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Abstract

We consider the problem of choosing the level of a public good when agents have single-peaked preferences. We search for solutions satisfying *Pareto-efficiency* and *population-monotonicity* (Thomson, 1983), the requirement that upon the arrival of additional agents, all agents initially present be made weakly worse-off, or alternatively, that upon the departure of some of the agents, all remaining agents be made weakly better-off. We characterize the class of solutions satisfying these requirements. It is a subclass of the class of generalized Condorcet-winner solutions (Moulin, 1980, 1984).

Keywords: Population-monotonicity, single-peaked preferences, generalized Condorcet-winner solutions

1 Introduction

We consider the problem, first analyzed by Moulin (1980), of choosing the level of a public good when agents have "single-peaked" preferences. We search for desirable methods, or ***solutions***, of associating a level of the public good with each economy, this level being interpreted as the recommendation for that economy. We consider the model in a variable population environment and investigate the existence of solutions satisfying "Pareto-efficiency" and "population-monotonicity." ***Population-monotonicity*** (Thomson, 1983) requires that upon the arrival of additional agents, all agents initially present be made weakly worse-off, or equivalently, that upon the departure of some of the agents, all remaining agents be made weakly better-off.

Such solutions do exist and our main result is a characterization of the class they constitute. Each member of this class can be described in terms of a "target" level: If that level is efficient, then it is chosen; otherwise the efficient level the closest to it is chosen. This is a subclass of the class of generalized Condorcet-winner solutions characterized by Moulin (1980) on the basis of "strategy-proofness."[1] The fixed-population counterparts of these solutions are characterized by Thomson (1993) on the basis of a certain property of "replacement-domination."[2]

2 The Model and the Result

We consider the problem of choosing the level of a public good in an interval $[0, M]$.[3] Our focus is on situations in which the number of agents may vary, situations also considered in this context by Moulin (1984). The formal model is as follows. There is an infinite number of "potential agents," indexed by the positive integers $\mathbb{N}$. Each agent $i \in \mathbb{N}$ is equipped with a continuous and "single-peaked" preference relation R_i defined over $[0, M]$. Let P_i be the strict relation associated with R_i, and I_i the indifference relation. The preference relation R_i is ***single-peaked:*** There is a number $p(R_i) \in [0, M]$ such that for all $x, y \in [0, M]$, if $y < x \leq p(R_i)$ or $p(R_i) \leq x < y$, then $x P_i y$. Let $\mathcal{R}$ be the class of all such preference relations. A preference relation $R_i \in \mathcal{R}$ can be described in

[1]The requirement that for every agent, revealing her true preferences be a dominant strategy in the direct revelation game associated with the solution.

[2]The requirement that when the preferences of one agent change, all other agents be affected in the same direction.

[3]We use a closed interval for technical convenience. Our result remains true if an open interval is used instead.

terms of the ***function*** $\boldsymbol{r_i : [O, M] \rightarrow [O, M]}$ defined as follows: For all $x \in [0, p(R_i)]$, $r_i(x) = y$ where $y \in [p(R_i), M]$ satisfies yI_ix if such y exists, and $r_i(x) = M$ otherwise; for all $x \in [p(R_i), M]$, $r_i(x) = y$ where $y \in [0, p(R_i)]$ satisfies yI_ix if such y exists, and $r_i(x) = 0$ otherwise. Let $\mathcal{Q}$ be the collection of all finite subsets of $\mathbb{N}$. Given $Q \in \mathcal{Q}$, we designate the preference profile $(R_i)_{i \in Q}$ by R_Q. Given $Q \in \mathcal{Q}$, let $\mathcal{R}^Q$ be the class of possible preference profiles for the group Q. Given $R_Q \in \mathcal{R}^Q$, let $\underline{p}(R_Q) = \min_{i \in Q}\{p(R_i)\}$ and $\overline{p}(R_Q) = \max_{i \in Q}\{p(R_i)\}$. Since the interval of possible levels of the public good is fixed, an ***economy*** is simply denoted by $R_Q \in \mathcal{R}^Q$, for $Q \in \mathcal{Q}$.

A solution is a function that associates a point in the interval $[0, M]$ with each economy.

Definition. A ***solution*** is a function $\varphi : \cup_{Q \in \mathcal{Q}} \mathcal{R}^Q \rightarrow [0, M]$.

A level of the public good is (Pareto)-efficient for an economy if there is no other level that is preferred by all agents and strictly preferred by at least one agent. Given $Q \in \mathcal{Q}$ and $R_Q \in \mathcal{R}^Q$, let $\boldsymbol{P(R_Q)}$ be the ***set of efficient levels for $\boldsymbol{R_Q}$:***

$$P(R_Q) = \{x \in [0, M] |\ \nexists y \in [0, M] \text{ s.t. } \forall i \in Q, yR_ix, \text{ and } \exists i \in Q, yP_ix\}.$$

The efficient set can be described in the following handy way:

$$P(R_Q) = [\underline{p}(R_Q), \overline{p}(R_Q)].$$

We will impose two axioms on solutions. The first axiom requires that the chosen level be efficient.

Efficiency. For all $Q \in \mathcal{Q}$, and for all $R_Q \in \mathcal{R}^Q$, $\varphi(R_Q) \in P(R_Q)$.

The second axiom requires that upon the arrival of new agents, all agents initially present be made weakly worse-off, or equivalently as we noted earlier, that upon the departure of some of the agents, all remaining agents be made weakly better-off. See Thomson (1992) for a survey of the various applications of this condition and of related conditions.

Population-monotonicity. For all $Q, Q' \in \mathcal{Q}$ such that $Q \subseteq Q'$, for all $R_{Q'} \in \mathcal{R}^{Q'}$, and for all $i \in Q$, $\varphi(R_Q) R_i \varphi(R_{Q'})$.[4]

[4] A weaker version of *population-monotonicity* is obtained by adding the hypothesis that $Q' = Q \cup \{i\}$ for some $i \in \mathbb{N}$. Our result remains true even for this weaker condition.

Thomson (1983) considers situations where the arrival of new agents is not accompanied by an expansion of the opportunities available to the agents initially present and proposes the requirement on solutions that all agents be made weakly worse-off by such arrivals. This is also the "right" population-monotonicity condition for the current model since the feasible set is unchanged when additional agents come in. In situations where the arrival of additional agents may entail a restriction ***or*** an expansion of the opportunities available to the agents initially present, the natural generalization of this requirement is that all agents initially present be affected in the same direction, as proposed by Chun (1986). One could argue, and the point is developed in Thomson (1992), that in order to maintain the conceptual distinction between efficiency and the normative condition of monotonicity, one should really use in general the condition that all agents initially present be affected in the same direction when new agents come in. In the present model, together with efficiency, such a condition would imply that the arrival of new agents makes all of the agents initially present ***weakly worse-off,*** the condition that we have adopted above for simplicity.

Our main result is a characterization of the following ***one-parameter class of solutions,*** $\Phi = \{\varphi^a | a \in [O, M]\}$: Given $Q \in \mathcal{Q}$ and $R_Q \in \mathcal{R}^Q$,

$$\varphi^a(R_Q) = \begin{cases} \underline{p}(R_Q) & \text{if } a < \underline{p}(R_Q) \\ a & \text{if } a \in P(R_Q) \\ \overline{p}(R_Q) & \text{if } \overline{p}(R_Q) < a. \end{cases}$$

Theorem 1. A solution satisfies *efficiency* and *population-monotonicity* if and only if it belongs to the class Φ.

Proof. Let φ be a solution satisfying *efficiency* and *population-monotonicity.* Let $Q^* \in \mathcal{Q}$ be such that $|Q^*| = 2$. Let $R_{Q^*} \in \mathcal{R}^{Q^*}$ be such that the preferred level of one agent is 0 and that of the other agent M. Let $a = \varphi(R_{Q^*})$. Let $Q \in \mathcal{Q}$ be such that $Q^* \cap Q = \emptyset$ and $R_Q \in \mathcal{R}^Q$. Starting from R_{Q^*}, consider now the arrival of the group Q. Then $\varphi(R_{Q^* \cup Q}) = a$; otherwise, the two agents in Q^* would be affected differently, in contradiction with *population-monotonicity.* We proceed by distinguishing between the following two cases:

Case 1: $\boldsymbol{a \in P(R_Q)}$**.** By *efficiency,* $\varphi(R_Q) \in P(R_Q)$. Suppose, without loss of generality, that $\underline{p}(R_Q) < \overline{p}(R_Q)$. Let $i \in Q$ with $p(R_i) = \underline{p}(R_Q)$ and $j \in Q$ with $p(R_j) = \overline{p}(R_Q)$. Starting from $R_{Q^* \cup Q}$, consider now the departure of the group Q^*. Then $\varphi(R_Q) = a$; otherwise, agents i and j

would be affected differently, in contradiction with *population-monotonicity.*

Case 2: $\boldsymbol{a \notin P(R_Q)}$. Suppose, without loss of generality, that $a < \underline{p}(R_Q)$ and, by contradiction, that $\varphi(R_Q) \neq \underline{p}(R_Q)$. By *efficiency,* $\underline{p}(R_Q) < \varphi(R_Q) \leq \overline{p}(R_Q)$. Let agent $k \notin Q^* \cup Q$ be such that $p(R_k) \in P(R_Q)$ and $aP_k\varphi(R_Q)$. It can be shown by an argument similar to the one made above that $\varphi(R_{Q^* \cup Q \cup \{k\}}) = \varphi(R_{Q^* \cup Q})$ and $\varphi(R_{Q \cup \{k\}}) = \varphi(R_Q)$. Let $j \in Q$ with $p(R_j) = \overline{p}(R_Q)$. Starting from $R_{Q^* \cup Q \cup \{k\}}$, consider now the departure of the group Q^*. Then, agents k and j are affected differently, in contradiction with *population-monotonicity.*

We omit the proof that, conversely, any solution in Φ satisfies the two axioms. ***Q.E.D.***

3 Concluding Remarks

3.1 Independence of the Axioms

(i) It is easy to construct solutions not in the class Φ that satisfy *efficiency* but not *population-monotonicity.* (ii) The members of the following class Ψ satisfy the form of *population-monotonicity* (discussed before the statement of Theorem 1), according to which, upon the arrival of new agents, all agents initially present should be affected in the same direction. Note that these solutions also satisfy "continuity" with respect to preferences, formulated by Sprumont (1991). Given $Q \in \mathcal{Q}$ and $R_Q \in \mathcal{R}^Q$, let $a \in [0, M]$ and

$$\psi(R_Q) \in \begin{cases} [0, \underline{p}(R_Q)] & \text{if } a < \underline{p}(R_Q) \\ \{a\} & \text{if } a \in P(R_Q) \\ [\overline{p}(R_Q), M] & \text{if } \overline{p}(R_Q) < a, \end{cases}$$

where ψ is continuous.

3.2 Adding a Neutrality Condition

The model discussed here can be seen as the reduced model of an underlying two-dimensional pure public good economy in which agents, with strictly convex and monotonic preferences, choose from a constraint set with a boundary which is concave toward the origin. When the boundary is linear, it is natural to require that the two goods be treated "neutrally," a requirement which for the reduced model, implies that the solutions be symmetric with respect to the mid-point.

Given $Q \in \mathcal{Q}$, $i \in Q$, and $R_i \in \mathcal{R}$, let R_i^π be defined by for all $x \in [0, M]$, $r_i^\pi(x) = M - r_i(M - x)$ and $R_Q^\pi = (R_i^\pi)_{i \in Q}$.

Neutrality. For all $Q \in \mathcal{Q}$ and for all $R_Q \in \mathcal{R}^q$, if $R_Q = R_Q^\pi$, then $\varphi(R_Q) = \frac{M}{2}$.

Corollary 1. The solution $\varphi^{\frac{M}{2}} \in \Phi$ is the only solution that satisfies *neutrality, efficiency,* and *population-monotonicity.*

3.3 Siting a Public Facility

Theorem 1 can be extended to provide a solution to the more general problem of choosing an alternative on a graph with a tree structure. An axiomatic analysis of this problem was carried out by Holtzman (1990). An economy here consists of a finite number of agents equipped with single-peaked preferences over the points of a tree. An application is when the tree represents a road network, and the problem is that of siting a facility on that network. Think of neighboring towns deciding on where to locate a facility, e.g., hospital, library, etc., that they will all use. In order to use the facility, agents have to travel to it. In this context, the assumption of single-peakedness is quite natural.

Now, consider the family of solutions defined as follows. Let a be a fixed point on the tree. Given a profile R_Q of preferences, let $\varphi(R_Q) = a$ if a is efficient for R_Q, and let $\varphi(R_Q)$ be the efficient point the closest to a otherwise (there is a unique such point). A simple adaptation of the proof of Theorem 1 reveals that the solutions so defined are the only solutions satisfying *efficiency* and *population-monotonicity.*

To prove this, start from a profile consisting of a number of agents equal to the number of endpoints of the tree and specify their preferences so that for each endpoint there be an agent whose preferred level is that endpoint. Let a be the choice for that profile. Now, the proof proceeds as in Theorem 1.

3.4 Related Literature

The class of solutions Φ characterized in Theorem 1 is a subclass of the class of "generalized Condorcet-winner solutions" introduced by Moulin (1980, 1984). To define that class, we first define the median of $(2q - 1)$ numbers, for $q \geq 1$. Given $X = \{x_1, \ldots, x_{2q-1}\}$, ***med X*** is the point $x^* \in X$ such that $\#\{x_i \in X | x_i \leq x^*\} \geq q$ and $\#\{x_i \in X | x^* \leq x_i\} \geq q$. Each ***generalized Condorcet-winner solution*** for the group $Q \in \mathcal{Q}$ is obtained by taking the median of the preferred levels of the members of Q and $q-1$ parameters. Given $Q \in \mathcal{Q}$ and $R_Q \in \mathcal{R}^q$,

$$\exists a_{Q,1}, \ldots, a_{Q,q-1} \in [0, M] \text{ s.t. } C(R_Q) = \text{med}\{p(R_1), \ldots, p(R_q), a_{Q,1}, \ldots, a_{Q,q-1}\}.$$

In the variable population context, the number of parameters has to vary with the number of voters and their values are in principle allowed to change. Accordingly, the first subscript of the number $a_{Q,l}$ indicates that $a_{Q,l}$ depends on the group Q.

Moulin (1980, 1984) characterizes the class of generalized Condorcet-winner solutions mainly by means of "anonymity,"[5] *efficiency,* and *strategy-proofness.* Barberà and Jackson (1991) generalize Moulin's result to the multi-dimensional case. Ching (1992) shows that *strategy-proofness* is solely responsible for the median formula. His result can also be used to fully separate out the implications of the different axioms in the earlier characterizations. Our approach leads to a more restricted class of solutions in which the $q-1$ parameters are equal to a given number that is independent of the cardinality.

[5]The one-man-one-vote requirement.

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