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Abstract

We consider the problem of allocating a single infinitely divisible commodity to agents with single-peaked preferences, and establish two properties of the rule that has played the central role in the analysis of this problem, the uniform rule. Among the efficient allocations, it selects (1) the one at which the difference between the largest amount received by any agent and the smallest such amount is minimal, and (2) the one at which the variance of the amounts received by all the agents is minimal. We also show that an important solution for bankrupcty problems, the constrained equal-award solution, can be characterized by analogous minimization exercises, subject to different constraints, and that a limited form of these results holds for another central solution, the Talmudic solution.

1 Introduction

We consider the problem of allocating a single infinitely divisible commodity to agents with single-peaked preferences. In previous studies of the problem, a solution known as the "uniform rule" was shown to satisfy a number of interesting properties, and several characterizations of the rule were established on the basis of the properties. Here, we point out two additional properties of the rule: among the efficient allocations, it selects (1) the unique allocation at which the difference between the largest amount received by any agent and the smallest such amount is smallest, and (2) the unique allocation at which the variance of the amounts received by all the agents is the smallest.

Similar results hold for the extension of the uniform rule to the class of economies obtained by specifying an individual endowment for each agent as well as an additional term interpreted as a collective obligation to or from the outside world. On this class, it is the transfers from the agents' endowments that are considered in specifying the two minimization exercises.

These results can be seen as "computational characterizations" of the uniform rule, but they can also be interpreted from the normative viewpoint, since they say in two different ways that agents should not get amounts that are too different from each other. In a related paper, de Frutos and Massó (1995) give a characterization of the uniform allocation as the only minimizer of the Lorenz ordering over the efficient set.

We also briefly discuss the problem of dividing the net worth of a bankrupt firm among its creditors (O'Neill, 1982), and show that a solution that has been extensively discussed for this class of problems can be described in parallel terms. The "constrained equal-award solution" is obtained by two similarly defined minimization exercises, subject to the constraint that no creditor receives more than his claim, as is natural to require. We also point out that another important solution for this class of problems, known as the Talmudic solution (Aumann and Maschler, 1985), is obtained by such minimization exercises, subject to the additional requirement that the problem of allocating the worth of the firm be considered in a symmetric way as the problem of allocating the amount of the agents' claims that they cannot receive, but only in "half" of the class of problems.

2 The model

Let $N = \{1, \ldots, n\}$ be a set of agents. Each $i \in N$ is equipped with a ***single-peaked*** preference relation over $\mathbb{R}_+$, R_i, with associated strict relation P_i: this means that there exists a non-negative number, denoted by $p(R_i)$, such that for all $x_i, x_i' \in \mathbb{R}_+$ if $x_i < x_i' < p(R_i)$ or $x_i > x_i' > p(R_i)$ then $x_i' P_i x_i$. Let $\mathcal{R}$ be the class of single-peaked preferences over $\mathbb{R}_+$, and $\mathcal{R}^n = \prod_N \mathcal{R}$. Let $M \in \mathbb{R}_+$ be a social endowment of an infinitely divisible commodity, to be divided among the members of N. An ***economy*** is a pair $e = (R, M) \in \mathcal{R}^n \times \mathbb{R}_+$. Let $\mathcal{E}^n$ denote the class of all economies.

A ***feasible allocation for*** $\boldsymbol{e = (R, M) \in \mathcal{E}^n}$ is a vector $x \in \mathbb{R}^n_+$ satisfying $\sum x_i = M$. Let $\boldsymbol{X(e)}$ denote the set of feasible allocations for e.

A feasible allocation, $x \in X(e)$, is (Pareto-)***efficient for e*** if there is no $x' \in X(e)$ such that for all $i \in N$ we have $x_i' R_i x_i$, and for some $j \in N$ we have $x_j' P_j x_j$. Let $P(e)$ denote the set of efficient allocations for e.

A ***solution*** is a function[1] $\varphi \colon \mathcal{E}^n \longrightarrow \mathbb{R}^n$ such that for all $e \in \mathcal{E}^n$, $\varphi(e) \in Z(e)$. Arguably, the most important solution is the uniform rule:

Definition Given $e = (R, M) \in \mathcal{E}^n$, the allocation $x \in X(e)$ is the ***uniform allocation of e*** if (i) when $\sum p(R_i) \geq M$, there is $\lambda \in \mathbb{R}_+$ such that $x_i = \min\{p(R_i), \lambda\}$ for all $i \in N$, and (ii) when $\sum p(R_i) \leq M$, there is

[1]Some authors have allowed solutions to be correspondences.

$\lambda \in \mathbb{R}_+$ such that $x_i = \max\{p(R_i), \lambda\}$ for all $i \in N$. The ***uniform rule*** selects for each economy its uniform allocation.

The first axiomatic analysis of this model is due to Sprumont (1991), who showed that the uniform rule is essentially the only efficient rule to be strategy-proof. The model has since been the object of a number of additional studies (Barberà, Jackson, and Neme, 1995; Ching, 1992, 1994; Gensemer, Hong, and Kelly, 1992, 1996; Otten, Peters, and Volij, 1996; Sönmez, 1994; Thomson, 1990, 1992a, 1994a,b, 1995), many of which have led to additional characterizations of the uniform rule based on normative properties of monotonicity and consistency on the one hand, and incentive-compatibility properties of strategy-proofness and implementability on the other.

3 The results

Our first result is a characterization of the uniform allocation of any problem as the unique minimizer among efficient allocations of the difference between the largest and smallest amounts received among the agents.[2]

Proposition 1 *For any economy, the difference between the largest amount received by any agent and the smallest such amount is strictly smaller at the uniform allocation than at any other efficient allocation.*

Proof: Let $e = (R, M) \in \mathcal{E}^n$. Define the ***range*** function for allocations, $r: \mathbb{R}^n_+ \to \mathbb{R}_+$, by $r(x) \equiv \max_i x_i - \min_i x_i$. Since $P(e)$ is compact and r is continuous, there exists $x \in \arg\min_{y \in P(e)} r(y)$. Suppose by contradiction that $x \neq U(e)$. We assume $\sum p(R_i) \geq M$, as the proof for the other case is similar.

[2]This result first appears in Thomson (1990).

Let $y^{\max} = \max_j x_j$. Since $x \neq U(e)$, there exists $i \in N$ such that $x_i < y^{\max}$ and $x_i < p(R_i)$. Let $\delta = \min\{p(R_i), y^{\max}\} - x_i$. Let $N' = \{j \in N : x_j = y^{\max}\}$.

Define $z \in X(e)$ as follows: $z_i = x_i + \delta/2$, for all $j \in N'$ $z_j = x_j - \delta/(2|N'|)$, and for all $k \notin N' \cup \{i\}$, $z_k = x_k$. It is straightforward to check that $z \in P(e)$ and $r(z) < r(x)$, contradicting our choice of x. Q.E.D.

Our second result is a characterization of the uniform allocation of any economy as the unique minimizer among all efficient allocations of the variance of the amounts received by all the agents.

Proposition 2 *For any economy, the variance of the amounts received by all the agents at the uniform allocation is strictly smaller than at any other efficient allocation.*

Proof: Let $e = (R, M) \in \mathcal{E}^n$. Define the ***variance*** function for allocations, $v: \mathbb{R}^n_+ \to \mathbb{R}_+$, by $v(x) \equiv \frac{1}{n}\sum_N (x_i - M/n)^2$. Since $P(e)$ is compact and v is continuous, there exists $x \in \arg\min_{y \in P(e)} v(y)$. Suppose $x \neq U(e)$. Then there exist $i, j \in N$ such that $x_i < U_i(e)$ and $x_j > U_j(e)$. The definition of the uniform rule and the efficiency of x then imply $x_i < U_i(e) \leq U_j(e) < x_j$.

Let $\delta = \min\{U_i(e) - x_i, x_j - U_j(e)\}$. Note that $\delta > 0$. Let $y \in X(e)$ be defined by $y_i = x_i + \delta$, $y_j = x_j - \delta$, and $y_k = x_k$ for all $k \in N \setminus \{i, j\}$. Note that $y_i \leq U_i(e)$ and $y_j \geq U_i(e)$, so that $y \in P(e)$. Letting $m = M/n$, we have

$$
\begin{aligned}
n \cdot v(x) - n \cdot v(y) &= (x_i - m)^2 + (x_j - m)^2 - (y_i - m)^2 - (y_j - m)^2 \\
&= ((x_i - m)^2 - (y_i - m)^2) + ((x_j - m)^2 - (y_j - m)^2) \\
&= (x_i - 2m + y_i)(x_i - y_i) + (x_j - 2m + y_j)(x_j - y_j) \\
&= (2x_i + \delta - 2m)(-\delta) + (2x_j - \delta - 2m)(\delta) \\
&= \delta(-2x_i + 2x_j - 2\delta) \\
&> 2\delta(x_j - x_i - (x_j - x_i)) = 0
\end{aligned}
$$

The inequality comes from the fact that $x_i < U_i(e) \leq U_j(e) < x_j$. This contradicts our choice of x. Q.E.D.

Remark on Generalized Economies

A ***generalized economy*** is a list $(R, \omega, M) \in \mathcal{R}^n \times \mathbb{R}^n_+ \times \mathbb{R}$ such that $\sum \omega_i + M \geq 0$. For each $i \in N$, ω_i is interpreted as agent i's endowment. The quantity M is interpreted as the population's obligation to or from the outside world. Let $\mathcal{G}^n$ be the class of generalized economies. A feasible allocation for $(R, \omega, M) \in \mathcal{G}^n$ is a list $x \in \mathbb{R}^n_+$ such that $\sum x_i = \sum \omega_i + M$. The inequality $\sum \omega_i \geq M$ guarantees the existence of a feasible allocation. A special case of this model is when $M = 0$, a situation examined by Klaus, Peters, and Storcken (1995, 1996). The notion of a generalized economy is introduced by Dagan (1995) and Thomson (1992b), and it is studied in the context of economies with single-peaked preferences by Thomson (1995c). It is argued there that the natural extension of the uniform rule to generalized economies as follows:

Definition Given $e = (R, \omega, M) \in \mathcal{G}^n$, the allocation $x \in X(e)$ is the ***extended uniform allocation of e*** if for some $\lambda \in \mathbb{R}$,

1. if $\sum \omega_i + M \leq \sum p(R_i)$, then for all $i \in N$, $x_i = \min\{\max\{\omega_i + \lambda, 0\}, p(R_i)\}$
2. if $\sum \omega_i + T \geq \sum p(R_i)$, then for all $i \in N$, $x_i = \max\{\omega_i - \lambda, p(R_i)\}$

We only note here that Theorems 1 and 2 extend to the class of generalized economies to give us characterizations of the extended uniform rule, by simply using the differences $x_i - \omega_i$ instead of the x_i in the minimization exercises.

4 Bankruptcy problems

In this section, we turn to bankruptcy problems and show that an important solution for this class of problems can be characterized in the same two ways in which we characterized the uniform rule.

A ***bankruptcy problem*** is a pair $(c, E) \in \mathbb{R}_+^{n+1}$ such that $\sum c_i \geq E$, interpreted as follows: there is a set N of n creditors, with each creditor $i \in N$ holding a claim of c_i against a firm. The net worth of the firm is E. Let $\mathcal{B}^n$ be the class of bankruptcy problems. A ***feasible allocation for*** $\boldsymbol{e = (c, E) \in \mathcal{B}^n}$ is a vector $x \in \mathbb{R}_+^n$ such that $\sum x_i = E$. Let $\boldsymbol{X(e)}$ denote the set of feasible allocations of e. Here, a ***solution*** is a function that associates with each $e = (c, E) \in \mathcal{C}^n$ a feasible allocation for e.

The first formal analysis of bankuptcy problems is due to O'Neill (1982). The following solution has played an important role in the literature that followed.[3]

Definition Given $(c, E) \in \mathcal{B}^n$, the allocation $x \in X(e)$ is the ***constrained equal-award allocation of*** $\boldsymbol{e}$, if there is $\lambda \in \mathbb{R}_+$ such that for all $i \in N$, $x_i = \min\{c_i, \lambda\}$.

Propositions 3 and 4 state characterizations of the constrained equal-award solution parallel to our earlier characterizations of the uniform rule. We omit their proofs, limiting ourselves to noting that the requirement that no agent receive more than his claim plays the role played by efficiency in the proofs of Propositions 1 and 2.

Proposition 3 *For any bankruptcy problem, the difference between the largest amount received by any agent and the smallest such amount is strictly smaller at the constrained equal-award allocation than at any other feasible allocation at which no agent receives more than his claim.*

[3] For a survey, see Thomson (1995b).

Proposition 4 *For any bankruptcy problem, the variance of the amounts received by all the agents at the constrained equal-award allocation is strictly smaller than at any other feasible allocation at which no agent receives more than his claim.*

Another important solution is the so-called Talmudic solution (Aumann and Maschler, 1985).

Definition Given $(c, E) \in \mathcal{B}^n$, the allocation $x \in X(e)$ is the ***Talmudic allocation of e,*** if (i) when $\sum c_i/2 \geq E$, there is $\lambda \in \mathbb{R}_+$ such that for all $i \in N$, $x_i = \min\{c_i/2, \lambda\}$ and (ii) when $\sum c_i/2 \leq E$, there is there is $\lambda \in \mathbb{R}_+$ such that for all $i \in N$, $x_i = c_i - \min\{c_i/2, \lambda\}$.

This rule satisfies ***self-duality***, that is, the way in which the worth of the firm is allocated is "dual" to the way in which the deficit (the sum of the agents' claims minus the worth of the firm) is allocated. It also satisfies ***E-monotonicity***, that is, an increase in the worth of the firm does not result in an agent receiving a smaller amount. Given the similarly between the equal-award solution and the Talmudic solution, it is natural to wonder whether the Talmudic solution enjoys the same two properties subject to the additional properties of *self-duality* and *E-monotonicity.* This idea is "half right":

Proposition 5 *Let a bankruptcy problem be such that the worth of the firm is no greater than half the sum of the agents' claims (i.e. $\sum c_i/2 \geq E$). Then the variance of the amounts received by all the agents at the Talmudic allocation is strictly smaller than at any other feasible allocation assigned by a solution satisfying* self-duality *and E*-monotonicity.

The following example shows that the Talmudic solution does not perform this variance minimization for the other half of the class of economies.

Example 1 *Let* $N = \{1, 2\}$, $c_1 = 1$, *and* $c_2 = 2$. *Let the solution* φ *be such that for any net worth* E,

$$\begin{array}{rl} \textit{if } E \leq 1 & \textit{then } \varphi_1(E) = \varphi_2(E) = E/2 \\ \textit{if } 1 \leq E \leq 2 & \textit{then } \varphi_1(E) = 1/2 \textit{ and } \varphi_2(E) = 1/2 + (E-1) \\ \textit{if } 2 \leq E & \textit{then } \varphi_1(E) = (E-1)/2 \textit{ and } \varphi_2(E) = (E+1)/2 \end{array}$$

Note that φ satisfies the *self-duality* and *E-monotonicity.* It is simple to check that when $E = 2$, the Talmudic solution assigns 1/2 to agent 1 and 1.5 to agent 2, but φ assigns 1 to each agent, hence minimizing variance.

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