naslund (1974) and Aubin and Clarke (1979). Their results would seem to be applicable since they do not require concave objective functions.

Even if all the conditions in Bensoussan, Hurst, and Naslund (1974) are satisfied, their results are not useful in solving our problem because the transversality condition that they obtain is different from the ones usually found in the literature. They show that under certain regularity conditions, $\lim_{t\to\infty} e^{-\alpha t}\|\mu(t)\|^2 = 0$, where $\|.\|$ is the L_2 norm in the space $L_2[0,t]$.[7] Since $\lim_{t\to\infty} e^{-\alpha t} = 0$ (as α is positive), $\lim_{t\to\infty} e^{-\alpha t}\|\mu(t)\|^2 = 0$ does not imply $\lim_{t\to\infty} \mu(t) = 0$, which is the transversality condition that we really need. The transversality condition that they derive seems to be a rather weak one and is not useful in our context.

Aubin and Clarke (1979) generalize the results of Bensoussan, Hurst, and Naslund (1974) and obtain stronger transversality conditions. In order to apply the Aubin-Clarke theorem, three more conditions have to be satisfied:

(C1) G is locally Lipschitz.

(C2) The derivatives of G satisfy a growth condition: there exist real numbers $c \geq 0$ and $k \geq 0$ such that

$$(G_h^2 + G_z^2)^{1/2} \leq c[1+(h^2+z^2)^{k/2}], \text{ i.e.,}$$

(9) $$e^{-z}[(\pi'(h))^2 + (\pi_2-\pi(h))^2]^{1/2} \leq c[1+(h^2+z^2)^{k/2}].$$

(C3) $\alpha > k+1$.

If $k > 0$, let q be defined by $1/q + 1/(k+1) = 1$. If all the three conditions (C1)-(C3) are satisfied, then the Aubin-Clarke theorem states that the following transversality conditions are necessary for optimality:

(10) $$\lim_{t\to\infty} e^{(q-1)\alpha t}|\mu(t)|^q = 0 \quad \text{if } k > 0,$$

(11) $$\lim_{t\to\infty} e^{\alpha t}\mu(t) = m \quad \text{if } k = 0 \text{ (m is some finite number).}$$

Clearly, both (10) and (11) imply (8) since $q > 1$ and $\alpha > 0$. Therefore, only the results of Aubin and Clarke (1979) are applicable to the class of nonconcave economic models considered in this paper.

We examine more closely the conditions (C1) to (C3). (C1) is easily satisfied because of the assumptions (A1) and (A4) that G is (twice) continuously differentiable, and hence locally Lipschitz. However, the remaining two conditions (C2) and (C3) are very restrictive. (C3) requires that $\alpha > k+1 \geq 1$, since $k \geq 0$. The condition $\alpha > 1$ means that from time zero to infinity, the incumbent firm's effective discount rate for future profits has to be strictly greater than one. Since $\alpha = r-g$, this implies that $r > 1$ if g is nonnegative. In short, (C3) requires either a high discount rate r and/or a negative growth rate g for all $t \in [0,\infty)$. If the model is applied to an individual instead of a firm, as in example I, then the individual has to be an impatient one. Aubin and Clarke (1979) are aware of this problem. They remark (p.569) that "the necessary conditions may be expected to hold ... provided the "discount rate" ... is sufficiently large." They also provide an example in which the transversality condition fails to hold when the discount rate is bounded between 0 and 1. Nevertheless, a discount rate greater than one is a rather unusual assumption in economics and seems to have little economic appeal.

The growth condition (9) does not allow $\pi(h)$ and $\pi'(h)$ to be unbounded at $p = 0$. This implies that $\pi_1(0) > -\infty$, $\pi_1'(0) < \infty$, and $h'(0) > 0$. There are three problems with these boundedness assumptions. First, they are too restrictive in many economic situations, a detailed discussion will be found in the next section. Second, there exist examples in which the growth condition (9) is violated but the transversality condition is still valid. One example is provided in the appendix. In that example, $\pi_1'(0) < \infty$ and $h'(0) = 0$, therefore $\pi'(0) = \pi_1'(0)/h'(0) = \infty$, but the transversality condition is shown to be valid. This example suggests that the growth conditions are too restrictive and can be removed. Third, it will be shown in the next section that the two assumptions $\pi_1'(0) < \infty$ and $h'(0) > 0$ are sufficient to justify the transversality condition. Thus, the restrictions that $\alpha > 1$ and $\pi_1(0) > -\infty$ are unnecessary.

Based on the available results in the literature, the above discussion shows that none of the existing results can give a satisfactory justification for the transversality condition. The structure of the KS class of models immediately rules out the relevance of the results of Bensoussan, Hurst and Naslund, Michel, Benveniste and Scheinkman, Araujo and Scheinkman. Only the Aubin-Clarke results are applicable, but it requires some unattractive economic assumptions such as a discount rate greater than unity, and several boundedness assumptions. The most general results to date seem to be Ekeland and Scheinkman (1986): there is no special restriction on the discount rate, the objective function can be nonconcave and some types of unbounded objective functions are allowed. The problem is, however, that they have only established the transversality conditions for discrete time optimization problems.

5. TWO SOLUTIONS

In this section, we will provide two approaches to solve the KS class of models. The first one is a straightforward approach to justify the transversality condition and the second one takes an entirely different route which does not involve the transversality condition. The second approach provides a complete solution to the KS class of models.

5.1 THE FIRST APPROACH

There is a straightforward way to justify the transversality condition

(7) if one is willing to make two additional assumptions, namely, for any $p \in [0,\bar{p}]$, there exist two constants k_1 and k_2 such that

(A5) $\pi_1'(p) < k_1 < \infty$, and

(A6) $h_1(p,g) > k_2 > 0$.

To see this, notice that (A6) implies that (5) can be written as $\lambda(t)(1-F(t)) = e^{-(r-g)t}[\pi_1'(p(t))(1-F(t))/h_1(p(t),g)]$. Since $\pi_1'(.)$ is bounded above (by (A5)), $h_1(.,.)$ is bounded away from zero (by (A6)), F(t) is bounded between 0 and 1, and $e^{-(r-g)t} \to 0$ as $t \to \infty$, therefore $e^{-(r-g)t}[\pi_1'(p(t))(1-F(t))/h_1(p(t),g)] \to 0$ as $t \to \infty$. Although $\lim_{t\to\infty} \lambda(t)(1-F(t)) = 0$ does not imply (7), it is easy to see that one can modify *[P1]* slightly by taking 1-F(t) (instead of F(t)) as the state variable so that the transversality condition becomes $\lim_{t\to\infty} \lambda(t)(1-F(t)) = 0$. The KS proof will still work with this modification, hence the KS solution is a valid one. Therefore, if one is willing to make these boundedness assumptions, then the transversality condition can easily be justified. Notice that in this case, the transversality condition is no longer a <u>separate</u> necessary optimality condition, it is just a result that follows directly from the first order condition (5).

It is useful to emphasize that the result obtained here is a general one and apparently has not been brought up in the literature. Consider a general infinite horizon control problem $\text{Max}_{C(.)} \int_0^\infty e^{-rt} g(X(t),C(t))dt$ subject to $X'(t)=f(X(t),C(t))$ and $X(0)=X_0$, which is the same problem studied in Michel (1982). Let $\beta(t)$ be the multiplier. If all feasible states X(t) are bounded, then the transversality condition $\lim_{t\to\infty} \beta(t)X(t) = 0$ will be verified if for all feasible (X,C), $g_2(X,C) < m_1 < \infty$ and $f_2(X,C) > m_2 > 0$, for some constants m_1 and m_2. This is because these assumptions and the first order condition $e^{-rt}g_2(X(t),C(t))+\beta(t)f_2(X(t),C(t)) = 0$ imply that $\lim_{t\to\infty} \beta(t) = 0$. If these two boundedness assumptions are made, then the Michel method of justifying the transversality condition is just redundant. It also shows that the Aubin-Clarke conditions are overly restrictive. The conditions $\alpha > 1$ and $\pi_1(0) > -\infty$ are simply redundant.[8]

This approach, though simple and straightforward, is not satisfactory because of the following considerations. First, assumption (A6) excludes a whole class of functions with $h_1(p,g) = 0$ for some p. For example, ordinary functions such as $h(p,g) = p^n$ or $\exp(p^n)$ $(n > 1)$ which satisfy assumption (A4) are excluded because $h_1(0,g) = 0$. In the context of limit pricing, this exclusion is too restrictive because it is reasonable to assume that there is a range of prices, say $[0,p_1]$ $(0 < p_1 < \bar{p})$, such that $h_1(p,g) = 0$ for $p\in[0,p_1]$. This is because potential entrants will not be attracted into the market if the price that the incumbent firm charges is only infinitesimally greater than zero. Potential entrants will consider entering the market only if the price is high enough to cover the costs of production and to signal that the market is an attractive one. In fact, the focus of the barriers to entry literature (see for example, Stigler, 1968) is to argue that there are significant costs (barriers) in entry so that it is impossible to enter a market with just an infinitesimal price. Let the "reservation" price be p_1, then $h(p,g) = 0$ for $p\in[0,p_1]$, which implies $h_1(p,g) = 0$ for $p\in[0,p_1]$. Similar criticisms of this assumption can be made in other applications of the KS class of models. For example, in example I above, there may be a range of pollution rate that cannot be detected due to the costs of measurement and enforcement by environmental agencies. In any case, the restriction that assumption (A6) imposes on the model is artificial and is not very appealing in economics. Second, assumption (A5) excludes functions with $\pi_1'(0) = \infty$, e.g., $\log(p)$ or p^n ($0 < n < 1$). If $\pi_1(p)$ denotes a profit function, as in the limit pricing model, then one may be able to rationalize the boundedness assumption. However, if $\pi_1(p)$ denotes a utility function, as in example I and in other applications of the KS class of models, then the boundedness assumption may be too restrictive. In consumption models (e.g., classical growth models), any reasonable consumption path should avoid zero consumption at any time. Hence, the assumption $\pi_1'(0) = \infty$ is made because it implies that with infinitely large marginal utility at zero consumption, one would always consume some positive amount, since the marginal cost of doing so is usually bounded in most economic situations. As Ekeland and Scheinkman (1986) have emphasized, zero consumption should be penalized much more heavily than by simply setting $\pi_1(0) = 0$ or some large negative value, therefore they argue that it is more appropriate to have $\pi_1(p) \to -\infty$ as $p \to 0$, which implies $\pi_1'(0) = \infty$.[9] In any case, the assumption $\pi_1'(0) = \infty$ is frequently used in economics to guarantee an interior solution. In fact, proving the transversality conditions for infinite horizon problems with unbounded objective functions is the main purpose of most work in this area because the unboundedness causes difficult technical problems, see for example, Araujo and Scheinkman (1983), and Ekeland and Scheinkman (1986). Lastly, there exist examples in which the transversality condition is verified even when assumption (A6) is violated. In the appendix, an example with a closed form solution is provided to illustrate the existence of such cases. This example suggests that assumption (A6) can be removed and the transversality condition is likely to be valid under more general conditions.

5.2 THE SECOND APPROACH

The first approach is not really satisfactory because it is no longer applicable with a slight change either in $\pi_1(.)$ or h(.). Furthermore, they cannot deal with more realistic economic situations. In this section, we will provide a complete solution to the problem. We have seen in the previous sections that with no extra assumptions, justifying the transversality condition for this class of nonconcave infinite horizon control problems is not a trivial task at all. Whether the transversality condition will hold without making the boundedness assumptions is still an open question. Given that there are still many unresolved technical problems in the justification of the transversality condition for the KS class of models, the method of solution offered by Kamien and Schwartz does not seem to be a fruitful way to tackle the problem. Instead, we choose an approach which is entirely different from the Kamien and Schwartz approach. Our approach does not require the justification of the transversality condition. No additional assumptions have to be made, and $\pi_1(0) = -\infty$, $\pi_1'(0) = \infty$, and $h_1(p,g) = 0$ are all allowed. The key is to make use of the uncertainty in the model and to reformulate the problem to time s ($s \geq 0$) instead of time zero.

Let $V(s,F(s))$ be the value function at time s ($s \geq 0$), given that the probability F(s) at time s is a given number, say F_s, then *[P1]* can be reformulated as:

$$V(s,F_s) = \max_{p(.)} \int_s^\infty e^{-(r-g)(u-s)}\left[\pi_1(p(u))\frac{1-F(u)}{1-F_s} + \pi_2(g)\frac{F(u)-F_s}{1-F_s}\right]du$$

$$\text{s.t.} \quad F'(u) = h(p(u),g)(1-F(u)) \qquad [P3]$$

$$F(s) = F_s$$

The main difference between [*P1*] and [*P3*] is that the probability terms are different. Notice that when $t = 0$, then $V(0,F_0) = V(0,0)$, and [*P3*] is just [*P1*]. At time s, given that there is no entry by time s, the probability of no entry by time u, $u > s$, is given by the conditional probability $[1-F(u)]/(1-F_s)$. This is because Prob(no entry by time u and no entry by time s|no entry by time s) = Prob(no entry by time u|no entry by time s), since $u > s$. Similarly, the conditional density that entry occurs at time u, given that no entry by time s, is given by $F'(u)/(1-F_s)$. Therefore, the second term, $\int_s^\infty e^{-(r-g)(u-s)}\pi_2(g)[F(u)-F_s)]/(1-F_s)du$, is obtained by using Fubini's theorem to interchange the double integrals $\int_s^\infty e^{-r(v-s)}[\int_v^\infty e^{-r(u-v)}e^{gu}\pi_2(g)du]F'(v)/(1-F_s)dv$.[10]

It is easy to see that the infinite horizon control problem [*P3*] is autonomous, therefore, $V(s,F_s)$ does not depend on s explicitly and can be abbreviated as $V(F_s)$. Now, let $t=u-s$, then by a change of variable, [*P3*] can be expressed as

$$V(F_s) = \max_{p(s+.)} \int_0^\infty e^{-(r-g)t}[\pi_1(p(s+t))\frac{1-F(s+t)}{1-F_s} + \pi_2(g)\frac{F(s+t)-F_s}{1-F_s}]dt$$

$$\text{s.t.} \quad F'(s+t) = h(p(s+t),g)(1-F(s+t))$$

$$F(s) = F_s,$$

$$= \max_{\hat{p}(.)} \int_0^\infty e^{-(r-g)t}[\pi_1(\hat{p}(t))(1-G(t)) + \pi_2(g)G(t)]dt$$

$$\text{s.t.} \quad G'(t) = h(\hat{p}(t),g)(1-G(t)) \qquad [P4]$$

$$G(0) = 0,$$

where $\hat{p}(t) = p(s+t)$ and $G(t) = [F(s+t)-F_s]/(1-F_s)$. Although the control variables $(p(t),\hat{p}(t))$ and the state variables $(F(t),G(t))$ in [*P1*] and [*P4*] are different, it is easy to see that the two control problems are actually mathematically identical, because the difference is only a matter of notations. Therefore, $V(F_s) = V(G(0)) = V(0) = V(F_0)$.[11] It is clear that if an optimal solution of *[P1]* exists, then *[P4]* will have the same optimal solution, i.e., $p(t) = \hat{p}(t)$ for any $t\in[0,\infty)$. Since $\hat{p}(t) = p(s+t)$, it follows that $p(t) = p(s+t)$. Since s is arbitrary, $p(t) = p(s+t)$ for any $s\in[0,\infty)$. This is possible if and only if $p(t)$ = constant. Therefore, if an optimal solution of *[P1]* exists, then it must be given by a constant that is independent of time. Differentiate (5) with respect to t and combine with (6), it is obvious that $dp/dt = 0$ if and only if (4) is true. In other words, the constant is the p^* that solves (4). The existence of p^* has been established by Kamien and Schwartz in the second step of their method. It follows immediately that the optimal solution of *[P1]* exists and is given by p^*.

As a byproduct of our approach, the transversality condition (7) is verified. This follows from (5), where $\lambda(t) = e^{-(r-g)t}\pi_1'(p^*)/h'(p^*)$, since $h'(p^*) \neq 0$. Clearly, $\lim_{t\to\infty} \lambda(t) = 0$. Notice that our approach does not require any extra assumptions to be imposed on the model. The function $\pi_1(p)$ can be unbounded at $p = 0$ and $h'(0)$ can be zero. Although the result of Ekeland and Scheinkman (1986) only applies to discrete time models, it is useful to make a comparison here. Their result cannot handle all kinds of unbounded objective functions, e.g., functions such as $-e^{1/p}$, which converges to $-\infty$ very fast as $p \to 0$, are not allowed. It is easy to see that our approach does not suffer from this problem since any kind of unbounded objective functions will not affect the proof.

Our proof is a very general one because it will work for any model in the KS class of models. We illustrate this point by briefly outlining the proofs for examples I and II. In example I, Davis' objective function can be expressed as $e^{-(r-g)t}\{[\pi_1(p(t))-\theta h(p(t))](1-F(t)) + \pi_2(g)F(t)\}$, since $F'(t) = h(p(t))(1-F(t))$. Let $\tilde{\pi}_1(p(t),\theta) = \pi_1(p(t))-\theta h(p(t))$, then the only difference between (1) and Davis' objective function is that $\tilde{\pi}_1(p(t),\theta)$ replaces $\pi_1(p(t))$ in (1), and the proof clearly carries over. One can also verify that the proof also works for the case when the fine θ and the income π_2 depend on the offense rate. In example II, the main difference is that $\pi_2(g)$ of *[P1]* is replaced by $rV(n+1)$. Our proof remains valid because the control problem is autonomous, hence the value function $V(n+1)$ does not depend on time explicitly. Consequently, the objective function does not depend on time explicitly. Our approach reveals clearly that the explicit independency of time in the objective function and the law of motion is one of the key driving forces for the constant solution of the infinite horizon control problem.

As a final remark, we give another example to illustrate the usefulness of our approach. We first show that Lucas' (1971) model of research and development can be expressed in the form of problem *[P1]* and can be solved by our approach. The control problem in Lucas' model (1971, p.686) is

$$\max_{c(.)} \int_0^\infty e^{-rt}[rRF(z(t)) - c(t)(1-F(z(t)))]dt$$

subject to

$$z'(t) = f(c(t))$$

$$z(0) = 0,$$

where r and R are some constants, $c(t)$ is the firm's expenditure on research and development, $f(.)$ is the production function, $z(t)$ is the firm's cumulated effort, and $F(z(t))$ is the probability that the research and development project will be completed when the cumulated effort is $z(t)$. One can reparametrize Lucas' model into *[P1]* by letting $M(t) = F(z(t))$. The law of motion becomes $M'(t) = F'(z(t))z'(t) = F'(F^{-1}(M(t)))f(c(t))$, since $z(t) = F^{-1}(M(t))$ and $z'(t) = f(c(t))$. As $F'(x) = h(x)[1-F(x)]$, where $h(x)$ is the hazard function, then the law of motion can be expressed as $M'(t) = h(F^{-1}(M(t)))[1-F(F^{-1}(M(t)))]f(c(t)) = h(F^{-1}(M(t)))[1-M(t)]f(c(t))$. The initial condition becomes $M(0) = 0$, since $F(0) = 0$. If F is an exponential distribution function, then the hazard function will be a constant, say K. In other words, $h(F^{-1}(M(t))) = K$, so the law of motion becomes $M'(t) = K[1-M(t)]f(c(t))$.[12] The Lucas' problem can then be expressed as

$$\max_{c(.)} \int_0^\infty e^{-rt}[rRM(t) - c(t)(1-M(t))]dt$$

subject to

$$M'(t) = Kf(c(t))[1-M(t)]$$

$$M(0) = 0,$$

which is similar to *[P1]* except that the state variable, the control variable, and the hazard function are now $M(t)$, $c(t)$, and $Kf(c(t))$ respectively. It follows from our approach that the optimal solution to the problem is a constant rate of research and development expenditure. It is useful to compare our method of solution with Lucas'. He makes three assumptions: a piecewise linear functional form for $f(c(t))$, F is a gamma distribution, and the transversality condition is a necessary optimality condition. Again, no justification for the transversality condition is provided. Similar to the discussions in Section 4, one can easily observe that the available results in the literature cannot justify the transversality condition in Lucas' model. Our approach does not rely on the functional form of f(c(t)) and does not rely on the transversality condition.[13] The solution that he obtains is different from the one obtained by our method, which implies that the solution depends on the assumption on the form of the distribution function F. The restrictions that he put on the parameters of the gamma distribution precludes the possiblity of obtaining the exponential distribution as a special case of the gamma distribution.[14]

6. CONCLUSION

In this paper, we examine a class of infinite horizon continuous time optimal control models which have been applied to study a wide range of economic problems. The solution of this class of infinite horizon optimal control problems proposed in the literature depends crucially on the assumption that the transversality condition is a necessary optimality condition. We demonstrate, however, that the transversality condition cannot be satisfactorily justified by the existing results in the literature. The unverified transversality condition calls into question the validity of the widely accepted solution for this class of models.

We provide two different approaches to solve the problem. The first approach is straightforward and requires two boundedness assumptions. These assumptions are restrictive in many economic situations and therefore they put a serious limitation on the applicability of the KS class of models. The second approach provides a complete solution to the KS class of models. It is an entirely different approach which obviates the difficult problem of verifying the transversality condition. This is possible because the approach solves the optimal control problem directly without involving any optimality condition. The approach does not require any additional assumption and yields the transversality condition as a byproduct. This suggests that the transversality condition is valid under very general conditions and further research is needed to demonstrate this conjecture directly. Although the second approach is successful in dealing with the KS class of models, we show that there are limitations in applying it to other models. For example, it solves Lucas' model of research and development only when the distribution function is exponential. To solve the Lucas model for other distribution functions requires the verification of the transversality condition, which has continually been ignored in the literature. Of course, one can always use the first approach to justify the transversality condition by making the boundedness assumptions. As these boundedness assumptions are restrictive in many economic situations, future research should be directed to verify the transversality condition without making these assumptions.

APPENDIX

In this appendix, an example is provided to show that the solution p* and the transversality condition are still valid even when $h_1(0,g)=0$. For brevity, the argument t is suppressed wherever necessary. Assume g = 0 and let

(E1) $$\pi_1(p) = Ap - Bp^2 \quad (A > 0,\ B > 0)$$

(E2) $$\pi_2 = rB/C \qquad (r > 0,\ C > 0)$$

(E3) $$h(p) = Cp^2$$

Assumptions (A1)-(A4) are satisfied. Clearly h'(0) = 0 and assumption (A6) is violated. The Hamiltonian is

(E4) $$H = e^{-rt}[(Ap - Bp^2)(1-F) + \pi_2 F] + \lambda Cp^2(1-F).$$

The necessary conditions are

(E5) $$H_p = [e^{-rt}(A-2Bp) + 2\lambda Cp](1-F) = 0,$$

(E6) $$\lambda' = -H_F = e^{-rt}(Ap-Bp^2-\pi_2) + \lambda Cp^2.$$

Since $F(t) \in [0,1)$ for $t \in [0,\infty)$, (E5) becomes

(E7) $$e^{-rt}(A-2Bp) + 2\lambda Cp = 0.$$

Differentiate (E7) with respect to t and combine with (E6), and use (E7) to eliminate λ, we get

(E8) $$p' = -[rAp + 2(C\pi_2-rB)p^2 - ACp^3]/A.$$

This is a first-order differential equation which does not have a nice closed form solution. By using the simplifying assumption (E2), (E8) becomes

(E9) $$p' = -rp + Cp^3.$$

Now (E9) is a Bernoulli's equation, which can be solved by a change of variable $y = p^{-2}$ [Coddington (1961, p.46)]. The solution is

(E10) $$p(t) = [C/r + Ke^{2rt}]^{-1/2},$$

where $K = [1/(p(0))^2] - (C/r)$. Thus, after some manipulations, we get

(E11) $$F(t) = 1 - \exp[-\int_0^t C(p(s))^2]ds = 1 - [e^{-rt}p(0)/p(t)].$$

Let V denote the value function (maximized value of (1) subject to (2) and (3)) for problem [*P1*]. Substituting (E10) and (E11) into (1) and integrating, we have a nice closed form solution for V:

(E12) $$V = Ap(0)/2r = (A/2r)(C/r + K)^{-1/2}.$$

Since p(t) must be a nonnegative real number, (E10) implies that K must also be nonnegative, otherwise $C/r + Ke^{2rt}$ will become negative for some sufficient large t. It follows from (E12) that V attains its maximum when K = 0, hence the optimal solution is $p^* = (C/r)^{-1/2}$, a constant. Clearly in this case, p^* is also the unique optimal solution.

Notice that we have not invoked any transversality condition to prove the optimality of the solution. On the other hand, the optimal solution p* obviously satisfies the transversality condition (7) since $\lim_{t\to\infty} \lambda(t)F(t) = \lim_{t\to\infty} -e^{-rt}[A-2Bp(t)]\{1 - [e^{-rt}p(0)/p(t)]\}/2Cp(t) = 0$ for $p(t) = p^*$.

FOOTNOTES

1. A nice review of the literature on limit pricing can be found in Kamien (1987).

2. See for example, Luenberger (1969) or Kamien and Schwartz (1981).

3. There are similar work in the discrete time literature. Since we only deal with continuous time models in this paper, we do not reference these work here. See Ekeland and Scheinkman (1986) for references.

4. A more familiar example may be income tax evasion. Let Y be a person's income, τ be the tax rate, and p(t) be the declared income, then $Y - \tau p(t)$ will be the person's post-tax income. The person derives utility $\pi_1(Y-\tau p(t))$ from the post-tax income. This example is a continuous time analog of the Allingham-Sandmo dynamic income tax evasion model, see Allingham and Sandmo (1972) for details.

5. This includes the books of Kamien and Schwartz (1981, p.206-208; 1982).

6. There is a subtle difference among these results. The transversality condition that Bensoussan, Hurst and Naslund, Aubin and Clarke, and Michel prove is that the multiplier converges to zero, while the transversality condition that Benveniste and Scheinkman, and Araujo and Scheinkman prove is that the product of the multiplier and the state variable converges to zero. Clearly, neither one implies the other. If the state variable is bounded (which is the case in this paper), then the two transversality conditions are equivalent.

7. See Bensoussan, Hurst, and Naslund (1974, p.274-275, eq. (7.34)).

8. If this is the case, then one may wonder why Aubin and Clarke need so many strong assumptions to obtain the transversality condition. The strong assumptions that they make allow them to deal with nondifferentiable nonconcave objective functions and derive stronger transversality conditions than the conventional ones found in the literature (such as the ones that we want to verify in this paper). They argue that the transversality conditions that they derive (which they call growth or dual conditions) are more natural than the conventional ones. See Aubin and Clarke (1979, p.569) for their arguments.

9. The condition $\pi_1(p) \to -\infty$ as $p \to 0$ is the classical assumption in growth theory first proposed by Koopmans (1965). His argument for this assumption (p.241) is that it provides "a strong incentive to avoid periods of very low consumption as much as is feasible."

10. One may compare the value function in [*P3*] with the one in Kamien and Schwartz (1971, p.450, eq. (30)). It is clear that their formulation is erroneous because they have ignored the additional information that there is no entry by time s.

11. Note that $V(F_s) = V(0)$ implies $\partial V(F_s)/\partial F_s = 0 \neq \lambda(s)$, which seems to be contradictory to the well known standard result that the multiplier should be equal to the derivative of the value function with respect to the state variable (if the derivative exists). This apparent contradiction can easily be resolved by noting that the objective function in [*P3*] also depends on F_s explicitly, which is different from the standard case. The contradiction will disappear when one takes care of these extra terms in the differentiation.

12. It is interesting to note that the exponential distribution is a common assumption in the research and development literature, see for example, Loury (1979). The exponential distribution is a key assumption that drives the results in Loury (1979).

13. One can also observe that Lucas' solution is very involved (e.g., it occupies 8 pages) while our approach is relatively much simpler.

14. From Lucas' model, one can observe the usefulness and also the limitation of our approach. One can check that our approach is applicable only if $M'(t)$ is proportional to $1-M(t)$. This implies that $h(F^{-1}(M(t)))$ must be a constant, which in turn implies that F must be an exponential distribution, because the exponential distribution is the only distribution with a constant hazard function.

REFERENCES

ALLINGHAM, M.G. and A. SANDMO (1972): "Income Tax Evasion: A Theoretical Analysis," Journal of Public Economics, 1, 323-338.

ARAUJO, A., and J. A. SCHEINKMAN (1983): "Maximum Principle and Transversality Condition for Concave Infinite Horizon Economic Models," Journal of Economic Theory, 30, 1-16.

ARROW, K. J., and M. KURZ (1970): Public Investment, the Rate of Return, and Optimal Fiscal Policy. Baltmore: Johns Hopkins.

AUBIN, J. P., and F. H. CLARKE (1979): "Shadow Prices and Duality for a Class of Optimal Control Problems, SIAM Journal of Control and Optimization, 17, 567-586.

BARON, D. P. (1972): "Limit Pricing and Models of Potential Entry," Western Economic Journal, 10, 298-307.

BARON, D. P. (1973): "Limit Pricing, Potential Entry, and Barriers to Entry," American Economic Review, 64, 666-674.

BENSOUSSAN, A., E. G. HURST, and B. NASLUND (1974): Management Applications of Modern Control Theory. Amsterdam: North-Holland.

BENVENISTE, L. M., and J. A. SCHEINKMAN (1982): "Duality Theory for Dynamic Optimization Models of Economics: The Continuous Time Case," Journal of Economic Theory, 27, 1-19.

CODDINGTON, E. (1961): An Introduction to Ordinary Differential Equations. New Jersey: Prentice-Hall.

DAVIS, M. L. (1988): "Time and Publishment: An Intertemporal Model of Crime," Journal of Political Economy, 96, 383-390.

DE BONDT, R. (1976): "Limit Pricing, Uncertain Entry, and the Entry Lag," Econometrica, 44, 939-946.

DESHMUKH, S. D., and S. D. CHIKTE (1976): "Dynamic Pricing with Stochastic Entry," Review of Economic Studies, 63, 91-97.

EKELAND, I. and J. SCHEINKMAN (1986): "Transversality Conditions for Some Infinite Horizon Discrete Time Optimization Problems," Mathematics of Operations Research, 11, 216-229.

HALKIN, H. (1974): "Necessary Conditions for Optimal Control Problems with Infinite Horizons," Econometrica, 42, 267-272.

KAMIEN, M. I. (1987): "Limit Pricing," The New Palgrave Dictionary of Economics. London: Macmillan.

KAMIEN, M. I. and N. L. SCHWARTZ (1971): "Limit Pricing and Uncertain Entry," Econometrica, 39, 441-454.

__________ (1972a): "Timing of Innovations under Rivalry," Econometrica, 40, 43-60.

__________ (1972b): "Uncertain Entry and Excess Capacity," American Economic Review, 62, 918-827.

__________ (1975): "Cournot Oligopoly with Uncertain Entry," Review of Economic Studies, 47, 125-131.

__________ (1981): Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management. New York: North-Holland.

__________ (1982): Market Structure and Innovation. Cambridge: Cambridge University Press.

KOOPMANS, T. (1965): "On the Concept of Optimal Economic Growth," in The Econometric Approach to Development Planning. Amsterdam: North Holland.

LIPPMAN, S. A. (1980): "Optimal Pricing to Retard Entry," Review of Economic Studies, 47, 723-731.

LOURY, G. (1979): "Market Structure and Innovation," Quarterly Journal of Economics, 93, 395-410.

LUCAS, R.E. Jr. (1971): "Optimal Management of a Research and Development Project," Management Science, 17, 679-697.

LUENBERGER, D. G. (1969): Optimization By Vector Space Methods. New York: John Wiley.

MICHEL, P. (1982): "On the Transversality Condition in Infinite Horizon Optimal Problems," Econometrica, 50, 975-985.

PHLIPS, L. (1980): "Intertemporal Price Discrimination and Sticky Prices," Quarterly Journal of Economics, 94, 525-542.

SHELL, K. (1969): "Applications of Pontryagin's Maximum Principle to Economics," in Mathematical Systems Theory and Economics I, ed. by H. W. Kuhn and G. P. Szego. Berlin: Springer-Verlag.

STIGER, G. (1968): The Organization of Industry. Chicago: University of Chicago Press.