Existence of Equilibrium with Nonconvexities and Finitely Many Agents

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ABSTRACT

Many economic models in the fields of public finance, location theory, and choice under uncertainty involve characteristic nonconvexities in either preferences or production sets for some types of commodities. One useful way to attack such nonconvexities is to employ the convexifying effect of large numbers of agents on demand for a finite number of commodities. The alternative proposed here relies on the convexifying effect of large numbers of commodities rather than agents. Sufficient substitutability and a large number of commodities can be used to replace some convexity assumptions. Existence of an equilibrium and the first welfare theorem are proved using Zame's existence theorem and Lyapunov's theorem as the key tools.

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I. Introduction

In many economic contexts, nonconvexities in preferences or technology arise naturally. These nonconvexities create well-known problems with the existence of an equilibrium. One useful way to attack such problems is to use the convexifying effect of large numbers of agents when there is a finite number of commodities. However, this method might not be available in some situations. For instance, there might be an infinity of commodities generated by the location, time, or state—dependence of commodities and preferences, so that these convexification arguments might not work. Another possibility is that the number of commodities grows with the number of agents, for example when there are externalities or public goods and personalized commodity markets. Finally, there are the problems discussed in Berliant (1985b) with the foundations of models having a continuum of consumers and land. In this paper, we show that equilibria might exist in these various contexts due to the convexifying effect of a large number of commodities. In this sense, our work can be viewed as dual to the large number of agents story told in the work of Hildenbrand (1974).

We will allow nonconvexities in preferences of any consumer, or alternatively, any producer, but not in both the consumption and production sides of the economy simultaneously. For the sake of brevity, we discuss in this paper only nonconvexities in preferences; the case of nonconvexities in production can be covered in an analogous manner. Consider a consumer who has non-convex preferences (made precise in the next section) over a continuous set of commodities T. For these commodities, it is assumed that there is sufficient substitutability in both the preferences of this non-convex consumer and production of some producer. Standard consumers and producers satisfying convexity restrictions can also be accommodated in the same economy with the same commodity space. These assumptions are made precise by specializing the framework of Zame (1987). In particular, our assumptions
on (non-convex) preferences and production sets have a hedonic interpretation. The commodities over which preferences are nonconvex are mapped into their characteristics. Preferences and technologies satisfy the usual convexity assumptions over these characteristics and the remaining (convex) commodities. The nonconvexities that we allow are introduced in the hedonic map from the nonconvex commodities to their characteristics. Heckman and Scheinkman (1987) is an example of a model with infinitely many inputs (in their case, labor supplied by different workers) where production depends on the characteristics of those inputs. They assume a linear map between inputs and characteristics. Our model allows for more general hedonic technologies.

For other examples of nonconvexities we refer to the public finance literature; see, for example, Baumol (1972), Starrett (1972), and Khan and Vohra (1987). Note that our results are more concerned with nonconvexities in isoquants and preferences than with increasing returns. However, our result does cover certain kinds of increasing returns and demonstrates the existence of a true competitive equilibrium. This is in contrast to the marginal cost pricing literature, where the notion of equilibrium merely requires firms to satisfy the first order conditions for profit maximization but not necessarily to choose a production plan that maximizes profits. The assumptions we use also have an expected utility interpretation, so that traders who prefer extreme over average lotteries can be accommodated.

The proofs of the existence of an equilibrium and of the first welfare theorem are straightforward applications of Zame's existence theorem and Lyapunov's theorem. For brevity, the only case considered here is that of nonconvexities in preferences. An analogous proof will work when there are similar nonconvexities in production. The existence proof begins by defining an artificial economy with piecewise linear utility for the nonconvex commodities. An extreme point of the set of equilibrium allocations is shown to be an equilibrium of the original economy. The moral of our
story is that sufficient substitutability and a large number of commodities can be used to replace some convexity assumptions.

The next section describes our model. Section III contains the statement of our existence theorem and its proof. Section IV concludes with a discussion of our assumptions and the potential extensions of our results.
II. The Model

Our focus in this paper is not to generalize existence theorems in convex economies with topological vector spaces as commodity spaces, but rather to examine existence of equilibrium with nonconvexities in such economies. For this reason, we wish to specialize the framework so that we can prove theorems.

We use the basic framework of Zame (1987) with some further assumptions. First, we define $L = (L^P)^k \times \hat{L}$, where $1 \leq p \leq w$, $k$ is a finite integer, $L^P$ has the norm topology, and $\hat{L}$ is a normed lattice. Then $L$ is a normed lattice with non-negative orthant $L_+$, which is assumed to be the consumption set for each consumer. We use $x$, $y$ and $z$ to denote generic elements of $L_+$. When we wish to decompose $x \in L$ into its two basic components, we write $x = (x^A, x^B)$, where $x^A \in (L^P)^k$ and $x^B \in \hat{L}$. Let the domain of functions in $(L^P)^k$ be given by $T$, and let $t$ be a generic element of $T$. $T$ will represent the set of commodities over which preferences can be nonconvex. We postulate existence of a $\sigma$-algebra on $T$, call it $\mathcal{F}$ so that we can discuss measurability. The measures on $T$ will be nonatomic. Let $q$ be such that $1/p + 1/q = 1$.

There are several possible interpretations of the elements in the consumption set $L_+$. The set $T$ could represent a set of differentiated commodities. Then $x^A$ would give the consumption of each of these commodities, which could be different in $k$ different locations, states of nature or time periods. A natural situation where nonconvexities occur would be when $T$ represents a set of indivisible commodities. If $k$ were equal to the number of consumers then $x^A$ could represent personalized commodities in the context of externalities. Alternatively, $T$ could represent continuous time, states of nature or locations. Then $x^A$ can be interpreted as a continuous-time consumption path, the state-contingent consumption or the location-specific consumption of $k$ ordinary commodities.
There are \( N \) consumers, indexed by \( i \), where \( N \) is a finite integer. There are \( M \) firms, indexed by \( j \), where \( M \geq 1 \) is a finite integer. Consumer \( i \) has an endowment \( e_i \in L_+ \) while the profit share of consumer \( i \) in firm \( j \) is given by \( \theta_{ij} \), with \( 0 \leq \theta_{ij} \leq 1 \) and \( \sum_{i=1}^{N} \theta_{ij} = 1 \) for all \( j \). The preferences of consumer \( i \) will be given by a strict preference correspondence \( P_i: L_+ \to 2^{L^+} \). The production set of firm \( j \) is called \( Y_j \subseteq L_+ \).

An allocation is given by an \( N + M - \) tuple \([(x_i),(y_j)]\) where \( x_i \in L_+ \) for all \( i \) and \( y_j \in Y_j \) for all \( j \); the allocation is feasible if \( \sum_{i=1}^{N} x_i = \sum_{i=1}^{N} e_i + \sum_{j=1}^{M} y_j \). An equilibrium is an \( N + M + 1 - \) tuple \([\pi,(x_i),(y_j)]\) where \( \pi \) is a non-zero continuous linear functional on \( L \) and \([(x_i),(y_j)]\) is a feasible allocation such that \( \pi(y_j) = \max \{ \pi(y'_j) \mid y'_j \in Y_j \} \), \( \pi(x_i) \leq \pi(e_i) + \sum_{j=1}^{M} \theta_{ij} \pi(y_j) \) for each \( i \), and if \( x' \in L_+ \) with \( x' \notin P_i(x_i) \), then \( \pi(x') > \pi(e_i) + \sum_{j=1}^{M} \theta_{ij} \pi(y_j) \). An equilibrium allocation is an allocation \([(x_i),(y_j)]\) such that there exists \( \pi \) that makes \([\pi,(x_i),(y_j)]\) an equilibrium. A Pareto optimum is a feasible allocation \([(x_i),(y_i)]\) such that there is no other feasible allocation \([(x'_i),(y'_i)]\) such that \( x'_i \in P_i(x_i) \) for all \( i \).

The economy is said to be irreducible if whenever \( i \neq i' \) and \([(x_i),(y_j)]\) is a feasible allocation, then there is a vector \( z \) in \( L \) such that \( z \leq e_i \), and \( x_i + z \in P_i(x_i) \).

Next we proceed to list the assumptions that distinguish our framework from that of Zame (1987), aside from the particular form of \( L \).

On the consumption side, we assume that for each consumer \( i \), either \( P_i \) satisfies all of the assumptions of Zame (1987), or \( i \) is a non-convex consumer, which means that consumer \( i \) has a complete preorder represented by a utility function of the

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\(^1\text{In particular, the Standard Assumptions on p. 1080 and the assumptions used in Theorem 1 on p. 1090.}\)
form:

\[ u_i(x_t) = f_i(\int_{T}^{T} h_{i}(x_i^A(t),t) \, dm_1, \ldots, \int_{T}^{T} h_{i}(x_i^A(t),t) \, dm_C, x_i^B) \]

where \( f_i : \mathbb{R}^C \times \mathbb{L}_+ \to \mathbb{R} \) is continuous, quasi-concave, and non-decreasing in its first \( C \) arguments (given a fixed argument \( C+1 \)); \( h_c \) is a measurable function mapping from \( \mathbb{R}^k \times T \) into \( \mathbb{R} \) that is continuous in its first \( k \) arguments for each fixed \( t \) and, for almost every \( t \), satisfies \( \operatorname{sup} \{ \limsup_{n \to \infty} h_c(a_n) \|a_n\| \mid \{a_n\}_{n=1}^\infty \subseteq \mathbb{R}^k, \lim_{n \to \infty} \|a_n\| = \omega \} < \omega \); and \( m_c \) is a nonatomic, positive measure on \( (T, \mathcal{F}) \) for each \( c \). Notice that \( h_c \) need not possess any concavity properties. We assume that the preferences induced by \( u_i \) are locally non-satiated in \( \mathbb{L} \): \( \forall x = (x^A, x^B) \in \mathbb{L}_+ \), \( \forall \varepsilon > 0 \), there exists \( x^B \in \mathbb{L}_+ \) with \( u_i(x) > u_i(\bar{x}) \) and \( \|x - \bar{x}\| < \varepsilon \), where \( \bar{x} = (x^A, x^B) \). Note that if we want the \( h_c \) functions to differ across consumers, we can simply include all of them in \( 1, \ldots, C \) and let the dependence of the utility functions on some of the functions \( h_c \) be trivial.

When the \( h_c \) are convex these assumptions allow non-convex preferences. Extremes can be preferred to averages. So consumers may prefer to concentrate their consumption on particular commodities, at particular times, in particular states of nature or locations depending on the interpretation of \( T \). The indifference curves describing trade-offs between commodities indexed by \( T \), and between those commodities indexed by \( T \) and those in \( \mathbb{L} \), can appear to be badly behaved. Note that the functional form of \( u_i \) can be given a hedonic interpretation, in which each integral aggregates some characteristic of a commodity bundle. The nonconvexities we allow arise in the hedonic map from commodities to characteristics and we require

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\(^2\)Alternatively, we could assume that \( f_i \) is increasing in its first \( C \) arguments and \( h_c \) is increasing in its first argument for each \( i \) and \( c \). This would place assumptions on preferences over \((\mathcal{L}_+)^k\) rather than preferences over \( \mathbb{L} \). We could not simply assume local nonsatiation for preferences, since those preferences generated by the "artificial" economy below do not necessarily inherit this property.
characteristics to be additive across the index t. Finally, note that externalities and personalized commodities can be handled by having an \( h_c \) depend only on the coordinates of \( x_1^A(t) \) that are associated with an externality (with \( k=N \)).

On the production side, we assume that if there is a non-convex consumer, the production set of at least one firm (firm 1) is of the form:

\[
Y_1 = \{ y \in L \mid (\int_T^{\beta_1(t)} y^A(t) \, d\lambda_1, \ldots, \int_T^{\beta_G(t)} y^A(t) \, d\lambda_G, y^B) \in Y_1^* \}
\]

where \( Y_1^* \) is a given closed, convex subset of \( \mathbb{R}^G \times \hat{L} \) containing 0; \( \beta_g \in (L^q)^k \forall g \); and \( \lambda_g \) is a nonatomic measure for each \( g \). For other producers \( (j > 1) \) we assume that \( Y_j \) is a convex subset of \( L \) containing 0.

The essential property of this assumption is that if there are commodities over which preferences are non-convex then there is a linear technology \( Y_1 \) over these commodities. For example, suppose \( T \) is a set of differentiated commodities with \( G = k = 1 \) and \( Y_1^* = 0 \). Then \( Y_1 \) represents a constant returns technology with \( \beta_1(t)/\beta_1(t') \) being the marginal rate of transformation between commodities \( t \) and \( t' \).

Another way of interpreting the technology described by \( Y_1 \) is that production depends on \( G \) characteristics and the commodities in \( \hat{L} \), where the characteristics are additive across the \( T \)-indexed non-convex commodities.

For example, suppose that housing density (per unit area of land) \( y^A(t) \) can be produced at each point \( t \in T \) with inputs \( y^B \) that must be allocated to \( G \) tasks (e.g. time spent building walls). The interpretation of \( \beta_1(t) \cdot y^A(t) \) would be the amount of task \( g \) (in terms of density per unit area) that is needed to produce housing density (per unit area) \( y^A(t) \). The densities are integrated over an area to obtain total inputs and outputs. The production set \( Y_1^* \) describes the technological relationship between the inputs \( y^B \) and the amount of each task that can be accomplished. For example, if \( y^B \) represents total labor time then \( Y_1^* \) might be defined by the constraint that the \( G \) integrals sum to this total time. On the
consumer side of the model, we can employ a hedonic interpretation: $x^A$ is the amount of housing per unit area (if $k > 1$ there could be several housing styles), and $h$ maps $x^A$ to characteristics, such as square feet of bedroom space per unit of housing area for the particular style. Other interpretations of the model can be constructed, but this should always be done carefully. See, for instance, the relevant examples of Jones (1984).

In the following, the roles of the special forms of production sets and utility functions can be interchanged without altering the proofs much. Hence, we could just as easily cover the case when the hedonic map for producers is nonconcave, and the hedonic map for consumers has a linear form.

As mentioned previously, our interest is in existence of equilibrium with nonconvexities. Thus, we alter Zame’s (1987) Main Theorem in that we relax the convexity/concavity assumptions, but we strengthen other assumptions. In particular, the assumptions about the existence and form of a utility function as well as the form of production are strengthened. Since we have postulated little about the form of the utility function acting on the well-behaved commodities in $\hat{L}$ as well as how production acts on $\hat{L}$, we must impose the same assumptions as Zame does in order to prove that an equilibrium exists.

As in Zame (1987), aside from the norm topology, we need a topology $\tau$ on $L$ that makes the feasible consumption and production sets compact. The most convenient topology to impose on $L$ for this purpose depends on the particular space employed. For the space $(L^p)^k$, we shall explicitly use the Mackey topology for the topology $\tau_1$. For the space $\hat{L}$, we shall leave the choice of the topology open for purposes of flexibility. Our prototypical model is when $\hat{L}$ is an $L^p$ or Euclidean space, and the topologies that are most suited to these spaces are discussed in Zame (1987, section 4). As this issue is not our focus here we shall simply assume that
there is an appropriate topology, $\tau_2$, on $\hat{L}$. Obviously, we impose the product topology $\tau = \tau_1 \times \tau_2$ on $L = (L^p)^k \times \hat{L}$.

We use the norm topology on our commodity space $L$ in order to be able to define the notions of extreme desirability and bounded marginal efficiency of production. They are needed for Zame's theorem. As these assumptions are not the focus of this paper, we do not dwell on this point.
III. Results

**Theorem:** Let \( L = (L^P)^k \times \hat{L} \) be a normed lattice satisfying the assumptions of Section II. Let \( \tau \) be a Haussdorff vector space topology on \( L \) which is weaker than the norm topology, and such that: (1) the feasible consumption and production sets are compact in the topology \( \tau \), (2) each order interval in \( L \) is compact in the topology \( \tau \), (3) for each nonconvex consumer \( i^3 \), \( u_i \) is continuous in the norm topology and \( f_i \) is upper-semicontinuous in the Euclidean \( \times \tau_2 \) topology, (4) for each \( j \) the production set \( Y_j \) is closed in the topology \( \tau \).

In addition, assume that for each \( i \) there is a \( \nu_i \) with \( 0 \leq \nu_i \leq \sum_{i=1}^{N} e_i \) and a relatively open subset \( W_i \) of \( L_+ \) which contains the feasible consumption set such that \( \nu_i \) is extremely desirable (see Zame (1987, p. 1086)) for \( P_i \) on \( W_i \). Assume also that the marginal efficiency of production is bounded (see Zame (1987, pp. 1088–1089)). Finally, if the economy is irreducible, then there exists an equilibrium.

**Proof:** We set up an artificial economy as follows. For \( W \subset \mathbb{R}^{k+1} \) let \( \text{co}(W) \) denote the convex hull of \( W \). Define:

\[
\text{Graph}_c(t) = \{(a,b) \mid a \in \mathbb{R}^k, b \in \mathbb{R}, b = h_c(a,t), a \geq 0\},
\]

for \( a \geq 0 \), \( h_c(a,t) = \sup \{ r \in \mathbb{R} \mid (a,r) \in \text{co}(\text{Graph}_c(t))\} \),

\[
\bar{u}_i(x) = f_i(\int_T h_1(x^A(t),t) \, dm_1, \ldots, \int_T h_C(x^A(t),t) \, dm_3, x^B) \quad \text{for each non-convex consumer } i.
\]

Since \( h_c \) is continuous in its first argument and \( \sup \{ \limsup_{n \to \infty} h_c(a_n,t)/\|a_n\| \mid \{a_n\}_{n=1}^{\infty} \subset \mathbb{R}^k, \lim_{n \to \infty} \|a_n\| = \infty \} < \infty \) a.s., \( h_c(a,t) < \infty \) a.s. Next we must check the standard assumptions of Zame (1987, p. 1080) for the artificial economy. First, \( L_+^{k+1} \)

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3See footnote 1 concerning other consumers.
is (norm) closed, convex, and $e_i \in L_+$ for each $i$. Second, it is easy to see that $\Phi_c(\cdot,t)$ is continuous a.s.($m_c$), and consequently that $\bar{u}_i$ is continuous in the norm topology. By definition, $\Phi_c(\cdot,t)$ is concave for each $c$ and $t$ a.s., so since $f_i$ is quasi–concave and nondecreasing in its first $C$ components, $\bar{u}_i$ is quasi–concave.

Local non–satiation has been assumed. The profit shares satisfy the usual conditions. Finally, $Y_j$ is (norm) closed and convex, and $0 \in Y_j$.

Given our assumptions, it is easy to verify that the artificial economy satisfies the conditions of Zame (1987, Theorem 1) and therefore has an (artificial) equilibrium $[\bar{\pi}(\bar{x}_i), (\bar{y}_j)]$. The remainder of the proof proceeds in a fashion similar to Berliant (1985a). Fix an equilibrium $[\bar{\pi}(\bar{x}_i), (\bar{y}_j)]$, and let $E(\bar{\pi}) \equiv \{[(x^A_i, y^A_i)] \in (L^P_+)^{kn} \times (L^P)^{km} | [\bar{\pi}(x^A_i - B_i), (y^A_j - y^B_j)]$ is an equilibrium $\} \neq \emptyset$. Notice that $E(\bar{\pi})$ is a closed and therefore Mackey compact subset of $(L^P_+)^{kn} \times (L^P)^{km}$ since $\bar{\pi}$ is a continuous linear functional, $Y_j$ is closed in the $\tau$ topology, and $u_i$ is upper semi–continuous in the $\tau$ topology on $L$. It is easy to see that $E(\bar{\pi})$ is also a convex subset of $(L^P_+)^{kn} \times (L^P)^{km}$. The Krein–Milman theorem (see Rudin (1973, p. 70)) tells us that a compact, convex set in $L$ is the closed, convex hull of its extreme points; so $E(\bar{\pi})$ has an extreme point. The next step is to show that any extreme point of $E(\bar{\pi})$ has the property that for all $c$, $h_c(x^A_i(t), t) = \Phi_c(\bar{x}^A_i(t), t)$, so $u_i(\bar{x}_i) = \bar{u}_i(\bar{x}_i)$ for any consumer $i$.

If this is not the case, then there is a consumer $i$ and a $c$ such that $h_c(x^A_i(t), t) < \Phi_c(x^A_i(t), t)$ for $t \in D$, where $m_c(D) > 0$. The next step is to show that there exists a set $W \in D$, $m_c(W) > 0$, and points $z_1, z_2 \in (L^P_+)^k$, $z_1 \neq z_2$, with $z_1(t) = z_2(t) = 0$ for $t \in T \setminus W$ a.s., $(1/2) \cdot z_1(t) + (1/2) \cdot z_2(t) = x^A_i(t)$ for $t \in W$ a.s. and $h_c(x^A_i(t), t) = (1/2) \cdot h_c(z_1(t), t) + (1/2) \cdot h_c(z_2(t), t)$ for $t \in W$ a.s. By definition of $h_c$, a.s.(t) there exist $\{(w^n_1, \ldots, w^n_b_n)_{n=1}^\infty\}$ (where $w^n_d \in (L^P_+)^k$ for all $d$ and $n$) and $\{(a^n_1, \ldots, a^n_n)_{n=1}^\infty\}$ (where $a^n_d \in L^\infty$ for all $d$ and $n$, for all $n \Sigma_{d=1}^b n a^n_d = 1$) such that...
\[
\tilde{x}_i^A(t) = \sum_{d=1}^{b_n} \alpha_d^n \cdot w_d^n \cdot h_c(\tilde{x}_i^A(t),t) - \sum_{d=1}^{b_n} \alpha_d^n \cdot h_c(w_d^n,t) < 1/n. \quad \text{(Note: To simplify notation, we omit the } t \text{ index in both } w \text{ and } \alpha, \text{ as well as the "almost surely in } t \in D" \text{ statements.)}
\]

Define \( \alpha_d^n = 0 \) if \( d > b_n \). Using the Banach–Alaoglu theorem, for each fixed \( d \), \( \{\alpha_d^n\}_{n=1}^\infty \) is contained in a weak* compact set in \( L_+^\infty \). Considering \( \{(\alpha_1^n,\alpha_2^n,\ldots)\}_{n=1}^\infty \) to be a sequence in the product space \( (L_+^\infty)^\infty \) (with the product topology), we can pass to a converging subsequence (without changing notation) with limit \( (\alpha_1,\alpha_2,\ldots) \). Now if for some \( d \), \( \alpha_d = 1 \) on a set \( S \subseteq D \) with \( m_c(S) > 0 \), then it must be the case that \( h_c(\tilde{x}_i^A(t),t) = h_c(\tilde{x}_i^A(t),t) \) for \( t \in S \), a contradiction. Hence for each \( d \), \( \alpha_d = 1 \) only on a set of measure zero. Also, \( \sum_{d=1}^\infty \alpha_d = 1 \), so there exists \( \mathcal{D} \) and \( V \in \mathcal{B} \) such that \( \alpha_d(t) > 0 \) for \( t \in V \), \( m_c(V) > 0 \). For some \( \epsilon > 0 \), there exists \( W \subseteq V \), \( m_c(W) > 0 \), with \( 1-\epsilon > \alpha_d(t) > \epsilon \) a.s. \( (t \in W) \). Let \( \alpha = \epsilon \), and let \( z_1^n(t) = \alpha_d^n(t) \cdot w_d^n(t)/\alpha \) for \( t \in W \), \( z_1^n(t) = 0 \) otherwise. Let \( z_2^n(t) = \sum_{d \neq \mathcal{D}} \alpha_d^n(t) \cdot w_d^n(t)/(1-\alpha) \) for \( t \in W \), \( z_2^n(t) = 0 \) otherwise. Now \( \{z_1^n\}_{n=1}^\infty \) and \( \{z_2^n\}_{n=1}^\infty \) are norm bounded sequences in \( (L^p)^k \) (for otherwise \( \|\bar{x}_i^A\| \) is infinite), so by the Banach Alaoglu theorem, we can pass to weak* convergent subsequences (without changing notation) with limits \( z_1 \) and \( z_2 \).

Without loss of generality, suppose \( \alpha < 1/2 \). Let \( z_1' = 2\alpha \cdot z_1 + (1-2\alpha) \cdot z_2 \), and let \( z_2' = z_2 \). Then \( z_1' \neq z_2' \), with \( z_1'(t) = 0 \) for \( t \in T \setminus W \) a.s., and
\[
(1/2) \cdot z_1'(t) + (1/2) \cdot z_2'(t) = \bar{x}_i^A(t), \quad \text{for } t \in W \text{ a.s. and } h_c(\bar{x}_i^A(t),t) = (1/2) \cdot h_c(z_1'(t),t) + (1/2) \cdot h_c(z_2'(t),t) \text{ for } t \in W \text{ a.s.}
\]
A symmetric argument works for \( \alpha > 1/2 \).

Then \( z_1' \) and \( z_2' \) have the desired properties.

Define a bounded, additive set function \( \mu \) on \( (T,\mathcal{F}) \) by \( \mu(Z) = \bar{\pi}(1_Z \cdot \bar{x}_i^A) \). Next we show that \( \mu \) is a nonatomic measure on \( (T,\mathcal{F}) \).\(^4\) The argument to show that \( \mu \) is countably additive is parallel to the proof in Bewley (1972, Theorem 2). If \( \mu \) is not

\(^4\)This part of the proof need only be used if \( p = \infty \).
countably additive on $T$, then there exists an increasing sequence of sets $F_j \in \mathcal{F}$ such that $\forall j$, $\mu(F_j) < \mu(\bigcup_{s=1}^{\infty} F_s) - \epsilon$, where $\mu(T) > \epsilon > 0$. Let $E_j = F_j \cup (T \setminus \bigcup_{s=1}^{\infty} F_s)$, where $\bigcup_{s=1}^{\infty} E_s = T$ and $\setminus$ means set-theoretic subtraction. So $\forall j$, $\mu(E_j) < \mu(T) - \epsilon$.

Let $v_j^A(t) \equiv \bar{x}_j^A(t) \cdot 1_{E_j}(t) \in (L^P)^k$ and $v_j \equiv (v_j^A, x_j^B) + \epsilon \cdot \nu_j / \pi(\nu_j) \in L$, where $\nu_j$ is the extremely desirable vector. Then $\pi(v_j) \leq \pi(\bar{x}_j)$, and for large $j$, $\bar{u}_i(v_j) > \bar{u}_i(\bar{x}_j)$, a contradiction, so $\mu$ is a measure. Hence, suppose that $\mu$ has an atom at $t^* \in T$.

Notice next that the integrals $\{\int \bar{h}_1(z_1'(t),t) \, dm_1, \ldots, \int \bar{h}_C(z_1'(t),t) \, dm_C\}$; $\int \bar{h}_G(z_2'(t),t) \, dm_G; \int \beta_1(t) \cdot z_1'(t) \, d\lambda_1, \ldots, \int \beta_G(t) \cdot z_1'(t) \, d\lambda_G\}$, along with $\mu$, form a finite set of nonatomic measures on $(T, \mathcal{F})$. In particular, they are nonatomic when restricted to the $\sigma$-algebra of measurable sets contained in $W$, $(W, \mathcal{W})$.

Using Lyapunov's theorem on the vector measure, there are disjoint sets $V, V' \in \mathcal{W}$ such that $\mu(V) = \mu(V') = (1/2) \cdot \mu(W)$, $\int \bar{h}_c(z_1'(t),t) \, dm_c = \int \bar{h}_c(z_2'(t),t) \, dm_c = (1/2)$ $\int \bar{h}_c(z_1'(t),t) \, dm_c$ for each $c$, $\int \bar{h}_c(z_2'(t),t) \, dm_c = \int \bar{h}_c(z_2'(t),t) \, dm_c = (1/2)$ $\int \bar{h}_c(z_1'(t),t) \, dm_c$ for each $c$, $\int \beta_g(t) \cdot z_1'(t) \, d\lambda_g = \int \beta_g(t) \cdot z_1'(t) \, d\lambda_g = (1/2)$ $\int \beta_g(t) \cdot z_1'(t) \, d\lambda_g$ for each $g$, $\int \beta_g(t) \cdot z_2'(t) \, d\lambda_g = \int \beta_g(t) \cdot z_2'(t) \, d\lambda_g = (1/2)$ $\int \beta_g(t) \cdot z_2'(t) \, d\lambda_g$ for each $g$. Now we can construct two allocations that yield the equilibrium allocation as a convex combination. Let $\hat{x}_i^A(t) \in (L^P)^k$, $\hat{x}_1^A(t) = \bar{x}_1^A$ for $t \notin W$, $\hat{x}_1^A(t) = z_1(t)$ if $t \in V$, $\hat{x}_1^A(t) = z_2(t)$ if $t \in V'$. Let $\hat{y}_1 \equiv (\hat{x}_1^A, x_1^B)$. Let the new production plan of firm 1 be $\hat{y}_1 \equiv (\hat{y}_1^A, x_1^B)$, where $\hat{y}_1^A \equiv \bar{y}_1^A + \hat{x}_1^A - \bar{x}_1^A$. All other agents get the same bundle or production plan in this new allocation.
[(\tilde{x}_1), (\tilde{y}_1)]. Let \( \tilde{x}^A_i \in (L^p)^k \), \( \tilde{x}^A_i (t) = \tilde{x}_i^A \) for \( t \notin W \), \( \tilde{x}_i^A (t) = z_\nu (t) \) if \( t \in V \), \( \tilde{x}^A_i (t) = z'_\nu (t) \) if \( t \in V' \). Let \( \tilde{\nu}_1 \equiv (\tilde{\nu}_1^A, \tilde{\nu}_1^B) \). Let the new production plan of firm 1 be \( \tilde{y}_1 \equiv (\tilde{\nu}_1^A, \tilde{\nu}_1^B \), where \( \tilde{\nu}_1^A = \tilde{\nu}_1^A + \tilde{x}_1^A - \tilde{x}_1^A \). All other agents get the same bundle or production plan in this new allocation \([(\tilde{x}_1), (\tilde{y}_1)]\) as in the equilibrium allocation \([(\tilde{x}_1), (\tilde{y}_j)]\). Given the form of firm 1's production set, both \( \tilde{y}_1 \) and \( \tilde{y}_1 \) are in the production set since \( \tilde{y}_1 \) is. Notice that for consumer \( i \), \( \tilde{u}_i (\tilde{x}_1) = \tilde{u}_i (\tilde{x}_1) = \tilde{u}_i (\tilde{x}_1) \), \( \tilde{\pi} (\tilde{x}_1) = \tilde{\pi} (\tilde{x}_1) = \tilde{\pi} (\tilde{x}_1) \), and profits are the same under all 3 production plans for each producer. The material balance conditions also hold for all three allocations. Hence, all three are equilibrium allocations with respect to prices \( \tilde{\pi} \). Finally, \([(\tilde{x}_1), (\tilde{y}_1)] = 1/2 \cdot [(\tilde{x}_1), (\tilde{y}_1)] + 1/2 \cdot [(\tilde{x}_1), (\tilde{y}_1)], [(\tilde{x}_1), (\tilde{y}_1)] \neq [(\tilde{x}_1), (\tilde{y}_1)]. \) Hence \([(\tilde{x}_1), (\tilde{y}_j)]\) is not an extreme point of \( E(\tilde{\pi}) \), a contradiction. So the hypothesis is false, and any extreme point of \( E(\tilde{\pi}) \) has the property that for any consumer \( i \), \( u_i (\tilde{x}_1) = \tilde{u}_i (\tilde{x}_1) \).

Finally, we claim that \( [\tilde{\pi}, (\tilde{x}_1), (\tilde{y}_j)] \), where \([(\tilde{x}_1), (\tilde{y}_j)]\) is an extreme point of \( E(\tilde{\pi}) \), is an equilibrium for the original economy. For each consumer \( i \), any bundle \( x' \) with \( u_i (x') > u_i (\tilde{x}_1) \) also has \( \tilde{u}_i (x') > u_i (\tilde{x}_1) = \tilde{u}_i (\tilde{x}_1) \), so \( \tilde{\pi} (x') > \tilde{\pi} (x_1) \). Feasibility and profit maximization follow from the fact that \( [\tilde{\pi}, (\tilde{x}_1), (\tilde{y}_j)] \) is an equilibrium for the artificial economy.

Q.E.D.

Finally, notice that by standard arguments any equilibrium allocation of an economy satisfying the assumptions of section II can be shown to be Pareto optimal.
IV. Conclusion — Discussion and Potential Extensions

We have shown that sufficient substitutability and a large number of commodities can be used to alter or replace some convexity assumptions. This provides a method dual to the continuum (of agent) models for dealing with the problems created by nonconvexities in preferences and technologies. The hedonic interpretation of some of our assumptions might make our model useful for various kinds of applications.

It would be desirable to consider formally the large, finite economies that approximate our continuum of commodities model. This might yield more insight into how the convexifying effect of a large number of commodities works. We would expect that the Shapley–Folkman theorem could be used to obtain approximate equilibria for the finite economies.

Note that one case in which our assumption might be reasonable is when the set T represents a set of indivisible commodities. These will generally be the same for all agents. One would want each indivisible commodity to be consumed by exactly one person, and this happens in any equilibrium of such a model.
References


