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*This paper will appear in a festschrift for John Chipman in celebration of his 65th birthday.
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The approach to competitive markets by way of bargaining among traders was developed by Edgeworth (1881). Walras (1874–77) considered markets led by market managers. In these markets prices are announced which lead to offers to buy and sell, which are aggregated over the market. Then price lists are revised in the light of supply and demand, upward for an excess demand, downward for an excess supply. Trades are carried out when demand and supply are in balance. On the other hand Edgeworth considered bargaining between individual traders in which no bargains are final until a point is reached where no group of traders can conclude a new bargain which they prefer to their existing bargains. Edgeworth showed in the simplest case of trading in two goods that such a situation, in which no new bargains are possible which are preferred by some participants, will approach a competitive equilibrium as the number of traders increases indefinitely. This result has been generalized to the case of many goods and to production economies in recent years. The path breaking paper was that of Debreu and Scarf (1963). Most of the results described in this paper may be found in McKenzie (1988 and 1990) and McKenzie and Shinotsuka (1991). There are also similar theorems in Hildenbrand and Kirman (1988). My purpose here is to show precisely where assumptions must be strengthened or may be weakened in establishing the major theorems of the subject for the case of economies with a finite number of agents and a finite number of goods.

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CORE ALLOCATIONS WHICH ARE COMPETITIVE ALLOCATIONS

Consider an economy where the consumption sets \( C^h \subseteq \mathbb{R}^n, \ h = 1, \ldots, H \), represent the sets of net possible trades by consumers. \( C = \sum_{h=1}^{H} C^h \) is the set of possible net trades by all consumers together. Positive quantities represent amounts received by the consumer and negative quantities represent quantities provided by the consumer. A strictly preferred set \( P^h(x) \) contains all commodity bundles preferred to \( x \). A weakly preferred set \( R^h(x) \) contains all commodity bundles to which \( x \) is not preferred. \( P^h \) is irreflexive if \( x \not\in P^h(x) \). From the interpretation of \( P^h \) it is clear that it is irreflexive. We will also use the expression \( xP^h y \) for \( x \in P^h(y) \) and correspondingly for \( R^h \). That is, \( P^h \) or \( R^h \) may denote a relation or a correspondence determined by the relation. There is a production possibility set \( Y \subseteq \mathbb{R}^n \). We will use the activities model of production, so \( Y \) will be a convex cone with vertex at the origin. However diminishing returns can be accommodated by introducing entrepreneurial factors (McKenzie (1959)). These activities are available to all consumers. An allocation of trades \( \tilde{x} \) is a list \((x^1, \ldots, x^H)\) such that \( x^h \in C^h \) for all \( h \). We may also write \( \{x^h\} \) for an allocation when no confusion will result. A feasible allocation is an allocation that satisfies \( \sum_{h=1}^{H} x^h \in Y \). If \( x \in C^h \) then the \( h \)th consumer is locally not satiated at \( x \) if an arbitrary neighborhood \( U \) of \( x \) contains \( y \in U \cap C^h \) with \( y \in P^h(x) \). The economy \( E_1 \) is given by the list \((C^1, \ldots, C^H, P^1, \ldots, P^H, Y)\).

In this section we make two assumptions on preferences.

1. \( C^h \) is convex and not empty for all \( h \).
2. For all \( h \), \( P^h \) is open valued relative to \( C^h \). If \( \tilde{x} \) is feasible, the \( h \)th consumer is locally not satiated at \( x^h \).

We also make two assumptions on production possibilities.

3. The production set \( Y \) is a convex cone with vertex at the origin.
4. \( Y \cap \mathbb{R}^n_+ = \{0\} \).
Finally two assumptions relate consumption sets and production sets.

Let \( x_S = \sum_{h \in S} x^h \) and \( C_S = \sum_{h \in S} C^h_S \) where \( S \) is a subset of consumers. An economy is said to be irreducible if for every nontrivial partition of the set of consumers into two subsets \( S_1 \) and \( S_2 \) the following condition holds. If \( (x^1, \ldots, x^H) \) is a feasible allocation then there is \( w \in C_{S_2} \) and \( y^r \in Y \) such that

\[
z_{S_1} = y^r - \alpha w \text{ for some } \alpha > 0 \text{ and } z^h \in p^h(x^h) \text{ for all } h \in S_1.
\]

5. \( Y \cap C^h \neq \emptyset \) and relative interior \( Y \cap \text{relative interior } C \neq \emptyset \).

6. The economy \( E_1 \) is irreducible.

Assumption 3 implies that production processes are independent and divisible. Assumption 4 is only a convenience since goods produced out of nothing need not be economized.

Irreducibility means that given any feasible allocation of trades if the consumers are divided into two nonempty sets each set will have a possible trade some multiple of which, together with a possible output variation, improves the allocation to the members of the other set. This idea and assumptions like 5 and 6 will play important roles whenever the existence of a competitive equilibrium is to be proved. The first part of Assumption 5 is a survival assumption for isolated individuals. The second part of Assumption 5 guarantees that there is \( z \in C^h \) with \( pz < 0 \) for some \( h \). Then Assumption 6 provides that if one consumer satisfies this condition all do. Assumption 6 together with Assumption 5 will imply that prices which support preferred sets together with the production set will be competitive equilibrium prices. We will mean by the relative interior of a set \( V \subset \mathbb{R}^n \) the interior of \( V \) relative to the smallest affine subspace containing \( V \). We use this notion only with reference to convex sets. If a convex set is not empty, its relative interior is not empty.

The central idea leading to the concept of the core is that of an improving
coalition. Given an allocation \((x^1, \cdots, x^H)\), an *improving coalition* is a subset of consumers \(B\) such that for each \(h \in B\) there is \(z^h \in C^h\) with \(\sum_{h \in B} z^h \in Y\) and \(z^h \in P^h(x^h)\) for all \(h\). Then the *core* of the economy is defined as the set of all feasible allocations for which there is no improving coalition. This concept corresponds to Edgeworth's idea of equilibrium since if an allocation is in the core there will be no group of traders who can negotiate a new bargain among themselves, taking account of their production possibilities, which will improve the position of all the traders in the group. We will show that when the consumers are duplicated without limit any allocation that remains in the core indefinitely can be realized as a competitive equilibrium. Also the part of the core that assigns the same allocation to the replicas of each person will shrink to the set of competitive equilibria of the original economy.

Under these assumptions there is not a set of firms which exist independently of the choices of the agents. Rather there is a set of production possibilities, which are generated by activities available to any subset of agents that chooses to use them. For this reason the profit condition of competitive equilibrium will require that no profits are available and the demand condition will assume that households receive no income net of their sales of goods and services. However the conditions for competitive equilibrium will be equivalent under the present assumptions to the usual ones. We will say that \(\{x^h\}\) is a *competitive allocation* if there is a price vector \(p\) and an output \(y\) such that

I. \(px^h \leq 0, \text{ and } z \in P^h(x^h) \text{ implies } pz > 0, \text{ for all } h.\)

II. \(y \in Y\) and \(py = 0\), while \(z \in Y\) implies \(pz \leq 0.\)

III. \(\sum_{h=1}^H x^h = y.\)

Since an activity can operate at any positive level a positive profit is inconsistent with profit maximization. Since an activity can operate at zero level a negative profit is also inconsistent with with profit maximization. This leads to Condition
II. Thus in equilibrium profits are zero and household income is 0 for all h. Zero income, along with maximization of preference within budget constraints, leads to Condition I. Condition III is the balance of supply and demand.

Local nonsatiation alone implies that a competitive allocation lies in the core. In a Pareto improvement it is only required that some consumers benefit, not necessarily all consumers, while none suffers. The stronger criterion for improving coalitions may be defended as providing all members of an improving coalition with an incentive to act. Also the weaker definition is not sufficient for our proof that the equal treatment core is closed. This result is needed to prove that the equal treatment core converges to the set of competitive equilibria of the original economy as the economy is replicated and also to prove existence of equilibrium.

**Theorem 1.** Local nonsatiation implies that a competitive allocation for $E_1$ is in the core.

**Proof.** Let $\{x^h\}$ be a competitive allocation. It is implied by local nonsatiation that $px^h = 0$. Suppose B is an improving coalition by means of the allocation $\{z^h\}_{h \in B}$ where $\sum_{h \in B} z^h = z \in Y$. Then $z^h \in P^h(x^h)$ for all $h \in B$. However $z^h \in P^h(x^h)$ implies that $pz^h > px^h$ from the demand condition of competitive equilibrium. Therefore $\sum_{h \in B} pz^h > \sum_{h \in B} px^h = 0$. But $z \in Y$ implies $pz \leq 0$ by condition II of competitive equilibrium. Thus no such improving coalition B can exist and $\{x^h\}$ lies in the core. $\Box$

For each $h$ let the number of consumers identical to the consumer with index $h$ be increased to $r$ by adding new consumers and index the expanded set by $hs$ where $h = 1, \ldots, H$ and $s = 1, \ldots, r$. The economy that is replicated $r$ times will be referred to as $E^r$. The economy $E_r$ for $r > 1$ is given by the list $(C^{11}, \ldots, C^{1r}, \ldots, C^{H1}, \ldots, C^{Hr}, P^{11}, \ldots, P^{Hr}, Y)$. Allocations for $E_r$ may be written $\{x^{hs}\}$. We will consider allocations $\{x^{hs}\}$ in the core in which $x^{h1} = x^{hs}$ for
s = 1 to r. These allocations form the equal treatment core. That is, the replicas of a given consumer receive the same allocation that he does. There will be no ambiguity if equal treatment allocations are indicated by the expression \( \{x^h_r\} \).

**Lemma 1.** As \( r \) increases the allocations \( \{x^h_r\} \) in the equal treatment core of \( E_r \) form a non-increasing sequence of nested sets.

**Proof.** If \( B \) is an improving coalition for the allocation \( \{x^h_r\} \) when \( r = s \), it is also an improving coalition when \( r = s + 1 \). Therefore as \( r \) increases no new allocations \( \{x^h_r\} \) can appear in the equal treatment core. \( \square \)

The basic result is

**Theorem 2.** Under Assumptions 1–6 if \( \{x^h_r\} \) is in the equal treatment core of \( E_r \) for all \( r \) as \( r \to \infty \) then \( \{x^h\} \) is a competitive allocation for \( E_1 \).

**Proof.** Assume that \( \{x^h_r\} \) is an allocation in the equal treatment core for all values of \( r \). For any \( x \geq 0 \) let \( P^h(x) \) be the set of trades preferred to the trade \( x \) by consumers who are duplicates of the original consumer with index \( h \). We may refer to them as consumers of the \( hth \) type. By Assumption 2, \( P^h(x) \) is not empty and \( P^h(x) \) is open relative to \( C^h \). Let \( P(x^1, \cdots, x^H) = P(x) \) be the convex hull of the \( P^h(x^h) \).

Suppose \( Y \cap P(x) \neq \emptyset \). Then there is a set of consumers \( B \) and weights \( \alpha_i \) such that \( \Sigma_{i \in B} \alpha_i z^i = y \in Y \), \( \alpha_i > 0 \), \( \Sigma_{i \in B} \alpha_i = 1 \), and \( z^i \in P^h(i)(x^h(i)) \) where the \( ith \) consumer of the set \( B \) is a replica of the \( h(i)th \) original consumer. The consumers may be chosen so that the number of consumers in \( B \) is less than or equal to \( n+1 \) (Fenchel (1953), p. 37). For any positive integer \( s \) let \( a^s_i \) be the smallest integer greater than or equal to \( s \alpha_i \). By the first part of Assumption 5, for each \( i \in B \) there is \( y^i \in C^h(i) \cap Y \). Let \( w^i_s = (s \alpha_i/a^s_i)(z^i - y^i) + y^i \). Since \( w^i_s \) is a convex combination of \( z^i \) and \( y^i \) it lies in \( C^h(i) \) by Assumption 1. Moreover \( w^i_s \to z^i \) as \( n \to \infty \). It is this argument that requires the convexity of
$C_h$ and the survival assumption for isolated individuals.

Since the preferred sets are open relative to the $C^h(i)$ by Assumption 2, we have $w^i_s \in P^h(i)(x^h(i))$ for all $i$ for some number $s$ which is large enough. Also

$$\Sigma_{i \in B} a^i_s w^i_s = \Sigma_{i \in B} \left( sa^i_s - sa^i_1 y^i + a^i_1 y^i \right) = s y + \Sigma_{i \in B} (a^i_s - sa^i_1) y^i.$$  

The fact that $0 \leq a^i_1 - sa^i_1$ and $y^i \in Y$ implies that $\Sigma_{i \in B} a^i_s w^i_s \in Y$, since $Y$ is a convex cone by Assumption 3. Let $T = \{ \tau \mid h(i) = \tau \text{ for some } i \in B \}$. Let $B_{\tau} = \{ i \in B \mid h(i) = \tau \}$. Put $r = \max_{\tau \in T} (\Sigma_{i \in B_{\tau}} a^i_s)$. If the economy has been replicated $r$ times there are enough traders of each type $\tau$ in $T$ to offer the net trade $\Sigma_{i \in B_{\tau}} a^i_s w^i_s$ needed from the $\tau$th type to achieve the improved allocation for all $i \in B$. Then the improving coalition can be formed if the original economy has been replicated $r$ times. This contradicts the hypothesis. Therefore $Y \cap P(x^1, \ldots, x^H) = \phi$. In other words \{x^h\}_T in the core for all $r$ implies that the production set $Y$ intersected with the convex hull of the $P^h(x^h)$, the sets of preferred trades of the original consumers, is empty.

Consider $P(x)$ and $Y$ in the smallest linear subspace $L$ that contains both $C$ and $Y$. By a separation theorem for convex sets (Berge (1963), p. 162) there is a vector $p \in L$, $p \neq 0$, such that $pz \geq 0$ for all $z \in P(x^1, \ldots, x^H)$ and $pz \leq 0$ for all $z \in Y$. Since $P^h(x^h) \subset P(x^1, \ldots, x^H)$, $pz \geq 0$ for all $z \in P(x^h)$. By Assumption 2 there is a point $z \in P^h(x^h)$ in every neighborhood of $x^h$. Thus $px^h \geq 0$ must hold for all $h$. Since $(x^1, \ldots, x^h)$ is an allocation it is also true that $p_{h=1}^H x^h = y \in Y$. Therefore $p \cdot \Sigma_{h=1}^H x^h \leq 0$. This implies that $px^h = 0$ must hold for all $h$.

To complete the proof it is necessary to show that in fact $pz > 0$ for any $z \in P^h(x^h)$ for all $h$. We first prove two lemmas.

**Lemma 2.** Let sets $A$ and $B$ be convex sets in $\mathbb{R}^n$. Suppose $0 \in A$. If relative interior $A \cap$ relative interior $B \neq \phi$ there is no hyperplane which
separates A and B in the smallest linear subspace containing both.

Proof. Let L be the smallest linear subspace containing A and B. If H is a hyperplane separating A and B in L then there is \( q \in L, q \neq 0 \), such that \( H = \{ z \in L \mid qz = \mu \} \) and H separates A and B in L. However, q may be chosen so that \( qz \leq \mu \) for all \( z \in A \) and \( qz \geq \mu \) for all \( z \in B \). Then \( y \in A \cap B \) implies \( qy = \mu \). But y ∈ relative interior A implies \( qz = \mu \) for all \( z \in A \). Similarly \( qz = \mu \) for all \( z \in B \). Then \( 0 \in A \) implies \( \mu = 0 \). Since A and B span L this implies \( qz = 0 \) for all \( z \in L \). Then \( q \in L \) implies that \( q = 0 \). Since this is a contradiction of the choice of q no such separation is possible. □

**Lemma 3.** If \( pz \geq 0 \) for all \( z \in P^h(x) \) where \( x \in C^h \), and there is \( w \in C^h \) such that \( pw < 0 \), then \( pz > 0 \) holds for all \( z \in P^h(x) \).

Proof. Assume there is \( z \in P^h(x) \) where \( pz = 0 \). Since \( P^h \) is open in \( C^h \) and \( C^h \) is convex by Assumption 1, there is \( z' \neq z \) on the line segment from \( w \) to \( z \) and close to \( z \) such that \( z' \in P^h(x) \). But \( pz' < 0 \) in contradiction to the hypothesis. Thus \( pz > 0 \) holds for all \( z \in P^h(x) \). □

The vector \( p \) supports Y. Thus \( pz \leq 0 \) for all \( z \in Y \). Since relative interior \( Y \cap \) relative interior C contains a point \( w \) by the second part of Assumption 5, and \( 0 \in Y \), it is implied by Lemma 2 that \( pw < 0 \). Therefore \( pw^h < 0 \) for some \( w^h \in C^h \) for some \( h \). Let \( pw^h < 0 \) hold for \( h \in S_1 \) and \( pw^h \geq 0 \) hold for \( h \in S_2 \) where \( S_2 \) and \( S_1 \) partition the set of consumers. Lemma 3 implies that \( pz > 0 \) holds for all \( z \in P^h(x^h) \) for \( h \in S_1 \). However by irreducibility there is \( w \in C^h \) and \( y' \in Y \) such that \( z_{S_1} = y' - \alpha w \) for \( \alpha > 0 \) and \( z^h \in P^h(x^h) \) for all \( h \in S_1 \). Since \( py' \leq 0 \) and \( pz_{S_1} > 0 \), we have \( pw < 0 \) in contradiction to the definition of \( S_2 \). Therefore \( S_2 \) must be empty. Then \( S_1 = (1,\ldots,H) \) and \( pz > 0 \) for \( z \in P(x^h) \) for all \( h \). Thus the demand condition of competitive equilibrium is met. Since \( px^h = 0 \) and \( y = \sum_{h=1}^{H} x^h \) we have
py = 0. But pz ≤ 0 for all z ∈ Y by the support property. Therefore the profit condition is met. The balance condition is implied by the definition of a feasible allocation. Thus \( \{x^h\}_1 \) is a competitive allocation for \( E_1 \).

2. CONVERGENCE OF THE EQUAL TREATMENT CORE

TO THE SET OF EQUILIBRIA

With somewhat strengthened assumptions it is possible to go further and prove convergence of the set of allocations in the equal treatment core to the set of competitive allocations. We introduce

1'. \( C^h \) is convex and not empty, and \( \text{closed and bounded from below} \), for all \( h \).

2'. Relative to \( C^h \), \( P^h \) is open valued with \( \text{open lower sections} \) for all \( h \). If \( x \) is feasible, the \( h \)th consumer is locally not satiated at \( x^h \).

3'. The production set \( Y \) is a \( \text{closed} \) convex cone with vertex at the origin.

The new assumptions are in italics.

Let \( K_r \) be the set of allocations \( \{x^h\}_r \) in the equal treatment core of \( E_r \) where there are \( r \) members of each type and let \( W \) be the set of competitive allocations \( \{x^h\} \) in \( E_1 \). Let the distance \( d(K_r, W) \) of the equal treatment core to the set of competitive allocations in \( E_1 \) be given by

\[
d(K_r, W) = \max_{x \in K_r} \min_{z \in W} |x - z| + \max_{z \in W} \min_{x \in K_r} |x - z|.
\]

Convergence of the equal treatment core to the set of competitive allocations of \( E_1 \) is defined by \( d(K_r, W) \to 0 \) as \( r \to \infty \).

Let \( T \) be the set of equal treatment allocations which are feasible, that is, \( T = \{\{x^h\}_r | \sum_{h=1}^H x^h \in Y\} \). Because of equal treatment and Assumption 3, that \( Y \) is a cone, the index \( r \) is irrelevant to feasibility. We first prove

**Lemma 4.** With Assumptions 1' and 3' the set \( T \) is compact.

**Proof.** Since \( T \) lies in a metric space, to prove compactness it suffices to
show that every infinite sequence of feasible allocations has a point of accumulation which is contained in $T$ (Berge (1963), p. 90). Let $	ilde{x}^s = (x_1^s, \ldots, x_{H_s}^s)$, $s = 1, 2, \ldots$, be a sequence of allocations with $\sum_{h=1}^{H} x_{h}^{s} = y^s \in Y$. I claim that $\tilde{x}^s$ is bounded. If not, since the $C^h$ are bounded below it must be that $x_{i}^{h} \to \infty$ for some $h$ and $i$. But for a subsequence (retain notation) $y^s/|y^s| \to y \geq 0$ and $\neq 0$. Since $Y$ closed implies that $y \in Y$ Assumption 4 is violated. This shows that $x^h$ is bounded for each $h$. Since $C^h$ is closed there is a point of accumulation $(\bar{x}^1, \ldots, \bar{x}^H)$ for the sequence $(x_1^s, \ldots, x_{H_s}^s)$, $s = 1, 2, \ldots$, with $\bar{x}^h \in C^h$ for all $h$. The point $(\bar{x}^1, \ldots, \bar{x}^H)$ represents a feasible allocation and lies in $T$. \qed

The new closedness and boundedness assumptions were needed for Lemma 4.

We may now prove

Theorem 3. Given Assumptions 1', 2', 3', 4, 5, 6, if the equal treatment core of $E_1$ is not empty, as the number of replications increases without limit, the equal treatment core of $E_1$ converges to the set of competitive allocations of the original economy $E_1$.

Proof. From the definition of a competitive allocation the competitive allocations of the original economy are the competitive allocations with equal treatment in the replicated economy. Since by Theorem 1 competitive allocations are always in the core, the second term of the distance formula is 0 for all $r$. Thus the theorem requires that the first term be shown to converge to 0. Suppose that $\{z^h\}_r$ is an allocation for which there is an improving coalition $B$. Recall $T$ is the set of equal treatment allocations which are feasible. Let the allocation to the replicas of the $h$th consumer be perturbed by $\Delta x^h$ where $\sum_{h=1}^{H} \Delta x^h = 0$, $|\Delta x^h| < \epsilon > 0$, and $x^h + \Delta x^h \in C^h$, for all $h$. By Assumption 2' the preference correspondences $p^h$ have open lower sections relative to $C^h$. Thus if $\epsilon$ is sufficiently small the new allocation $\{x^h + \Delta x^h\}_{h \in B}$ is still
dominated by the same net trades achievable within B alone that dominated \( \{x^h\}_{h \in B} \). Note that this argument uses the strong definition of an improving coalition. Then the set of allocations \( T \setminus K_t \) for which there is an improving coalition is open relative to the set of feasible allocations. Also the set \( T \) of feasible equal treatment allocations \( T \) is closed in \( \mathbb{R}^n \) by Lemma 4. Therefore the set of allocations \( K_t \) is closed in \( \mathbb{R}^n \). This is the argument that requires open lower sections for the preference correspondences \( P^h \).

By Lemma 1 an allocation in the equal treatment core for the sth replica economy is in the core for all \( r < s \). Suppose there are allocations \( \{w^h_r\}_r \) in \( K_t \) and at a distance of at least \( \epsilon > 0 \) from any allocation in \( W \) for indefinitely large \( r \). Since \( T \) is compact by Lemma 4, there would be an accumulation point \( \{z^h\} \) of the sequence \( \{w^h_r\}_r \) where \( \{z^h\} \) lies at least \( \epsilon \) from any allocation in \( W \). But Lemma 1 implies that a subsequence \( \{w^h_r\}_r \) (save notation) which converges to \( \{z^h\} \) as \( r \to \infty \) provides for any \( s \) a subsequence (save notation) along which \( r > s \) holds which converges to \( \{z^h\} \) as \( r \to \infty \) and which is contained in \( K_s \). Since \( K_s \) is closed \( \{z^h\} \) lies in \( K_s \) for all \( s \). Therefore \( \{z^h\} \) is a competitive allocation of \( E_1 \) by Theorem 2, contradicting the inference from the definition of the sequence \( \{w^h_r\}_r \) that \( \{z^h\} \) lies at least \( \epsilon \) from the set of competitive allocations of \( E_1 \). Thus no such sequence can exist and it must be that \( K_t \) converges to \( W \) as \( r \to \infty \). □

The Pareto optimum is an allocation for which the coalition of the whole is not an improving coalition, even in a weak sense. No other improving coalitions are considered, in particular not the coalitions composed of single consumers. On the other hand the competitive equilibrium does not allow any improving coalition. Moreover competitive equilibrium requires that each consumer's allocations have zero value at some price vector. In a sense the core is an intermediate notion, especially if we take a weaker definition of an improving
coalition and thus a stronger definition of the core. This definition would serve
for the proof of Theorem 2 with no major change, but not for the proof of
Theorem 3. Let a weakly improving coalition \( B \) for an allocation \( \{x^h\} \) be a
coalition for which there is an allocation \( \{z^h\} \) with \( \sum_{h \in B} z^h = \sum_{h \in B} x^h \) such
that \( x^h \not\in P^h(z^h) \) for any \( h \in B \) and \( z^h \in P^h(x^h) \) for some \( h \in B \). Then the
strong core is the set of allocations for which there is no weakly improving
caliation. If an allocation is in the strong core then for any coalition there is no
allocation within its feasible set which is a Pareto improvement over its core
allocation. The analogous relation holds between the ordinary core allocation and
the weak Pareto optimum, defined as an allocation such that no reallocation can
improve the position of every consumer.

The set of competitive equilibrium allocations is the subset of the strong
core in which the core allocations have a zero value at a supporting price vector.
Thus although a limit of equal treatment allocations in the strong core as the
number of replications increases without limit is a competitive equilibrium, it is
not clear that this limit lies in the strong core. However we will find that a
slight strengthening of the irreducibility assumption, which we need for proving
that the equal treatment core is not empty, will imply that the strong and weak
cores coincide.

3. NONEMPTINESS OF THE CORE

It has been proved that every competitive equilibrium is in the core. We
will show that the existence of an allocation in the core may be proved
independently of the existence of equilibrium allocations. Then the existence of
an allocation in the equal treatment core can be used together with Theorem 2 to
prove the existence of a competitive equilibrium for \( E_1 \). The economy \( E_1 \) is
defined, as described, by means of consumption sets \( C^h \subset \mathbb{R}^n \), strict preference
correspondences \( P^h \), for \( h = 1, \ldots, H \), and a production set \( Y \). \( C = \bigcup_{h=1}^H C^h \).
The assumptions are strengthened in some respects and weakened in others. Assumptions 1', 3', and 4 are retained. The graph of $P^h$ is the set \[(x,y) \in C^h \times C^h \mid y \in P^h(x)\]. Assumption 2' is replaced by the slightly stronger assumption

2''. For all $h$, $P^h$ is convex valued. The graph of $P^h$ is open relative to $C^h \times C^h$.

Assumption 6 is not needed and Assumption 5 is replaced by the weaker assumption

5'. For all $h$, $C^h \cap Y$ is not empty.

It will be noted that $P^h$ is now assumed to be convex valued although this assumption was not needed in Theorem 3 to prove convergence of the equal treatment core to the set of competitive equilibria for $E_1$. This assumption is also critical for the existence of competitive equilibrium. On the other hand irreducibility, interiority, and local nonsatiation are not needed here. The assumption that the graph of $P^h$ is open is a slight strengthening of the assumption in 2' that $P^h$ is open valued and has open lower sections. On the other hand, irreducibility and interiority assumptions are important for the existence of a competitive equilibrium but not for the existence of a point in the core.

We first record some properties of the set of feasible allocations $F_S$ where $S \subset \{1, \cdots, H\}$. Let $x_S = (x^h)_{h \in S}$, $F_S = \{(x_h)_{h \in S} \mid x^h \in C^h \text{ for all } h \in S \text{ and } \Sigma_{h \in S} x^h \in Y\}$.

**Lemma 5.** $F_S$ is nonempty, compact, and convex.

**Proof.** Assumption 5' and the fact that $Y$ is a cone imply that $F_S$ is not empty. $F_S$ is compact by the argument of the proof of Lemma 4. Convexity of $F_S$ follows from the convexity of the $C^h$ and $Y$. \(\Box\)

Let $\mathcal{B}$ be a nonempty family of subsets of $\{1, \cdots, H\}$. Define
\( \mathcal{A}_h = \{ S \in \mathcal{B} \mid h \in S \} \). A family \( \mathcal{B} \) is balanced if there exist nonnegative weights \( \{ \lambda_S \} \) with \( \sum_{S \in \mathcal{A}_h} \lambda_S = 1 \) for all \( h \). Let \( I = \{1, \ldots, H\} \). The economy is said to be \( O \)-balanced, if for any balanced family \( \mathcal{B} \) with balancing weights \( \{ \lambda_S \} \) which satisfies \( x_S \in F_S \) for all \( S \in \mathcal{B} \) it follows that \( x_I \in F_I \) where

\[
x_I^h = \sum_{S \in \mathcal{A}_h} \lambda_S x^h_S.
\]

**Lemma 6.** The economy \( E_1 \) is \( O \)-balanced.

Proof. To show that \( x_I \) is feasible it is necessary and sufficient to show that \( x_I^h \in C^h \) for all \( h \) and \( \sum_{h \in S} x_I^h \in Y \). Since \( x_S^h \in C^h \) for each \( S \in \mathcal{A}_h \) by the feasibility of \( x_S \) and the convexity of \( C^h \) implies that \( \sum_{h \in S} x^h_S \in Y \) for all \( S \in \mathcal{B} \), \( x_I^h \) lies in \( C^h \) by the convexity of \( C^h \). On the other hand, since \( Y \) is a cone with vertex at the origin \( \sum_{h \in S} x^h_S \in Y \) implies that \( \sum_{h \in S} \lambda_S x^h_S \in Y \) and \( \sum_{S \in \mathcal{B}} \sum_{h \in S} \lambda_S x^h_S \in Y \). But \( \sum_{S \in \mathcal{B}} \sum_{h \in S} \lambda_S x^h_S = \sum_{h \in I} \sum_{S \in \mathcal{A}_h} \lambda_S x^h_S = \sum_{h \in I} x_I^h \).

This completes the proof. \( \Box \)

The following theorem is from Border (1985). It is rephrased to accord with our terminology.

**Theorem (Border).** Let \( E \) be an economy given by the list

\( (C^1, \ldots, C^H, P^1, \ldots, P^H, Y) \) satisfying

1. For each \( h = 1, \ldots, H \), \( C^h \) is a nonempty convex subset of \( \mathbb{R}^n \).
2. For any \( S \subset I \), \( F_S \) is a nonempty compact subset of \( \prod_{h \in S} C^h \).
3. For each \( h \), (a) \( P^h \) has an open graph in \( C^h \times C^h \), (b) \( x \notin P^h(x) \), (c) \( P^h \) is convex valued (but possibly empty).
4. \( E \) is \( O \)-balanced.

Then the core of \( E \) is not empty.

We may now assert

**Theorem 4.** Under Assumptions 1', 2', 3', 4, and 5' the economy \( E \) has a nonempty core.
Proof. Condition 1 in Border's theorem is contained in Assumption 1. Lemma 5 implies Condition 2. Condition 3 is implied by Assumption 2'' and the definition of $P^h$. Condition 4 is provided by Lemma 6. Applying Border's theorem the conclusion follows. □

4. EXISTENCE OF COMPETITIVE EQUILIBRIUM

We have proved the core not to be empty under assumptions which are weaker in some respects and stronger in other respects than the assumptions used to show that allocations that remain in the equal treatment cores of replicated economies are competitive allocations. We also found that the equal treatment core converges to the set of competitive allocations. This suggests that if the nonemptiness of the core can be extended to the equal treatment core it will be possible to prove that a competitive equilibrium exists by using the strongest form of each assumption from earlier sections. However to prove that the equal treatment core is not empty it is necessary to strengthen the irreducibility assumption although no irreducibility assumption was needed to prove the core nonempty.

If $C$ is a convex set, $x \in C$ is an extreme point of $C$ if $x$ is not a convex combination of two points of $C$ distinct from $x$. We will say that the economy is strongly irreducible if it is irreducible and, whenever $S_1, S_2$ is a nontrivial partition of $\{1, \cdots, H\}$, and $x_{S_1} + x_{S_2} \in Y$ where $x_{S_2}$ is not an extreme point of $C_{S_2}$, there are $z_{S_1} + z_{S_2} \in Y$ with $z^h P^h x^h$ for $h \in S_1$. Another version of this concept was introduced in Boyd and McKenzie (1993).

By Theorem 2 we have found that under weak conditions the equal treatment core $K_\tau$ converges to the set of competitive equilibria $W$ of $E_1$. Bounding and closing $C^h$ and closing $Y$, and introducing convexity of preference with an open graph, it was possible to prove that the core of $E_\tau$ is not empty.
This result did not require irreducibility. However in order to obtain the existence of competitive equilibrium we will require stronger forms of both Assumptions 2'' and 6.

Assumption 2'''. For all h, \( P^h \) is convex valued and transitive. For all \( x \in C^h \), \( R^h(x) \) is the closure of \( P^h(x) \). The graph of \( P^h \) is open relative to \( C^h \times C^h \).

Assumption 6'. The economy is strongly irreducible.

Recall that \( x \) is indifferent with \( y \), \( xI^h y \), if \( xR^h y \) and \( yR^h x \) where \( xR^h y \) means not \( yP^h x \). We define \( I^h(x) = \{ y \mid yI^h x \} \). Thus \( y \in I^h(x) \) means \( x \not\in P^h(y) \) and \( y \not\in P^h(x) \). The strengthening of the assumption on preferences in 2''' and the introduction of strong irreducibility in 6' are needed to prove that equal treatment allocations in the core for replicates of a given consumer are indifferent. This result is needed in turn to show that \( K_r \) is not empty if \( E_r \) is not empty.

**Lemma 8.** For \( x \in C^h \), \( x \) is locally nonsatiated for all \( h \).

**Proof.** Since \( x^h \in R^h(x^h) \) Assumption 2''' implies there is \( z \) in every neighborhood of \( x^h \) with \( z \in P^h(x^h) \). \( \Box \)

Lemma 8 allows us to appeal to results which depend on local nonsatiation.

**Lemma 9.** If \( x \in P^h(z) \) and \( z \in R^h(y) \) then \( x \in P^h(y) \).

**Proof.** Since \( R^h(y) \) is the closure of \( P^h(y) \) there is \( z' \to z \) where \( z' \in P^h(y) \). Since the graph of \( P^h \) is open, for \( \nu \) large enough \( x \in P^h(z') \). Thus by transitivity \( x \in P^h(y) \). \( \Box \)

Lemma 9 allows us to prove

**Lemma 10.** Under Assumptions 1', 2''', and 6', suppose \( \{ x^h \}_r \), \( h = 1, \ldots, H \), and \( k = 1, \ldots, r \), is an allocation in the core of \( E_r \). Then for given \( h, x^{hk}_h x^{hk'} \) holds for all \( k \) and \( k' \).

**Proof.** Let the allocation \( \{ x^{hk}_r \} \) where \( h = (1, \ldots, H) \) and \( k = (1, \ldots, r) \) lie in
the core for the economy $E_r$. I claim that $x^{hk} x^{hj}$ for all $h, k, j$. Suppose not. Consider a replica with index $hj(h)$, for each original consumer with index $h$, where $hj(h)$ satisfies $x^{hk} R^h x^{hj(h)}$ for all $k = 1, \ldots, r$. That is, $hj(h)$ has an allocation which no better, and perhaps poorer, than the allocation of any other replica of $h$. The existence of $hj(h)$ is guaranteed by the irreflexivity and transitivity of preference, and it is for this reason that transitivity is introduced. Consider the coalition $B = \{1j(1), \ldots, Hj(H)\}$ and the allocation to the member of $B$ with index $hj(h)$ of $(1/r) \sum_{k=1}^{r} x^{hk} = x^h$. Since $R^h$ is the closure of $P^h$ by Assumption 2', $R^h$ is convex valued. Therefore for each $h$ we have $x^h R^h x^{hj(h)}$. Also if $x^{ik} P^i x^{ij(i)}$ for some $i,k$, then $x^i$ is a convex combination which includes an interior point of $R^i(x^{ij(i)})$, so $x^i$ cannot lie on the boundary of $R^i(x^{ij(i)})$ relative to $C^h$. Therefore $x^{ik} P^i x^{ij(i)}$ holds.

Now $\sum_{h=1}^{H} x^h = \frac{1}{r} \sum_{h=1}^{H} x^{hk}$ which lies in $Y$ since $\{x^{hk}\}$ is feasible and $Y$ is a cone. Thus $\{x^h\}_1$ is a feasible allocation. By strong irreducibility and convexity we will see that it is possible to spread the gain received by $i$ from the allocation $\{x^h\}_1$ to all $h$. Let $S_1 = \{h \mid h \neq i\}$ and $S_2 = \{i\}$. Since $x^i$ is not an extreme point of $C^i$, strong irreducibility implies that there is a feasible allocation $\{z^h\}$ with $z^h P^h x^h$ for $h \in S_1$. Take the convex combination $w^h = \{\lambda x^h + (1-\lambda)z^h\}$ for $0 \leq \lambda \leq 1$. $\{w^h\}_{h=1}^{H}$ is a feasible allocation. Moreover $w^h$ is preferred by all $h \in S_1$ to $x^h$ and, using Lemma 9, to $x^{hj(h)}$ as well. Since $x^{ik} P^i x^{ij(i)}$ and the graph of $P^i$ is open, for $\lambda$ sufficiently close to 1, $w^i$ is preferred by $i$ to $x^h$. Thus $B$ is an improving coalition and $\{x^{hk}\}_r$ is not an allocation in the core of $E_r$ contrary to the assumption. Therefore $x^{hk} x^{hk'}$ must hold for all $k,k'$. □

Note that Lemma 10 implies that the weak and strong cores coincide under strong irreducibility.

**Lemma 11.** Under the assumptions of Lemma 10 the equal treatment core
$K_r$ of $E_r$ is nonempty if the core of $E_r$ is nonempty.

Proof. According to Lemma 10 for any allocation in the core the allocations received by the replicas of a given $h$ in the original economy are indifferent. By the convex valuedness of the correspondence $R^h$ the equal treatment allocation in which each replica of $h$ receives $x^h$, as defined in the proof of Lemma 10, satisfies $x^h R^h x^{hk}$ for all $h$, $k$. Then by Lemma 9 there is no improving coalition for the allocation $\{x^{hk}\}_r$ there is also no improving coalition for the allocation $\{y^{hk}\}_r$ in which $y^{hk} = x^h$ for all $h$, $k$. Therefore $\{y^{hk}\}_r$ is in the core of $E_r$. $\Box$

Lemma 12. Under Assumptions 1', 2', 3', 4, 5', 6', the equal treatment core $K_r$ of $E_r$ is not empty.

Proof. By Theorem 4 the core of $E_r$ is not empty. By Lemma 11 this implies that the equal treatment core $K_r$ of $E_r$ is not empty. $\Box$

We may now prove

Theorem 5. Under Assumptions 1', 2', 3', 4, 5, and 6' the economy $E_1$ has a competitive equilibrium, and $K = W$.

Proof. The assumptions imply the assumptions of Section 3, so by Theorem 4 the core of $E_r$ is not empty for any $r$. Since the assumptions also imply the assumptions of Lemma 12, the set of equal treatment allocations in the core is not empty for any $E_r$. From the proof of Theorem 3 the set $K_r$ is closed. Also the $K_r$ are nested by Lemma 1. Therefore $K = \cap_{r=1}^\omega K_r$ is not empty. But the assumptions imply the assumptions of Section 1, so, by Theorem 2, $K$ is included in the set of competitive allocations for $E_1$. Indeed, by Theorem 3 the equal treatment core of $E_r$ converges to the set of competitive allocations of $E_1$ as $r \to \omega$. Since both sets are closed $K$ is precisely the set $W$ of competitive allocations. Finally the proof of Theorem 2 provides a price vector $p$ which supports the allocations of $K$ in a competitive equilibrium given by $(p, y, x)$ where $y = \sum_{h=1}^\omega x^h$. $\Box$
Once transitivity is introduced the Scarf theorem (1967) becomes available for proving that the core is not empty, so it would be enough to stay with open lower sections and open values, rather than requiring an open graph. On the other hand, use of the weak core requires strong irreducibility, which is needed in the proof that the equal treatment core is closed. However we know from the theorem of Gale and Mas–Colell (1975) that existence of competitive equilibrium can be proved without transitivity. Indeed we know from Moore (1975) and McKenzie (1981) that the survival of isolated individuals is also not needed for proving the existence of competitive equilibrium. Indeed individual survival, convexity of $R^h$, and transitivity may be dispensed with together (see McKenzie (1981)).

The line of proof that we have followed is that used by Boyd and McKenzie to prove a theorem for the case of an infinite number of goods, except that they assume transitivity already at the stage where it is proved that the core is not empty. This is to allow the use of the Scarf theorem for a nonempty core, which is proved in the utility space, rather than the Border theorem, which is proved in the goods space. The Border proof makes essential use of the finite dimensionality of the goods space (see Yannelis (1991)). Yannelis provides a proof of nonemptiness of the core for the case of a goods space with infinite dimension. However the proof of existence is still blocked by need for transitivity to establish that consumers of the same type have allocations in the core which are indifferent. Whether the line of proof we have used here for the finite case can be improved to match the results of Gale and Mas–Colell, Moore, and McKenzie, is an open question so far as I know.

A further strengthening of the assumptions can reduce the core of $E_r$ to the equal treatment core. Assume that preferences are strictly convex in the sense that $x^h y$ and $z = \alpha x + (1-\alpha)y$, for $x,y \in C^h$ and $0 < \alpha < 1$ implies $z^h x$. 
Then we may prove a result used by Debreu and Scarf (1963).

Lemma 13. Assume $P^h$ is strictly convex, $R^h(x^h)$ is the closure of $P^h(x^h)$ when $x$ is feasible, and $E_1$ is strongly irreducible. Then all allocations in the core of $E_r$ are equal treatment allocations.

Proof. The conclusion of the lemma is equivalent to the statement that an allocation in the core of $E_r$ has $x^{hk} = x^{hk'}$ for all $k, k' = 1, \ldots, r$. Suppose not. For any $h$ let $x^{hj(h)}$ satisfy $x^{hk} R^h x^{hj(h)}$ for all $k$. Then the consumer with index $hj(h)$ is a worse off consumer among the replicates of the $h$th original consumer. By strict convexity of preference $(1/r) \sum_{k=1}^{r} x^{hk} R^h x^{hj(h)}$ for $h = 1, \ldots, H$ with $P^h$ in place of $R^h$ at least once. Strong irreducibility implies that $P^h$ can be realized for all $hj(h) \in B = (1j(1), \ldots, Hj(H))$. But $x^h = (1/r) \sum_{k=1}^{r} x^{hk} \in C^h$ by convexity of $C^h$ and $(1/r) \sum_{h=1}^{H} \sum_{k=1}^{r} x^{hk} \notin Y$ since $Y$ is a cone. Thus $B$ is an improving coalition with the allocation $\{x^h\}$, so $\{x^{hk}\}_r$ is not in the core of $E_r$, contrary to assumption. This implies that $x^{hk} = x^{hk'}$ for all $k, k' = 1, \ldots, r$, and all allocations in the core of $E_r$ are equal treatment allocations. □

REFERENCES


