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1. Introduction

Much work has been done to modify Dickey and Fuller (1979, 1981) tests for unit roots to allow for general distributions and serial correlation of the disturbance term and various order of deterministic trends (see, e.g., Said and Dickey (1984), Phillips (1987), Phillips and Perron (1988), Ouliaris, Park, and Phillips (1988)). Dejong, Nankervis, Savin, and Whiteman (1988) study small sample properties of Said-Dickey and Phillips-Perron tests and conclude that these tests are practically useless against trend stationary alternatives which are plausible for annual, quarterly, and monthly macroeconomic time series. The problem of these tests is that they are not powerful when the autoregressive root is close to one and the sample size is small. Rather than modifying Dickey-Fuller tests, we develop a χ^2 test for a unit root that has greater power than Dickey-Fuller tests when the autoregressive root is close to one and the sample size is small. Though it is beyond the scope of the present paper to develop tests that are practically useful in the sense of DeJong *et al*, it is important to investigate alternative tests to Dickey-Fuller tests given that modified versions of Dickey-Fuller tests do not have high power in small samples.

Our test does not rely on a functional central limit theorem as most of the other existing tests do, but on a general central limit theorem. Though functional central limit theorems apply under general conditions (see, e.g., McLeish (1975)), central limit theorems apply under even more general conditions. For example, we can allow for deterministic seasonal fluctuations in the time series to be tested while Dickey-Fuller tests do not. This is useful because seasonally unadjusted data are of interest in most applications. We exploit some discontinuities when autoregressive root test with a null of stationarity utilizing these discontinuities.

The null of stationarity is much more attractive than the null of unit roots in many cases as emphasized by Hatanaka and Fukushige (1989) among others. This is especially true in the context of cointegration theory. However, most tests for unit roots employ the null of unit roots, and hence most tests for cointegration take the null of no cointegration against cointegration or the null of a smaller number of cointegrating vectors against a larger number of cointegrating vectors.

2. Regression Properties

Consider a stochastic process $\{x_t : t \geq 1\}$ generated in discrete time according to

$$(1) \qquad x_t = \alpha x_{t-1} + u_t \qquad (t=1,2,\ldots)$$

where $\{u_t\}$ is a sequence of random variables with mean zero. We are interested in the following two hypotheses:

HU: $\alpha=1$,

and

HS: $|\alpha|<1$.

The initial value, x_0, is allowed to be any random variable.

To motivate the tests we develop, let us consider the regression

$$(2) \qquad x_t = \beta \Delta x_t + e_t .$$

First, let us consider the case where hypothesis HS is true. For now, assume that x_t is covariance stationary. In this case the best linear predictor of x_t given Δx_t is $0.5\Delta x_t$. To see this, define

$$(3) \qquad \eta_t = \Delta x_t [x_t - 0.5\ \Delta x_t] = 0.5(x_t^2 - x_{t-1}^2).$$

Then $E(\eta_t)=0$ as long as x_t is covariance stationary so that $E(x_t^2)=E(x_{t-1}^2)$. Thus $\beta\Delta x_t$ satisfies the orthogonality condition $E\{\Delta x_t[x_t-\beta\Delta x_t]\}=0$ at $\beta=0.5$. This result is convenient to treat a composite null hypothesis of $|\alpha|<1$ because the true value of β does not depend on the value of α.

Second, let us consider the case where hypothesis HU is true. For now, assume that u_t is serially uncorrelated and that x_0 has a finite second moment, so that the least square prediction theory is applicable. In this case $E(\eta_t)\neq 0$ because $E(x_t^2)\neq E(x_{t-1}^2)$. Hence $\beta\neq 0.5$. In fact, $E(x_t\Delta x_t)/E[(\Delta x_t)^2] = E(x_t u_t)/E[(u_t)^2] = 1$ if $E(x_0 u_t)=0$ for $t\geq 1$. In this sense, $\beta=1$ when $\alpha=1$. This discontinuity of β as α approaches one motivates our tests.

3. Asymptotic properties of OLS Estimators

In this section, properties of the OLS estimator of β in equation (5) is studied. Let b_T be the OLS estimator:

$$(4) \qquad b_T = [\sum_{t=1}^{T} (\Delta x_t)^2]^{-1}[\sum_{t=1}^{T} x_t \Delta x_t].$$

Since $\sum_{t=1}^{T}\eta_t = 0.5(x_T^2 - x_0^2)$,

(5) $$(b_T - 0.5) = [T^{-1} \sum_{t=1}^{T} (\Delta x_t)^2]^{-1} 0.5\ T^{-1}(x_T^2 - x_0^2).$$

The key requirement for our test, when the null is that of a unit root (hypothesis HU), is that a central limit theorem applies to the partial sum of u_t. To be precise, let

(6) $$P_t = \sum_{j=1}^{t} u_j$$

be the partial sum. We require that $(1/T)^{1/2}P_T$ converges in distribution to $N(0,\sigma^2)$, where

(7) $$\sigma^2 = \lim_{T\to\infty} E(T^{-1}P_T^2) > 0$$

as the sample size T approaches ∞ (all the limits in this paper is taken as $T\to\infty$). This requirement is satisfied under very general conditions. In the following, we assume that u_t is stationary and ergodic with finite second moments, and apply Gordin's Theorem (see, e.g., Hansen [1985]) so that

(8) $$\sigma^2 = \lim_{\ell\to\infty} \sum_{\tau=-\ell}^{\ell} [(\ell - |\tau|)/\ell] E(u_t u_{t-\tau}).$$ [1]

Alternatively, we can use a central limit theorem for weakly dependent random variables (see, e.g., Serfling [1968] and White and Domowitz [1984]).

Noting that the seasonal dummy can be artificially viewed as stationary and ergodic stochastic process (see, e.g., footnote 1 on page 26 of Ogaki (1988)), analysis in frequency domain by Hansen (1985) shows Gordin's Theorem is applicable to stochastic processes with deterministic seasonal fluctuations.

Let (Ω,F,P) be the underlying probability space. Let S denote a measurable transformation mapping Ω into itself. Then an arbitrary random variable, say h_0, and S together generates a time series via the relation $h_t(\omega)=h_0(S^t(\omega))$. We assume that S is constructed so that $u_t(\omega)=u_0(S^t(\omega))$. Let B_0 be a subsigma algebra of F and $B_t=\{b_t$ in F: $b_t=\{\omega:S^t(\omega)$ is in $b_0\}$ for some b_0 in $B_0\}$. Define the set $G=\{g$: g is a random variable that is measurable with respect to B, has a finite second moment, and $E(g|B_{-\tau})=0$ for some nonnegative $\tau\}$, and the notation $P_T(h_0)=\sum_{t=1}^{T}h_t$

Assumption 1: The transformation S is stationary and ergodic, u_0 is measurable with respect to B_0, u_0 has a finite second moment, $\inf_{g\in G}\lim\sup_{T\to\infty} E[T^{-1}\{P_T(g-u_0)\}^2]=0$, and $\sigma^2=\lim_{\ell\to\infty}\sum_{\tau=-\ell}^{\ell}[(\ell-|\tau|)/\ell]E(u_tu_{t-\tau})>0$.

Assumption 1 allows us to apply Gordin's theorem.

The next result shows that b_T does not converge in probability, but converges in distribution to a nondegenerate random variable under hypothesis HU. Let $\sigma_u^2 = E(u_t^2)$.

Theorem 1: Suppose that assumption 1 is satisfied and hypothesis HU is true. Then $\{b_T-0.5:T\geq1\}$ converges in distribution to $0.5(\sigma^2/\sigma_u^2)\chi_1^2$, where χ_1^2 is a random variable with the Chi-square distribution with one degree of freedom.

Proof: Set $P_0=0$. Then $x_t=x_0+P_t$, and $T^{-1}P_T$ converges to zero in distribution $(1/T)^{1/2}\sigma^{-1}P_T$ converges in distribution to a normally distributed random variable with mean zero and variance one. Since $x_T^2-x_0^2=P_T^2+x_0T^{-1}P_T$, by (5)

$$b_T-0.5 = (T^{-1}\sum_{t=1}^{T}u_t^2)^{-1}0.5\{\sigma^2(T^{-1/2}\sigma^{-1}P_T)^2 - x_0(T^{-1}P_T)\} \tag{9}$$

Thus b_T-0.5 converges in distribution to $0.5(\sigma^2/\sigma_u^2)\chi_1^2$. Q.E.D.

It should be noted that the term which involves x_0 in (9) is negligible asymptotically but may have a strong impact on the small sample properties of b_T.

The next theorem shows that b_T is a consistent estimator of β under hypothesis HS.

Theorem 2: Suppose that assumption 1 is satisfied and that hypothesis HS is true. Then $\{T^{\epsilon}(b_T-0.5):T\geq1\}$ converges to zero in probability for any $\epsilon<1$.

Proof: In equation (5), $T^{-1}\sum_{t=1}^{T}(\Delta x_t)^2$ converges almost surely to $E[(\Delta x_t)^2]$, and $T^{1-\epsilon}(x_T^2-x_0^2)$ converges in probability to zero. The conclusion follows immediately from equation (5). Q.E.D.

There are two types of discontinuity when α approaches one from below which we exploit in this paper. First, b_T converges in probability to 0.5 when $|\alpha|<1$, while b_T has mean $0.5(1+\sigma^2/\sigma_u^2)$ when $\alpha=1$. Second, b_T-0.5 is of order T^{-1} in probability if $|\alpha|<1$, while b_T-0.5 is of order 1 in probability if $\alpha=1$.

4. A Consistent Test for a Unit Root

Theorem 1 and 2 suggest a simple χ^2 test based on the OLS estimator b_T with the null of a unit root (hypothesis HU). Under the null of a unit root, $2(\sigma_u^2/\sigma^2)(b_T-0.5)$ converges in distribution to χ_1^2. Since the ratio σ_u^2/σ^2 is an unknown parameter in general, we need to estimate this parameter to construct a test statistic.

Suppose that s_{uT}^2 and s_T^2 are consistent estimators for σ_u^2 and σ^2 under the null hypothesis HU. We also require that s_{uT}^2/s_T^2 converges in probability to a positive real number under the alternative hypothesis HS. These requirements are met by the following estimators. Let a_T be the OLS estimator of α in equation (1) and

$$(10) \qquad s_{uT}^2 = T^{-1}\sum_{t=1}^{T}(x_t - a_T x_{t-1})^2$$

$$(11) \qquad s_T^2 = T^{-1}\sum_{\tau=-\ell(T)}^{\ell(T)}[\ell(T)-|\tau|)/\ell(T)]\sum_{t=1}^{T}[(x_t - a_T x_{t-1})(x_{t-\tau} - a_T x_{t-1-\tau})]$$

where $\ell(T)$ is the lag truncation number that satisfies $\ell(T)=O(T^{1/4})$. The test statistic we propose is

$$(12) \qquad J_T = 2(s_{uT}^2/s_T^2)(b_T - 0.5).$$

Under the null of a unit root, J_T converges in distribution to χ_1^2 in the light of theorem 1. This test rejects the null of a unit root when J_T is smaller than some critical value.

Now let us consider the asymptotic property of J_T under the alternative hypothesis HS. Theorem 2 implies that J_T is $O_p(T^{-1})$ in this case. Hence this test is consistent.

5. A Consistent Test for Stationarity

In this section, we develop a consistent test with the null of stationarity. We need a much stronger assumption for this purpose:

Assumption 2: $\{u_t: t \geq 1\}$ is a sequence of independent normal random variables with mean zero and variance σ_u^2 (i.e., u_t NID(0, σ_u^2)).

For hypothesis HS, we require that the process $\{x_t: t \geq 0\}$ is stationary. Thus if hypothesis HS is true, then $|\alpha|<1$ and the initialization, x_0, is a random variable with the unique stationary distribution, $N[0, \sigma_u^2/(1-\alpha^2)]$. When $\alpha=1$ as in hypothesis HU, x_0 is allowed to be any random variable.

The next theorem shows that b_T is a consistent estimator of β under hypothesis HS and gives asymptotic distribution of b_T under HS.

Theorem 3: Suppose that assumption 2 is satisfied and that hypothesis HS is true. Then $\{T^{\epsilon} b_T: T \geq 1\}$ converges to zero in probability for any $\epsilon<1$, and $\{T(b_T-0.5): T \geq 1\}$ converges in distribution to $\{1/[4(1-\alpha)]\}(y_1-y_2)$ where y_1 and y_2 are independent Chi-square variates with one degree of freedom.

Proof: In equation (5) $T^{-1} \sum_{t=1}^{T} (\Delta x_t)^2$ converges almost surely to $E[(\Delta x_t)^2]=[2/(1+\alpha)]\sigma_u^2$, and x_T converges in distribution to a random variable with the stationary distribution, $N[0, \sigma_u^2/(1-\alpha^2)]$ that is independent of x_0. Hence $(x_T^2-x_0^2)/[\sigma_u^2/(1-\alpha^2)]$ converges in distribution to y_1-y_2. The conclusion follows immediately from equation (5). Q.E.D.

Theorem 3 shows that $4(1-\alpha)T(b_T-0.5)$ converges in distribution to the difference of two independent Chi-square variates with one degree of freedom, whose density function was derived in Miller (1964, Corollary 3 on p.65). Since α is unknown, we replace α by the OLS estimator of α in equation (1), which we denote by a_T. However, a_T may not satisfy the condition $|a_T|<1$. Hence we choose a constant c_T depending on the sample size that is smaller than one in absolute value, and we replace α by c_T

instead of a_T if $|a_T|>c_T$. When we make c_T approach one at a slow enough rate, we obtain a consistent test. Specifically, we choose a sequence of real numbers $\{c_T:T\geq 1\}$ that satisfies the following two conditions: (i) $|c_T|<1$, and (ii) $1-c_T = O(T^{-(1-\delta)})$ and $\lim_{T\to\infty}T^{1-\delta+\epsilon}(1-c_T)=\infty$ for any $\epsilon>0$. Define a sequence of functions

$$(13) \qquad \phi_T(a)=\begin{cases} a & \text{if } -1<a<c_T \\ c_T & \text{otherwise.}\end{cases}$$

for any real number a. The test statistic we propose is

$$(14) \qquad K_T = 4\{1-\phi_T(a_T)\}T(b_T-0.5).$$

If $|\alpha|<1$, then $\{K_T:T\geq 1\}$ converges in distribution to a random variable that is the difference between two independent chi-square variates with one degree of freedom. This follows from the fact that $\phi_T(a_T)$ converges in probability to α because asymptotically $|a_T|<c_T$ and hence $\phi_T(a_T)=a_T$.

Next, consider the case where $\alpha=1$ to show that the K_T test is consistent against the alternative of $\alpha=1$. In this case r_T converges to one, and $a_T-1=O_p(T^{-1})$. Since a_T converges to one faster than c_T does, asymptotically $c_T<a_T$ and $\phi_T(a_T)=c_T$. Thus $1-\phi_T(a_T)=O_p(T^{-(1-\delta)})$ and $T(1-\phi_T(a_T))$ diverges. Since $b_T-0.5$ converges in distribution as shown in Theorem 1, $K_T=O_p(T^{\delta})$ and K_T diverges if $\alpha=1$. Thus the test based on K_T is consistent against the alternative of a unit root.

We have shown the following result.

Theorem 4: Under hypothesis HS, K_T converges in distribution to y_1-y_2 where y_1 and y_2 are independent Chi-square variates with one degree of freedom.

Under the hypothesis HU, $K_T=O_p(T^{\delta})$, and this test is consistent.

6. Finite Sample Properties

We compare finite sample properties of the J_T tests with those of conventional Dickey-Fuller tests, using simulations based on 40,000 replications. Data were generated by the model (1) with the u_t independent and identically distributed N(0,1).[2] We consider a situation in which an econometrician knows that u_t is serially uncorrelated. Since $\sigma^2=\sigma_u^2$ in this case, we define $J_T = 2(b_T-0.5)$. Two versions of Dickey-Fuller tests used are the test statistic $T(a_T-1)$, which we denote by $\alpha(1)$, and the t test for the hypothesis $\alpha=1$ in the regression (1), which we denote by t(1). In the following, x_0 is an $N(0,1/(1-\alpha^2))$ random variable that is independent of the u_t (t=1,...,T) when $|\alpha|<1$, so that x_t is stationary.

Table 1 reports finite sample powers when the five percent critical values implied by asymptotic theories are used. The results are relevant when the econometrician does not know the true distributions and does not correct for small sample size distortions. In this case, our K_T test has much greater power than the $\alpha(1)$ and t(1) tests when α is close to one and the sample size is small. On the other hand, Dickey-Fuller tests have higher power than the K_T test when the sample size is 200 and α is 0.95.

Table 2 reports finite sample size calculations when the five percent critical values implied by asymptotic theories are used. The initial value, x_0, was either fixed to zero or a $N(0,1/(1-\rho^2))$ random variable with $\rho=0.95$ or $\rho=0.99$. When the initial value is fixed to zero, there are little size

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[2]We used the RNDN function of GAUSS to create normal (pseudo) random variables.

distortions for the K_T and t(1) tests while the $\alpha(1)$ test is conservative when the sample size is small. When the initial value is a $N(0,1/(1-\rho^2))$ random variable, the K_T test has significant size distortions and is very liberal. The $\alpha(1)$ test is very conservative while the t(1) test has little size distortions.

Table 3 reports size adjusted powers. The K_T test has higher power than Dickey-Fuller tests when $\alpha=0.99$ and the sample size is 50 or 100. The $\alpha(1)$ and t(1) tests have higher power then the K_T test when $\alpha=0.95$ for any sample size and when $\alpha=0.99$ and the sample size is 200.

7. Concluding Comments

The present paper developed a chi-square test for a unit root that has higher power than Dickey-Fuller tests when the sample size is small and the autoregressive root is close to one. Since this is exactly where unit root tests have most difficulties, our test may have some merit. It also developed a consistent test for the null of stationarity. Phillips and Ouliaris (1988) discuss difficulties in testing the null of cointegration against the alternative of no cointegration. Because of these difficulties, we impose stringent assumptions about distributions and serial correlations of the disturbance term to develop a test for stationarity. This is in contrast to our test for the null of a unit root, which requires only very mild conditions.

Hatanaka and Fukushige (1989) developed a test with the null of stable autoregressive roots against a unit or explosive root. Their test allow more general serial correlation. However, the Hatanaka and Fukushige's test does not take the whole region of stable roots as the null hypothesis and leaves out of the null hypothesis roots close to one. Our test takes the entire region of a stable root as the null.[3]

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[3]Park, Ouliaris, and Choi (1988), Park and Choi (1988), and Park (1988) have independently developed different tests for the null of stationarity and cointegration.

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TABLE 1

EMPIRICAL POWERS USING FIVE PERCENT NOMINAL CRITICAL VALUES

T	α	K_T	$T(a_T-1)$	t_1
50	0.95	0.518	0.096	0.184
50	0.99	0.505	0.021	0.085
100	0.95	0.533	0.282	0.373
100	0.99	0.507	0.041	0.103
200	0.95	0.552	0.745	0.791
200	0.99	0.514	0.083	0.146

NOTE: Estimates obtained from 40,000 replications.

TABLE 2

EMPIRICAL SIZES USING FIVE PERCENT NOMINAL VALUES

T		K_T	$T(\alpha_T-1)$	t_1
50	$x_0=0$	0.049	0.042	0.051
50	$\rho=0.95$	0.241	0.028	0.051
50	$\rho=0.99$	0.356	0.016	0.051
100	$x_0=0$	0.051	0.045	0.050
100	$\rho=0.95$	0.193	0.038	0.051
100	$\rho=0.99$	0.308	0.024	0.051
200	$x_0=0$	0.051	0.048	0.050
200	$\rho=0.95$	0.148	0.043	0.050
200	$\rho=0.99$	0.261	0.032	0.050

NOTE: Estimates obtained from 40,000 replications.

TABLE 3

SIZE ADJUSTED POWERS OF FIVE PERCENT TESTS

T	$\alpha=\rho$	K_T	$T(\alpha_T-1)$	t_1
50	0.95	0.116	0.166	0.180
50	0.99	0.098	0.068	0.083
100	0.95	0.141	0.354	0.365
100	0.99	0.109	0.085	0.101
200	0.95	0.174	0.789	0.790
200	0.99	0.117	0.131	0.148

NOTE: Estimates obtained from 40,000 replications.