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Abstract

Recently, Moulin gave various axiomatic characterizations of solutions to quasi-linear social choice problems. He used a consistency axiom, which relates solutions for societies of different sizes, in addition to some basic axioms. In this paper, we introduce another axiom relating solutions for societies of different sizes, called the "Solidarity Axiom". This axiom demands that when additional agents enter the scene, all of the original agents be affected in the same direction, i.e., all of them gain or all of them lose. Our main result is a complete characterization of solutions satisfying the solidarity axiom, in addition to Pareto optimality, anonymity and two normalization axioms. All solutions satisfying these five axioms are in the egalitarian spirit; each solution assigns to every agent an equal share of the surplus over some reference level, but uses a different method to compute the reference level. Then, using additional milder axioms, we give further characterization results concerning various subfamilies.

1. Introduction.

We consider the following class of quasi-linear social choice problems. A society must choose one among a finite number of public decisions; money is available to perform side payments. Each agent has quasi-linear preferences (separably additive with respect to the public decisions and money and linear with respect to money). We are interested in determining what decision should be chosen and what side payments among agents should be performed.

In [3], Moulin proposes various axiomatic characterizations of solutions to quasi-linear social choice problems. In addition to four basic axioms, Pareto optimality, anonymity and two normalization conditions, he uses a consistency axiom,[1] which relates solutions for societies of different sizes, to characterize various subfamilies of solutions in the egalitarian spirit; each solution assigns to every agent an equal share of the surplus over a reference level, but uses a different method of computing the reference level.

In this paper, we introduce another axiom relating solutions for societies of different sizes, called the "Solidarity Axiom". It is a natural variant of an axiom first introduced by Thomson [5] in the framework of bargaining theory under the name of "monotonicity with respect to changes in the number of agents" and used by him [5,6] to characterize the Kalai-Smorodinsky solution and the egalitarian solution. More specifically, the solidarity axiom demands that when additional agents enter the scene, all of the original agents be affected in the same direction, i.e., all of them gain or all of them lose. The purpose of this paper is to explore the implications of the solidarity axiom for solving quasi-linear social choice problems.

Our main result is a complete characterization of solutions satisfying the solidarity axiom, in addition to the four basic axioms of Moulin's. All solutions satisfying these five

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1 Moulin called this the "Separability axiom".

axioms are also in the egalitarian spirit, but they constitute a family that is different from the family identified by Moulin. Then, using additional axioms introduced by Moulin [3], we give further characterization results concerning various subfamilies, most of which have been identified and alternatively characterized by Moulin.

The paper is organized as follows. Section 2 contains some preliminaries, and introduces the concept of a solution and the basic axioms. Section 3 presents the solidarity axiom and section 4 contains the main characterization result and its proof. In section 5, we introduce additional axioms and give further characterization results.

2. Preliminaries.

The framework of analysis is taken from Moulin [3] (it is also related to Green [1] and Moulin [2]).

Let $I \equiv \{1,2,...\}$ be the (infinite) universe of "potential" agents. Agent i in I is indexed by the subscript i. $\mathcal{N}$ is the class of subsets of I. Let $N \in \mathcal{N}$ be a society with members 1,...,n, and A be the set of public decisions. Each decision a has a cost c(a) that must be covered by a vector $t \equiv (t_1,...,t_n)$ of monetary transfers across agents. An outcome, which is chosen by the society, is a pair (a,t) where $\Sigma_{i \in N}\, t_i + c(a) = 0$. Every agent i in N has "quasi-linear preferences" over the set $A \times \mathbb{R}$, i.e., agent i's preferences can be described by a utility vector $u_i \equiv (u_i(a))_{a \in A}$ in $\mathbb{R}^A$,[2] so that his utility for outcome (a,t) is $u_i(a)+t_i$. Let $c \equiv (c(a))_{a \in A}$ in $\mathbb{R}^A$ be the cost vector and $u \equiv (u_1,...,u_n)$ in $[\mathbb{R}^A]^n$ be the utility profile.

Also, let $\Pi \in \mathbb{R}^A$ be defined by $\Pi(a) = 1$ for all $a \in A$.

Definition. Given a society $N = \{1,...,n\}$ and a finite set A of public decisions, a solution

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[2] For convenience, we use $\mathbb{R}^A$ instead of $\mathbb{R}^{|A|}$

for the society N is a function $S^N : [\mathbb{R}^A]^{(n+1)} \rightarrow \mathbb{R}^N$, which associates to any (n+1)-tuple made up of a utility profile u and a cost vector c, a vector $S^N(u,c) \equiv (S_1^N(u,c),...,S_n^N(u,c))$ of utility levels. A solution is a list $\{S^N : N \in \mathcal{N}\}$.

We impose the following axioms on solutions;

Pareto optimality (PO). For all societies N, for all profiles u and for all cost vectors c,

$$\Sigma_{i \in N} S_i^N(u,c) = \max_{a \in A} \{ \Sigma_{i \in N} u_i(a) - c(a)\}.$$

Anonymity (AN). For all societies N, for all permutations ρ of N, for all $i \in N$, for all profiles u and for all cost vectors c,

$$S_i^N(u,c) = S_{\rho(i)}^N(u',c) \quad \text{where } u' = (u_{\rho(i)})_{i \in N}.$$

Independence of the individual utilities' zero (IND1). For all societies N, for all $i \in N$, for all profiles u, v, for all cost vectors c and for all $\alpha \in \mathbb{R}$, if $v_i = u_i + \alpha\Pi$ and $v_j = u_j$ for all $j \neq i$, then $S_i^N(v,c) = S_i^N(u,c) + \alpha$ and $S_j^N(v,c) = S_j^N(u,c)$ for all $j \neq i$.

Independence of the cost vector zero (IND2). For all societies N, for all profiles u, for all cost vectors c, c' and for all $\alpha \in \mathbb{R}$, if $c' = c + \alpha\Pi$, then $S_i^N(u,c') = S_i^N(u,c) - \frac{\alpha}{n}$ for all $i \in N$.

PO requires that a society picks a decision which maximizes the difference between the sum of individual utilities and the cost. AN says that the solution should be a symmetrical function of individual utility vectors. IND1 requires that the zero of individual utility vectors does not play any role. Finally, IND2 requires that the zero of cost vectors does not play any role.

In what follows, solutions are assumed to satisfy these four axioms. Since S^N satisfies the anonymity axiom, from now on we use the notation S^n instead of S^N. So a solution S is denoted $\{S^1,...,S^n,...\}$. Given $x \in \mathbb{R}^A$, we define $x^{max} \equiv \max_{a \in A} x(a)$ and for any coalition $T \subseteq N$, we define $S_T = \Sigma_{i \in T} S_i$, $u_T = \Sigma_{i \in T} u_i$, and so on.

Below, we define several families of solutions. All these solutions are in the egalitarian spirit; each solution assigns to every agent an equal share of the surplus over a "reference level", but uses a different method of computing the reference level.

Definitions.

(a) A solution belongs to the O-family if there exists a function $g : [\mathbb{R}^A]^2 \to \mathbb{R}$ such that

(i) $g(x+\alpha\Pi,z) = g(x,z) + \alpha$ for all $x, z \in \mathbb{R}^A$ and all $\alpha \in \mathbb{R}$,

(ii) $g(0,z) = 0$ for all $z \in \mathbb{R}^A$,

(iii) $g(x,z+\alpha\Pi) = g(x,z)$ for all $x, z \in \mathbb{R}^A$ and all $\alpha \in \mathbb{R}$,

and such that for all n, S^n is defined by

$$S_i^n(u,c) = \frac{1}{n}(u_N - c)^{max} + \frac{1}{n}\{(n-1)g(u_i,c) - \Sigma_{j\neq i}\, g(u_j,c)\}$$

for all i, for all profiles u and for all cost vectors c.

(b) A solution belongs to the I-family if the function g defined above does not depend on its second argument, that is, for some function $\tilde{g} : \mathbb{R}^A \to \mathbb{R}$ satisfying

$$\tilde{g}(x+\alpha\Pi) = \tilde{g}(x) + \alpha \quad \text{for all } x \in \mathbb{R}^A \text{ and } \alpha \in \mathbb{R},$$

$$\tilde{g}(0) = 0,$$

the solution can be written as

$$S_i^n(u,c) = \frac{1}{n}(u_N - c)^{max} + \frac{1}{n}\{(n-1)\tilde{g}(u_i) - \Sigma_{j\neq i}\, \tilde{g}(u_j)\} \quad \text{for all n, i, u and c.}$$

(c) A solution belongs to the M-family if it belongs to the I-family with $\tilde{g}$ satisfying one additional property, namely,

$$\text{for all } x, y \in \mathbb{R}^A,\ x \leq y \text{ implies } \tilde{g}(x) \leq \tilde{g}(y).$$[3]

(d) A solution belongs to the Q-family if, for some vector $\sigma = (\sigma_a)_{a\in A}$ such that $\Sigma_{a\in A}\, \sigma_a = 1$ and $\sigma_a \geq 0$ for all $a \in A$,

$$g(u_i,z) = u_i\cdot\sigma \quad \text{for all i and } u_i.$$

[3] Given two vectors x and y in $\mathbb{R}^A$, $x \geqq y$ means $x_i \geqq y_i$ for all i, $x \geq y$ means $x \geqq y$ and $x \neq y$, $x > y$ means $x_i > y_i$ for all i.

Then the solution can be written as

$$S_i^n(u,c) = \frac{1}{n}(u_N - c)^{max} + \frac{1}{n}\{(n-1)u_i \cdot \sigma - \Sigma_{j \neq i} u_j \cdot \sigma\} \quad \text{for all n, i, u and c.}$$

The relationships between the families are as follows; O-family ⊇ I-family ⊇ M-family ⊇ Q-family.[4]

3. The Solidarity Axiom.

Now we are ready to introduce our main axiom. Let $u' \equiv (u, u_{n+1})$ in $[\mathbb{R}^A]^{n+1}$.

Solidarity (SOL). For all n, for all profiles u' and for all cost vectors c, if $S_i^n(u,c) > S_i^{n+1}(u',c)$ for some $i = 1,\ldots,n$, then $S_j^n(u,c) \geq S_j^{n+1}(u',c)$ for all $j = 1,\ldots,n$.

One can imagine a decision problem faced by the original group $N = \{1,\ldots,n\}$. The solution S is first applied to this problem. Then one additional agent enters the scene and it is recognized that he has the same rights as the members of N, so the solution is again applied to the decision problem faced by the enlarged group of agents. The solidarity axiom requires that the members of N all be better off (in the weak sense) or that they all be worse off (also in the weak sense) at the new decision than they were before. Note that, although this axiom is stated assuming the arrival of only one more agent, it could be stated assuming that an arbitrary number of agents come in. Such a strengthening of SOL would not affect the analysis of this paper

In [5,6], Thomson introduced the axiom of "monotonicity with respect to changes in the number of agents (MON)" in the framework of bargaining theory, as formalized by Nash [4], but generalized to allow for variations in the number of agents. Thomson used

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4 Moulin gives axiomatic characterizations of the M-family and the Q-family. He called
M-family Equal sharing from an individual reference level,
Q-family Equal sharing above a convex status quo.

the axiom to characterize the Kalai-Smorodinsky solution and the egalitarian solution. MON requires that if the number of agents were to increase but not the resources at their disposal, then all of the agents originally present should contribute (perhaps not strictly) to the support of the newcomers. SOL is stronger in that it demands solidarity regardless of whether resources change, but it is weaker in that the original agents can gain or lose.

4. The Main Characterization Result.

Now we identify all solutions satisfying the four basic axioms, as well as the solidarity axiom.

Theorem 1. A solution $S \equiv \{S^1,\ldots,S^n,\ldots\}$ satisfies PO, AN, IND1, IND2 and SOL if and only if it belongs to the O-family, that is, there exists a real valued function g(x,z) on the domain $[\mathbb{R}^A]^2$ satisfying (i) - (iii),

(i) $g(x+\alpha\Pi,z) = g(x,z) + \alpha$ for all $x, z \in \mathbb{R}^A$ and all $\alpha \in \mathbb{R}$,

(ii) $g(0,z) = 0$ for all $z \in \mathbb{R}^A$,

(iii) $g(x,z+\alpha\Pi) = g(x,z)$ for all $x, z \in \mathbb{R}^A$ and all $\alpha \in \mathbb{R}$,

and such that for all n, S^n is defined by, for all i, for all profiles u and for all cost vectors c,

$$(1)\quad S_i^n(u,c) = \frac{1}{n}(u_N - c)^{max} + \frac{1}{n}\{(n-1)g(u_i,c) - \Sigma_{j\neq i}\, g(u_j,c)\}.$$

<u>Proof</u>. Given g satisfying the properties (i) - (iii), consider first the function S^n defined by (1). Routine checking shows that S^n satisfies PO, AN, IND1 and IND2 So we only show that it satisfies SOL.

For given $i \in N$, let $\phi_i : [\mathbb{R}^A]^{(n+2)} \to \mathbb{R}$ be defined by

$$\phi_i(u',c) \equiv S_i^{n+1}(u',c) - S_i^n(u,c)$$

where $u \equiv (u_1,\ldots,u_n)$ and $u' \equiv (u,u_{n+1})$.

Let $N' \equiv N \cup \{n+1\}$. Then

$$\phi_i(u',c) = \frac{1}{n+1}(u_{N'} - c)^{max} + \frac{1}{n+1}\{ng(u_i,c) - \Sigma_{j\neq i, j\varepsilon N'}\, g(u_j,c)\}$$

$$- \frac{1}{n}(u_N - c)^{max} - \frac{1}{n}\{(n-1)g(u_i,c) - \Sigma_{j\neq i, j\varepsilon N}\, g(u_j,c)\}$$

$$= \frac{1}{n+1}(u_{N'} - c)^{max} - \frac{1}{n+1} g(u_{n+1},c) - \frac{1}{n}(u_N - c)^{max}$$

$$+ \left(\frac{n}{n+1} - \frac{n-1}{n}\right) g(u_i,c) + \left(-\frac{1}{n+1} + \frac{1}{n}\right) \Sigma_{j\neq i, j\varepsilon N}\, g(u_j,c)$$

$$= \frac{1}{n+1}(u_{N'} - c)^{max} - \frac{1}{n+1} g(u_{n+1},c)$$
$$- \frac{1}{n}(u_N - c)^{max} + \frac{1}{n(n+1)} \Sigma_{j\varepsilon N}\, g(u_j,c).$$

Since ϕ_i is identical for all $i = 1,\ldots,n$, the O-family satisfies the solidarity axiom.

The proof of the converse statement is divided into 4 steps. Let S be a solution satisfying the 5 axioms. Also let N, N', u, u' and ϕ_i, for $i = 1,\ldots,n$, be defined as before.

Step 1. $\phi_i(u',c) = \phi_j(u',c)$ for all $i,j \,\varepsilon\, N$, for all u' and for all c.

Proof. We argue by contradiction, assuming, without loss of generality, that for some $(u',c) \,\varepsilon\, [\mathbb{R}^A]^{n+2}$, $\phi_1(u',c) > \phi_2(u',c)$. By SOL, either

(a) $0 \geq \phi_1(u',c) > \phi_2(u',c)$,

or

(b) $\phi_1(u',c) > \phi_2(u',c) \geq 0$

Assume that (a) holds. For $\alpha \,\varepsilon\, \mathbb{R}$, we have

$\phi_1(u',c+\alpha\Pi)$

$= S_1^{n+1}(u',c+\alpha\Pi) - S_1^n(u,c+\alpha\Pi)$ (By def of ϕ_1)

$= S_1^{n+1}(u',c) - S_1^n(u,c) - \frac{1}{n+1}\alpha + \frac{1}{n}\alpha$ (By IND2)

$= \phi_1(u',c) - \frac{1}{n+1}\alpha + \frac{1}{n}\alpha$ (By def of ϕ_1)

$= \phi_1(u',c) + \frac{1}{n(n+1)}\alpha.$

Similarly, we have

$$\phi_2(u',c+\alpha\Pi) = \phi_2(u',c) + \frac{1}{n(n+1)}\alpha.$$

By choosing α such that $-n(n+1)\phi_1(u',c) < \alpha < -n(n+1)\phi_2(u',c)$, we obtain

$$\phi_1(u',c+\alpha\Pi) > 0,$$

and

$$\phi_2(u',c+\alpha\Pi) < 0.$$

in violation of SOL. A similar argument can be developed for (b). This proves Step 1.

Remark 1. By AN, step 1 implies that for given N, the arrival of agent j, $j \notin N$, yields equal gains or equal losses to all members of N.

For n=1, (1) is just a restatement of PO, so we consider the first non-trivial case, that of n=2.

Step 2. There exists a function $g : [\mathbb{R}^A]^2 \to \mathbb{R}$ such that

$$g(0,c) = 0 \quad \text{for all } c \in \mathbb{R}^A,$$

and that for all i,j = 1,2, $i \neq j$,

$$S_i^2(u_1,u_2,c) = \frac{1}{2}(u_1+u_2-c)^{max} + \frac{1}{2}\{g(u_i,c) - g(u_j,c)\}.$$

Proof. Let $u' = (u_1,u_2,u_3)$. By step 1 applied 3 times and AN, we obtain

$$S_1^3(u',c) - S_1^2(u_1,u_2,c) = S_2^3(u',c) - S_2^2(u_1,u_2,c),$$

$$S_2^3(u',c) - S_1^2(u_2,u_3,c) = S_3^3(u',c) - S_2^2(u_2,u_3,c),$$

$$S_3^3(u',c) - S_1^2(u_3,u_1,c) = S_1^3(u',c) - S_2^2(u_3,u_1,c).$$

Summing up these three equations yields,

$$S_1^2(u_1,u_2,c) + S_1^2(u_2,u_3,c) + S_1^2(u_3,u_1,c)$$

$$= S_2^2(u_1,u_2,c) + S_2^2(u_2,u_3,c) + S_2^2(u_3,u_1,c).$$

On the other hand, by PO,

$$S_1^2(u_1,u_2,c) + S_2^2(u_1,u_2,c) = (u_1+u_2-c)^{max},$$

$$S_1^2(u_2,u_3,c) + S_2^2(u_2,u_3,c) = (u_2+u_3-c)^{max},$$

$$S_1^2(u_3,u_1,c) + S_2^2(u_3,u_1,c) = (u_3+u_1-c)^{max}.$$

Therefore,

$$S_1^2(u_1,u_2,c) + S_1^2(u_2,u_3,c) + S_1^2(u_3,u_1,c)$$
$$= \frac{1}{2}\{(u_1+u_2-c)^{max} + (u_2+u_3-c)^{max} + (u_3+u_1-c)^{max}\}.$$

Fix c, and let $f : [\mathbb{R}^A]^2 \to \mathbb{R}$ be defined by

$$f(x,y) \equiv S_1^2(x,y,c) - \frac{1}{2}(x+y-c)^{max}.$$

Then

(2) $\quad f(u_1,u_2) + f(u_2,u_3) + f(u_3,u_1) = 0.$

(2) holds for an arbitrary $u'=(u_1,u_2,u_3)$ if and only if there exists a function $h : \mathbb{R}^A \to \mathbb{R}$ such that

$$f(u_1,u_2) = \frac{1}{2}[h(u_1) - h(u_2)].$$ [5]

Therefore, for some function $k : [\mathbb{R}^A]^2 \to \mathbb{R}$,

$$S_1^2(u_1,u_2,c) = \frac{1}{2}(u_1 + u_2 - c)^{max} + \frac{1}{2}[k(u_1,c) - k(u_2,c)]$$

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5 Set $u_3 = 0$, so that for some functions $h, \tilde{h} : \mathbb{R}^A \to \mathbb{R}$

$$f(u_1,u_2) = (1/2)\,[h(u_1) - \tilde{h}(u_2)].$$

Substituting this into (2) gives

$$[h(u_1) - \tilde{h}(u_2)] + [h(u_2) - \tilde{h}(u_3)] + [h(u_3) - \tilde{h}(u_1)] = 0$$

for all u_1, u_2 and u_3

Since this holds for all $u' = (u_1,u_2,u_3)$, by setting $u_1 = u_2 = u_3 = 0$, we get $h(0) = \tilde{h}(0)$.
Now let $u_2 = u_3 = 0$, and obtain

$$h(u_1) = \tilde{h}(u_1) \quad \text{for all } u_1.$$

I borrowed this argument from the proof of Lemma 1 in Moulin [3].

Define $g : [\mathbb{R}^A]^2 \to \mathbb{R}$ by $g(u_i,c) \equiv k(u_i,c) - k(0,c)$. We note that

$$g(0,c) = 0 \quad \text{for all } c \in \mathbb{R}^A$$

and

$$S_1^2(u_1,u_2,c) = \frac{1}{2}(u_1+u_2-c)^{max} + \frac{1}{2}[g(u_1,c) - g(u_2,c)].$$

This proves Step 2.

Step 3. g satisfies properties (i) and (iii).

Proof. Let $u^* \equiv (u_1,0)$.

(a) $$S_1^2(u_1+\alpha\Pi,0,c) = \frac{1}{2}(u_1+\alpha\Pi - c)^{max} + \frac{1}{2}g(u_1+\alpha\Pi,c) \quad \text{(By step 2)}$$
$$= \frac{1}{2}(u_1-c)^{max} + \frac{\alpha}{2} + \frac{1}{2}g(u_1+\alpha\Pi,c)$$
$$= S_1^2(u^*,c) + \frac{\alpha}{2} + \frac{1}{2}g(u_1+\alpha\Pi,c) - \frac{1}{2}g(u_1,c). \quad \text{(By step 2)}$$

This gives

$$g(u_1+\alpha\Pi,c) - g(u_1,c) = \alpha \quad \text{for all } u_1 \text{ and } c. \quad \text{(By IND1)}$$

This proves that g satisfies property (i).

(b) $$S_1^2(u^*,c+\alpha\Pi) = \frac{1}{2}(u_1-c-\alpha\Pi)^{max} + \frac{1}{2}g(u_1,c+\alpha\Pi) \quad \text{(By step 2)}$$
$$= \frac{1}{2}(u_1-c)^{max} - \frac{\alpha}{2} + \frac{1}{2}g(u_1,c+\alpha\Pi)$$
$$= S_1^2(u^*,c) - \frac{\alpha}{2} + \frac{1}{2}g(u_1,c+\alpha\Pi) - \frac{1}{2}g(u_1,c). \quad \text{(By step 2)}$$

This gives

$$g(u_1,c+\alpha\Pi) - g(u_1,c) = 0 \quad \text{for all } u_1 \text{ and } c. \quad \text{(By IND2)}$$

This proves that g satisfies property (iii).

We have obtained the desired conclusion for the case of $n \leq 2$. In the next step, we will consider an arbitrary n

Step 4. The conclusion in steps 2 and 3 are true for all n. That is, there exists g satisfying (i) - (iii) and S^n satisfying (1) for all n.

Proof.[6] As usual, let $N \equiv \{1,...,n\}$ and $u \equiv (u_1,...,u_n)$. By repeated application of step 1, we obtain

$$\begin{aligned} S_1^n(u,c) - S_2^n(u,c) &= S_1^{n-1}(u_1,...,u_{n-1},c) - S_2^{n-1}(u_1,...,u_{n-1},c) \\ &= S_1^{n-2}(u_1,...,u_{n-2},c) - S_2^{n-2}(u_1,..,u_{n-2},c) \\ &= S_1^2(u_1,u_2,c) - S_2^2(u_1,u_2,c). \end{aligned}$$

Thus, by step 2, we have

$$S_1^n(u,c) - S_2^n(u,c) = g(u_1,c) - g(u_2,c).$$

Similarly, we have

$$S_1^n(u,c) - S_i^n(u,c) = g(u_1,c) - g(u_i,c) \quad \text{for all } i.$$

Summing up these n equations, we obtain

$$n\, S_1^n(u,c) - S_N^n(u,c) = n\, g(u_1,c) - \Sigma_{j\neq 1}\, g(u_j,c).$$

By PO, we have

$$S_1^n(u,c) = \frac{1}{n}(u_N-c)^{max} + \frac{1}{n}\{(n-1)g(u_1,c) - \Sigma_{j\neq 1}\, g(u_j,c)\}.$$

This completes the proof. QED.

Remark 2. Since the function g is arbitrary except for the properties (i) - (iii), the O-family is fairly large. However, the O-family excludes some otherwise interesting solutions. For example, the utilitarian and the equal allocation of non-separable cost solutions, characterized in Moulin [3], do not belong to the O-family.

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6 I am grateful to Prof. Hervé Moulin for suggesting this proof that simplifies my original proof considerably.

5. Further Results.

Now we introduce some additional axioms of Moulin's to get further characterization results.

Cost monotonicity (CM). For all n, for all i, for all profiles u and for all cost vectors c, c', if $c \leq c'$, then $S_i^n(u,c') \leq S_i^n(u,c)$.

CM requires that no agent be hurt by an improvement in technology that would reduce the cost of some or all public decisions.

By imposing CM in addition to PO, AN, IND1, IND2 and SOL, we can characterize the I-family.

Theorem 2. A solution $S \equiv \{S^1,\ldots,S^n,\ldots\}$ satisfies PO, AN, IND1, IND2, SOL and CON if and only if it belongs to the I-family, that is, there exists a real valued function $\tilde{g}(x)$ on the domain $\mathbb{R}^A$ satisfying

(i') $\tilde{g}(x+\alpha\Pi) = \tilde{g}(x) + \alpha$ for all $x \in \mathbb{R}^A$ and all $\alpha \in \mathbb{R}$,

(ii') $\tilde{g}(0) = 0$,

and such that for all n, S^n is defined by, for all i, for all profiles u and for all cost vectors c,

$$S_i^n(u,c) = \frac{1}{n}(u_N - c)^{max} + \frac{1}{n}\{(n-1)\tilde{g}(u_i) - \Sigma_{j\neq i}\,\tilde{g}(u_j)\}.$$

Proof. It is immediate that the I-family satisfies all 6 axioms. So we prove only the "only if" part of the Theorem. From step 1 in the proof of Theorem 1 in [3],[7] if a solution

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[7] For some fixed profile u and cost vector c,

$$u_N - (u_N - c)^{max} \leq c.$$

From CM and IND2,

$$S_i^n(u,c) \leq S_i^n(u,u_N-(u_N-c)^{max}\Pi) = S_i^n(u,c) + (1/n)(u_N - c)^{max}.$$

On the other hand, by PO,

$$S_N^n(u,c) = (u_N - c)^{max} = S_N^n(u,u_N-(u_N-c)^{max}\Pi).$$

satisfies PO, IND2 and CM, then we have

$$S_1^n(u,c) = S_1^n(u,u_N) + \frac{1}{n}(u_N-c)^{max} \quad \text{for all i, u and c.}$$

Let $u_i = 0$ for all $i \neq 1$. By (1), we get

$$g(u_1,c) - g(u_1,u_1) = 0 \quad \text{for all } u_1 \text{ and c.}$$

This means that g is independent of its second argument and we can replace g by $\tilde{g}$ where

$$\tilde{g}(u_1) = g(u_1,c) \quad \text{for all } u_1 \text{ and c.} \qquad \text{QED.}$$

Another characterization of the I-family can be achieved by using Moulin's main axiom of consistency, which relates solutions for societies of different sizes.

Consistency (CON). For all n, for all i = 1,...,n, for all profiles u', and for all cost vectors c,

$$S_i^{n+1}(u',c) = S_i^n(u,c')$$

$$\text{where } c'(a) \equiv c(a) - u_{n+1}(a) + S_{n+1}^{n+1}(u',c) \quad \text{for all } a \in A.$$

The consistency axiom demands that the restriction of a solution to some subgroup of agents is itself the solution of the restricted decision problem.

Theorem 3. A solution satisfies PO, AN, IND1, IND2, SOL and CON if and only if it belongs to the I-family.

Proof Since the proof for the "if" part of the Theorem is straightforward, we prove only the "only if" part of the theorem. By Theorem 1, the solution can be written as

(1) $$S_i^n(u,c) = \frac{1}{n}(u_N - c)^{max} + \frac{1}{n}\{(n-1)g(u_i,c) - \Sigma_{j\neq i}\, g(u_j,c)\}$$

where g satisfies (i) - (iii) of Theorem 1.

Therefore,

$$S_i^n(u,c) = S_i^n(u,u_N) + (1/n)(u_N - c)^{max} \quad \text{for all i, u and c.}$$

Let $i \equiv 1$ and $u^* \equiv (u_1,0,...,0,u_n)$. Then from property (ii) of g, we have

(3) $S_1^n(u^*,c) = \frac{1}{n}(u_1+u_n-c)^{max} + \frac{n-1}{n} g(u_1,c) - \frac{1}{n} g(u_n,c),$

(4) $S_n^n(u^*,c) = \frac{1}{n}(u_1+u_n-c)^{max} + \frac{n-1}{n} g(u_n,c) - \frac{1}{n} g(u_1,c).$

On the other hand, also by (1), we get

(5) $S_1^{n-1}(u_1,0,...,0,c-u_n+S_n^n(u^*,c)\Pi) = \frac{1}{n-1}(u_1-c+u_n-S_n^n(u^*,c)\Pi)^{max}$

$+ \frac{n-2}{n-1} g(u_1,c-u_n+S_n^n(u^*,c)\Pi).$

By CON, we obtain

(6) $S_1^n(u^*,c) = S_1^{n-1}(u_1,0,...,0,c-u_n+S_n^n(u^*,c)\Pi).$

Substitute (3)-(5) into (6) and get from property (iii),

$g(u_1,c) = g(u_1,c-u_n)$ for all u_1, u_n and c.

This implies that g is independent of its second argument, and we can define

$\tilde{g}(u_1) \equiv g(u_1,c)$ for all u_1 and c. QED.

Next axiom prevents a strategic behavior by agents.

No disposal of utility (NDU). For all n, for all profiles u, v and for all cost vectors c, if $u_1 \leq v_1$ and $u_i = v_i$ for all $i \neq 1$, then $S_1^n(u,c) \leq S_1^n(v,c)$.

If a solution does not satisfy NDU, then an agent may sometimes benefit from disposing of his utility. This possibility is discussed in other contexts under the name of the "destruction paradox". NDU requires that the solution should be free from the destruction paradox.

Theorem 4. A solution satisfies PO, AN, IND1, IND2, SOL and NDU if and only if it belongs to the O-family with g satisfying one additional property, namely,

(iv) g is monotonic in x: for all x, y, z in $\mathbb{R}^A$ $x \leq y$ implies $g(x,z) \leq g(y,z)$.

Proof. For any profile u and non-negative vector $\delta \in \mathbb{R}_+^A$, we get from NDU

$S_1^n(u,c) \leq S_1^n(u_1+\delta,u_2,...,u_n,c)$

By Theorem 1,

$$\frac{1}{n}(u_N-c)^{max} + \frac{n-1}{n} g(u_1,c) \leq \frac{1}{n}(u_N+\delta-c)^{max} + \frac{n-1}{n} g(u_1+\delta,c)$$

$$(n-1)\{g(u_1+\delta,c) - g(u_1,c)\} \geq -(u_N+\delta-c)^{max} + (u_N-c)^{max}.$$

Let $u_2 = ... = u_n$, divide both sides by (n-1), and let n go to infinity. We obtain

$$g(u_1+\delta,c) - g(u_1,c) \geq 0 \qquad \text{for all } u_1, c \text{ and } \delta.$$

This means that g is monotonic in x. QED.

As a direct consequence of Theorems 2-4, we obtain the following two characterization results for the M-family.

Corollary 1. A solution satisfies PO, AN, IND1, IND2, SOL, CM and NDU if and only if it belongs to the M-family.

Corollary 2. A solution satisfies PO, AN, IND1, IND2, SOL, CON and NDU if and only if it belongs to the M-family.

Let $h^n(u_i,c) \equiv \inf S_i^n(u_i,u_{-i},c)$, where the infimum is taken over all (n-1)-tuples $u_{-i} \in [\mathbb{R}^A]^{n-1}$. h^n denotes the agent's guaranteed utility level, as a function of u_i and c.

<u>Individual rationality relative to σ (IRσ)</u>. For all n, for all i, for all profiles u, for all cost vectors c and for some vector $\sigma \equiv (\sigma_a)_{a \in A}$ such that $\Sigma_{a \in A}\, \sigma_a = 1$ and $\sigma_a \geq 0$ for all a,

$$h^n(u_i,c) \geq (u_i - \frac{c}{n}) \cdot \sigma$$

Suppose that the public decisions are drawn at random with probabilities $\sigma \equiv (\sigma)_{a \in A}$. Then the status quo outcome amounts to picking a decision according to σ and sharing its cost equally. IRσ requires that the solution should guarantee each agent at least the status quo utility level.

By imposing IRσ in addition to PO, AN, IND1, IND2 and SOL, we characterize the

Q-family.

Theorem 5. A solution satisfies PO, AN, IND1, IND2, SOL and IRσ if and only if it belongs to the Q-family.

Proof.[8] Since it is easy to show the "if" part of the Theorem, we prove only the "only if" part of the Theorem. Let S be a solution satisfying PO, AN, IND1, IND2 and SOL. By Theorem 1, S is given by (1) for some function g(x,z) satisfying (i)-(iii). From IRσ, we have

$$S_1^n(u,c) \geq (u_1 - \frac{c}{n})\cdot\sigma \quad \text{for all } u \text{ and } c.$$

Equivalently,

$$n\, S_1^n(u,c) \geq (nu_1 - c)\cdot\sigma \text{ for all } u \text{ and } c.$$

From (1), we obtain

$$(u_N - c)^{max} + (n-1)g(u_1,c) - \Sigma_{j=2}^n g(u_j,c) \geq nu_1\cdot\sigma - c\cdot\sigma - u_N\cdot\sigma + u_N\cdot\sigma$$

$$\text{(7)} \quad (n-1)g(u_1,c) - \Sigma_{j=2}^n g(u_j,c) \geq [(n-1)u_1 - \Sigma_{j=2}^n u_j]\cdot\sigma - c\cdot\sigma + u_N\cdot\sigma - (u_N - c)^{max}.$$

Let $h : [\mathbb{R}^A]^2 \to \mathbb{R}$ be defined by $h(x,z) \equiv g(x,z) - x\cdot\sigma$. Then (7) is equivalent to

$$(n-1)h(u_1,c) - \Sigma_{j=2}^n h(u_j,c) \geq (u_N - c)\cdot\sigma - (u_N - c)^{max},$$

which holds for all n, c and u. Choose now $u_2 = \ldots = u_n$, divide both sides by (n-1) and let n go to infinity. We obtain

$$h(u_1,c) - h(u_2,c) \geq 0 \quad \text{for all } u_1, u_2, \text{ and } c.$$

This implies that h(x,z) does not depend on x, so that $g(x,z) = x\cdot\sigma + h(0,z)$. Adding to g a function of the variable z only leaves invariant (1) so we can take $g(x,z) = x\cdot\sigma$. QED.

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8 Even though we use SOL while Moulin [3] uses CON, the proof of Theorem 5 is very similar to that of Moulin's Theorem 2.

The Q-family can be characterized in a different way by using the following axiom.

Guaranteed utility level size-independence (GU). For all n, for all i, for all profiles u_i, and for all cost vectors c,

$$h^n(u_i,nc) = h^2(u_i,2c).$$

GU requires that the guaranteed utility level to each agent depends only on his utility vector and per-capita cost vector, but does not depend on the size of the society to which he belongs.

For completeness, we reproduce Moulin's Lemma 3 as our Lemma 4 before stating our final results.

Lemma 4. A solution in the M-family satisfies GU if and only if it belongs to the Q-family.

As a direct consequence of Corollaries 1, 2 and Lemma 4, we obtain the following two characterization results for Q-family.

Corollary 3. A solution satisfies PO, AN, IND1, IND2, SOL, CM, NDU and GU if and only if it belongs to the Q-family.

Corollary 4. A solution satisfies PO, AN, IND1, IND2, SOL, CON, NDU and GU if and only if it belongs to the Q-family.

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