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THE REPLACEMENT PRINCIPLE IN PUBLIC GOOD ECONOMIES WITH SINGLE-PEAKED PREFERENCES

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Abstract

We study the implications for a simple public good model with single–peaked preferences of a "replacement principle" proposed in Thomson (1990). We apply the principle to situations when the preferences of one of the agents may change. The principle says that in such circumstances, all other agents should be affected in the same direction. We characterize the class of efficient solutions satisfying the property. It is a subclass of the class of strategy–proof solutions introduced by Moulin (1980) under the name of "generalized Condorcet–winner solutions".

Key–words. Replacement principle. Single–peaked preferences. Generalized Condorcet–winner solutions.

1. ***Introduction.*** The purpose of this paper is to study the implications in a simple model of a principle introduced in Thomson (1990) under the name of "replacement principle". This principle is intended to help in the evaluation of decision rules in a wide variety of contexts and we start with a general formulation : essentially, it states that if the value of one of the components of the data entering in the description of the problem to be solved is replaced by another admissible value, all "relevant" agents should be affected in the same direction : all gain or all lose as a result of the replacement.[1]

Here, we consider the replacement of the preferences of one agent by some other admissible preferences and we name the form taken by the principle in this application, "replacement monotonicity". The issue we address is that of choosing a public good level from some interval when agents have single–peaked preferences. Our main result is a characterization of a subfamily of the family of "generalized Condorcet–winner solutions", introduced by Moulin (1980, 1984). Each of the members of this subfamily is defined by first choosing a "target" public good level in the interval, selecting that target if it is efficient and selecting the preferred consumption that is the closest to that target otherwise. We show that the members of this subfamily are the only efficient solutions to be replacement–monotonic.

As all generalized Condorcet–winner solutions, these solutions have the very attractive feature of being strategy–proof (Moulin, 1980; Barbera and Jackson, 1991; Ching, 1992). They also satisfy a certain property of population–monotonicity and they can be characterized on that basis (Ching and Thomson, 1992).

In related work (Thomson, 1992), we apply the replacement principle in the context of the the fair allocation of an infinitely divisible private good in economies with single–peaked preferences and obtain a characterization of a unique efficient

[1]For a more extensive discussion of the principle and of its relation to other principles that have been studied in the literature, and in particular to notions of monotonicity, see Thomson (1992).

selection from the solution that chooses, for each economy, its set of envy–free allocations. The results presented here and in that companion paper suggest that the replacement principle may be a fruitful notion. This author is currently engaged in an analysis of its implications for other classes of decision problems.

2. ***The model.*** The choice of the level of a public good in some interval $[0,M]$ has to be made. There is a set $N = \{1,...,n\}$ of agents, indexed by i, each agent i being equipped with a continuous preference relation R_i defined over $[0,M]$. Let P_i denote the strict preference relation associated with R_i, and I_i the indifference relation. These preference relations are ***single–peaked***: for each R_i, there is a number $p(R_i) \in [0,M]$ such that for all $x, x' \in [0,M]$, if $x' < x \leqq p(R_i)$, or $p(R_i) \leqq x < x'$, then xP_ix'. Let $\mathscr{R}$ be the class of all such preference relations. We write $R = (R_i)_{i\in N}$ and $p(R) = (p(R_i))_{i\in N}$. An ***economy*** is a list $R \in \mathscr{R}^n$.

A ***decision for*** $R \in \mathscr{R}^n$ is simply a point $x \in [0,M]$. We wish to identify desirable decision procedures. A ***solution*** is a mapping $\varphi\colon \mathscr{R}^n \to [0,M]$ which associates with each economy $R \in \mathscr{R}^n$ a non–empty subset of $[0,M]$, denoted by $\varphi(R)$. Each of the points in this subset is interpreted as a recommendation for the economy. An example is the solution that associates with each economy its set of public good levels such that there is no other level that every agent prefers and at least one agent strictly prefers.

Pareto solution, P: $x \in P(R)$ if $x \in [0,M]$ and there is no $x' \in [0,M]$ with $x'R_ix$ for all $i \in N$ and $x'P_ix$ for some $i \in N$.

It is clear that $P(R) = [\min\{p(R_i) \mid i \in N\}, \max\{p(R_i) \mid i \in N\}]$.

The following one–parameter family of solutions indexed by the parameter $a \in [0,M]$ will play the central role in our analysis.

Family $\Phi = \{\varphi^a \mid a \in [0,M]\}$. Given $a \in [0,M]$, let $\varphi^a(R) = a$ if $a \in P(R)$; $\varphi^a(R) = \min\{p(R_i) \mid i \in N\}$ if $a < \min\{p(R_i) \mid i \in N\}$; and $\varphi^a(R) = \max\{p(R_i) \mid i \in N\}$ if $a > \max\{p(R_i) \mid i \in N\}$.

The family Φ is a subfamily of the family of generalized Condorcet–winner solutions characterized by Moulin (1980), Barbera and Jackson (1991), and Ching (1992) on the basis of considerations of strategy–proofness (the property that, for each agent, announcing his true preferences is a dominant strategy in the associated direct revelation game). In the context of a variable population, where solutions are defined over classes of problems of arbitrary cardinality, the solutions obtained from the above definition by choosing the same parameter a for all cardinalities have been characterized by Ching and Thomson (1992) on the basis of considerations of population–monotonicity.[2]

We next turn to our central condition of replacement–monotonicity : replacing the preference relation of one agent by another preference relation in the admissible class affects all other agents in the same direction. Although we could easily write the condition for solution correspondences, we find it somewhat more natural to assume that the solution is actually a function and our formulation below is chosen accordingly.[3]

Note that the condition is ordinal : it is expressed in terms of preferences, not in terms of "utilities". It does not involves measuring, let alone comparing, the gains or losses agents experience. However, we will see that the condition is already quite powerful. Indeed, it is possible to completely determine which subsolutions of the pareto solution satisfy it.

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[2]In this context, a solution is a list $\{\varphi^Q | \ Q \subseteq \mathbb{N}\}$, where $\mathbb{N}$ is interpreted as a set of "potential agents" and for each finite group of agents $Q \subseteq \mathbb{N}$, φ^Q is a solution defined on $\mathscr{R}^Q$, the class of economies involving the group Q. Population–monotonicity is a property that helps relate the components of φ across cardinalities.

[3]The most appealing formulation for solution correspondences would perhaps be to say that, when an agent's preferences change, then at **each** of the decisions made initially, either each of the other agents is better off than at **each** of the decisions made afterwards, or the opposite holds. In the case of single–valued solutions, such a condition would coincide with the one we use here. Weaker conditions would be obtained by choosing the quantifiers differently.

Replacement–monotonicity. For all R, $R' \in \mathscr{R}^n$, for all $i \in N$, if $R_{-i} = R'_{-i}$, then either $\varphi(R) R_j \varphi(R')$ for all $j \in N\backslash\{i\}$ or $\varphi(R') R_j \varphi(R)$ for all $j \in N\backslash\{i\}$.

Note that the condition has no power if $n \leqq 2$. Note also that a stronger condition is obtained by requiring that replacing the preferences of the members of an arbitrary ***group*** of agents affects all of the members of the complementary group in the same direction. The solutions characterized below satisfy the stronger condition.

We are now ready for our main result.

Theorem 1. Suppose $n \geqq 3$. The members of the family Φ are the only efficient solutions to be *replacement–monotonic.*

Proof. We omit the proof that all members of Φ satisfy the two requirements of the theorem. To prove the converse, let $R \in \mathscr{R}^n$ and in order to simplify the notation, suppose that $p(R_1) \leqq p(R_2) \leqq \dots \leqq p(R_n)$. Let $R^0 \in \mathscr{R}^n$ be defined by $p(R^0) = (M,M,\dots,M,0)$, and let $a = \varphi(R^0)$.

Case 1: $a \in P(R)$. We consider the following sequence of economies : R^0, $R^1 = (R_1,R_2^0,R_3^0,\dots,R_{n-1}^0,R_n^0)$, $R^2 = (R_1,R_2^0,R_3^0,\dots,R_{n-1}^0,R_n)$, $R^3 = (R_1,R_2,R_3^0,\dots,R_{n-1}^0,R_n)$, ..., $R^{n-1} = (R_1,R_2,R_3,\dots,R_{n-1}^0,R_n)$, $R^n = R$, in which first agent 1's and then agent n's preferences are changed from R_1^0 and R_n^0 to R_1 and R_n respectively, and then for each $i \in \{2,\dots,n-1\}$, agent i's preferences are changed from R_i^0 to R_i. We show next that at each step, the choice cannot change.

Since $p(R_n^1) = 0$ and $p(R_{n-1}^1) = M$, we cannot have $\varphi(R^1) < a$ since agent n would gain and agent n–1 would lose, in contradiction with *replacement–monotonicity.* Similarly, we cannot have $\varphi(R^1) > a$. Therefore $\varphi(R^1) = a$.

Since $\varphi \subsetneq P$, $\varphi(R^2) \in P(R^2) = [p(R_1^2),p(R_{n-1}^2)] = [p(R_1),M]$. We cannot have $\varphi(R^2) < a$ since agent 1 would gain and agent n–1 would lose, in contradiction with *replacement–monotonicity.* Similarly, we cannot have $\varphi(R^2) > a$. From $k = 3$ to $k = n-1$, the conclusion $\varphi(R^k) = a$ is obtained in an identical way.

Finally, $P(R^n) = [p(R_1^n),p(R_n^n)] = [p(R_1),p(R_n)]$. We cannot $\varphi(R^n) < a$ since agent 1 would gain and agent n would lose, in contradiction with *replacement–monotonicity.* Similarly, we cannot have $\varphi(R^n) > a$. Therefore, $\varphi(R^n) = \varphi(R) = a$.

Case 2: $a \notin P(R)$. Suppose, without loss of generality, that $p(R_n) < a$, and by contradiction, that $\varphi(R) \neq \varphi^a(R) = p(R_n)$. Since $\varphi \subseteq P$, $p(R_1) \leqq \varphi(R) < p(R_n)$. Let $R' \in \mathscr{R}^n$ be such that $\varphi(R) < p(R_2') \leqq p(R_n)$ and $aP_2'\varphi(R)$, and $R'_{-2} = R_{-2}$. Since $\varphi \subseteq P$, $\varphi(R') \in P(R') = P(R)$. We cannot have $\varphi(R') < \varphi(R)$, since agent 1 would gain and agent n would lose, in contradiction with *replacement–monotonicity.* Similarly, we cannot have $\varphi(R') > \varphi(R)$. Now, let $R'' \in \mathscr{R}^n$ be such that $R''_{-n} = R'_{-n}$ and $p(R''_n) \geqq a$. We cannot have $\varphi(R'') = a$ since agent 1 would lose and agent 2 would gain, in contradiction with *replacement–monotonicity.* However, we now have a contradiction to Case 1 since $a \in P(R'')$ and yet $a \neq \varphi(R'')$.

Q.E.D.

To show the independence of the requirements used in Theorem 1, we first note that the members of the generalized Condorcet–winner family that are not in the family Φ are selections from the pareto solution that violate *replacement–monotonicity.* We skip the straightforward proof, which would require a formal introduction of this wider family.

Also, if efficiency is not imposed, many other solutions become available. However, provided a continuity condition is added, the class of admissible solutions can still be completely described. This class is closely related to a class characterized in Ching and Thomson (1992) in their search for (not necessarily efficient) population–monotonic solutions. It is defined as follows. Let $a \in [0,M]$. If $a \in P(R)$, $\varphi(R) = a$; if $a < \min\{p(R_i) \mid i \in N\}$, $\varphi(R)$ is a point of $[0, \min\{p(R_i) \mid i \in N\}]$ which depends continuously on R; if $a > \max\{p(R_i) \mid i \in N\}$, $\varphi(R)$ is a point of $[\max\{p(R_i) \mid i \in N\}, M]$ which depends continuously on R. The proof that there is no other *replacement–monotonic* and *continuous* solution is relegated to the appendix.

We conclude by noting that, although in Theorem 1 we did not impose the requirement that "all agents be treated the same" (such a condition of "anonymity" is often used in social choice), we nevertheless obtained solutions satisfying this property.

3. Conclusion. We have provided a characterization of the class of solutions satisfying a property of *replacement–monotonicity* for a simple model concerning the choice of the level of a public good in an economy with single–peaked preferences. *Replacement–monotonicity* is a strong property of solidarity among agents. However, in our model, the property is compatible with other interesting properties, such as strategy–proofness and population–monotonicity. This is also the case for the private good model with single–peaked preferences, as we show in Thomson (1992). It remains to be seen whether the same holds true for other models.

Appendix. In this appendix, we investigate the consequences of dropping efficiency as one of the requirements of Theorem 1. Instead, we impose a condition of "continuity", which says that small changes in preferences should not affect the choice by much.

For a formal statement, we note first that a preference relation $R_i \in \mathscr{R}$ can be described in terms of the function r_i: $[0,M] \to [0,M]$ defined as follows: given $x \leqq p(R_i)$, $r_i(x) \geqq p(R_i)$ and $x I_i r_i(x)$ if such a number exists and $r_i(x) = M$ otherwise; given $x \geqq p(R_i)$, $r_i(x) \leqq p(R_i)$ and $x I_i r_i(x)$ if such a number exists and $r_i(x) = 0$ otherwise. (The number $r_i(x)$ is the public good level on the other side of agent i's preferred level that he finds indifferent to x, if such a level exists; it is 0 or M otherwise.).

Given R_i and $R_i' \in \mathscr{R}$, let $d(R_i,R_i') = \max\{|r_i(x)-r_i'(x)| \mid x \in [0,M]\}$. ***The sequence*** $\{R^\nu\}$ ***of elements of*** $\mathscr{R}^n$ ***converges to*** $R \in \mathscr{R}^n$ if for all $\epsilon > 0$, there is $\nu \in \mathbb{N}$ such that for all $i \in N$ and for all $\nu' > \nu$, $d(R_i^{\nu'},R_i) < \epsilon$.

Continuity. For all sequences $\{R^\nu\}$ of elements of $\mathscr{R}^n$, and for all $R \in \mathscr{R}^n$, if $\{R^\nu\}$ converges to R, then $\varphi(R^\nu)$ converges to $\varphi(R)$ (This is the notion of *continuity* used by Sprumont, 1991).

Let us designate by Φ' the family described in the penultimate paragraph of section 2.

Theorem 2. Suppose $n \geqq 3$. The members of the family Φ' are the only solutions to be *replacement–monotonic* and *continuous.*

Proof. We omit the proof that all members of Φ' satisfy the requirements of Theorem 2. To prove the converse, let R^0 be the profile used in the proof of Theorem 1. Let $a = \varphi(R^0)$. Then, given $R \in \mathscr{R}^n$, with $p(R_1) \leqq p(R_2) \leqq ... \leqq p(R_n)$, we distinguish two cases.

Case 1: $a \in P(R)$. We first consider the subcase $a \in$ rel.int$\{P(R)\}$. We proceed as in Case 1 of Theorem 1. The conclusion that the choice remains the same when agent 1's preferences change from R_1^0 to R_1 is as in that theorem. Then, we change agent n's preferences continuously from R_n^0 to R_n and obtain by *replacement–monotonicity* and *continuity* that the choice remains the same along the way. A similar continuous replacement, for each $i = 2,...,n-1$, of agent i's preferences from R_i^0 to R_i is handled in a similar way. Altogether, we obtain $a = \varphi(R)$. The second subcase is $a \in P(R)\backslash$rel.int$\{P(R)\}$. Here, the desired conclusion follows from the first subcase and *continuity.*

Case 2: $a \notin P(R)$. Suppose, without loss of generality, that $p(R_n) < a$. We only need to show that $\varphi(R) \in [p(R_n),M]$. Suppose not. First, we consider the subcase $p(R_1) < p(R_n)$. We change agent 2's preferences continuously from R_2 to $R_2' \in \mathscr{R}$ such that $p(R_2') \geqq a$. By *continuity*, the choice changes continuously but it cannot increase in the non–degenerate interval $[\max\{\varphi(R),p(R_1)\},p(R_n)]$ without a violation of *replacement–monotonicity*. Therefore, $\varphi(R_1,R_2',...,R_n) \neq a$. However, $a \in P(R_1,R_2',...,R_n)$, and by Case 1, $\varphi(R_1,R_2',...,R_n) = a$. If $p(R_1) = p(R_n)$, we obtain by *continuity* that there is $R'' \in \mathscr{R}^n$ such that $\varphi(R'') < p(R_1'') < p(R_n'') < a$, and we are back to the previous subcase.

Q.E.D.

The fact that *continuity* is necessary for this characterization is established by the following example, which is discussed in Ching and Thomson (1992) : Let $a \in [0,M]$. Given $R \in \mathscr{R}^n$ such that $\max\{p(R_i) \mid i \in N\} < a$ and for each $i \in N$, there is $x_i^* < a$ such that $x_i^* I_i a$, let $\varphi(R)$ be an arbitrary point of $[0, \min\{x_i^* \mid i \in N\}]$; given $R \in \mathscr{R}^n$ such that $\min\{p(R_i) \mid i \in N\} > a$ and for each $i \in N$, there is $x_i^* > a$ such that $x_i^* I_i a$, let $\varphi(R)$ be an arbitrary point of $[\max\{x_i^* \mid i \in N\}, M]$; let $\varphi(R) = a$ in all other cases. We omit the proof that any solution φ so defined does satisfy *replacement–monotonicity,* but violates $\varphi \subseteq P$ and *continuity.*

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