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Abreu, Dilip and Prajit K. Dutta

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Dilip Abreu and Prajit Dutta

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Dilip Abreu
Princeton University
and
Prajit Dutta
Columbia University

The Rochester Center for Economic Research
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Abstract: The Fudenberg and Maskin folk theorem for discounted repeated games assumes that the set of feasible payoffs is full dimensional. We obtain the same conclusion using a weaker condition which we term payoff asymmetry. This condition is natural, and also almost necessary.

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1. Introduction

Fix a finite stage game $G(A_i, \pi_i; i = 1,..n)$ where A_i is a finite action set, Σ_i is the associated set of mixed actions and π_i is the payoff function, for player i. Player i's minmax payoff is denoted $\underline{u}_i = \min_{\sigma_{-i}} \max_{a_i} \pi_i(a_i, \sigma_{-i})$. This is the lowest payoff a maximizing player can be forced down to. It is important that the minimization by other players be over mixed strategies. Note also the order of the min and max operators: player i's maximizing action is chosen "after" the minmaxing mixed strategy choice of other players. Let F be the convex hull of the set of feasible payoffs. A payoff vector $v = (v_1,...v_n)$ is strictly (resp. weakly) individually rational if for all i, $v_i >$ (resp. $\geq$) $\underline{u}_i$. Folk theorems assert that any feasible and individually rational payoff vector is a (subgame perfect) equilibrium payoff in the associated repeated game with little or no discounting. It is obvious that feasibility and individual rationality are necessary conditions for a payoff vector to be an equilibrium payoff. The (surprising) content of the folk theorems is that these conditions are also (almost) sufficient.

The earliest folk theorems are due to Aumann and Shapley (1976) and Rubinstein (1977, 1979). These results assume that payoff streams are undiscounted. Fudenberg and Maskin (1986) establish an analogous result for discounted repeated games. For the two player case their result is complete (modulo the requirement of strict rather than weak individual rationality, which we retain in this note) and does not employ additional conditions. Their result for three or more players relies on a full dimensionality condition: the dimension of the set of feasible payoff vectors must equal the number of players. This condition is sufficient. They present an example in which the conclusion of the folk theorem is false. In this example players' payoffs are perfectly positively correlated: all players receive the same payoffs in all contingencies. The full dimensionality assumption, of course, rules out examples of this sort.

We present a weaker sufficient condition which is also almost necessary. Thus our characterization is essentially complete. The condition is that there exist n feasible payoff vectors $\bar{v}^1,...\bar{v}^n$ such that $\bar{v}^i_i < \bar{v}^j_i$ for all i, j, $i \neq j$ where $\bar{v}^j_i$ is the i-th component of the j-th payoff vector. We may express this condition verbally as follows: there exist n "punishment" payoff vectors, one for each player, such that player i's payoff in his own "punishment" is strictly worse than his payoff in any other player's punishment. We term this condition payoff asymmetry.[1] Note that this condition would be trivially satisfied if we replaced strictly by weakly in the preceding description.

Payoff asymmetry is a natural condition. It is clearly implied by full-dimensionality. For $n \geq 3$ it implies that the set of payoffs is at least two-dimensional but does not imply, nor is implied by (n-1) dimensionality. The following example demonstrates this point.

Example Let $n = 4$ and take $\bar{v}^1 = (0, 1, 0, -2)$, $\bar{v}^2 = (1, 0, -1, 1)$, $\bar{v}^3 = (2, 1, -2, 0)$ and $\bar{v}^4 = (1, 2, -1, -3)$. Let $F = \text{co}\,(\bar{v}^i;\ i = 1,..4)$.

This example satisfies payoff asymmetry. Consequently the conclusion of the folk theorem holds. However $\bar{v}^3 = \bar{v}^1 + 2\bar{v}^2$ and $\bar{v}^4 = 2\bar{v}^1 + \bar{v}^2$ so that F is two-dimensional. It is interesting to note that adding any constant to player 3's payoffs yields an F which is three-dimensional. Thus dimensionality is not invariant with respect to the origins of players' utility functions although the latter are clearly irrelevant strategically. It is therefore not surprising that the notion of dimension does not neatly capture the essence of what is necessary to prove a folk theorem.

Our result is for the standard case in which mixed strategies are unobservable. But to develop some feeling for our condition and the argument assume, for the moment, that mixed strategies are ex-post observable or simply confine attention to pure strategies (and define individual rationality in the latter case through pure strategies). Let w be a strictly individually rational and feasible payoff vector. Then, as we argue explicitly below, payoff asymmetry implies that there exist strictly individually rational vectors $v^1,..v^n$ such that $v_i^i < v_i^j$ $\forall i,j,\ i \neq j$ and $v_i^i < w_i$. Let a and a^i be (correlated) action profiles which yield the payoff vectors w and v^i, $i=1,..n$ respectively. Let Q^0 be the path in which a is played in every period and Q^i the path in which player i is minmaxed for T periods (during which he plays a best response) followed by the action a^i forever after. Consider the simple profile (Abreu (1988)) in which Q^0 is played initially and any deviation by player i alone from an ongoing path is responded to by imposing Q^i (and simultaneous deviations are ignored). Let T satisfy $T(v_i^i - \underline{u}_i) > \max_{a_i} \pi_i(a_i, a_{-i}^i) - \pi_i(a^i)$, for all i. Then, using the criterion of "unimprovability" (which only checks one-shot deviations) it can be directly verified that for high enough discount factors, the described simple profile is a subgame perfect equilibrium. (Since $v_i^i < w_i$ a one-shot deviation from Q^0 is unprofitable. Player i will not

[1]This condition was first presented at the International Game Theory Conference at Ohio State in July 1989. Our proof then did not however cover unobservable mixed strategies.

deviate from Q^i since by the above inequality any one-period gain is wiped out by T periods of minmaxing. Player $j \neq i$ will not deviate from Q^i since $v^j_j < v^i_j$). That is, with observable mixed strategies and given payoff asymmetry the extension of the undiscounted folk theorem to the discounted case is straightforward. The subtleties in the argument derive primarily from the consideration of mixed strategies.

This note is organized as follows. Section 2 presents the model. Section 3 contains our results on the necessity and sufficiency of payoff asymmetry. Section 4 concludes.

2. The Model

We consider a finite n-player game in normal form defined by $< A_i, \pi_i; i=1,..n >$ where A_i is the i-th player's finite set of actions, and let $A = \prod_{i=1}^{n} A_i$. The i-th player's payoff is $\pi_i : A \to \mathbb{R}$. Let Σ_i be the set of player i's mixed strategies and let $\Sigma = \prod_{i=1}^{n} \Sigma_i$. Abusing notation, we write $\pi_i(\sigma)$ for i's expected payoff under the mixed strategy $\sigma = (\sigma_1,..\sigma_n) \in \Sigma$. For any n-element vector $x = (x_1,..x_n)$ let x_{-i} denote the corresponding (n-1) element vector with the i-th element missing. Let $\pi^*_i(\sigma_{-i}) = \max_{a_i} \pi_i(a_i, \sigma_{-i})$ be player i's best response payoff against the mixed profile σ_{-i}. Denote by $m^i = (m^i_1,.. m^i_n) \in \Sigma$ a mixed strategy profile which satisfies $m^i_{-i} \in \operatorname{argmin}_{\sigma_{-i}} \pi^*_i(\sigma_{-i})$ and $m^i_i \in \operatorname{argmax}_{a_i} \pi_i(a_i, m^i_{-i})$. In words, m^i_{-i} is a (n-1) profile of mixed strategies which minmax player i and m^i_i is a best response for i when being minmaxed. We adopt the normalization $\pi_i(m^i) = 0$ for all i. Let $F = co\{\pi(\sigma): \sigma \in \Sigma\}$ and denote by F^* the set of feasible and (strictly) individually rational payoffs; $F^* = \{w \in F: w_i > 0, \text{ for all } i\}$, .

We will analyze the associated repeated game with perfect monitoring. That is, for all i, player i's action in period t can be conditioned on the past actions of all players. In addition, we permit public randomization. That is, in every period players publicly observe the realization of an exogeneous continuous random variable and can condition on its outcome. This assumption can be made without loss of generality. A result due to Fudenberg and Maskin (1991) shows explicitly how public randomization can be replaced by "time-averaging".

Denote by $\alpha_i = (\alpha_{i1},..\alpha_{it},..)$ a (behavior) strategy for player i in the repeated game and by $\pi_{it}(\alpha)$ his expected payoff in period t given the strategy profile α. Player i's average discounted payoff under the (common) discount factor δ is: $v_i(\alpha) = (1-\delta) \sum_0^\infty \delta^t \pi_{it}(\alpha)$. Let $V(\delta)$ denote the set of subgame perfect equilibrium payoffs.

3. The Theorems

The new condition we propose is

Payoff Asymmetry There exist payoff vectors $\bar{v}^i \in F$, for $i = 1,..n$, such that $\bar{v}^i_i < \bar{v}^j_i$, for all i and j, $i \neq j$.

Let F_{ij}, $j \neq i$ denote the projection of F on the i-j coordinate plane. An implication of payoff asymmetry is:

Lemma Under payoff asymmetry $\dim(F_{ij}) \geq 1$ for all i and j, $j \neq i$. Furthermore, if $\dim(F_{ij}) = 1$ for some i and j, then $\dim(F_{ik}) = 2$ for all $k \neq i,j$.

Proof: It is immediate that $\dim(F_{ij}) \geq 1$ for all i and j, $j \neq i$. Suppose now that $\dim(F_{ij}) = \dim(F_{ik}) = 1$ for some $j \neq k$. Payoff asymmetry applied to the payoffs of players i and j (and similarly players i and k) implies that these pairs of payoffs are negatively correlated. This in turn implies that the payoffs of players j and k are perfectly positively correlated. Such correlation is ruled out by payoff asymmetry.•

Theorem 1 **Under payoff asymmetry, any point in F^* is a subgame perfect equilibrium payoff when players are sufficiently patient. That is, for any $w \in F^*$ there exists $\underline{\delta} < 1$ such that $w \in V(\delta)$ for all $\delta \geq \underline{\delta}$.**

Proof: Fix $w \in F^*$. Let w^i denote the payoff vector which yields player i his lowest payoff in the game; i.e. $w^i_i \equiv \min \{ v_i: (v_{-i}, v_i) \in F\}$. Define

$$v^i = \beta_1 w^i + \beta_2 \bar{v}^i + (1-\beta_1-\beta_2)\, w$$

where $\beta_1>0$, $\beta_2>0$ are convexifying weights. Clearly one can pick these so that the payoffs v^i satisfy $\forall i,j,\ i \neq j$

Strict individual rationality	$v^i >> 0$
Asymmetry	$v^i_i < v^j_i$

Target payoff domination $v^i_i < w_i$

(The w^i's are needed because it might be the case that $\bar{v}^i_i > w_i$ for some i).

We will assume without loss of generality that in addition the v^i's lie in the relative interior of F^*. We now specify a strategy profile which yields payoff w and which for sufficiently high δ is also a subgame perfect equilibrium. The strategy is as follows. Players play the (correlated) action that generates w at t=1 and continue to do so unless some player i deviates singly in some period t. The key element of the specification is the "punishment" for player i which is invoked for any (single person) deviation by player i from prescribed behavior. Assume that for some $k \neq i$, $\dim(F_{ik}) = 1$. (The proof when there is no such k is a corollary). Then by the Lemma, $\dim(F_{ij}) = 2$, for $j \neq i,k$. Furthermore, by payoff asymmetry F_{ik} is a straight line with negative slope. Let m^i minmax player i. Then, fixing m^i_j for $j \neq i,k$ induces a zero-sum game between i and k and we may assume without loss of generality that m^i_i and m^i_k are equilibrium minmax strategies in this induced game. The lemma and this observation are the key new elements of our proof.

The punishment for player i consists of T periods of play of m^i as specified above. The difficulty now is to induce minmaxing players $j \neq i,k$ to play pure strategies in the support of their mixed strategies with the appropriate probabilities. As noted by Fudenberg and Maskin (1986) the only way to do so is to make them indifferent across their pure strategies. Note that player k's mixed strategy m^i_k is a best response to m^i_{-k}, and that for $j \neq i,k$ deviations outside the support of m^i_j are easily deterred by directly punishing player j.

For $j \neq i,k$, since $\dim(F_{ij}) = 2$ and the v^i's are in the relative interior of F^*, there exist payoff vectors $u^{ij} \in F^*$ such that

Asymmetry	$v^l_l < u^{ij}_l, \quad l \neq i$
Indifference for i	$v^i_i = u^{ij}_i$
Differential for j	$v^i_j > u^{ij}_j$

At the end of T periods of minmaxing player i, play moves probabilistically (via the public randomization device) to v^i or to the u^{ij}'s, where the probabilities are chosen to maintain the incentives of minmaxing players $j \neq i$. Let $p^{ij}_t(a_j)$ satisfy for all a_j, a'_j in the support of m^i_j, and $t \leq T$,

$$(1-\delta)\,\pi_j(a_j, m^i_{-j}) + \delta^{T-t+1}\, p^{ij}_t(a_j)(u^{ij}_j - v^i_j) = (1-\delta)\pi_j(a'_j, m^i_{-j}) + \delta^{T-t+1}\, p^{ij}_t(a_j')(u^{ij}_j - v^i_j)$$

If the realized sequence of action profiles is a_t, t = 1,..T then for all $j \neq i,k$ play proceeds to u^{ij} with probability $\Sigma_1^T p_t^{ij}(a_{jt})$. Notice that the p_t^{ij}'s are independent across players $j \neq i$ and across periods and are chosen to make players indifferent across the support of their minmaxing strategies. Let $P^{ij}(a_1,...a_T) = \Sigma_1^T p_t^{ij}(a_{jt})$. Then with probability $1 - \Sigma_j P^{ij}(a_1,..a_T)$ play goes to v^i. For high enough δ, the P^{ij}'s defined above can be made small and positive for all possible realizations of $a_1,...a_T$ and so we indeed have probabilistic transitions. Since all the relevant inequalities are strict, no detectable one-shot deviation is profitable. By construction, minmaxing players are indifferent over the support of their mixed minmaxing strategies. It follows that the profile specified is a subgame perfect equilibrium.•

We turn now to the <u>necessity</u> of payoff asymmetry. To avoid trivialities, from here on $F^* \neq \phi$. Recall that minmax payoffs have been normalized to zero. The necessity of payoff asymmetry is shown for games in which no two players can be minmaxed simultaneously.

<u>No Simultaneous Minmaxing</u> For all $\sigma \in \Sigma$ such that $\pi_i^*(\sigma_{-i}) = 0$ for some i, it is the case that $\pi_j^*(\sigma_{-j}) > 0$ for all $j \neq i$.

Under the above assumption we obtain a <u>complete</u> characterization.

<u>Theorem 2</u> **Suppose no simultaneous minmaxing is possible. Then, payoff asymmetry is both necessary and sufficient for the conclusion of the folk theorem.**

Proof: Since the conclusion of the folk theorem is valid, the set of subgame perfect equilibrium payoffs is non-empty for sufficiently high δ. Denote by $W^i(\delta)$ an equilibrium payoff vector which yields player i his lowest subgame perfect equilibrium payoff. By the arguments of Abreu (1988), $W^i(\delta)$ exists; denote by α^i an equilibrium strategy profile that generates $W^i(\delta)$. By definition, $W_j^j(\delta) \leq W_j^i(\delta)$ for all i,j. By the folk theorem hypothesis, $W_i^i(\delta) \to 0$, as $\delta \to 1$. Clearly, $W_i^i(\delta) \geq (1-\delta)\, \pi_i^*(\gamma_{-i}^i(\delta)) + \delta W_i^i(\delta)$, or equivalently $W_i^i(\delta) \geq \pi_i^*(\gamma_{-i}^i(\delta))$, where $\gamma^i(\delta)$ is the first period strategy vector in the play of α^i. It then follows that $\pi_i^*(\gamma_{-i}^i(\delta)) \to 0$, as $\delta \to 1$ and hence any subsequential limit of $\gamma_{-i}^i(\delta)$ minmaxes player i.

Clearly $W_j^i(\delta) \geq (1-\delta)\, \pi_j^*(\gamma_{-j}^i(\delta)) + \delta W_j^j(\delta)$. Hence if $W_j^i(\delta) = W_j^j(\delta)$ it follows that $W_j^j(\delta) \geq \pi_j^*(\gamma_{-j}^i(\delta))$. We claim now that there is $\underline{\delta} < 1$ such that $W_j^j(\underline{\delta}) < W_j^i(\underline{\delta})$, for all i,j,

$j \neq i$. A contradiction to this claim implies the existence of a sequence $\delta_m \to 1$ and fixed indices i,j, $j \neq i$ such that $W^j_j(\delta_m) \geq \pi^*_j(\gamma^i_{-j}(\delta_m))$. The left hand side of the inequality goes to zero while the right hand side is strictly positive since $\lim \gamma^i(\delta)$ minmaxes player i and simultaneous minmaxing is impossible. This yields the desired contradiction. Now take $\bar{v}^i = W^i(\underline{\delta})$.•

Remark: For general games a condition slightly weaker than payoff asymmetry can be shown to be a necessary condition for the discounted folk theorem. For any player i, let N_i denote the players who cannot be simultaneously minmaxed with player i. That is $N_i = \{j \neq i$: there is no $\sigma \in \Sigma$ such that $\pi^*_i(\sigma_{-i}) = \pi^*_j(\sigma_{-j}) = 0\}$. Then a (weak) payoff asymmetry condition is: there are feasible payoff vectors $\bar{v}^i \in F$, for i = 1,..n, such that $\bar{v}^i_i \leq \bar{v}^j_i$ and whenever $j \in N_i$, $\bar{v}^i_i < \bar{v}^j_i$. In other words, this condition requires the strict inequality of payoff asymmetry to hold only if players cannot be simultaneously minmaxed. Arguments analogous to those above establish that weak payoff asymmetry is a general necessary condition for the discounted folk theorem. This then is the sense in which payoff asymmetry is almost necessary.

4. Discussion

A recent paper by Smith (1990) weakens full dimensionality of F to dim $(F_{ij}) = 2$, for all i,j, $j \neq i$. However his result does not generalize Fudenberg and Maskin (1986) in that it confines attention to pure strategies, or equivalently observable mixed strategies. As noted earlier, a "pure" folk theorem follows directly under payoff asymmetry. Smith's condition can be shown to imply payoff asymmetry but the converse is not true, as can be immediately ascertained either from the two player case or from the example above where $\dim(F_{13}) = 1$. Dutta (1991) uses some of the ideas presented here in proving a folk theorem for the more general class of stochastic games. An interesting question is whether payoff asymmetry has a natural extension in other environments, such as imperfect monitoring, in which full dimensionality has been invoked (see Fudenberg, Levine and Maskin (1989)) to prove folk theorems.

References

Abreu, D., 1988, On the Theory of Infinitely Repeated Games with Discounting, Econometrica 56, 383 - 396.

Aumann, R. and L.Shapley, 1976, Long Term Competition: A Game Theoretic Analysis, mimeo, Hebrew University.

Dutta, P.K., 1991, A Folk Theorem for Stochastic Games, University of Rochester Working Paper # 293.

Fudenberg, D., D. Levine and E.Maskin, 1989, The Folk Theorem with Imperfect Public Information, mimeo, UCLA.

Fudenberg, D. and E.Maskin, 1986, The Folk Theorem in Repeated Games with Discounting or Incomplete Information, Econometrica, 54, 533 - 554.

Fudenberg, D. and E.Maskin, 1991,On the Dispensability of Public Randomization in Discounted Repeated Games, Journal of Economic Theory, 53, 428-438.

Rubinstein, A., 1977, Equilibrium in Supergames, Research Memorandum #25, Center for Research in Mathematical Economics and Game Theory, Hebrew University.

Rubinstein, A., 1979, Equilibrium in Supergames with the Overtaking Criterion, Journal of Economic Theory, 31, 227-250.

Smith, L., 1990, Folk Theorems: Two-Dimensionality is (Almost) Enough, mimeo, University of Chicago.