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Abstract

Population-monotonicity is a property of resource allocation rules which says that if the number of agents increases while the resources at their disposal remain fixed, so that the profile of welfare levels chosen for the initial group remains feasible only by ignoring the newcomers, then none of the agents initially present should gain. The implications of abstract versions of this requirement have been investigated in game-theoretical models such as bargaining problems and coalitional form games; the requirement, and a number of variants, have also been the object of several recent studies in the context of resource allocation, in classical models as well as in non-classical models, including economies with public goods, economies with indivisible goods, and economies with single-peaked preferences, both in the private good and in the public good cases. The purpose of this paper is to survey this literature.

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Contents

List of Figures

1 Introduction

In the recent developments that have occurred in axiomatic analysis, several requirements designed to ensure the good behavior of decision rules, or ***solutions***, when the number of individuals involved varies, have been formulated. Among them, a principle of ***population-monotonicity*** has been playing an increasingly important role. It is the objective of this paper to present this principle, to survey the various applications that have been made of it, and finally, to provide a guide to future research.

Like several of the principles that have been considered in normative studies of solutions, *population-monotonicity* expresses the solidarity of agents as their circumstances change; in this case, what changes is how many of them there are. Here is a statement for the canonical problem of fairly dividing a vector of resources among agents with equal rights on them, a problem that has provided the inspiration for the introduction of the principle. Think for instance of the division of an estate among several heirs: the division having been performed by applying some solution, an additional claimant appears and his rights are recognized to have equal validity as those of anyone else; the estate has to be redivided. We say that the solution is *population-monotonic* if none of the heirs originally present gains as a result of this arrival. Equivalently, if some of the heirs relinquish their rights, none of the remaining heirs loses.

Our starting point in this survey will be the following more abstract form of the requirement: when additional agents arrive, and the profile of welfare levels chosen by the solution for the initial group remains feasible only by "ignoring the newcomers", then none of the agents initially present gains. Conversely, the departure of some of the agents, if it permits a Pareto improvement for the remaining agents, is indeed accompanied by such an improvement.

We first apply the principle to two standard models of game theory — bargaining problems and coalitional form games — and after examining a somewhat more concrete model of cost allocation, we turn to the classical problem of allocating a bundle of infinitely divisible goods. Then, we examine several special classes of problems of allocation of goods: economies with public goods, economies with indivisible goods, and finally economies with single-peaked preferences, both in the private good case and in the public good case.

As we will see, much remains to be done. Ideally, we would like to know how restrictive the condition is, by completely describing the class of solutions satisfying it, of course together with standard conditions. Such characterizations are available for only a few models. For a number of other models, *population-monotonic* solutions have been identified, but we are still far from fully understanding the implications of the property.

We conclude by outlining the various components that a systematic analysis of *population-monotonicity* on a given domain should comprise: first, identifying the most natural form of the property for the domain; finding out if the main solutions for the domain satisfy the property; describing the class of solutions satisfying it, together with standard properties; clarifying the extent to which the requirement is compatible with other properties of interest; when solutions exist that do satisfy *population-monotonicity*, formulating criteria permitting to evaluate how well these solutions perform the job: if all agents lose, how "evenly" are the losses distributed across them. On the other hand, when it is too restrictive, one has to be satisfied with weaker properties and the task then is formulating weakenings that retain as wide a range of applicability as possible; alternatively, identifying subdomains of interest on which the main property can still be met; but also, defining criteria to evaluate how frequently solutions fail to satisfy the property.

Such a program was pursued by Thomson (1983*a*, *b*, *c*, 1984*a*, *b*,1987*a*), and in joint work with Lensberg (1983) and Chun (1988, 1989, 1992), in the context of abstract bargaining problems, but recent studies by Chun (1986), Alkan (1989), Sprumont (1990), Moulin (1990a, 1992a), Thomson (1991) and others will permit us in the final section of this survey to reformulate in more precise terms and by reference to some specific models a number of its components.

2 Population-monotonicity; a general formulation

In most axiomatic studies, the number of agents is held fixed. Here, we allow it to vary, and require of solutions that they provide recommendations for economies of all admissible cardinalities. The axiom of *population-monotonicity* is meant to help us relate the recommendations made by solutions as the number of agents varies.

The formulation given above for the problem of fair division is the one that has been adopted in most of the applications since typically, and here we deliberately use vague language so as obtain a statement that is meaningful for as wide a range of models as possible, the arrival of newcomers is indeed a "burden" on the agents initially present. It implies a "restriction of their opportunities", in the sense that the list of welfare levels initially selected for them is feasible only by "ignoring these newcomers", but the newcomers should of course not be ignored.

In some situations of interest however, the arrival of the newcomers permits a "sufficient" expansion of opportunities, by which we mean that the list of welfare levels initially selected is now Pareto dominated by the list of welfare levels attained at a decision at which the rights of all agents, including those of the newcomers, are fully acknowledged. Here, the requirement will be that none of the agents initially present be made to lose.

In a third class of models, whether the arrival of newcomers is beneficial or not depends on their characteristics, and the natural formulation of the property is that all agents initially present be affected in the same direction by this arrival: none of them loses or none of them gains (Chun, 1986).

We believe that an essential part of what is generally, altough often implicitly, understood by the phrase "economic justice" is that "agents be affected in the same direction as their circumstances change", provided no one bears any special responsibility for these changes. Changes in circumstances might be increases or decreases in the quantities of the goods available for consumption, or of the inputs to be used in production; or they might be improvements in technology, fluctuations in climate; or finally, as we consider here, variations in the population. In this latter case, we can imagine actual changes due to immigration or emigration, epidemics or births; or the changes may be hypothetical : we often evaluate decision rules in terms of what they would recommend in situations other than the one we actually face. Some of the changes just listed could of course be due to some particular action that some agents may have taken but we repeat that here, eschewing issues of incentives and responsabilities, we will limit ourselves to situations when they have occurred independently of the will of the agents whose welfare levels are to be chosen.

It is particulary illuminating to test rules by having some agents *leave* the scene. Then, two possibilities arise. Either a decision has already been made and a commitment to certain payoffs for the departing agents has to be honored. In most models, the requirement that all remaining agents be affected in the same direction by this departure, together with efficiency, implies that the remaining agents also end up with the same payoffs. This gives us a form of the condition of "consistency" that has been the object of a considerable literature in the last few years[1]. Alternatively, in situations where it is natural to assume that agents relinquish their rights when they leave, we obtain the requirement on the rule that is being studied here.

Population-monotonicity is an ordinal requirement, that is, it depends only on agents' preferences. Its application does not rely on the social planner's ability to measure, let alone compare and even less, equate, sacrifices or gains. But it is conceptually compatible with the use of such operations, and in any case, if so desired — we will see several examples of this — *population-monotonicity* can be applied in models specified in utility space. In such models it can be complemented with additional requirements based on cardinal information.

In order to be able to deal with a variable number of agents, we need a sufficiently general formulation. We assume that there is an infinite number of "potential agents", indexed by the positive integers, $\mathbb{N}$, but we only consider problems involving finite groups. Let $\mathcal{Q}$ be the class of all finite subsets of $\mathbb{N}$, with generic elements Q, Q'... A ***solution*** is a correspondence which associates with every problem in the class that the group Q may face, where $Q \in \mathcal{Q}$, a set of outcomes in the feasible set of that problem, each of which being interpreted as a recommendation for that problem (if we assumed the set of potential agents to be finite, the statements of most of the characterization results would have to be weakened). Our generic notation for a solution is the letter φ.

When we apply *population-monotonicity* to a solution correspondence, we require to be able to compare ***all*** of the allocations chosen in the initial economy to ***all*** of the allocations chosen in any larger economy. We could imagine weaker statements allowing the comparison of at least one allocation from each set, or one allocation from one set to all of the allocations from

[1]This connection is made by Chun (1985) in the context of rights problems. See Thomson (1993) for a survey of the literature devoted to the analysis of *consistency*. We will encounter the condition on several other occasions below.

the other set. In the case of *single-valued* solutions, all of these conditions are of course equivalent.

We are now ready to turn to applications of the principle.

3 Bargaining problems

Bargaining problems are decision problems specified as subsets of utility space satisfying certain regularity conditions. A familiar concrete application of this abstract model is to the distribution of goods, when consumers are equipped with utility functions satisfying appropriate assumptions: then the subset in question is the image in utility space of the set of feasible allocations. If utility information is available, any allocation problem can be so represented, but it is important to emphasize that when the analysis of a class of allocation problems is limited to their representations in utility space, information about their concrete structure is ignored, information that in some situations might be quite relevant. Later on, we will explore the implications of *population-monotonicity* in concretely specified economic models. Also, we should note that not all allocation problems give rise to feasible sets satisfying the assumptions that have been imposed in the study of bargaining problems.

3.1 Population-monotonic solutions

A group of agents $Q \in \mathcal{Q}$ can attain any of the points of a ***feasible set*** $\boldsymbol{S}$, a subset of their utility space $\mathbb{R}^Q$, by unanimously agreeing on it[2]. If they fail to reach an agreement, they get a particular outcome $\boldsymbol{d} \in \boldsymbol{S}$, the ***disagreement point***. A ***bargaining problem*** is a pair $(S, d) \in 2^{\mathbb{R}^Q} \times \mathbb{R}^Q$. We make the standard assumptions that S is convex and compact, and that there exists at least one point of S that strictly dominates d. We also assume that S is "d-comprehensive", (if a point x is feasible, then any point y such that $d \leqq y \leqq x$ is also feasible); this mild assumption is imposed to guarantee that the solutions that we will want to consider always select outcomes that are at least weakly Pareto-optimal[3]. In order to simplify the exposition, and

[2]By the notation $\mathbb{R}^Q$, we designate the cartesian product of $|Q|$ copies of $\mathbb{R}$, indexed by the members of Q.

[3]For a formal definition, see below.

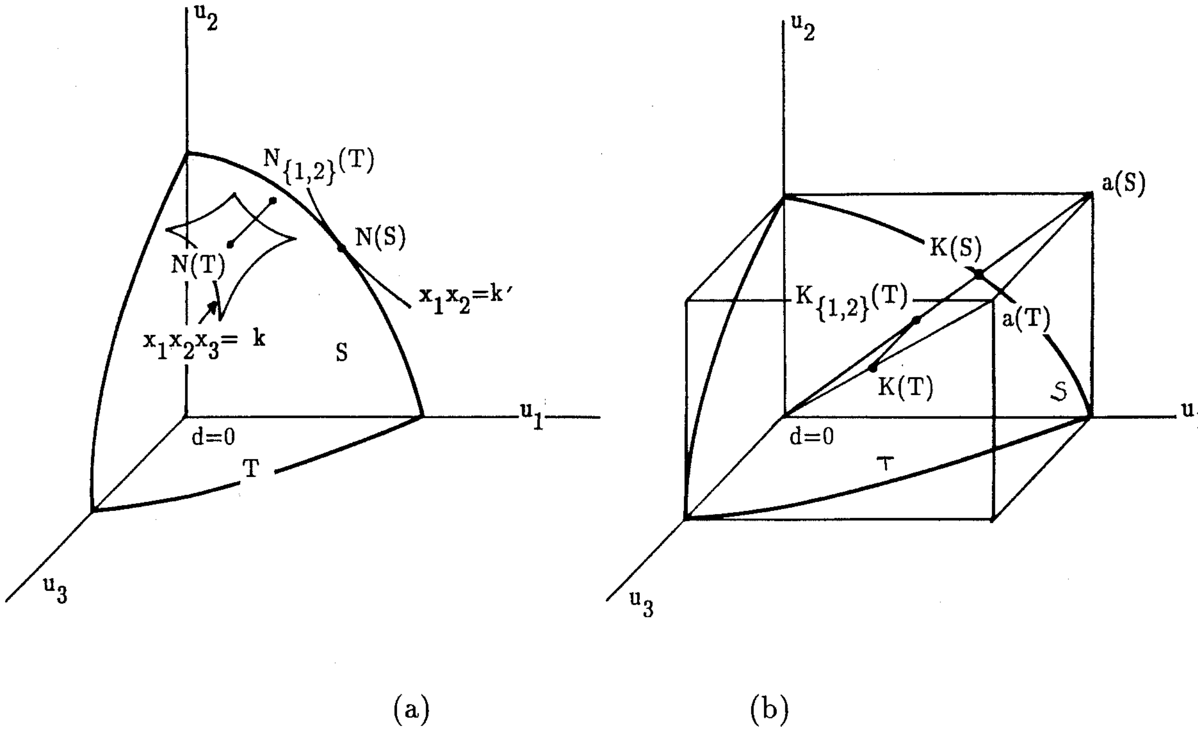


Figure 1: Population-monotonicity in bargaining theory. (a) The Nash solution is not *population-monotonic* since agent 2 receives less in the two-person problem S than in the three-person problem T that results from the arrival of agent 3. (b) The Kalai-Smorodinsky solution is *population-monotonic*: neither agent 1 nor agent 2 gains upon the arrival of agent 3.

with no loss of generality since the requirement that solutions be invariant with respect to changes in the origin of the utility scales is used in all of the literature reviewed below, we assume $d = 0$, and we write S instead of $(S, 0)$. Finally, we assume that all points of S dominate d. For each $Q \in \mathcal{Q}$, let Σ^Q be the class of all problems satisfying all of the above assumptions. A ***solution*** is a function that associates with every $S \in \Sigma^Q$, where $Q \in \mathcal{Q}$, a unique alternative in S. Depending upon the context, this alternative is interpreted as the recommendation that an impartial arbitrator would make, or as a prediction as to which alternative the agents would select on their own. The Nash solution (1950) chooses the feasible alternative at which the product of utility gains from the disagreement point is maximal.

Consider the two-person problem S, involving agents 1 and 2, which is represented in Figure 1 in the coordinate subspace relative to them. Solve it by applying the Nash solution. Then, imagine that an additional agent,

agent 3, comes in. In this context, there is a natural way of specifying the resulting three-person problem T so that the arrival of the new agent can be described as a "burden on the initial group"; it is to require that its intersection with the coordinate subspace pertaining to the initial group (this is the set of alternatives at which the new agent receives his coordinate of the disagreement point, here 0), be the initial problem S. Note that if the problem is derived from the distribution of private goods, this is exactly what happens. Now, it is easy to specify S and T so that the projection of the Nash solution outcome of T, $N(T)$, onto the two-dimensional subspace in which S lies, $N_{\{1,2\}}(T)$, is not dominated by the Nash solution outcome of S, $N(S)$. Any such example reveals that the Nash solution does not satisfy *population-monotonicity*, which on this domain takes the following form:

Population-monotonicity for bargaining problems: For all Q, $Q' \in \mathcal{Q}$ with $Q \subset Q'$, for all $T \in \Sigma^{Q'}$, and for all $i \in Q$, $\varphi_i(T) \leqq \varphi_i(T_Q)$ (where T_Q is the projection of T onto the coordinate subspace relative to Q).

On the other hand, consider the solution introduced by Kalai and Smorodinsky (1975): this solution picks the maximal feasible alternative proportional to the ***ideal point***, the point whose i^{th} coordinate is equal to the maximal feasible utility for agent i. The fact that it is *population-monotonic* is illustrated in Figure 1b. Indeed, the projection of the ideal point of the three-person problem T (this is the point marked $a(T)$) onto the two-dimensional space in which S lies coincides with the ideal point of S (the point marked $a(S)$), so that the projection $K_{\{1,2\}}(T)$ of $K(T)$ onto that subspace is collinear with $K(S)$; the fact that $K(S)$ dominates $K_{\{1,2\}}(T)$ coordinate by coordinate follows from the comprehensiveness of T.

Another *population-monotonic* solution is the ***egalitarian solution*** (Kalai, 1977), which selects the maximal feasible point of equal coordinates. In fact, so is any member of the following family of ***monotone path solutions***. A ***monotone path in*** $\mathbb{R}_+^Q$ is the graph of a continuous function $f : \mathbb{R} \to \mathbb{R}_+^Q$, each component of which is increasing, and such that $f(0) = 0$ and $\| f(\lambda) \| \to \infty$ as $\lambda \to \infty$. A ***monotone path solution on*** Σ^Q is a solution such that for some monotone path in $\mathbb{R}_+^Q$, and given any $S \in \Sigma^Q$, the solution outcome of S is the maximal point of intersection of the weak Pareto-optimal boundary of S with the path. Finally, consider a list of monotone paths indexed by $Q \in \mathcal{Q}$, $G = \{G^Q \mid Q \in \mathcal{Q}\}$, such that for all Q, $Q' \in \mathcal{Q}$

with $Q \subset Q'$, the projection of $G^{Q'}$ onto $\mathbb{R}^Q$ coincides with G^Q (this property of projections is crucial). Then, the ***monotone path solution relative to*** $\boldsymbol{G}$ is the solution that, for each $Q \in \mathcal{Q}$, coincides on Σ^Q with the monotone path solution associated with G^Q.

To present the results we need to impose a few other conditions on solutions. ***Weak Pareto-optimality*** says that there is no feasible outcome that all agents prefer to the solution outcome. ***Anonymity*** says that the solution is invariant under exchanges of the names of the agents. ***Scale-invariance*** says that a linear rescaling, independent agent by agent, of the utility functions, is accompanied by a similar rescaling of the solution outcome. ***Contraction-independence***[4] says that the elimination of alternatives that were not chosen by the solution does not affect the choice, if this choice is still feasible. ***Continuity*** says that small changes in problems do not produce large changes in solution outcomes.

The final condition, just like *population-monotonicity*, pertains to possible changes in the number of agents. It says that if a point x is the solution outcome of some problem, then the restriction of x to any subgroup of agents is the solution outcome of the "reduced problem" they face: this is the problem comprising all the alternatives at which the utilities of the other agents are given their values at x. (In the introduction, we presented the requirement as a form of "conditional" *population-monotonicity*). We will use, under the name of ***weak consistency***, the slightly weaker version of the condition that is obtained by requiring domination coordinate by coordinate of the restriction of x by the solution outcome of the reduced problem[5] (the egalitarian solution may violate the strong form of the condition for problems whose weak Pareto-optimal boundary contains non-degenerate parts parallel to a coordinate subspace.)

We are now in a position to state the results. They are characterizations of two of the most important solutions in the theory of bargaining.

Theorem 1 (Thomson, 1983a) The Kalai-Smorodinsky solution is the only

[4]This condition is more commonly known under the name of "independence of irrelevant alternatives".

[5]This condition being somewhat less transparent than the others, we give it in full: For all $Q, Q' \in \mathcal{Q}$ with $Q \subset Q'$, and for all $T \in \Sigma^{Q'}$, we have $\varphi(r^x_Q(T)) \geqq x_Q$, where $x = \varphi(T)$ and $r^x_Q(T) = \{y \in \mathbb{R}^Q_+ | (y, x_{Q'\setminus Q}) \in T\}$.

solution satisfying *weak Pareto-optimality, anonymity, scale-invariance, continuity*, and *population-monotonicity.*

Theorem 2 (Thomson, 1983b) The egalitarian solution is the only solution satisfying *weak Pareto-optimality, symmetry, contraction-independence, continuity*, and *population-monotonicity*[6].

Theorem 3 (Thomson, 1984b) The egalitarian solution is the only solution satisfying *weak Pareto-optimality, anonymity, continuity, population-monotonicity*, and *weak consistency.*

It is worth noting that the egalitarian solution, in fact all monotone path solutions, satisfy an even stronger monotonicity condition: the arrival of any group of agents, no matter how it influences the shape of the feasible set, affects all agents initially present in the same direction. This is certainly not true for the Kalai-Smorodinsky solution.

Chun and Thomson (1992) consider the class of problems obtained by specifying, in addition to the data needed to define a bargaining problem, a "claims point", representing prior claims that agents may have. The conjunction of these claims results in an infeasible point, but the claims are made in good faith, or represent commitments that cannot be jointly met, and a "good" solution should take them into account. Chun and Thomson provide characterizations of the solution that selects the maximal point on the segment connecting the disagreement point to the claims point. One of these characterizations is based on the natural form of *population-monotonicity* for that domain.

3.2 Guarantee structures

Although *population-monotonicity* is designed to ensure that when additional agents arrive, none of the agents initially present gain, it would be unfortunate if the burden fell disproportionately on some of them. If a solution satisfies *weak Pareto-optimality* and *population-monotonicity*, the arrival of

[6]Variants of this theorem are given in Thomson (1984a). There, it is shown that if *weak Pareto-optimality* is dropped, a certain family of "truncated egalitarian" solutions obtains. Also, a characterization of the monotone path solutions obtains if *anonymity* is dropped.

new agents has to hurt each of the agents initially present. If all of them are negatively affected, it is natural next to want them to be affected in a "quantitatively similar" way. In the context of bargaining, losses can be easily measured and compared by means of the cardinal information conveyed by the utility functions. Here, we propose to use this information to evaluate how well solutions distribute the burden due to the arrival of newcomers accross the agents initially present, and to compare them on that basis.

Given Q, $Q' \in \mathcal{Q}$ with $Q \subseteq Q'$, let $S \in \Sigma^Q$ and $T \in \Sigma^{Q'}$ be such that $S = T_Q$. Given $i \in Q$, the proportional loss incurred by agent i upon the arrival of the group $Q' \backslash Q$ is equal to $1 - \varphi_i(T)/\varphi_i(S)$. Now, calculate the smallest value taken by this ratio as S and T vary subject to the conditions stated above. Let this number be denoted by α. Seen positively, the number α can be interpreted as a "guarantee" offered by the solution to agent $i \in Q$ that, as the group enlarges from Q to Q', his final utility will be at least α times his initial utility. When a solution φ is *anonymous*, this guarantee depends only on the cardinalities of Q and Q' and it can be written as $\alpha_\varphi^{qq'}$, where $q = |Q|$ and $q' = |Q'|$. We call the collection of all the numbers $\alpha_\varphi^{qq'}$, for q, $q' \in \mathbb{N}$, $q' > q$, the ***guarantee structure of*** φ. This notion can be used to obtain a partial ordering on the space of solutions. Solutions that offer greater guarantees are of course more desirable. One would perhaps not expect to find maximal elements in the space of solutions, and therefore the following may be surprising:

Theorem 4 (Thomson and Lensberg, 1983) The guarantee structure of the Kalai-Smorodinsky solution is greater than the guarantee structure of any *weakly Pareto-optimal* and *anonymous* solution.

Naturally, offering high guarantees to individuals might be costly to the groups to which they belong. In order to understand the tradeoffs between protection of individuals and protection of groups, we define the ***collective guarantee structure*** of a solution by considering, for each pair S, T as specified above, the arithmetic average of the proportional losses incurred by the members of the initial group upon the arrival of additional agents, and again, calculating the smallest value taken by this ratio (of course, the geometric average or some other measure could be considered). Now, we find that the Nash solution performs better than any *weakly Pareto-optimal* and *anonymous* solution (in particular, better than the Kalai-Smorodinsky solution (Thomson, 1983c)).

Instead of focusing on how much agents may lose, one could alternatively focus on how much they may gain, and rank solutions on the basis of the extent to which they allow such gains. Here too, the Kalai-Smorodinsky solution performs better than any *weakly Pareto-optimal* and *anonymous* solution when individuals are examined, but it is the Nash solution that performs the best in that class when groups are examined (Thomson, 1987b).

Finally, we could compare how any two agents initially present fare, using the ratio of their relative losses, and look for solutions for which this ratio is as close to 1 as possible. Here, the Kalai-Smorodinsky solution performs better than any *weakly Pareto-optimal*, *anonymous*, and *scale invariant* solution, and the egalitarian solution performs better than any *weakly Pareto-optimal, anonymous*, and *contraction independent* solution (Chun and Thomson, 1989).

4 Games in coalitional form

The next class of problems that we will examine is richer than the class of bargaining problems because their specification involves a description of the opportunities available to each group, or ***coalition***, of agents. Two subclasses of such problems are usually considered. In a "transferable utility" game, what a coalition can achieve is given as a single number. In a "non-transferable utility" game, it is given as a subset of utility space. As an example of application of this model, consider an economy in which agents can form productive units. The productivity of each subgroup depends on the complementarities between the skills of the agents composing it. It is measured by the output that they can jointly produce, or the value of this output at some given prices. Another standard application of this model is to cost allocation, where each coalition is characterized by the cost of providing a certain service to its members when it is isolated from the rest of the economy.

4.1 The transferable utility case

We start with the class of ***transferable utility (TU)*** games. There is a group $Q \in \mathcal{Q}$ of agents whose members may gather in ***coalitions***[7]. What

[7] A coalition is a non-empty subset of Q.

each coalition can achieve on its own is its ***worth***. A ***game in coalitional form*** is a vector $v \in \mathbb{R}^{2^{|Q|}-1}$, the worth of each coalition being one of the coordinates of v. Let $v(S)$ denote the worth of coalition S. Restrictions may be imposed on v making the game monotonic (if $S \supset T$, then $v(S) \geqq v(T)$), or super-additive (the worth of a coalition is greater than the sum of the worths of the coalitions comprising a partition, no matter what that partition is). For all $Q \in \mathcal{Q}$, let $\mathcal{G}^Q$ be a class of admissible games for the group Q.

4.1.1 Population-monotonic solutions

We would like to reward agents as a function of the worths of the various coalitions. A ***solution*** is a correspondence that associates with every $v \in \mathcal{G}^Q$, where $Q \in \mathcal{Q}$, a non-empty set of vectors $x \in \mathbb{R}^Q$ such that $\sum_{i \in Q} x_i \leqq v(Q)$. The i^{th} coordinate of such a vector represents one of the possible payments to agent $i \in Q$ for being involved in the game. Already in this model, we have to allow for multi-valuedness; although some interesting *single-valued* solutions exist, many others are multi-valued.

A well-known solution is the ***core***: given $Q \in \mathcal{Q}$ and $v \in \mathcal{G}^Q$, it recommends any payoff vector $x \in \mathbb{R}^Q$ such that $\sum_{i \in Q} x_i = v(Q)$ and for no $S \subset Q$, $v(S) > \sum_{i \in S} x_i$; another important solution is the ***Shapley value*** (Shapley, 1953); it recommends, for each $i \in Q$, the payoff $x_i = \sum_{S \subset Q, S \ni i} k_S(v(S) - v(S \backslash \{i\}))$, where $k_S = [(|S|-1)!(|Q|-|S|)!]/|Q|!$; finally, evaluate the "dissatisfaction" of coalition S at $x \in \mathbb{R}^Q$ by the number $v(S) - \sum_{i \in S} x_i$. Then, the ***nucleolus*** (Schmeidler, 1969) selects the payoff vector $x \in \mathbb{R}^Q$ with $\sum_{i \in Q} x_i = v(Q)$ at which the dissatisfactions of coalitions are minimized in a lexicographic way in the set $\{x' \in \mathbb{R}^n | x'_i \geqq v(\{i\})$ for all $i \in N$, and $\sum_{i \in Q} x'_i = v(N)\}$, starting with the most dissatisfied coalition.

In the study of coalitional form games, most investigators have limited their attention to situations where the arrival of new agents is beneficial to the agents originally present. A central issue in the literature is identifying when these benefits are sufficient to ensure that none of them loses. We will refer to the requirement that none loses from the arrival of the newcomers as *population-monotonicity*$_+$. Letting the newcomers in is of course the socially efficient choice, so the property helps bring about coincidence of individual and social interests. Given $Q' \in \mathcal{Q}$, $v \in \mathcal{G}^{Q'}$ and $Q \subseteq Q'$, let $v_Q \in \mathcal{G}^Q$ be defined by $v_Q = (v(S))_{S \subseteq Q}$.

Population-monotonicity$_+$ for coalitional form games: For all Q, $Q' \in \mathcal{Q}$ with $Q \subset Q'$, for all $v \in \mathcal{G}^{Q'}$, $\varphi_i(v) \geqq \varphi_i(v_Q)$ for all $i \in Q$.

As we will see, in order to obtain the property, it is necessary to limit oneself to classes of games with a non-empty core, but much more is needed. An important class of games admitting *population-monotonic$_+$* solutions is the class of convex games, that is, games v such that for all S, $S' \subseteq Q$, $v(S \cup S') + v(S \cap S') \geqq v(S) + v(S')$. For such games the returns to co-operation increase quite fast with the size of coalitions. Conversely, on the class of concave games (games for which the inequality written above goes the other way), *population-monotonicity* can be met. This result is based on an observation due to Ichiishi (1988) that in a concave game, given any fixed order of the players, the payoff vectors obtained by paying each player his contribution to the coalition made up of the players preceding him in the ordering meet the *population-monotonicity* inequalities.

Proposition 1 (Sprumont, 1990; Rosenthal, 1990b) On the class of concave games, the Shapley value is a *population-monotonic* solution. On the class of convex games, it is a *population-monotonic$_+$* solution.

Sprumont identifies another interesting class of games admitting *population-monotonic$_+$* solutions, and for that class he exhibits a *population-monotonic$_+$* solution which bears a certain relationship to the Shapley value: it is the class of games for which, given any two coalitions S and T with $S \subset T$, the ***average*** contribution of the members of S to the worth of S is less than the comparable quantity for T.

Next, we present results obtained for three special classes of games. Sönmez (1993) shows that the nucleolus is not *population-monotonic$_+$* on the class of convex games, and that neither are the solutions known as the separable cost remaining benefit (Moulin, 1988), nor the τ-value (Tijs, 1981). He also studies a class of games that are exemplified by the well-known "airport problem" (Littlechild and Owen, 1973; Littlechild, 1974): each agent is characterized by a number, which can be interpreted as the cost of a public project (the length of the runway) when provided at the appropriate level for him. The corresponding cost for each coalition is defined to be the greatest cost associated with any of its members. Sönmez shows that on this subclass of the class of convex games, the nucleolus is a *population-monotonic*$_+$ solution (on the other hand, the separable cost remaining benefit solution and the τ-value continue to fail the test).

Rosenthal (1990b) considers a class of games on graphs, called "flow games", in which players control edges. To each edge is associated a number interpreted as the value that is created by transport along this edge. The worth of a coalition is defined to be the maximal flow along the subgraph consisting of the edges controlled by the members of the coalition. He shows that on this class the Shapley value is not *population-monotonic*$_+$, but he identifies characteristics of the new agents for which the payoffs attributed to the agents initially present by both the Shapley value and a certain selection from the core satisfy the inequalities required by the property. Then, a "conditional" form of the requirement is met.

Grafe, Iñarra and Zarzuelo (1992) analyze the simple class of games defined as follows: for each $i \in \mathbb{N}$, there is a coefficient $\beta_i \in \mathbb{R}_+$, which can be interpreted as a measure of agent i's "usefulness"; there is also an increasing function $r : \mathbb{N} \to \mathbb{R}$ indicating the productivities of groups as a function of their sizes; these data are combined so as to give the worth of each coalition S by the formula $(\sum_{i \in S} \beta_i) r(|S|)$. They also consider the special case when $r(|S|)$ takes the form $|S|^\sigma$ for $\sigma \in [0, 1]$. They study the *population-monotonicity*$_+$ of the Shapley value and of the nucleolus on this class of games. The results are negative. However, the rule that divides the worth of the grand coalition proportionally to the coefficients β_i clearly has the property.

4.1.2 Population-monotonic payoff configurations

Instead of searching for *population-monotonic*$_+$ solutions, we now limit ourselves to the less ambitious task of searching, ***game by game***, for *population-monotonic*$_+$ ***payoff configurations***: given $Q \in \mathcal{Q}$ and $v \in \mathcal{G}^Q$, a ***payoff configuration for v*** (Hart, 1985) is a list $(x^S)_{S \subseteq Q}$, where for each $S \subseteq Q$, $x^S \in \mathbb{R}^S$ and $\sum_{i \in S} x_i^S = v(S)$. A payoff configuration provides a recommendation for each coalition S, should it form, on the division of its worth $v(S)$ among its members; this recommendation may depend on the components of v pertaining to coalitions that are not subsets of S.

Population-monotonic$_+$ payoff configuration: Given $Q \in \mathcal{Q}$ and $v \in \mathcal{G}^Q$, the payoff configuration for v, $(x^S)_{S \subseteq Q}$, is *population-monotonic*$_+$ if for all S, $S' \subseteq Q$ with $S \subset S'$, and for all $i \in S$, $x_i^S \leqq x_i^{S'}$.[8]

To better understand the relationship between the concept of a *population-monotonic*$_+$ solution on some domain and that of a *population-monotonic*$_+$ payoff configuration for a game, note first that if a game and all of its subgames belong to a domain on which there exists a *population-monotonic*$_+$ solution, then of course, the game has a *population-monotonic*$_+$ payoff configuration; simply apply the solution to the game and its subgames. On the other hand, a domain of games each of which has a *population-monotonic*$_+$ payoff configuration does not necessarily admit a *population-monotonic*$_+$ solution. To see this, consider the domain consisting of the following two three-person games v and w and their subgames: $Q = \{1,2,3\}$, $v(i) = 0$ for all $i \in Q$, $v(23) = 0$, $v(S) = 1$ for all other $S \subseteq Q$, $w(i) = 0$ for all $i \in Q$, $w(13) = 0$, and $w(S) = 1$ for all other $S \subseteq Q$. The game v has a unique *population-monotonic*$_+$ payoff configuration, at which all players always get 0 except player 1 in the subgames $v_{\{1,2\}}$, $v_{\{1,3\}}$, and v itself, where he gets 1; a similar statement holds for w, where it is player 2 who gets the non-zero payoffs. Since the subgames of v and w relative to players 1 and 2 are the same, we obtain a contradiction[9].

Obtaining *population-monotonic*$_+$ payoff configurations requires that restrictions be imposed on the game. As a preparation for the main result on this issue, first note that as announced earlier any such game has a non-empty core. Indeed, let $(x^S)_{S \subseteq Q}$ be a payoff configuration for v. Given $S \subseteq Q$, if $x_i^Q \geqq x_i^S$ for all $i \in S$, then $\sum_{i \in S} x_i^Q \geqq \sum_{i \in S} x_i^S = v(S)$ so that S cannot improve upon x_Q. A similar inequality can be established for any pair S, S' with $S \subseteq S'$ (and not just pairs where $S' = Q$), so that the cores of all subgames of v are non-empty as well[10]. If $|Q| = 3$, this condition turns out to be sufficient, but if $|Q| > 3$, it is not, as shown by the following example:

Example (Sprumont, 1990). Let $Q = \{1,2,3,4\}$. Each of players 1 and 2 owns a left glove. Each of players 3 and 4 owns a right glove. A pair of gloves has value 1. A single glove has value 0.

This situation can be described by the following game:

$v(i) = 0$ for all $i \in Q$

[8]Note that the set of *population-monotonic*$_+$ payoff configurations is convex.

[9]I am grateful to Y. Sprumont for providing me with this example.

[10]This is the condition known as "total balancedness".

$v(12) = v(34) = 0,\ v(13) = v(14) = v(23) = v(24) = 1$
$v(ijk) = 1$ for all $i, j, k \in Q$
$v(Q) = 2$

The core of each three-person subgame of v is a single point, at which the odd man out gets 1 and the others get 0. Since each player is involved in a three-person game at which his payoff is 1, for the configuration $(x^S)_{S \subseteq Q}$ to be *population-monotonic*$_+$ for v, we need $x_i^Q \geqq 1$ for all $i \in Q$, but since $v(Q) = 2$, this is impossible. The cores of the two-person subgames (and of course, of each one-person subgame), are also non-empty. Sprumont generalizes this example to show that no assignment game with at least two buyers and two sellers such that every buyer-seller pair derives some benefit from trade has a *population-monotonic*$_+$ payoff configuration.

To state the main result of this section, which is a characterization of the class of games admitting *population-monotonic*$_+$ payoff configurations, we need a few additional definitions: The game $v \in \mathcal{G}^Q$ is ***simple*** if $v(S) = 0$ or 1 for all $S \subseteq Q$; ***monotonic*** if for all $S, S' \subseteq Q$ with $S \subset S'$, $v(S) \leqq v(S')$; ***additive*** if there is $a \in \mathbb{R}$ such that $v(S) = a|S|$ for all $S \in \mathcal{S}$; finally, player $i \in Q$ is a ***veto player in*** $\boldsymbol{v \in \mathcal{G}^Q}$ if for all S such that $i \notin S$, $v(S) = 0$.

Theorem 5 (Sprumont, 1990) A game has a *population-monotonic*$_+$ payoff configuration if and only if it is the sum of an additive game and a positive linear combination of monotonic simple games with veto players[11].

Moulin (1990a) investigates the existence of *population-monotonic*$_+$ payoff configurations satisfying certain individual upper bounds. In the economic application motivating his work, the bound relative to a given agent is defined to be the maximal payoff he would obtain under the assumption that all other agents had the same preferences as his, and under the requirements of efficiency and equal treatment of equals[12]. Formally, and returning to our abstract model, given $Q \in \mathcal{Q}$, and $v \in \mathcal{G}^Q$, an ***aspiration for v*** (Bennett, 1983) is a vector $y \in \mathbb{R}^Q$ such that for all $S \subseteq Q$, $\sum_{i \in S} y_i^S \geqq v(S)$. Similarly, an ***aspiration configuration for v*** is a list $(y^S)_{S \subseteq Q}$ such that for

[11]Sonn (1990) shows that whether or not a game can be decomposed as stated in the theorem can be determined by solving a simple linear program.

[12]We will come back to this bound, defined by reference to economies made up of identical agents.

all $S \subseteq Q$, y^S is an aspiration for $v(S)$. The configuration is *population-monotonic*$_+$ if for all $i \in Q$, for all S, $S' \subset Q$ with $i \in S \subseteq S' \subseteq Q$, we have $y_i^S \leqq y_i^{S'}$.

Proposition 2 (Moulin, 1990a) Let $Q \in \mathcal{Q}$ and $v \in \mathcal{G}^Q$ be a convex game. Let $(y^S)_{S \subseteq Q}$ be a *population-monotonic*$_+$ aspiration configuration such that for all $i \in Q$, and for all S, $S' \subseteq Q$ with $i \in S \cap S'$ and $|S| = |S'|$, $y_i^S = y_i^{S'}$. Then, v has a *population-monotonic*$_+$ payoff configuration bounded above by y.

Moulin shows that the result does not necessarily hold if the uniformity assumption (the condition that for all $i \in Q$, and for all S, $S' \subseteq Q$ with $i \in S \cap S'$ and $|S| = |S'|$, $y_i^S = y_i^{S'}$) is not made. But in his application of the result to economies with public goods, and if aspirations are defined as explained above by reference to economies made up of agents with identical preferences, the uniformity assumption does hold (see Section 8 for additional results on public good economies).

4.2 The non-transferable utility case

Consider now the richer model in which what each coalition $S \subseteq Q$ can achieve is given as a ***subset*** $\boldsymbol{V(S)}$ of the utility space $\mathbb{R}^S$ pertaining to that coalition. Each $V(S)$ is of course required to satisfy certain regularity conditions. These games are called ***non-transferable utility***, or NTU, games. For each $Q \in \mathcal{Q}$, let $\mathcal{H}^Q$ be a class of admissible NTU games involving the group Q. A ***solution*** associates with every $V \in \mathcal{H}^Q$, where $Q \in \mathcal{Q}$, a non-empty subset of $V(Q)$.

In this model also, we are even further from a full understanding of the implications of *population-monotonicity*$_+$. However, a result is available which concerns a special subclass that has recently been the object of some attention. This is the class of "hyperplanes games", games $V \in \mathcal{H}^Q$ such that for all $S \subseteq Q$, $V(S)$ is a hyperplane (a TU game can be represented as a hyperplane game in which the normals to the hyperplanes are vectors of ones). Maschler and Owen (1989), who introduced this class of games, proposed for it an extension of the Shapley value which is defined as follows. Given $Q \in \mathcal{Q}$, $v \in \mathcal{H}^Q$, and an ordering $\{i_1, i_2, \ldots, i_{|Q|}\}$ of the players, consider the payoff vector x obtained by giving (1) to agent i_1 the most that he can get on his own, $x_{i_1} = \max\{x \in V(i_1)\}$, (2) to agent i_2 the most he can get in $V(i_1 i_2)$ subject to agent i_1 getting $x_{i_1}, \ldots$, (k) to agent i_k the most he can get in $V(i_1 i_2 \ldots i_k)$ subject to each of the preceding agents i_ℓ getting x_{i_ℓ}... Finally, the ***Maschler-Owen solution*** of the game is the average of the payoff vectors so obtained when all orders are equally likely. We will consider the subclass of hyperplanes games satisfying a certain property of "strong cardinal convexity" (Sharkey, 1981), which ensures that the feasible sets expand sufficiently as the number of agents increases. Now, we have the following positive result :

Proposition 3 (Rosenthal, 1990b) On the class of strongly cardinally convex hyperplane games the Maschler-Owen solution is *population-monotonic*$_+$.

Concerning the existence, for each given game, of *population-monotonic*$_+$ payoff configurations, we can only report that the counterpart of the remark preceding Proposition 1 that one might have hoped for does not hold: there are ordinally convex games (Peleg, 1986) for which the payoff configuration that consists of the marginal contributions vectors are not *population-monotonic*[13].

5 Quasi-linear cost allocation problems

We now turn to a family of somewhat more concrete decision problems: imagine a society that must select one among a finite number of projects; with each project is associated a certain level of utility for each agent and a certain cost. Which project should be selected and how should its cost be allocated among the agents? This class of problems differs from the classes examined so far in that information is retained on the manner in which utility levels are generated. However, no particular structure is imposed on the physical nature of the options available. In the sections to follow, we will keep on record a complete description of the physical features of the alternatives of which the feasible set is comprised.

Formally, let A be a finite set of ***public projects***. A ***quasi-linear cost allocation*** problem is a pair $((u_i)_{i\in Q}, C) = (u_Q, C) \in \mathbb{R}^{|A|Q} \times \mathbb{R}^{|A|}$. Here, C is the ***cost vector***, each coordinate of C being the cost of the corresponding

[13]Observation attributed to Moulin in Sprumont (1990).

project. In addition, there is a private good called "money" which can be used for compensations. The preferences of agent $i \in \mathcal{Q}$, defined over the product $A \times \mathbb{R}$, admit a quasi-linear numerical representation u_i: given the project $a \in A$ and given agent i's holdings of money $m_i \in \mathbb{R}$, his utility is $u_{ia} + m_i$. For each $Q \in \mathcal{Q}$, let $\mathcal{M}^Q$ be the class of these problems. ***A solution*** is a function that associates with every $(u_Q, C) \in \mathcal{M}^Q$, where $Q \in \mathcal{Q}$, a vector $x \in \mathbb{R}^Q$ such that $\sum_{i \in Q} x_i \leqq \max_{a \in A}(\sum_{i \in Q} u_{ia} - C_a)$. A family of examples are obtained by first selecting the project for which the difference between sum of utilities and cost is the highest, and then choosing contributions so that all agents receive an equal share of the surplus over some reference level so generated.

Chun (1986) proposed for this model the condition that all agents be affected in the same direction by the arrival of additional agents, which is indeed the appropriate form of the principle of *population-monotonicity* for this model. This is the first time in this survey that we find it necessary to use this condition.

Weak population-monotonicity for quasi-linear cost allocation problems. For all Q, $Q' \in \mathcal{Q}$ with $Q \subset Q'$, for all $(u_{Q'}, C) \in \mathcal{M}^{Q'}$, for all $z \in \varphi(u_Q, C)$ and $z' \in \varphi(u_{Q'}, C)$, either $z_i \geqq z'_i$ for all $i \in Q$ or $z_i \leqq z'_i$ for all $i \in Q$.

Chun searched for solutions satisfying the following additional requirements (formulated by Moulin, 1985a, 1985b, in his extensive analysis of this class of problems.) ***Pareto-optimality*** says that the decision maximizes the net aggregate benefit. ***Anonymity*** says that the solution is invariant under exchanges of the names of agents. ***Independence of the zero of the utility functions*** says that the solution is invariant under the addition of an arbitrary constant to the agents' utilities. ***Independence of the zero of the cost function*** says that an increase in the cost function, uniform across all alternatives, is distributed evenly among the agents. ***Cost monotonicity*** says that an increase in the cost function is borne by all agents.

The following theorems give a very complete picture of the implications of *weak population-monotonicity* in this model. Essentially, all admissible solutions can be described as "egalitarian", as they consist in dividing equally a surplus over some vector of reference utility levels, but they differ in the way the reference levels are computed. Let e be the vector of all ones in $\mathbb{R}^{|A|}$.

Theorem 6 (Chun, 1986)
A solution φ satisfies *Pareto-optimality, anonymity,* the two *independence axioms* and *weak population-monotonicity* if and only if there is a function $g : [\mathbb{R}^A]^2 \to \mathbb{R}$ satisfying
(i) $g(x + \alpha e, z) = g(x, z) + \alpha$ for all x, $z \in \mathbb{R}^A$ and for all $\alpha \in \mathbb{R}$
(ii) $g(0, z) = 0$ for all $z \in \mathbb{R}^A$
(iii) $g(x, z + \alpha e) = g(x, z)$ for all x, $z \in \mathbb{R}^A$ and for all $\alpha \in \mathbb{R}$
and such that for all $Q \in \mathcal{Q}$, for all $i \in Q$, and for all $(u_Q, C) \in \mathcal{M}^Q$,
$\varphi_i(u_Q, C) = (1/|Q|)\max_{a\in A}\{\sum_{i\in Q} u_{ia} - C_a\} + (1/|Q|)\{(|Q| - 1)g(u_i, C) - \sum_{j\in Q\setminus\{i\}} g(u_j, C)\}$.

Theorem 7 (Chun, 1986) A solution φ satisfies the five axioms of Theorem 6 and *cost-monotonicity* if and only if there is a function $\tilde{g} : \mathbb{R}^A \to \mathbb{R}$ satisfying
(i) $\tilde{g}(x + \alpha e) = \tilde{g}(x) + \alpha$ for all $x \in \mathbb{R}^A$ and for all $\alpha \in \mathbb{R}$
(ii) $\tilde{g}(0) = 0$
and such that for all $Q \in \mathcal{Q}$, for all $i \in Q$, and for all $(u_Q, C) \in \mathcal{M}^Q$,
$\varphi_i(u_Q, C) = (1/|Q|)\max_{a\in A}\{\sum_{i\in Q} u_{ia} - C_a\} + (1/|Q|)\{(|Q| - 1)\tilde{g}(u_i) - \sum_{j\in Q\setminus\{i\}} \tilde{g}(u_j)\}$.

Alternatively, an axiom of *consistency* (informally described in section 2) can be used in Theorem 7 instead of *cost-monotonicity.* Additional characterizations are offered in Chun (1986). The requirements that no agent be able to gain by disposing of utility, or that the solution provide a minimal reference utility to each individual, place further restrictions on the g functions. These restrictions can be completely described.

6 Fair allocation in economies with private goods

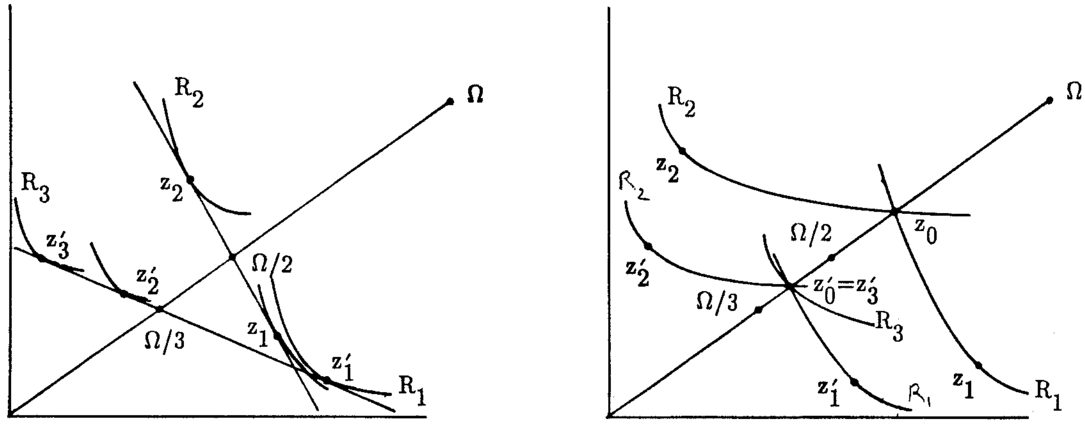
We now apply the idea of *population-monotonicity* to one of the most commonly studied problems, that of allocating a fixed bundle of goods among a group of agents with equal rights on these goods.

6.1 Population-monotonicity for the classical problem of fair division

The model is as follows. There are $\ell \in \mathbb{N}$ goods and a group $Q \in \mathcal{Q}$ of agents; for each $i \in Q$, R_i is agent i's continuous, convex and monotone preference relation defined over $\mathbb{R}^\ell_+$ with I_i designating the indifference relation associated with R_i. Let $\mathcal{R}_{cl}$ be the class of all such "classical" preference relations. Also needed is $\Omega \in \mathbb{R}^\ell_+$, the ***social endowment***. A ***problem of fair division*** is a pair $((R_i)_{i\in Q}, \Omega) \in \mathcal{R}^Q_{cl} \times \mathbb{R}^\ell_+$, or simply (R_Q, Ω). This formulation is to be distinguished from formulations in which each agent is entitled to a particular share of the social endowment, his "individual" endowment (see below); here, we assume instead that agents are ***collectively*** entitled to the resources Ω. For each $Q \in \mathcal{Q}$, let $\mathcal{E}^Q$ be a class of admissible problems involving the group Q. A ***solution*** is a correspondence that associates with every $(R_Q, \Omega) \in \mathcal{E}^Q$, where $Q \in \mathcal{Q}$, a non-empty subset of the set of feasible allocations of (R_Q, Ω), $Z(e) = \{z \in \mathbb{R}^{\ell|Q|}_+ | \sum_{i\in Q} z_i \leqq \Omega\}$. We have already discussed the condition of *population-motonicity* for this model in the introduction and we will not give a formal statement since it is straightforward.

A simple example of a *population-monotonic* solution is the solution that always chooses equal division: $z_i = \Omega/|Q|$ for all $i \in Q$. Of course, this solution suffers from the major drawback of not being efficient. To obtain efficiency, give the entire social endowment to the agent with the lowest index in Q and nothing to the others. This *population-monotonic* solution is efficient but quite unappealing from the distributional viewpoint.

The suggestion is often made to solve problems of fair division by operating the Walrasian mechanism from equal division. Is the solution so defined *population-monotonic*? The example of Figure 2a, which is taken from Chichilnisky and Thomson (1987), shows that it is not. In the two-person economy consisting of agents 1 and 2 in which the vector $\Omega \in \mathbb{R}^2_+$ has to be divided, it leads to the allocation (z_1, z_2). After the arrival of agent 3, it leads to the allocation (z'_1, z'_2, z'_3). Since agent 1 prefers z'_1 to z_1, *population-monotonicity* fails. We add that the example can be specified with homothetic preferences; this assumption often has regularizing implications (together with the assumption of equal endowments, it implies uniqueness and stability of the Walrasian equilibrium), but it does not prevent the un-



(a) (b)

Figure 2: Population-monotonicity in exchange economies. (*a*) The Walrasian solution operated from equal division is not *population-monotonic.* (*b*) The "egalitarian" solution defined by requiring all agents to be indifferent between their consumptions and the same scale multiple of the social endowment is *population-monotonic.*

desirable possibility that the presence of one more claimant benefits one of the original agents. Quasi-linearity of preferences would not help either.

The following theorem provides additional information about the circumstances in which violations occur, as it relates the likelihood of *population-monotonicity* being violated by the Walrasian solution operated from equal division to the likelihood of the solution being subject to the "transfer problem", a problem that has been the object of a considerable amount of attention in the international trade literature: a solution is said to be subject to the (strong form of the) transfer problem if the transfer of part of some agent's endowment to another agent ultimately benefits him and hurts the recipient:

Proposition 4 (Jones, 1987) Consider the class of economies with homothetic preferences. In the absence of substitution effects, the Walrasian solution from equal division is subject to the transfer paradox if and only if it violates *population-monotonicity.* In the presence of substitution effects, it may violate *population-monotonicity* even in situations where no transfer paradox would occur.

For a positive result, we have the ***"egalitarian" solution*** (Pazner and Schmeidler, 1978), which is defined by selecting the efficient allocation(s) at which utilities are equal, using the utility representations obtained by calibrating along the ray through the aggregate bundle. Figure 2b illustrates the fact that this solution is *population-monotonic*: in the two-person economy (R_1, R_2, Ω) the efficient allocation (z_1, z_2) is egalitarian since there exists z_0 proportional to Ω such that both agents are indifferent between their consumptions and z_0. In the enlarged three-person economy (R_1, R_2, R_3, Ω), the efficient allocation (z_1', z_2', z_3') is egalitarian with a reference bundle z_0' that can only be lower than z_0 on the ray through Ω. Therefore, both agent 1 and agent 2 lose upon the arrival of agent 3.

This solution is only one example in a large family of *population-monotonic* solutions defined as follows (Thomson, 1987). Let $\mathcal{B}$ be a family of choice sets $\{B(\lambda) \subseteq R_+^\ell | \lambda \in \mathbb{R}_+\}$ with the following properties

(i) $B(\lambda)$ is closed for all $\lambda \in \mathbb{R}_+$

(ii) $B(0) = \{0\}$

(iii) $B(.)$ is upper-semi-continuous

(iv) $B(.)$ is monotonic : $B(\lambda) \subseteq B(\lambda')$ whenever $\lambda \leqq \lambda'$

(v) $B(.)$ is unbounded : for all $r \in \mathbb{R}_+$, there is λ such that $r(1, \ldots, 1) \in B(\lambda)$

Given an economy $e = (R_Q, \Omega) \in \mathcal{R}_{cl}^Q \times \mathbb{R}_+^\ell$ in which preferences are strictly monotone, let $\varphi_{\mathcal{B}}(e)$ be the set of efficient allocations z of e such that for some $\lambda \in \mathbb{R}_+$, and for all $i \in Q$, $z_i I_i z_i^*$ where $z_i^* R_i z_i'$ for all $z_i' \in B(\lambda)$. It is easy to see that any such ***equal-opportunity equivalent*** solution $\varphi_{\mathcal{B}}$ is *population-monotonic.* The following proposition is a straightforward generalization of an observation made in Thomson (1987b):

Proposition 5 *On the domain of classical economies in which preferences are strictly monotone, the equal-opportunity equivalent solutions are* population-monotonic *selections from the Pareto solution.*

These solutions are directly inspired by the monotone path solutions of bargaining theory (Section 3). Here, the monotone path is obtained by choosing for each agent a continuous numerical ("utility") representation of his preferences and tracing out the image in utility space of the list of maximizers $(z_i^*)_{i \in Q}$ in $B(\lambda)$ of the preference relations $(R_i)_{i \in Q}$ as λ varies. Under our assumptions on preferences, the image of the feasible set is comprehensive (Section 3.1). For $\lambda = 0$, a point in the feasible set is obtained and by the unboundedness of $B(.)$, for λ large enough, a point outside of the feasible set results. The fact that the solutions are well-defined follows from these observations. Proving that they are *population-monotonic* is straightforward.

Examples of interesting families satisfying the above conditions are the following :

$B_1(\lambda) = \{x \in \mathbb{R}^\ell | 0 \leqq x \leqq \lambda\Omega\}$. Note that the solution associated with this family is the egalitarian solution defined above.

$B_2(\lambda) = \{x \in \mathbb{R}^\ell | 0 \leqq x \leqq d\}$ for some $d \in \mathbb{R}^\ell_+$.

$B_3(\lambda) = \{z \in \mathbb{R}^\ell_+ | pz \leqq \lambda\}$ for fixed $p \in \Delta^{\ell-1}$.

As already noted, we are not interested only in *population-monotonicity.* We certainly want our allocation rule to be efficient, that is, to be a subsolution of the ***Pareto solution***; but we also want it to satisfy some distributional requirements. The distributional requirements that have played the main role in the literature are embodied in the following solutions: the ***no-envy solution*** (Foley 1967) selects the set of feasible allocations z such that for no pair $\{i, j\} \subseteq Q$, we have $z_j P_i z_i$ - at such an allocation no agent would want to exchange bundles with anyone else; the ***individual rationality from equal division solution*** selects the set of feasible allocations that all agents prefer to equal division; the ***egalitarian-equivalence solution*** (Pazner and Schmeidler, 1978) selects the set of feasible allocations z such that for some $z_0 \in \mathbb{R}^\ell_+$ and for all $i \in Q$, $z_0 I_i z_i$.

The egalitarian solution is a selection from both the individual rationality from equal division solution and the egalitarian-equivalence solution and it is *population-monotonic.* On the other hand, the Walrasian solution from equal division is a selection from both the no-envy solution and the individual-rationality from equal division solution but, as we saw earlier, it is not *population-monotonic.*

Is there any *population-monotonic* selection from the no-envy and Pareto solution? The next theorem states that the answer is no. We should point out however that its proof relies on having access to economies with a large number of agents (a continuum), and this leaves open the question whether the impossibility holds for the small number case. (All of the other negative results reviewed here are proved by means of examples with a small number of agents).

Theorem 8 (Moulin, 1990c) There is no *population-monotonic* selection from the no-envy and Pareto solution.

In order better to understand the strength of *population-monotonicity*, the following result is useful: A solution satisfying *consistency* (see Section 2 for a general statement of this property and Section 3 for an application to bargaining), and *resource-monotonicity* (any increase in the resources to divide benefits everyone) automatically satisfies *population-monotonicity* (Fleurbaey, 1992c, 1993; Chun, 1985, establishes this fact for bankruptcy problems). Consequently, as Fleurbaey (1992) notes, in exchange economies satisfying the familiar gross substitutability assumption and in which all goods are normal, the Walrasian rule from equal division is a single-valued *population-monotonic* solution. This is because under those assumptions, it is single-valued, as is well-known; it is also *resource-monotonic* (this follows from Polterovich and Spivak, 1983, as observed in Moulin and Thomson, 1988). It is *consistent* in general (Thomson, 1988).

6.2 A generalized notion of population-monotonicity

If the commodity to divide is a "bad", the counterpart of *population-monotonicity* is that all agents benefit from the arrival of additional agents, the requirement that we considered for coalitional form games under the name of *population-monotonicity*$_+$. Moulin (1989) points out that for the quasi-linear domain (Section 6.5) the egalitarian solution of Pazner and Schmeidler (1978) is a *population-monotonic*$_+$ selection from the individual rationality from equal division and Pareto solution. On the negative side, it remains true that there exists no *population-monotonic*$_+$ selection from the no-envy and Pareto solution (Moulin, 1990d).

It can be argued that it is really when we have in mind efficiency that in the classical problem of fair division we require that none of the agents initially present gains upon the arrival of additional claimants, and in the model with bads that none of them loses. If it is found desirable to keep efficiency considerations separate from fairness considerations, *weak population-monotonicity* should be used instead. In the classical case, and when efficiency is imposed, *weak population-monotonicity* reduces to *population-monotonicity*, and in the case of bads, it reduces to *population-monotonicity*$_+$. Another advantage of using *weak population-monotonicity* is that it is better adapted to non-classical models. We have already seen its relevance in the analysis of quasi-linear cost allocation problems. Here are several other illustrations for general resource allocation problems. First, suppose that the mere presence of additional agents affects positively the agents initially present. Then it might be possible to make the latter better-off in spite of the fact that resources have not changed. More generally, an agent's welfare may depend on the consumptions of the others (and not just on his own consumption) and here too, if these external effects are positive and strong enough, it may be possible to make all agents initially present gain when new agents come in.

Or suppose that it is not aggregate resources that are constant, but ***resources per capita***. Then, depending upon the preferences, one may well be able to make all agents initially present gain when new agents come in.

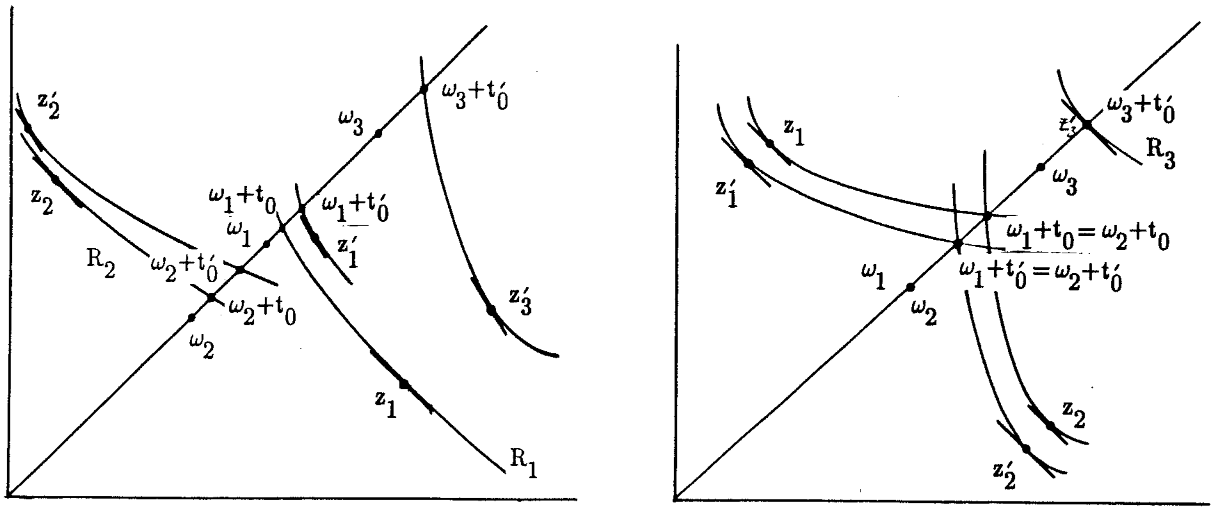
The case of the division of a good when preferences are single-peaked will be discussed in Section 10. We only note here that this case is a mixture of the classical case and of the case of bads: the situation is sometimes like the classical one, and the arrival of additional agents is bad news, but sometimes the arrival of additional agents is good news. In this model, the appropriate requirement is *weak population-monotonicity.*

6.3 Economies with individualized endowments

A further generalization would allow agents to be differentially endowed of the various goods and here too, the most that one can legitimately require is *weak population-monotonicity.*

Formally, an economy is now a pair (R_Q, ω_Q) where $Q \in \mathcal{Q}$, R_Q is as before a profile of preference relations, and ω_Q is a profile of endowments: $\omega_Q = (\omega_i)_{i \in Q} \in \mathbb{R}^{\ell}_{+}$. Let $\mathcal{F}^Q$ be the class of all such economies. Solutions are defined on the union of all $\mathcal{F}^Q$, for $Q \in \mathcal{Q}$.

That the Walrasian solution does not satisfy the property can be proved by a simple modification of Figure 2a. An example of a *weakly population-monotonic* solution is the solution that picks any allocation such that all agents be indifferent between their net trade and a reference trade proportional to some multiple of a fixed vector. The family so defined is the natural



(a) (b)

Figure 3: ***The d-egalitarian trade solution is weakly population-monotonic.*** (a) In the example depicted here, both agents 1 and 2 gain upon the arrival of agent 3. (*b*) In this example, both agents 1 and 2 lose upon the arrival of agent 3.

counterpart for economies in which individual endowments are specified, of the equal-opportunity equivalence solutions associated with the family of choice sets B_2 (see above). For a formal definition, let $d \in \mathbb{R}^\ell_+$. Given $Q \in \mathcal{Q}$ and $e = (R_Q, \omega_Q) \in \mathcal{F}^Q$, the ***d-egalitarian trade solution*** picks the efficient allocations z of e such that there is $\lambda > 0$ with, for all $i \in Q$, $z_i I_i (\omega_i + \lambda d)$. It is well-defined when preferences are strictly monotonic. Figure 3 illustrates the fact that the solution is *weakly population-monotonic* and that indeed both cases may occur: upon the arrival of new agents, all agents initially present may gain (Figure 3a) or they may all lose (Figure 3b).

6.4 Bargaining solutions used as resource allocation rules

The relevance of bargaining theory to the resolution of concretely specified problems of resource allocation is discussed by Roemer (1986*a*, *b*, 1988) and Chun and Thomson (1988). The theory of bargaining relies on the requirement that two "concrete" problems having the same image in utility space be treated the same. Roemer spells out informational assumptions under which results obtained in the abstract theory of bargaining can be rewritten for concrete economic problems. The crucial property he uses is designed to relate the recommended allocations as the number of goods varies. This property, together with standard efficiency and symmetry conditions, and *population-monotonicity*, allows him to derive from the characterization of the egalitarian solution to the bargaining problem (Theorem 2) a characterization of the counterpart of that solution for the economic domain.

The advantage of the abstract bargaining model is that many concrete situations can be represented in utility space as objects satisfying the assumptions typically made in the theory of bargaining. In particular, if utility functions are continuous, monotone increasing and concave, the images in utility space of two problems of fair division differing only in the number of agents are pairs of problems satisfying the hypotheses of the axiom of *population-monotonicity* relevant for that theory. Is the converse true? Given a pair of problems satisfying the assumptions of *population-monotonicity*, are they the images in utility space of a pair of economies differing only in the number of agents? If all such pairs of problems could be so derived then of course, the characterization results of bargaining theory would be directly applicable. However, and not surprisingly, the answer is no. Yet, not all pairs of problems are actually needed in the characterization proofs, and enough richness could remain for these characterizations to hold. It is intuitive that this richness depends on the number of goods. The question is then whether some bounds on the number of commodities can be identified that help predict whether the property can be met. The answer is yes, as explained next.

Chun and Thomson (1988) show that in the one-commodity case, the Nash solution is *population-monotonic* (recall that the Nash solution does not have this property on the general domain). The proof is simple: let each agent $i \in \mathbb{N}$ be equipped with a concave utility function $u_i : \mathbb{R}_+ \to \mathbb{R}$. Allocating Ω units of the unique good among the members of a group $Q \in \mathcal{Q}$ according to the Nash solution means finding $(x_i)_{i \in Q} \in \mathbb{R}^Q_+$ maximizing the product $\prod_{i \in Q} u_i(x'_i)$ with respect to $(x'_i)_{i \in Q} \in \mathbb{R}^Q_+$ subject to the condition $\sum_{i \in Q} x'_i = \Omega$. Assuming for simplicity that all utilities are differentiable, this exercise is solved by equating the ratios $u'_i(x_i)/u_i(x_i)$ across all agents. But since each ratio is a decreasing function of its argument, it follows that when there are more agents, the common value of the ratios solving this exercise can only be greater, which results in a smaller consumption for all agents initially present.

However, it turns out that as soon as there are two goods, the class of admissible problems is sufficiently rich to produce the same behavior as on arbitrary problems. The next proposition summarizes these results.

Proposition 6 (Chun and Thomson, 1988) In the one-commodity case, the Nash solution (used as a resource allocation rule) is *population-monotonic.* For any greater number of commodities, it is not.

Note that in the one-commodity case any solution defined by maximizing a sum $\sum_{i \in Q} f_i(x_i)$ subject to the condition $\sum_{i \in Q} x_i = \Omega$, where, for each $i \in N$, f_i is continuous, monotone increasing and concave, also is *population-monotonic.*

6.5 Solutions to games in coalitional form used as resource allocation rules

Recall that on the class of TU concave coalitional form games the Shapley value is *population-monotonic* (Section 4). This result can be applied to economies that can be represented as concave games. What are these economies ?

A "TU" economy is one in which agents can be described in terms of "utility functions", utility being transferable from any agent to any other agent at a one-to-one ratio. This is mathematically equivalent to adding an "accounting" good such that each agent's preferences can be represented by a function that is separable additive in the accounting good on the one hand, and some function of the others on the other, and linear in the accounting good. If this formulation is adopted, the social endowment of the accounting good is of course equal to zero. Alternatively, and in fact more generally, we can assume that this special good is an actual good, without necessarily requiring the social endowment to be equal to 0. We will adopt this formulation and refer to the good as "money": to summarize, we say that agent i's preferences are ***quasi-linear*** if they admit a numerical representation that is separable additive in money on the one hand and all the other goods on the other, and linear in money: $u_i(x_i, y_i) = x_i + v_i(y_i)$ where $x_i \in \mathbb{R}$ is agent i's consumption of money and $y_i \in \mathbb{R}_+^{\ell-1}$ is the vector of his consumptions of the other goods. It is standard to assume that x_i is unconstrained in sign and unbounded below. These assumptions are usually imposed for mathematical convenience, but they are of particular significance for our problem. For each $Q \in \mathcal{Q}$, let $\mathcal{E}_{ql}^Q$ be the class of quasi-linear economies. Since the social endowment of money may be negative, there is of course no reason to expect *population-monotonicity*, and in these cases, we will limit ourselves to the search for *weakly population-monotonic* solutions.

On the domain of quasi-linear economies with a social endowment of money equal to 0, the Walrasian solution from equal division is still not *population-monotonic*. In fact, the property cannot be met even if no distributional requirements are imposed:

Theorem 9 (Moulin, 1992b) On the domain of quasi-linear economies in which the social endowment of money is 0 (and even if the functions v_i are concave), there is no *population-monotonic* selection from the Pareto solution.

However the counterpart of the egalitarian solution of Pazner and Schmeidler (1978) (see above), which is obtained by requiring the reference bundle to be proportional to the social endowment vector, is obviously *weakly population-monotonic*. This solution is a selection from the individual rationality from equal division and Pareto solution. Another *weakly population-monotonic* solution is obtained by choosing allocations at which the surplus measured in terms of money is divided equally (Moulin, 1992b). This solution only satisfies a weaker condition of individual rationality, which says that each agent's utility level, evaluated in terms of the quasi-linear function that represents his preferences normalized so that the utility of the zero bundle be equal to zero, is at least as large as his utility, divided by the number of agents, from consuming the social endowment.

Population-monotonic solutions do exist if preferences are appropriately restricted. Given $Q \in \mathcal{Q}$ and $e = (R_Q, \Omega) \in \mathcal{E}_{ql}^Q$, where R_i is represented by the function $u_i : \mathbb{R} \times \mathbb{R}_+^{\ell-1} \to \mathbb{R}$ such that $u_i(x_i, y_i) = x_i + v_i(y_i)$, consider the coalitional form game $w_e = (w_e(S))_{S \subseteq Q} \in \mathcal{G}^{|Q|}$ defined by : $w_e(S) = \max\{\sum_{i \in S}(x_i + v_i(y_i)) | \sum_{i \in S}(x_i, y_i) = \Omega\}$. This is the ***stand alone game*** associated with the economy. Finally, let $Sh^*(e)$ be the set of allocations $z = (x_i, y_i)_{i \in Q} \in Z(e)$ such that for all $i \in Q$, $x_i + v_i(y_i) = Sh_i(w_e)$, the i^{th} coordinate of the payoff vector chosen by the Shapley value for the game w_e.

To specify the restriction on the domain that will be useful, first say that two goods j and k are ***substitutes for the function*** $v_i : \mathbb{R}^{\ell-1} \to \mathbb{R}$ if for all $y_i \in \mathbb{R}_+^{\ell-1}$, and for all $a, b \in \mathbb{R}_+$, $v_i(y_i + be^k) - v_i(y_i) \geqq v_i(y_i + ae^j + be^k) - v_i(y_i + ae^j)$, where e^j denotes the j^{th} unit vector: this says that the marginal benefit of an additional unit of good k decreases as the consumption of good j increases. Writing the condition for $j = k$ means that the function v_i is concave in y_j. Also, a function v_i satisfies ***substitutability*** if any two goods are substitutes in v_i. Now, we have:

Proposition 7 (Moulin, 1992b) *On the domain of quasi-linear economies in which the social endowment of money is any non-negative number, and such that for each $S \subseteq Q$ the function $w_e(S)$ (see above) satisfies substitutability, the Shapley value Sh^* (used as a resource allocation rule) is population-monotonic.*

Note that on this domain, the Shapley value only satisfies the weak individual rationality condition described above. However, the stronger condition of individual rationality from equal division can be met together with *population-monotonicity.* Moulin (1990b) proposes a constructive algorithm producing such a solution[14].

If there is only one good in addition to money, the hypothesis of substitutability is equivalent to concavity of the function v_i. Moulin gives other examples of application of Proposition 7. For instance, the result holds if there are only two goods in addition to money, and each function v_i is concave and submodular over $\mathbb{R}_+^2$. It also applies in the ℓ-good case if each function v_i is twice continuously differentiable in the interior of $\mathbb{R}_+^{\ell-1}$, strictly concave, exhibits gross substitutability and has an infinite marginal utility of each good at 0.

Finally, we note that on the domain of public good economies (Section 8), the associated stand-alone game is convex without having to impose additional assumptions on preferences beyond quasi-linearity, so that the Shapley value is then a *population-monotonic*$_+$ solution (Moulin, 1990a).

[14]A disadvantage of the solution is that it does not respond well to changes in resources, in contrast with the Shapley value.

7 Fair allocation in economies with production

Generalizing even further, we turn next to economies with production. We first augment the description of an economy by a production set, to be interpreted as being jointly owned by everyone. To complete the specification of the model, we have several choices. One is to add a social endowment of goods that can be used as inputs or distributed for private consumption. The other and more interesting choice is to endow each agent with some amount of "time": time also can be used as an input or it can be consumed as a private good, in the form of leisure, but it is not transferable across agents. We will make the second choice, which raises a number of interesting issues. Therefore, an economy is a list (R_Q, ω_Q, Y), where (R_Q, ω_Q) are as in the specification of economies with individual endowments, and $Y \subseteq \mathbb{R}^\ell$ is a ***production set***. Let $\mathcal{P}^Q$ be the class of all economies so specified. A ***solution*** is a mapping defined on the union of all such $\mathcal{P}^Q$, where $Q \in \mathcal{Q}$, which selects for each economy a non-empty subset of its feasible set.

Here too, depending upon the nature of the technology and depending on agents' endowments, the arrival of new agents might be good news or bad news for the agents initially present and we will use *population monotonicity*, as well as *population monotonicity*$_+$, and *weak population-monotonicity.*

It is easy to see that the equal-opportunity equivalent solutions $\varphi_{\mathcal{B}}$ associated with monotonic families $\mathcal{B}$ of choice sets as described earlier (Section 6.1) are still well-defined here. They continue to be *population-monotonic* when production sets are convex, and they are *weakly population-monotonic* in general.

Moulin (1988, 1990c,d) studies the considerably weaker form taken by *population-monotonicity* when the small group contains only one agent, under the name of ***free access upper bound***. The one-person components of all efficient solutions agree, so that instead of pertaining to the comparison of the recommendations made by a solution for two economies of different sizes, the axiom essentially reduces to a "one-economy axiom" (just as individual rationality from equal division in the exchange case). Therefore, the following negative result in which, since agents may be differentially endowed, the no-envy requirement should be understood to apply to trades, is all the more disapointing: on the class of one-input, one-output economies with concave production functions, there is no subsolution of the no-envy (for trades) and Pareto solution satisfying the *free access upper bound* (Moulin, 1990c). However, when the no-envy requirement is dropped, not only the *free access upper bound*, but in fact *population-monotonicity* itself can be met, in particular by the ***constant returns-to-scale equivalent solution***, defined as follows: given $e = (R_Q, \omega_Q, Y)$, this solution selects any $z \in Z(e)$ such that for some reference constant returns-to-scale technology, and for all $i \in Q$, $z_i I_i z_i^*$, where z_i^* maximizes R_i under the assumption that agent i has access to that reference technology.

Proposition 8 (Moulin, 1990c) On the class of one-input, one-output production economies with concave production functions, the constant returns-to-scale equivalent solution satisfies *population-monotonicity*[15].

In the case of quasi-linear economies in which the social endowment of money is equal to 0, Moulin (1990c) notes the existence of a selection from the Pareto solution satisfying *population-monotonicity* and the ***identical preferences upper bound***: no agent is better off than at the allocation that would be chosen if all others had preferences identical to his, under the requirements of efficiency and "equal treatment of equals".

In economies with increasing returns-to-scale technologies, there is no selection from the Pareto solution satisfying no-envy (for trades) and the requirement that all agents prefer what they receive to the best they could achieve if given free access to the technology, a requirement which, in view of our earlier terminology, it is natural to call the ***free access lower bound*** (Moulin, 1990d). This requirement is a special case of *population-monotonicity*$_+$. However, the constant returns-to-scale equivalent solution is a *population-monotonic*$_+$ selection from the Pareto solution (Moulin, 1990c).

On the domain of economies such that the marginal rate of substitution between the input and the output increases along each ray through the origin, the "proportional benefit" solution (Roemer and Silvestre, 1992), which selects the efficient allocation such that all consumptions are proportional to each other, is single-valued and *population-monotonic* (Fleurbaey, 1992c).

[15] It is in fact the only selection from the Pareto solution satisfying *technological-monotonicity* and the *free access upper bound*. See Maniquet (1993) for a formulation of an alternative bound of "Pareto domination of average cost equilibrium" and a discussion of its compatibility with *population- monotonicity*.

Fleurbaey (1992c) gives some insight about the shape of the newcomers' preferences that are likely to cause various solutions, in particular the equal-income Walrasian solution, to violate *population-monotonicity.*

8 Fair allocation in economies with public goods

Next, we consider economies with public goods. Here too, the implications of *population-monotonicity* (and of its variants) are not well understood. For the case of general preferences, a brief discussion appears in Thomson (1987c) who observes that certain egalitarian type solutions have the property. We also have the following positive result for the case of one public good.

Proposition 9 (Moulin, 1992a) The selection from the egalitarian-equivalence and Pareto solution obtained by requiring the reference bundle to be proportional to the unit vector corresponding to the public good is *population-monotonic*$_+$.

We have already noted that on the class of quasi-linear economies in which the social endowment of money is 0, the Shapley value used as a resource allocation rule and applied to the stand-alone game associated with each economy is a *population-monotonic* selection from the Pareto solution (Section 6.5; Moulin, 1990a). On that domain, and applying the concept of *population-monotonic payoff configuration* of cooperative games (Section 6.5), Moulin (1990a) also shows the existence of what could be called ***population-monotonic*$_+$ *allocation configurations*.** The component allocations are selected from the Pareto solution and meet the *identical preferences upper bound.*

Consider now the following model of an economy with a public "bad": there is one private good which is produced according to a technology with an input consumed at the same level by all agents (think of a productive activity that creates pollution). Moulin (1990d) notes that for such a model, the selection from the egalitarian-equivalence and Pareto solution obtained by requiring the reference bundle to be proportional to the unit vector corresponding to the public good is *population-monotonic*$_+$. Also, a selection from the Pareto solution satisfying *population-monotonicity*$_+$ and the *identical preferences upper bound* (see above) can be defined as follows: Given a parameter $\lambda \in \mathbb{R}_+$, determine for each agent the highest welfare he could obtain subject to the condition that x_i units of the input would give him $f(\lambda x_i)$ units of the output, where f is the production function. Then, select the efficient allocation such that for some λ, each agent's welfare be equal to his welfare at the solution of this maximization exercise (Moulin, 1990c).

9 Fair allocation in economies with indivisible goods

We now turn to the problem of allocating jobs and salaries among workers with equal seniorities and qualifications. The jobs are not identical and the workers' preferences for the various job-salary packages differ. Each job is to be assigned to only one worker and the sum of the salaries is not to exceed a certain budget. How should the job-salary packages be defined and assigned?

The formal model is as follows. There is a group $Q \in \mathcal{Q}$ of agents and a collection A of ***objects***. An amount $\Omega \in \mathbb{R}_+$ of an infinitely divisible good, called ***money***, is also available for distribution. Each agent $i \in Q$ has a preference relation R_i defined over the space $A \times \mathbb{R}$. It is strictly monotonic in its second argument and such that for all α, $\beta \in A$ and for all $m \in \mathbb{R}$, there is $m' \in \mathbb{R}$ such that $(\beta, m')R_i(\alpha, m)$. Let $\mathcal{R}_{ind}$ be the class of all such preference relations. Each agent should receive at most one object. An allocation is a pair $z = (\sigma, m)$ of a function $\sigma : Q \to A$ specifying which object each agent receives, and a vector $m \in \mathbb{R}^A$ such that $\sum_{\alpha \in A} m_\alpha = \Omega$ specifying how much money is associated with each object. The bundle received by agent $i \in Q$ at z is $(\sigma(i), m_{\sigma(i)})$. A ***problem of fair allocation with indivisible goods*** is a triple $(R_Q, A, \Omega) \in \mathcal{R}_{ind}^Q \times A \times \mathbb{R}_+$, where $Q \in \mathcal{Q}$. Let $\mathcal{I}^Q$ be the class of these problems. A ***solution*** is a correspondence that associates with every $e = (R_Q, A, \Omega) \in \mathcal{I}^Q$, where $Q \in \mathcal{Q}$, a non-empty subset of the feasible set of e.

In Figure 4, along each of the axes, indexed by the jobs, is measured the salary that is associated with the corresponding job. To keep track of which job-salary combinations an agent finds indifferent to each other, we connect them by an "indifference curve". A few sample indifference curves

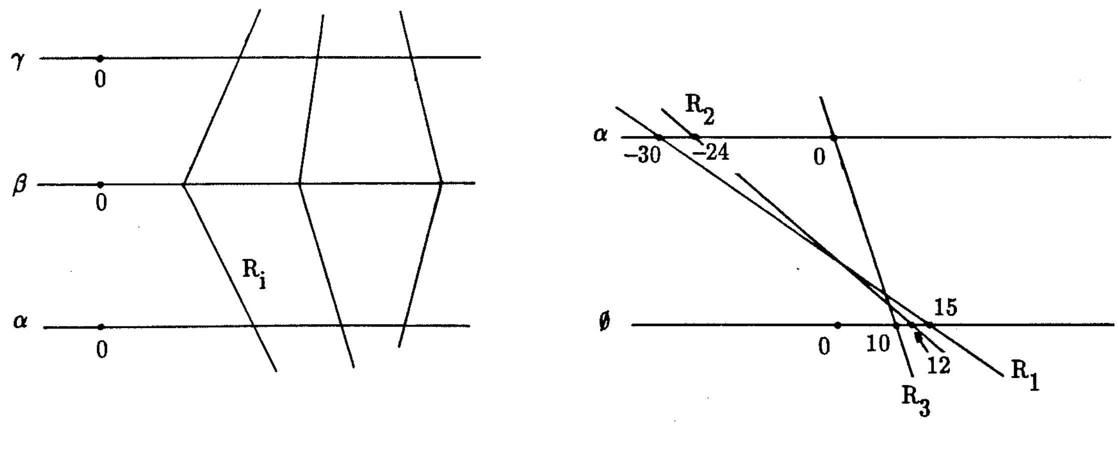


Figure 4: The allocation of indivisible objects when monetary compensations are feasible. (a) Representation of preferences. (b) Upon the arrival of agents 1 and 2, agent 3 may gain.

are indicated for agent 1. The notion of an envy-free allocation applies to this situation just as well as in the previous section. Under the above assumptions, envy-free allocations exist. In fact, there usually is a continuum of them and here too, a natural question is how to make selections from this continuum.

A special case of the model just described is when the objects are all identical. For instance, consider the allocation of jobs on an assembly line when there are more workers than jobs, all extra workers being allocated the "null" object, denoted $\emptyset$, which corresponds to unemployment. Finally, we have the even more special situation in which there is a single object and some money to allocate. To represent preferences in either one of these two situations, we need only two axes, one indexed by the object, which will be received by the "winners", or "the unique winner" in the one-object case, and the other indexed by the "null" object, which will be received by all the other agents, the "losers". For the winners not to envy each other, when there are several of them, they should receive the same amount of money. Similarly, for the losers not to envy each other, they should also receive the same amount of money.

9.1 Population-monotonicity in the one-object case

We start with the one-object case. First, we note that in that case *population-monotonicity* is incompatible with no-envy, even when preferences are quasi-linear (Alkan, 1988, Moulin, 1990b).

Example (Moulin, 1990c). See Figure 4b. Let $Q = \{1,2,3\}$, $A = \{\emptyset, \alpha\}$

and $\Omega = 0$. Agents have quasi-linear preferences such that $(\emptyset, 45)I_1(\alpha, 0)$, $(\emptyset, 36)I_2(\alpha, 0)$, and $(\emptyset, 10)I_3(\alpha, 0)$. Let $e = (R_Q, A, \Omega)$ and $z \in F(e)$. Then, efficiency requires that agent 1 gets the object. Agents 2 and 3 both get the null object and t units of money satisfying $12 \leqq t \leqq 15$. Thus, the worst bundle agent 3 receives is $(\emptyset, 12)$ and he prefers it to $(\alpha, 0)$, the bundle that he would receive if he were alone.

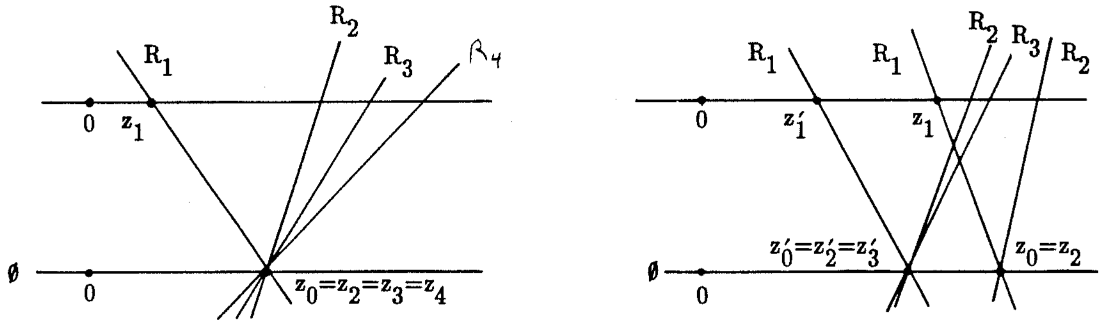
In the example, agent 3 is better-off at any envy-free allocation of the three-person economy than if he were alone, so that in fact a violation of the counterpart of the *free access upper bound* introduced earlier (see Section 7) is unavoidable if no-envy is insisted upon.

However, since here consumption spaces are unbounded below, it may be unnatural to require agents to lose when new agents come in. This is because the model is essentially equivalent to a production model. Receiving the object is similar to being given a chance to produce "utility" by using the object. When new agents come in with "good" production functions, they may be able to use the object very productively and the agents originally present may be made to benefit from it. To be ready to deal with that case and with the case when the new agent has a poor production function, we return to the condition of *weak population-monotonicity*.

Then, consider the solution φ^* that systematically selects the envy-free allocation that is the least favorable to the winner, as illustrated in Figure 5. At this allocation, the winner's indifference curve through his bundle passes through the losers' common bundle. It is easy to see that this solution is *weakly population-monotonic*.

The solution φ^* is a selection from the egalitarian-equivalence solution introduced in Section 6.1 (It is because there is only one object that egalitarian-equivalence is compatible with no-envy[16].) Moreover, a characterization of the solution can be obtained on the basis of *weak population-monotonicity*. To formally state the result, we need the following very mild condition of ***neutrality***: if an allocation obtained by exchanges of bundles from one that is chosen by the solution leaves unaffected the welfares of all agents, then it is also chosen by the solution. We will also use the condition of ***translation invariance*** which says that if all the preference maps are translated by some amount $t \in \mathbb{R}$, and the social endowment of money is changed by t times the

[16]In general, the two distributional requirements of no-envy and egalitarian-equivalence are incompatible (Thomson, 1990).



(a) (b)

Figure 5: A weakly population-monotonic selection from the no-envy solution in the one-object case. (a) Definition of the solution: it picks the allocation that is the worst for the winner in the set of envy-free allocations, so that all indifference curves pass through a common "reference bundle". (b) Its *weak population-monotonicity*: as the number of agents changes, the reference bundle moves to the left or to the right for all agents. In the example, as agent 3 arrives, the reference bundle moves from z_0 to z_0'.

number of agents, then the recommended bundle for each agent is obtained from his old one by increasing its money component by t.

Theorem 10 (Tadenuma and Thomson, 1993) In the one-object case, the solution φ^* is the only *weakly population-monotonic*, *neutral* and *translation invariant* selection from the no-envy solution.

In economies with indivisible goods, there is no meaning to "equal division" but a particularly useful distributional requirement is obtained by insisting that each agent be made at least as well off as at the only envy-free allocation that would exist in an economy in which all agents had the same preference as his; this is the *identical preferences lower bound* encountered earlier. In the two-person case, meeting this bound is actually the same condition as no-envy, but if there are more than 2 agents, the *identical preferences lower bound* is weaker than no-envy. The next few results pertain to economies in which the object is a "good": an agent would always need to be compensated to give it up. Then, in economies with quasi-linear preferences in which the social endowment of money is 0, Moulin (1990c) shows that the Shapley value applied to the associated stand alone game is a *population-monotonic* selection from the Pareto solution that meets the *identical preferences lower bound* (this solution differs from φ^*.) A generalization of this result to the multiple object case appears below. Another generalization, to economies where preferences are not necessarily quasi-linear, (and under the assumption that the social endowment of money is any positive number,) is given by Bevia (1992), who constructs an extension of this solution having these same properties.

9.2 Population-monotonicity in the multiple-object case

In the multiple-object case, the selection from the egalitarian-equivalence and Pareto solution obtained by requiring the reference bundle to contain a fixed object is *weakly population-monotonic* but it is not guaranteed to be a selection from the no-envy solution anymore (Thomson, 1990). In fact, if no-envy is insisted upon, we have the following impossibility which even holds on the quasi-linear domain:

Theorem 11 (Tadenuma and Thomson, 1992) In the multiple-object case, there is no *weakly population-monotonic* selection from the no-envy solution.

The following positive result is available however :

Proposition 10 (Moulin, 1992b) In the multiple-object case, if preferences are quasi-linear and the social endowment of money is non-negative, the Shapley value, applied to the stand alone game associated to each economy (see section 6.5), is a *population-monotonic* selection from the Pareto solution[17].

Finally, we turn to the more general case when each agent can be assigned several objects. A general analysis of this case has been carried out by Bevia (1993).

Proposition 11 (Bevia, 1993) Consider the multiple-object case when each agent can be assigned several objects. Then, even if preferences are quasi-linear and if the social endowment of money is 0, there is no *population-monotonic* selection from the Pareto solution. However, if preferences are quasi-linear and satisfy the counterpart of the substitutability assumption

[17]This result should be compared to Proposition 7.

of section 6.5., and if the social endowment of money is non-negative, the Shapley-value applied to the stand alone game associated with each economy defines a *population-monotonic* selection from the Pareto solution.

9.3 Locally extendable allocations

Theorem 11 shows that *weak population-monotonicity* is a very strong requirement in the present context and it is therefore natural to investigate the possibility of satisfying weaker requirements. Alkan (1989)'s contribution is along those lines. He asks the following question: For each given economy, is there ***some*** allocation such that the arrival of another agent can be made to affect all agents initially present in the same direction, and such that the departure of an agent can be made to affect all remaining agents in the same direction.

Locally extendable allocation. Let φ be a solution. Let $Q \in \mathcal{Q}$, $e = (R_Q, A, \Omega) \in \mathcal{I}^Q$, and $z \in \varphi(e)$. Let m_M be the maximal amount of money received by anyone at z. ***The solution φ permits the local upper-extendability of z*** if for any $i \in Q$, there is $z' \in \varphi(R_{Q\setminus\{i\}}, A, \Omega)$ such that (i) $z'_j P_j z_j$ for all $j \in Q\setminus\{i\}$ if $m_M > 0$, (ii) $z'_j I_j z_j$ for all $j \in Q\setminus\{i\}$ if $m_M = 0$, and (iii) $z_j P_j z'_j$ for all $j \in Q\setminus\{i\}$ otherwise. ***It permits the local lower-extendability of z*** if for all $i \notin Q$, there is $z' \in \varphi(R_{Q\cup\{i\}}, A, \Omega)$ such that (i) $z_j P_j z'_j$ for all $j \in Q$ if $z_j P_j(\emptyset, 0)$ for all $j \in Q$, (ii) $z'_j I_j z_j$ for all $j \in Q$ if $z_j I_j(\emptyset, 0)$ for all $j \in Q$, and (iii) $z'_j P_j z_j$ for all $j \in Q$ otherwise.

It turns out that the no-envy solution F permits only limited *local extendability.* The allocations that can be so extended are defined as follows : given $e = (R_Q, A, \Omega)$ and $z \in \varphi(e)$, define the ***welfare of agent $i \in Q$ at z*** to be the amount of money that by itself would constitute a bundle that the agent finds indifferent to z_i. Now, say that z is a ***maximin welfare allocation of $F(e)$*** if the agent with the lowest welfare at z has the highest possible welfare in $F(e)$. Also, say that z is a ***minimax money allocation of $F(e)$*** if the maximal amount of money received by anyone at z is the smallest among all allocations in $F(e)$.

Theorem 12 (Alkan, 1989) The no-envy solution permits the *local upper-extendability* of the minimax money allocation and the *local lower-extendability* of the maximin welfare allocation. It permits the *local upper-extendability* of the minimax money allocation only.

Alkan also shows that *local lower-extendability* is an easier requirement to meet than *local upper-extendability*: indeed there are economies in which the no-envy solution permits the *local lower-extendability* of all of its allocations.

Fleurbaey (1993) considers a version of this model in which the indivisible goods are interpreted as non-transferable talents or handicaps, and establishes the *population-monotonicity* of several solutions.

10 Fair division in private good economies with single-peaked preferences

Examples abound of activities that one enjoys up to a point, any time spent on it beyond that point decreasing one's overall satisfaction; in many cases, there may very well be a point beyond which one wishes that one would not have started at all. Consider such an activity to be divided up among the members of a team, and assume that the activity has to be completed[18]. How should this division be done?

Formally, there is a group $Q \in \mathcal{Q}$ of agents among whom to allocate $\Omega > 0$ units of an infinitely divisible commodity; for each $i \in Q$, R_i is agent i's continuous and ***single-peaked*** preference relation defined over $\mathbb{R}_+$ i.e., there is a number $p(R_i) \in \mathbb{R}_+$ such that for all x_i, $x'_i \in \mathbb{R}_+$, if $x'_i < x_i \leqq p(R_i)$ or $p(R_i) \leqq x_i < x'_i$, then $x_i P_i x'_i$. Let $\mathcal{R}_{sp}$ be the class of all such preference relations. A ***problem of fair division with single-peaked preferences*** is a pair $(R_Q, \Omega) \in \mathcal{R}^Q_{sp} \times \mathbb{R}_+$. For each $Q \in \mathcal{Q}$, let $\mathcal{S}^Q$ be the class of all problems involving the group Q. A ***solution*** associates with every $(R_Q, \Omega) \in \mathcal{S}^Q$, where $Q \in \mathcal{Q}$, a non-empty subset of the set of feasible allocations of (R_Q, Ω), $\{z \in \mathbb{R}^Q_+ | \sum_{i \in Q} z_i = \Omega\}$. Note that feasibility is defined with an equality sign, reflecting the fact that the commodity is not freely disposable. The axiomatic analysis of this class of problems was initiated by Sprumont (1991).

Efficiency is easily understood. If the amount to divide is larger than the sum of the preferred amounts — we will say that there is "too much" of the commodity — efficiency requires that each agent consumes more than he would prefer, and if the opposite holds — then, we will say that there is "not

[18]Another situation with an identical mathematical representation is rationing in a two-good economy.

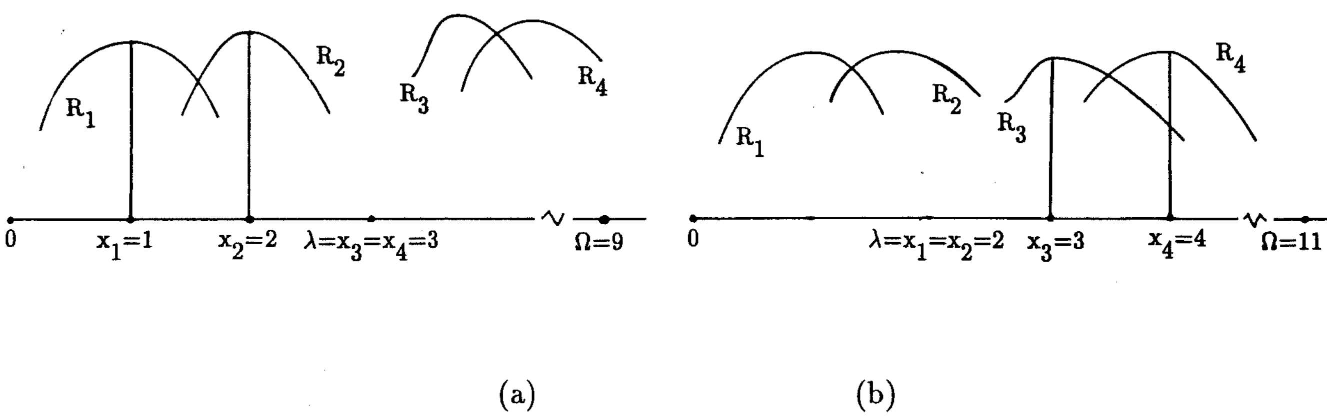


Figure 6: The uniform rule illustrated for $Q = \{1,2,3\}$. (a) The case $\Omega \leqq \sum_{i \in Q} p(R_i)$: each agent whose preferred consumption is smaller than λ gets his preferred consumption; each of the others gets λ. (b) The case $\sum_{i \in Q}(R_i) \leqq \Omega$: each agent whose preferred consumption is greater than λ gets his preferred consumption; each of the others gets λ.

enough" — efficiency requires that each agent consumes less than he would prefer. Fairness, in addition to efficiency, is one of our objectives and we will consider the same distributional requirements as in Section 6, no-envy and individual rationality from equal division. Just as in classical economies, there is a continuum of efficient allocations satisfying these distributional requirements and the question of selection arises.

An appealing selection is the ***uniform rule,*** introduced in the fix-price literature and recently characterized by Sprumont (1991) on the basis of incentive considerations. It is defined as follows: Let $Q \in \mathcal{Q}$ and $(R_Q, \Omega) \in \mathcal{S}^Q$. Then, $x \in \mathbf{R}_+^Q$ is the ***uniform allocation of*** $(\boldsymbol{R_Q}, \boldsymbol{\Omega})$ if when $\Omega \leqq \sum_{i \in Q} p(R_i)$, there is $\lambda \in \mathbf{R}_+$ such that for all $i \in Q$, $x_i = \min\{p(R_i), \lambda\}$, and when $\sum_{i \in Q} p(R_i) \leqq \Omega$, there is $\lambda \in \mathbf{R}_+$ such that for all $i \in Q$, $x_i = \max\{p(R_i), \lambda\}$, in each case λ being chosen so as to ensure feasibility. It is easy to see that the uniform rule selects efficient allocations that are both envy-free and individually rational from equal division. The rule is illustrated in Figure 6 for each of the two cases.

In this model, the arrival of additional agents may be either good news (in cases when there is too much of the good to begin with) or it may be bad news, and the natural property to consider is *weak population-monotonicity.* Unfortunately, the property is quite demanding. For instance, the equal division rule does not satisfy it. To see this, let $Q = \{1, 2\}$ and $\Omega = 6$;

then, equal division is (3,3). Now, let $Q' = \{1, 2, 3\}$; here, equal division is (2,2,2). If $p(R_1) = 2$ and $p(R_2) = 3$, (3,3) is worse for agent 1 and better for agent 2 than (2,2,2). If in addition $p(R_3) = 2$, the allocations $(3, 3)$ and $(2, 2, 2)$ are the uniform allocations of the economies (R_Q, Ω) and $(R_{Q'}, \Omega)$, so the example shows that the uniform rule is not *weakly population-monotonic* either.

These negative results extend much further, as stated in the next proposition, in which we use the requirement, under the name of ***symmetry*** , that identical agents be treated identically, reserving the term "anonymity" for the requirement that the solution be invariant under exchanges of the names of agents.

Theorem 13 (Thomson, 1991) There is no *weakly population-monotonic* selection from the no-envy solution, nor from the individual rationality from equal division solution. Also, there is no *weakly population-monotonic* and *symmetric* solution that depends only on preferred consumptions.

These are disappointing results. Note in particular that they obtain even though efficiency is not required. However, if the distributional requirements of no-envy and individual rationality from equal division are dropped, *anonymous* and *weakly population-monotonic* selections from the Pareto solution can be found on large subdomains of the primary domain. For example, certain solutions based on equating "sacrifices" as measured by the size of upper contour sets satisfy the property. These "egalitarian" type solutions do provide appealing ways of solving the problem.

Moreover, upon close examination of the proofs of the negative results stated in Theorem 13, one discovers that they involve comparing economies where initially there is too much of the commodity and after the arrival of the new agents there is not enough of it, or conversely. This naturally suggests limiting one's attention to situations in which changes in the population are not so disruptive, that is, situations where there is too much before ***and*** after, or there is not enough before ***and*** after:

One-sided population-monotonicity: For all Q, $Q' \in \mathcal{Q}$ with $Q' \subset Q$, for all $(R_Q, \Omega) \in \mathcal{S}^Q$, if $\sum_{i \in Q'} p(R_i) \leqq \Omega$ or if $\sum_{i \in Q} p(R_i) \geqq \Omega$, then $\varphi_i(R_Q, \Omega) R_i \varphi_i(R_{Q'}, \Omega)$ for all $i \in Q$.

The property retains a wide range of relevance and fortunately it is satisfied by a number of interesting solutions, including the uniform rule, the proportional rule (which allocates the commodity in proportion to the preferred amounts), and others. However, among these, the uniform rule is close to being the only one to satisfy the no-envy requirement, as stated in the next theorem, which also makes use of the requirement of ***replication-invariance***: if an allocation is chosen for some economy, then for any order of replication, the replica allocation is chosen for the replica economy. This is a weak requirement, being satisfied by most of the solutions that have been proposed for this model.

For this next result, we impose on preferences the requirement that there be a finite consumption indifferent to the zero consumption.

Theorem 14 (Thomson, 1991) The uniform rule is the only *replication-invariant* and *one-sided population-monotonic* selection from the no-envy and Pareto solution.

11 Public decision in economies with single-peaked preferences

The public good version of the model discussed in the previous section is examined by Ching and Thomson (1993). In brief, there is an interval $[0, \Omega]$ of possible levels of a public good, all potential agents having single-peaked preferences over it. A ***solution*** associates with each profile of preferences a single level of the public good.

It is easy to check that the following solutions, indexed by the parameter $a \in [0, \Omega]$, are *population-monotonic*: given some profile of preferences, select the level a if it is efficient (that is, if it is between the smallest and the largest preferred levels in the profile of preferences); if not, select the preferred level in the profile the closest to a (note that the parameter a is required to be the same for all cardinalities). These solutions constitute a subfamily of a family introduced by Moulin (1984) and characterized by him on the basis of strategy-proofness. Let C_a be the solution associated with the parameter a.

Theorem 15 (Ching and Thomson, 1992) The solutions $\{C_a | a \in [0, \Omega]\}$ are the only *population-monotonic* selections from the Pareto solution.

Since here the only good present is a public good and the feasible set does not change as the number of agents enlarges (this model should be compared with the model of section 8), there is no means of compensating an agent for any change in the chosen level that the arrival of newcomers might cause. As a result, *population-monotonicity* is quite a strong condition.

12 Conclusion

Although the principle of *population-monotonicity* is well understood in some of the models in which it has been investigated, not much is known about its implications in a number of other important contexts. It is usually easy to find out whether a given solution does or does not satisfy the property. However, when it comes to characterizing the class of well-behaved solutions that satisfy the property, relatively little has been accomplished. We hope that this review will stimulate the search for answers to the numerous questions that are still open. To help in this work, we suggest that the analysis of *population-monotonicity* (or its variants *population-monotonicity*$_+$ and *weak population-monotonicity*) in a given model should have the following components:

A. Identifying *population-monotonic* solutions. In particular, finding out whether the most widely used solutions for the model satisfy the property.

- If such solutions do exist, characterizing all of them. Of course, the class of *population-monotonic* solutions might be large, and a characterization possible only if the solutions are also required to satisfy minimal requirements of efficiency and distribution.

- Studying the compatibility of the property with other properties of interest.

- Formulating criteria that would help evaluate how well *population-monotonic* solutions do perform the job and compare them on that basis. Obtaining *population-monotonicity* is only the first step. In a second step, one may want to ensure that sacrifices (or gains) not only be in the same direction but also be distributed "evenly" (see the notion of *guarantee structure* used in bargaining theory).

B. If *population-monotonic* solutions do not exist on the primary domain of interest, identifying interesting domain restrictions that would help recover existence (Quasi-linearity of preferences has been a useful restriction in the study of exchange economies and in the case of indivisible goods; in the latter case, so has allowing only one object. In production economies, allowing only two goods has been a useful assumption.)

- Formulating weaker monotonicity requirements:

(i) Conditions involving a given problem and all of its subproblems instead of a general class of problems (see the notion of a *population-monotonic* payoff configuration used in the context of coalitional form games).

(ii) Conditions involving only unit changes in the population (see the notion of *local upper-* or *lower-extendability* introduced in economies with indivisible goods).

(iii) Conditions based on comparing what each agent gets in some problem to the average of what he gets in the subproblems[19].

(iv) Conditions based on specifying an order in which new agents arrive. When the space of characteristics is endowed with an order structure, limiting one's attention to situations where agents' arrivals are in agreement with that order might be interesting.

(v) Finally, special features of the model might be relevant in formulating conditions restricting the applicability of the condition (as we saw on the single-peaked domain).

- Formulating criteria that would help evaluate how far from *population-monotonic* a given solution may be. If some agents gain when they should lose, taking the maximal (or average) gain they might incur could provide the basis for the comparisons of solutions.

[19]Similarly to the way Maschler and Owen had suggested weakening *consistency*.

13 References

Alkan, A., Private communication, 1988.

________, "Monotonicity and fair assignments," Bogaziçi University mimeo, 1989.

________, G. Demange, and D. Gale, "Fair allocation of indivisible objects and criteria of justice," **Econometrica** 59 (1991), 1023-1039.

Bennett, E., "The aspiration approach to predicting coalition formation and payoff distribution in sidepayment games," **International Journal of Game Theory** 12 (1983), 1-28.

Bevia, C., "Equal split guarantee solution in economies with indivisible goods. Consistency and population monotonicity", University of Alicante mimeo, 1992.

________"Fair allocation in a general model with indivisible goods," University of Alicante mimeo, 1993.

Chichilnisky G. and W. Thomson, "The Walrasian mechanism from equal division is not monotonic with respect to variations in the number of agents," **Journal of Public Economics** 32 (1987), 119-124.

Ching, S. and W. Thomson, "Population-monotonic solutions in public good economies with single-peaked preferences", University of Rochester mimeo, 1993.

Chun, Y., University of Rochester notes (1985).

________, "The solidarity axiom for quasi-linear social choice problems," **Social Choice and Welfare** 3 (1986), 297-320.

________ and W. Thomson, "Monotonicity properties of bargaining solutions when applied to economies," **Mathematical Social Sciences** 15 (1988), 11-27.

________ and ________, "Bargaining solutions and relative guarantees," **Mathematical Social Sciences** 17 (1989), 285-295.

________ and ________, "Bargaining problems with claims," **Mathematical Social Sciences** 24 (1992a), 19-33.

Fleurbaey, M., "Reward patterns of fair division," University of California — Davis working paper no. 398, June 1992b.

____________, "Preference responsability and monotonicity in fair division", INSEE mimeo, 1992c.

____________, "Three solutions for the compensation problem", INSEE mimeo, May 1993, forthcoming in the **Journal of Economic Theory**.

Foley, D., "Resource allocation and the public sector," **Yale Economic Essays** 7 (1987), 45-98.

Grafe, F., E. Iñarra and J.M. Zarzuelo, "On externality games", SEEDS discussussion paper 102, December 1992.

Hart, S., "An axiomatization of Harsanyi's nontransferable utility solution," **Econometrica** 53 (1985), 1295-1313.

Ichiishi, T., "The cooperative nature of the firm", Ohio State University mimeo, 1988.

Jones, R., "The population monotonicity property and the transfer paradox," **Journal of Public Economics** 32 (1987), 125-132.

Kalai, E., "Proportional solution to bargaining problems: interpersonal utility comparisons," **Econometrica** 45 (1977), 1023-1030.

________ and M. Smorodinsky, "Other solutions to Nash's bargaining problem," **Econometrica** 43 (1975), 513-518.

Littlechild, S.C., "A simple expression for the nucleolus in a special case", **International Journal of Game Theory** 3 (1974), 21-29.

______________, and G. Owen, "A simple expression fo the Shapley value for a special case", **Management Science** 20 (1973), 370-372.

Maniquet, F., "Allocation rules for a commonly owned technology: the average cost lower bound", University of Namur discussion paper, 1993.

Maschler, M. and G. Owen, "The consistent Shapley value for hyperplane games", **International Journal of Game Theory** 18 (1989), 389-406.

Moulin, H., "Generalized Condorcet-winners for single-peaked and single-plateau preferences", **Social Choice and Welfare** 1 (1984), 127-147.

__________ "The separability axiom and equal sharing methods," **Journal of Economic Theory** 36 (1985a), 120-148.

__________, "Egalitarianism and utilitarianism in quasi-linear bargaining," **Econometrica** 53 (1985b), 49-67.

__________, "Equal or proportional division of a surplus, and other methods," **International Journal of Game Theory** 16 (1987), 161-186.

__________, **Axioms of cooperative decision-making**, Cambridge University Press, Cambridge, 1988.

__________, "Cores and large cores when population varies," **International Journal of Game Theory** 19 (1990a), 219-252.

__________, "Joint ownership of a convex technology: comparisons of three solutions", **Review of Economic Studies** 57 (1990b), 439-452.

__________, "Fair division under joint ownership : recent results and open questions" **Social Choice and Welfare** 7 (1990c), 149-170.

__________, "Interpreting common ownership", **Recherches Economiques de Louvain** 56 (1990d), 303-326.

__________, "All sorry to disagree : a general principle for the provision of non-rival goods", **Scandinavian Journal of Economics** (1992a), 37-51.

__________, "An application of the Shapley value to fair division with money," **Econometrica** 60 (1992b), 1331-1349.

__________ and W. Thomson, "Can everyone benefit from growth? Two difficulties", **Journal of Mathematical Economics**, 17 (1988), 339-345.

Nash, J.F., "The bargaining problem," **Econometrica** 18 (1950), 155-162.

Pazner, E. and D. Schmeidler, "Egalitarian equivalent allocations: A new concept of economic equity," **Quarterly Journal of Economics** 92 (1978), 671-687.

Polterovich, V, and V. Spivak, "Gross substitutability of point-to-set correspondences", **Journal of Mathematical Economics** 92 (1983), 117-140.

Roemer, J., "The mismarriage of bargaining theory and distributive justice," **Ethics** 97 (1986a), 88-110.

__________, "Equality of resources implies equality of welfare," **Quarterly Journal of Economics** 101 (1986b), 751-784.

__________, "Axiomatic bargaining theory on economic environments," **Journal of Economic Theory** 45 (1988), 1-31.

__________ and J. Silvestre, "The proportional solution for economies with both private and public ownership", **Journal of Economic Theory** (1992), 426-444.

Rosenthal, E., "Monotonicity of solutions in certain dynamic cooperative games," **Economics Letters** 34 (1990a), 221-226.

__________, "Monotonicity of the core and value in dynamic cooperative games" **International Journal of Game Theory** 19 (1990b), 45-57.

Schmeidler, D., "The nucleolus of a characteristic function game", **SIAM Journal of Applied Mathematics** 17 (1969), 1163-1170.

Shapley, L., "A value for n-person games", in **Contributions to the theory of games II**, Annals of Mathematics Studies 28, (H. Kuhn and A.W. Tucker, eds), Princeton University Press, Princeton, NJ, 1953, 307-317.

Sharkey, W., “Convex games without side-payments”, **International Journal of Game Theory** 10 (1981), 101-106.

Sönmez, T., “Population-monotonicity of the nucleolus on a class of public good problems”, University of Rochester mimeo, 1993.

Sonn, S., “A note on Sprumont’s characterization”, University of Rochester mimeo, 1990.

Sprumont, Y., “Population-monotonic allocation schemes for cooperative games with transferable utility,” **Games and Economic Behavior** 2 (1990), 378-394.

____________, “The division problem with single-peaked preferences: A characterization of the uniform allocation rule,” **Econometrica** 59 (1991), 509-519.

Tadenuma, K. and W. Thomson, “Solutions to the problem of fair allocation in economies with indivisible goods,” University of Rochester mimeo, 1992, forthcoming in **Theory and Decision**.

____________ and ____________,“The fair allocation of an indivisible good when monetary compensations are possible,” **Mathematical Social Sciences** 25 (1993), 117-132.

Thomson, W., “The fair division of a fixed supply among a growing population,” **Mathematics of Operation Research** 8 (1983a), 319-326.

____________, “Problems of fair division and the egalitarian solution,” **Journal of Economic Theory** 31 (1983b), 211-226.

____________, “Collective guarantee structures,” **Economics Letters** 11 (1983c), 63-68.

____________, “Truncated egalitarian solutions,” **Social Choice and Welfare** 1 (1984a), 25-32.

____________, “Monotonicity, stability and egalitarianism,” **Mathematical Social Sciences** 4 (1984b), 15-28.

____________, “Individual and collective opportunities,” **International Journal of Game Theory** 16 (1987a), 245-252.

____________, “Monotonic allocation rules,” University of Rochester mimeo, 1987b, revised August 1993.

____________, “Monotonic allocations rules in economies with public goods,” University of Rochester mimeo, 1987c, revised August 1993.

____________, “Notions of equal, and equivalent, opportunities”, University of Rochester mimeo, 1987d, revised April 1993, forthcoming in **Social Choice and Welfare**.

____________, "A study of choice correspondences in economies with a variable number of agents", **Journal of Economic Theory** 46 (1988), 237-254.

____________, "On the non-existence of envy-free and egalitarian-equivalent allocations in economies with indivisibilities", **Economics Letters**, 34 (1990), 227-229.

____________, "Population-monotonic solutions to the problem of fair division when preferences are single-peaked," University of Rochester discussion paper (1991, revised June 1992), forthcoming in **Economic Theory**.

____________, "Consistent allocation rules", University of Rochester mimeo, 1993.

____________ and T. Lensberg, "Guarantee structures for problems of fair division," **Mathematical Social Sciences** 4 (1983), 205-218.

Tijs, S.,"Bounds for the core and the τ-value", in **Game Theory and Mathematical Economics** (O. Moeschlin and Pallaschke, Eds.) North-Holland, Amsterdam, 1981, 123-132.