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AGAINST THE ALTERNATIVE OF A UNIT ROOT

by

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Abstract

This paper constructs a consistent test for the null of stationarity against the alternative of a unit root, utilizing the regression properties investigated by Kahn and Ogaki (1990).

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1. Introduction

This paper constructs a consistent test for stationarity against unit root nonstationarity, utilizing the regression properties we investigated in Kahn and Ogaki (1990). In Kahn and Ogaki (1990), we used the regression properties to develop a chi-square test for the null of a unit root. We found that our test is more powerful than standard Dickey-Fuller tests when the sample size was small and the autoregressive root is close to one. Thus it is of interest to see if these regression properties will lead us to a powerful test in small samples when we flip the null to stationarity.

The null of stationarity is more attractive than the null of unit roots in many applications, as Fukushige, Hatanaka, and Koto (1990) and Ogaki and Park (1990),among others, emphasized. This is especially true in the context of cointegration. However, most tests for unit roots employ the null of unit roots, and hence most tests for cointegration take the null of no cointegration against cointegration or the null of a smaller number of cointegrating vectors against a larger number of cointegrating vectors.

2. Regression Properties

This section reviews some of the results in Kahn and Ogaki (1990) in the context of the set of assumptions that the present paper use. Consider a stochastic process $\{x_t:t\geq1\}$ generated in discrete time according to

$$(1) \qquad x_t = \alpha x_{t-1} + u_t \qquad (t=1,2,...)$$

where $\{u_t:t\geq1\}$ is a sequence of independent normal random variables with mean zero and variance σ_u^2 (i.e., u_t NID(0, σ_u^2)). As Phillips and Ouliaris (1990) pointed out, it is not simple to construct a consistent test for the null of stationarity against the unit root nonstationarity. As a result, we need this stringent distributional assumption that Kahn and Ogaki (1990) did not require to construct a test for the null of stationarity.

We assume that the initial value, x_0, is $N[0, \sigma_u^2/(1-\rho^2)]$, where ρ is an unknown real number with $|\rho|<1$. The null hypothesis of our test is $\alpha=\rho$, so that x_t is strictly stationary under the null. The alternative hypothesis is $\alpha=1$.

To motivate the test we develop, let us consider the regression

(2) $$x_t=\beta\Delta x_t+e_t \ .$$

Then $\beta=0.5$ under the null hypothesis and $\beta=1$ under the alternative hypothesis of a unit root in the sense discussed in Kahn and Ogaki (1990). It should be noted that $\beta=0.5$ for any value of α as long as x_t is stationary. This invariance property is useful in constructing a test for the null of stationarity.

3. A Consistent Test for Stationarity

In this section, we develop a consistent test with the null of stationarity. Let b_T be the OLS estimator:

(3) $$b_T = [\sum_{t=1}^{T} (\Delta x_t)^2]^{-1}[\sum_{t=1}^{T} x_t\Delta x_t].$$

Since $\sum_{t=1}^{T}\eta_t = 0.5(x_T^2 - x_0^2)$,

(4) $$(b_T-0.5) = [T^{-1} \sum_{t=1}^{T} (\Delta x_t)^2]^{-1}0.5\ T^{-1}(x_T^2 - x_0^2).$$

Kahn and Ogaki (1990) showed that $\{b_T-0.5:T\geq 1\}$ converges in distribution to $0.5\chi_1^2$ under the unit root hypothesis, where χ_1^2 is a random variable with the Chi-square distribution with one degree of freedom.

Under the null hypothesis of stationarity, it is easy to see that (i) $\{T^{\epsilon}b_T:T\geq 1\}$ converges to zero in probability for any $\epsilon<1$, and that (ii) $\{T(b_T-0.5):T\geq 1\}$ converges in distribution to $\{1/[4(1-\alpha)]\}(y_1-y_2)$ where y_1

and y_2 are independent Chi-square variates with one degree of freedom.

To see this result, note that $T^{-1} \sum_{t=1}^{T} (\Delta x_t)^2$ converges almost surely to $E[(\Delta x_t)^2]=[2/(1+\alpha)]\sigma_u^2$, and x_T converges in distribution to a random variable with the stationary distribution, $N[0, \sigma_u^2/(1-\alpha^2)]$ that is independent of x_0. Hence $(x_T^2-x_0^2)/[\sigma_u^2/(1-\alpha^2)]$ converges in distribution to y_1-y_2. The conclusion follows immediately from equation (4).

Thus $4(1-\alpha)T(b_T-0.5)$ converges in distribution to the difference of two independent Chi-square variates with one degree of freedom, whose density function was derived in Miller (1964, Corollary 3 on p.65). Since α is unknown, we replace α by the OLS estimator of α in equation (1), which we denote by a_T. However, a_T may not satisfy the condition $|a_T|<1$. Hence we choose a constant c_T depending on the sample size that is smaller than one in absolute value, and we replace α by c_T instead of a_T if $|a_T|>c_T$. When we make c_T approach one at a slow enough rate, we obtain a consistent test. Specifically, we choose a sequence of real numbers $\{c_T:T\geq1\}$ that satisfies the following two conditions: (i) $|c_T|<1$, and (ii) $1-c_T = O(T^{-(1-\delta)})$ and $\lim_{T\to\infty} T^{1-\delta+\epsilon}(1-c_T)=\infty$ for any $\epsilon>0$. Define a sequence of functions

(6) $$\phi_T(a)=\begin{cases} a & \text{if } -1<a<c_T \\ c_T & \text{otherwise.} \end{cases}$$

for any real number a. The test statistic we propose is

(7) $$K_T = 4\{1-\phi_T(a_T)\}T(b_T-0.5).$$

If $|\alpha|<1$, then $\{K_T:T\geq1\}$ converges in distribution to a random variable that is the difference between two independent chi-square variates with one degree of freedom. This follows from the fact that $\phi_T(a_T)$ converges in probability to α because asymptotically $|a_T|<c_T$ and hence $\phi_T(a_T)=a_T$.

Next, consider the case where $\alpha=1$ to show that the K_T test is consistent against the alternative of $\alpha=1$. In this case r_T converges to one, and $a_T-1=O_p(T^{-1})$. Since a_T converges to one faster than c_T does, asymptotically $c_T<a_T$ and $\phi_T(a_T)=c_T$. Thus $1-\phi_T(a_T)=O_p(T^{-(1-\delta)})$ and $T(1-\phi_T(a_T))$ diverges. Since $b_T-0.5$ converges in distribution as shown in Kahn and Ogaki (1991), $K_T=O_p(T^{\delta})$ and K_T diverges if $\alpha=1$. Thus the K_T test is consistent against the alternative of a unit root when the null is rejected when K_T is large.

4. *Finite Sample Properties*

We compare finite sample properties of the K_T test with those of Park and Choi's (1988) $G(0,q)$ tests based on 3000 replications. Recently, Fukushige, Hatanaka, and Koto (1990), Kwiatkowski and Schmidt (1990), and Saikkonen and Luukkonen (1989) also developed tests for the null of stationarity. We chose to compare our test with Park and Choi's test because Park and Choi's test and its extension for the null of cointegration have been used in several recent applications (see, e.g., Costello (1990), Ogaki and Park (1990), Ogaki (1990), and Cooley and Ogaki (1990)). Data were generated by the model (1) with the u_t independent and identically distributed N(0,1).[1] The RNDN function of GAUSS was used to create (pseudo) random variables.

Table 1 reports our Monte Carlo results. For our K_T test, we used $c_T=0.95$ for all experiments. For the G(0,q) test, an estimate of the long run variance is necessary. For this purpose, we used Andrews and Monahan's (1990) prewhitened QS kernel estimator based on AR(1) prewhitening (the test

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[1] We used the RNDN function of GAUSS to create pseudo random variables.

is denoted $G_P(0,q)$ in this case) and Andrews's (1990) nonprewhitened QS kernel estimator (the test is denoted $G_N(0,q)$). Andrews's automatic bandwidth parameter estimate based on AR(1) is used in both cases.[2]

The first panel reports empirical sizes when the five percent critical values implied by asymptotic theories are used. Here $\rho=\alpha$, so that x_t is stationary. We estimate the nominal critical value for the K_T test at the 5-percent level to be 3.2 from 3000 replications of $y_1^2-y_2^2$ where y_1 and y_2 are independent normal random variables. The K_T test has little size distortion. The $G_p(0,q)$ tests are conservative especially when q is large. The $G_N(0,5)$ test is as conservative as the $G_p(0,5)$ test when T=100, but the size distortion begins to disappear quickly when the sample size is increased to 200. The $G_N(0,1)$ and $G_N(0,3)$ tests are liberal.

The second panel of Table 1 reports powers when the nominal critical values are used. The third panel of Table 1 reports size corrected powers. Size is adjusted by taking the empirical critical value from the experiment with $\alpha=\rho$ reported in the first panel for each ρ. The $G_p(0,q)$ and $G_N(0,5)$ tests have too small powers when T=100 to be practically useful when the nominal critical values are to be used. The K_T test is more powerful than $G_p(0,q)$ and $G_N(0,q)$ tests when T=100. When the sample size is increased to 200, it is less powerful than the $G_N(0,3)$ test. The $G_N(0,1)$ test is more powerful than $G_N(0,3)$ test when T=100 while $G_N(0,1)$ test is less powerful than $G_N(0,3)$ test when T=200. Considering this result and the size distortion problem, it is recommended to increase q for the $G_N(0,q)$ test when the sample size is increased.

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[2]The bandwidth parameter is bounded by the square root of T. When the prewhitened QS kernel estimator is used, we bounded the AR coefficient to be less than 0.95.

5. Conclusions

The present paper developed a test for the null of stationarity against the alternative of a unit root. We used the regression property previously employed in Kahn and Ogaki (1990) in constructing a test for the null of a unit root. Our test is more powerful than Park and Choi's (1988) tests (which have been used by several authors) when the sample size is small.

References

Andrews, Donald K. 1991, Heteroskedasticity and autocorrelation consistent covariance matrix estimation, Econometrica 58, 817-858.

Andrews, Donald K. and J. Christopher Monahan, 1990, An improved heteroskedasticity and autocorrelation consistent covariance matrix estimator, manuscript, Yale University.

Cooley, Thomas F., and Masao Ogaki, 1991, A Time Series Analysis of Real Wages, Consumption, and Asset Returns: A Cointegration-Euler Equation Approach, 1991, Paper presented at the NBER Fluctuations Research Meeting in August 1991.

Costello, Donna Marie, 1990, The dynamic adjustment of producitivity growth, Paper presented at the 1990 North American Winter Meeting of the Econometric Society.

Fukushige, Mototsugu, Michio Hatanaka, and Yasuji Koto, 1990, A test on the attainability of equilibrium and its applications to international capital markets, Invited paper at the 6th World Congress of Econometric Society, August, 1990.

Kahn, James A. and Masao Ogaki, 1990, A chi-square test for a unit root," *Economics Letters*, 34, 37-92.

Kwiatkowski, Denis, and Peter Schmidt, 1990, Testing the null hypothesis of stationarity against the alternative of a unit root: how sure are we that economic time series are nonstationary?" manuscript.

Miller, K. S., 1964, *Multivariate Gaussian Distributions*, New York: Wiley.

Ogaki, Masao, 1990, Engel's law and cointegration, Rochester Center for Economic Research Working Paper No. 228.

Ogaki, Masao and Joon Y. Park, 1990, A cointegration approach to estimating preference parameters, Paper presented at the Conference on Empirical Applications of Structural Models organized by Econometrica and Social Systems Research Institute, University of Wisconsin, Madison, Wisconsin.

Park, Joon Y., and Buhmsoo Choi, 1988, A new approach to testing for a unit root," CAE Working Paper No. 88-23, Cornell University.

Phillips, P. C. B. and S. Ouliaris, 1990, Asymptotic properties of residual based tests for cointegration, *Econometrica* 58: 165-193.

Saikkonen, Pentti and Ritva Luukkonen, 1989, Testing for a moving average unit root, manuscript.

Table 1

Monte Carlo Results

T	ρ	K_T	$G_P(0,1)$	$G_N(0,1)$	$G_P(0,3)$	$G_N(0,3)$	$G_P(0,5)$	$G_N(0,5)$
Empirical sizes using five percent nominal critical values								
100	0.85	0.041	0.029	0.111	0.007	0.027	0.002	0.002
100	0.90	0.044	0.023	0.170	0.002	0.046	0.001	0.001
100	0.95	0.066	0.024	0.295	0.003	0.114	0.001	0.001
200	0.85	0.050	0.040	0.096	0.016	0.056	0.004	0.010
200	0.90	0.041	0.028	0.130	0.005	0.101	0.001	0.012
200	0.95	0.072	0.042	0.242	0.009	0.249	0.001	0.053
Empirical powers using five percent nominal critical values								
100	0.85	0.564	0.211	0.547	0.073	0.384	0.018	0.000
100	0.90	0.567	0.217	0.577	0.079	0.405	0.020	0.002
100	0.95	0.533	0.016	0.553	0.062	0.388	0.016	0.001
200	0.85	0.519	0.533	0.640	0.507	0.767	0.409	0.480
200	0.90	0.685	0.519	0.625	0.499	0.759	0.393	0.460
200	0.95	0.667	0.519	0.625	0.506	0.755	0.401	0.467
Size adjusted powers of five percent tests								
100	0.85	0.580	0.254	0.448	0.170	0.510	0.111	0.210
100	0.90	0.576	0.282	0.396	0.208	0.422	0.153	0.143
100	0.95	0.508	0.272	0.242	0.223	0.241	0.184	0.097
200	0.85	0.677	0.544	0 569	0.609	0.753	0.545	0.761
200	0.90	0.695	0.547	0.519	0.635	0.651	0.598	0.670
200	0.95	0.649	0.537	0.394	0.625	0.488	0.625	0.461

NOTE: Estimates obtained from 3000 replications.