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Bidding Rings in Repeated Auctions

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Abstract

This paper considers the question of tacit collusion in repeated auctions with independent private values. McAfee and McMillan show that the extent of collusion is limited by the availability of transfers. If no transfers are possible, the private information of bidders precludes any collusive scheme beyond bid rotation (BRS), even when the cartel has unlimited enforcement. In the absence of side payments, asymmetric continuation values in the repeated game may be a partial substitute for monetary transfers. For repeated auction games with communication and where players observe all bids, a folk theorem proved by Fudenberg et al. implies that collusive schemes better than BRS can be supported as subgame perfect equilibrium. In absence of communication, and when players can only observe the identity of the winners, the possibilities for collusion are more limited. This paper considers the possibilities for collusion in this setup. We give conditions such that collusion over and above BRS is possible. For large number of bidders and low discount rates, perfect collusion can be approached.

1 Introduction

How effectively can a cartel collude when its participating members are heterogeneous? A fundamental problem that the cartel must overcome is how to distribute the surplus generated from collusion. In their analysis of bidding rings, McAfee and McMillan (M&M) show that, when the values of cartel members are private, the success in colluding and splitting this surplus is intimately connected to the possibility of ex-post transfers. Total surplus is maximized when the highest valuation bidder gets the object and the auctioneer gets paid the reserve price. If cartel members can engage in side payments, a mechanism such as holding an internal auction and then bidding at reserve price, can effectively sustain perfect collusion (we call it an efficient collusion). But in the absence of such transfers, M&M show that a cartel can do no better than have all its members bid the reserve price, thus splitting an inefficiently low surplus. Moreover, enforcement problems arise, as cartel members have strong incentives to deviate from this mechanism.

As pointed out by M&M, *bidders may prefer to collude implicitly if the risk of detection and severity of punishment of explicit collusion are sufficiently great.* Their analysis suggests that in such case, the best that the cartel can hope for is a bid rotation scheme, where cartel members take turns in bidding independently of their valuations. In this paper we study the problem of collusion in bidding rings by explicitly modelling this repeated auctions game. Our analysis broadens that of M&M in two important ways. First, we show that even in the absence of side-payments, implicit transfers are still possible in this repeated game, as (asymmetric) continuation equilibria may depend on the past history of the game. Second, we address explicitly the enforcement issue, as members of the cartel are not bound to their agreements. As usual in repeated games, we impose the restriction on enforcement that any agreement must be a subgame perfect equilibrium of this game.

An important consideration for collusion, is how much information players need in order to play the associated equilibrium strategies. By limiting the information released, the auctioneer may be able to restrict cooperation. In all our analysis, we consider equilibria that depend only on the identity of winners in past auctions and no information on actual bids. It is hard to imagine that such information would not be publicly available. A second consideration is whether bidders have an opportunity to exchange private information to coordinate their bidding behavior. Assuming players meet to exchange such information and yet are not able to engage in monetary transfers, seems quite implausible, since both actions would entail violations to antitrust. However, as M&M show, in absence of transfers there is not much to gain from information exchange: implementing a bid rotation scheme does not require such exchange. In most of our results, we assume no information exchange. Players' bidding strategies are functions only of the history of wins and their own current private values.

As mentioned above, the existence of asymmetric equilibria of the repeated game is essential to sustain collusion over and above bid rotation, but it is no guarantee of an overall improvement, since the asymmetric continuation equilibria may be associated with inefficiently low average payoffs. This points to a fundamental difference between monetary transfers, and implicit transfers through continuation payoffs: while the former have no effect on total surplus, the latter do. If one considers total expected discounted payoffs of the game as the criteria for ranking equilibria, gains in the first period could be offset by losses in the future. In the first part of the paper we provide sufficient (and almost necessary) conditions on the set of continuation payoffs, such that there exist equilibria that improve on competitive bidding and on a bid rotation scheme. This result is given for a large class of auctions, which we call *standard auctions*, in a symmetric independent values environment. We then provide conditions under which there exist such continuation equilibria for first and second price auctions.

Fudenberg, Levine and Maskin (1994) show that when players have values with finite support, can communicate and observe each others' bids, a folk theorem applies. In paricular this implies that for sufficiently high δ BRS and competitive bidding can be improved upon. Our setup is more restrictive since we don't allow for communication and bids are privately observed, so their theorem does not apply.

The second part of the paper considers the question of efficiency. As seen above, implicit transfers in the form of asymmetric continuation payoffs are essential to support cooperation. The larger is the set of equilibrium payoffs, the better are the incentives that players can be given in any given auction. Yet in providing incentives for cooperation in a stage game, a cost is borne in terms of future surplus. We first show that for any given discount factor, there is no subgame perfect equilibrium that achieves full efficiency for the cartel. The idea is quite simple. Any efficient equilibrium must give in all auctions the object to the player with the highest value, who in turn must pay the auctioneer no more than the reserve price. In a symmetric environment, this implies that players expected payoffs should be identical in all component auctions of the repeated game. In particular, this implies that continuation payoffs after any auction must be the same for all players, regardless of the outcome of that auction. But then players bid competitively in this and every auction.

The asymmetric continuation equilibria has the cost of inefficient selection: sometimes bidders with values below the highest one must win. As the number of players increases, the extent of this inefficiency decreases. On the other hand, with more players the individual incentives for cooperation are reduced. In the final section we show that if we allow the number of bidders to grow together with the discount factor, full efficiency can be achieved in the limit.

Though in general implicit transfers through equilibrium payoffs are less effective than direct transfers, they do have an advantage. As M&M point out, if non-serious bidders -bidders with zero value for any good in any auction- cannot be excluded from participating in the cartel, direct transfers have no use and collusion is not possible. When transfers take the form of payoffs associated to future wins, non-serious bidders are automatically excluded.

For most of the paper, our proofs are constructive. Though this makes for longer proofs, at the same time it provides added insights into the structure of asymmetric equilibria in repeated auctions. Of particular interest, is the asymmetric equilibria of first price auctions which relies on *stick-and-carrot* type of strategies. This involves a phase of 'cooperation' and a 'punishment' phase. For the latter, we construct a bidding war, where players' bids even exceed their highest possible valuation. Bidders engage in this war, as the winner is selected to be the player entitled to higher future continuation payoffs in the 'cooperative' phase. The threat of this war is, of course, what provides incentives to restrict bids in the 'cooperative' phase.

The paper is organized as follows. In Section 1 we describe the basic setup and provide conditions under which there exist subgame perfect equilibria that improve upon bid rotation and competitive bidding. Section 2 considers the question of efficiency. Section 3 concludes. A final appendix contains most proofs.

2 Improving on Bid Rotation and Competitive Equilibrium

In this section we study bidding rings in repeated general standard auctions with finite number of players. We begin with describing the setup and notation and then present two propositions stating that there exist mechanisms dominating bid rotation and competitive bidding (at least under some conditions). In these mechanisms players condition only upon the identities of winners in past auctions, as this is the only publicly available information. The bidding ring does not use any monetary transfers, but instead chooses asymmetric continuation equilibria to provide similar incentives.

2.1 The setup

There are N players participating in infinitely (countably) many auctions for separate objects. The valuation player i has for an object is denoted by v_i. These valuations are distributed independently across players and across time and are drawn from the same stationary distribution with a *c.d.f.* function $F(v)$, which is a common knowledge. We assume that F is continuously increasing and twice differentiable, with a density function $f(v)$ strictly positive over interval $[0, v^h]$.

At time t players know only the individual valuation of the good currently being sold. The participation in the auction is limited to the N players, which is also common knowledge.

All players are risk neutral and have a common discount factor δ.

We denote one player's expected utility from a single auction by x. The total expected payoff is an expected discounted sum of such one-period payoffs. We denote it by w.

All the objects are sold using some standard auction with reserve price equal to zero. We define the class of standard auctions as:

Definition 1 ***Standard auction*** *is any set of auction rules which satisfies the following 4 conditions:*

1) A buyer can make any nonnegative bid.

2) The buyer that submits the highest bid is awarded the good.

3) The auction rules are anonymous, so each player is treated symmetrically.

4) There exists a common equilibrium bidding strategy, which is strictly increasing in players' values and with zero expected payoff for a player with value 0*.*

First price sealed bid auction (FPA) and second price sealed bid auction (SPA) are commonly studied examples of standard auctions.

Denote by $H(t)$ the history of the game up to time t. This history contains information about the bids of the players in all auctions up to time t and the history of identities of winners in the corresponding auctions.

Denote by $h(t)$ the public history of the game up to time t. This contains only information that is publicly available. So each player knows $h(t)$ and his own past bids and probably has partial information about other elements of $H(t)$[1] In this paper we restrict our attention to models in which $h(t)$ contains only the history of identities of winners (called shortly a history of wins from now on) and to equilibria in which players condition their strategies only on $h(t)$ and not on any other information.

This can be partly supported by the fact, that in many sealed bid auctions not all the bids are revealed ex-post, but only the identity of the winner and probably the winning bid or the payment.

This restriction on strategies allows us to construct a lower bound on what a cartel can achieve when more information is publicly available.[2]

In all that follows we allow the players to use external randomization. If $\delta > \frac{1}{2}$, this randomization could be replaced by some (possibly complicated) deterministic alternating of the equilibria that gives the same expected payoffs.

It's important to notice, that in this setup a bid rotation scheme (BRS) in every period, based on randomization by the auctioneer cannot be supported as an equilibrium. In such a mechanism all players are supposed to bid the reserve price and the winner is picked randomly. But as the players do not observe each others' bids, deviations cannot be observed. However, there are

[1]For example if a player bids b and wins, he knows that all other bids were below b. If he loses he knows that there was at least one bid above b. Furthermore, in some auctions (like in SPA) the payment of the winner can depend on the bids of other players, so the payment may convey some information about other elements of $H(t)$.

[2]The question, how the amount of publicly released information influences the set of possible equilibria (and particularly set of equilibrium payoffs) is very interesting in itself, as it has important policy implications and we hope to return to it in future research.

other ways to support expected payoffs equal to those in BRS, which are used in the following proofs.

2.2 Structure of improved equilibria

We define a cartel to be efficient if it maximizes the sum of its members ex-post profits, i.e. if it makes the player with the highest value win the object[3] and pay the auctioneer only the reserve price[4].

In the seminal paper McAfee and McMillan (1992)[5] show that if the parties can exclude nonserious bidders (with values always smaller than reserve price) and if side transfers and communication are available, the cartel can achieve efficiency.

M&M also prove that in case no transfers are used, or the cartel cannot exclude nonserious bidders, the best the collusion can hope to obtain is either competitive bidding or a bid rotation scheme (BRS) - the winner is decided randomly and independently of her value and pays the reserve price. Which of these two mechanisms is better depends on the particular distribution of values.

These two mechanisms are somehow two extremes in the attempt to obtain efficiency: In a competitive equilibrium the good always goes to the player with highest valuation, but the payment is higher than the reserve price, while in BRS the payment is equal to the reserve price but the good does not necessarily go to the highest bider.

M&M suggest that even in a dynamic game these mechanisms cannot be improved upon. As we show now, this observation depends on the assumption of symmetric continuation payoffs and is not correct when asymmetric payoffs are available. We argue that asymmetric continuation equilibria can work partly as transfers in their setup. If the continuation payoff for a player is smaller when he wins than when he loses, it can provide incentives for him not to cheat on the collusive agreement.

The continuation payoffs can work as transfers only to a certain extent. First, the amount of a feasible transfer is constrained by the fact that the

[3]This is important because it is assumed that ex-post trade of the good cannot take place.

[4]For more about efficiency in a dynamic setup see next section.

[5]For previous studies on collusion in English and Second Price Auctions see for example Graham and Marshall (1987), Mailath and Zemsky (1991),as well as von Ungern-Stenberg (1988).

continuation payoffs must correspond to a SPNE of the continuation game. Second, transfers are based on asymmetric equilibria and asymmetry means a loss in efficiency of selection. So although transfers through continuation payoffs may allow improvements over the competitive or BRS mechanisms, they do not support full efficiency, as we explain in Section 3.

Fudenberg, Levine and Maskin (1994) show that when players have values with finite support, can communicate and observe each others' bids, a folk theorem applies. In paricular this implies that for sufficiently high δ BRS and competitive bidding can be improved upon. Our setup is more restrictive since we don't allow for communication and bids are privately observed, so their theorem does not apply.

We develop sample cartel schemes that improve upon both BRS and competitive bidding in repeated Second and First Price Auctions. The analysis is in two parts. First, we prove that if certain asymmetric continuation equilibria exist, then both BRS and competitive bidding are dominated by some cartel schemes. This result applies to any repeated standard auction. Second, we construct examples of such asymmetric equilibria for SPA and FPA.

We restrict ourselves to equilibria in which players condition only on the history of wins so in a symmetric equilibrium the expected continuation payoffs for all losers in the first auction must be the same. Denote by w^1 and w^2 the expected continuation payoffs conditional on winning and losing in the first auction respectively. Each player chooses her bid in order to solve:

$$\underset{b\geq 0}{Max} \quad vQ(b) - P(b) + \delta(Q(b)w^1 + (1 - Q(b))w^2) \tag{1}$$

where $Q(b)$ is probability of winning given bid b and other players' equilibrium bidding functions and $P(b)$ is expected payment in the current auction given the bid b. This problem can be rewritten as:

$$\underset{b\geq 0}{Max} \quad (v - c)Q(b) - P(b) + \delta w^2 \tag{2}$$

where $c = \delta(w^2 - w^1)$. If the continuation payoffs after winning and losing are the same, then $c = 0$ and players behave in the first auction as in a one-shot game. If, however, the losers are rewarded with higher continuation payoffs than the winner, then $c > 0$ and the dynamic game is equivalent to a one shot game in which the support of players' values is shifted to the left by c.

Looking at the rewritten problem we can see that players with realizations $v < c$ strictly prefer to lose than to win even if they could pay nothing for the good in the current auction. As we can see, the imposed information structure leads to a very limited class of bidding functions: for a range of values $[0, c)$ the players choose not to bid at all (or if forced to bid they bid 0) and for a range of values $[c, v^h]$ the bidding functions are strictly increasing as in a competitive equilibrium with values distributed over $[-c, v^h - c]$.

Notice that efficient selection of bidders takes place only in the region of values $v > c$. But when all bidders have values below c, no selection takes place. Thus choosing c involves a trade-off between efficient selection and payment minimization. The larger c is, the smaller are the payments to the auctioneer, but at the same time the length of the region over which players bid 0 and do not achieve selection increases.

Let w_* be the payoff in a base symmetric equilibrium which we seek to improve upon. Given a pair of asymmetric continuation payoffs $\underline{w} < \overline{w}$ that are used to support this equilibrium, pick w^2 and w^1 to be:

$$\begin{aligned} w^1 &= \alpha \underline{w} + (1-\alpha) w_* \\ w^2 &= \alpha \overline{w} + (1-\alpha) w_* \end{aligned} \tag{3}$$

By choosing α we control c, the shift in the support of the values.

When w_* corresponds to the payoffs from repeated competitive equilibrium, then setting $\alpha = 0$ gives competitive bidding in every period. As we show in Proposition 2 increasing α results in a second order loss due to decreased selection and a first order gain from lower payments to the auctioneer. A similar idea is used when w_* corresponds to a BRS.

Proposition 1 *Consider any infinitely repeated standard auction with N players. If for $\delta < 1$ there exist asymmetric equilibria with payoffs $w_i = \underline{w}$ and $w_{-i} = \overline{w}$ for any i such that $\delta(\overline{w} - \underline{w}) \geq v^h$ and $\frac{1}{N}((N-1)\overline{w} + \underline{w}) \geq \frac{1}{N}\frac{ev}{1-\delta}$, then there exists a symmetric equilibrium with players conditioning only on the history of wins and with expected payoffs higher than in the repeated bid rotation scheme BRS.*

Proof. See Appendix. ■

Proposition 2 *Consider any infinitely repeated standard auction with N players. If for $\delta < 1$ there exist asymmetric equilibria with payoffs $w_i = \underline{w}$ and $w_{-i} = \overline{w}$ such that $\overline{w} > w_{ce}$ and $\overline{w} > \underline{w}$, then there exists a symmetric equilibrium with players conditioning only on the history of wins and with payoffs higher than in a repeated competitive equilibrium w_{ce}.*

Proof. See Appendix. ■

Our objective in this section has been to show the existence of equilibria that improve over the base payoffs (BRS and competitive equilibrium). We have thus restricted the analysis to strategies that involve higher expected surplus in the first period only. It is obvious that we could iterate this procedure achieving improvements in additional periods in each step. Moreover, as the set of equilibrium payoffs is enlarged, new asymmetric payoffs $\overline{w}$ and $\underline{w}$ can be obtained which provide more efficient rewards and punishments. Characterizing the extreme set is an interesting question which is beyond the scope of this paper. A straightforward extension of the APS algorithm can be used to identify this set.

2.3 Sufficient Conditions for Second and First Price Auctions.

The propositions presented above hold for any standard auction and require particular continuation equilibrium payoffs. This section provides so that such continuation payoffs can be supported by SPNE in repeated First and Second Price Auctions. This implies that in a dynamic context, a weak cartel can collude more effectively in these auctions than in the static setup described in previous literature.

We begin the results with Second Price Auction, as the asymmetric equilibria are easier to construct in that case and in general collusion is easier to sustain[6].

Corollary 1 *In Repeated Second Price Auctions for large enough $\delta < 1$, there exists a symmetric equilibrium with players conditioning only on the*

[6] The mechanisms we propose rely on the use of Nash Equilibria in which players use weakly dominated strategies which are simple to construct. As will be clear in the discussion of FPA it is possible to construct other sufficient asymmetric equilibria without relying on these NE.

history of wins with payoffs higher than in the repeated bid rotation scheme BRS.

Proof. By the above proposition we only need to show that we can construct the asymmetric continuation equilibria. Consider the following asymmetric bid rotation scheme. Call player i to be excluded player and all other players to be included players.

Before each auction one of the included players is chosen to be the winner. He bids v^h in the given auction and all other players (including the excluded one) bid 0. This set of strategies constitutes a Nash Equilibrium in each stage game so it generates a SPNE. In this equilibrium the expected payoff for the excluded player is $\underline{w} = 0$. This equilibrium is symmetric for the $N-1$ included players and the expected payoff for any of them is: $\overline{w} = \frac{1}{1-\delta}\frac{ev}{N-1}$.

Now let's verify the conditions of the proposition.

Clearly $\frac{1}{N}((N-1)\overline{w}+\underline{w}) = \frac{1}{N}\frac{ev}{1-\delta}$. Furthermore $\delta(\overline{w}-\underline{w}) = \frac{\delta}{1-\delta}\frac{ev}{N-1}$ grows to infinity as δ grows to 1. So there exists $\delta_0 < 1$ s.t. $\delta(\overline{w} - \underline{w}) \geq v^h$ for all $\delta \geq \delta_0$. ■

Corollary 2 *For any $\delta > 0$ in Repeated Second Price Auctions there exists a symmetric equilibrium with players conditioning only on the history of wins and with payoffs higher than in repeated Competitive Equilibrium w_{ce} if $N = 2$ or N is large.*

Proof. Again, it's sufficient to show that there exist asymmetric equilibria satisfying the conditions of Proposition 2. Take simply the equilibrium constructed in the above corollary: $\overline{w} = \frac{1}{1-\delta}\frac{ev}{N-1}$ and $\underline{w} = 0 < w_{ce}$. To show that $\overline{w} > w_{ce}$ notice that for $N = 2$, a player that achieves $\overline{w}$ wins in every auction while the other never wins. So clearly $\overline{w} > w_{ce} > \underline{w}$. Furthermore, notice that $N\overline{w}$ decreases in N to $\frac{ev}{1-\delta}$ while Nw_{ce} decreases in N to zero. So for large N again $\overline{w} > w_{ce} > \underline{w}$. ■

Corollary 3 *For sufficiently large δ in Repeated Second Price Auctions with $N > 2$ players there exists a symmetric equilibrium with players conditioning only on the history of wins and with payoffs higher than in repeated Competitive Equilibrium ,w_{ce}.*

Proof. A detailed proof is presented in the appendix. The reason the case N >2 is different from $N = 2$ is that while for $N = 2$ we have $\frac{1}{1-\delta}\frac{ev}{N-1} >$

$w_{ce}(N)$ this inequality does not necessarily hold for intermediate N. So we construct an alternative asymmetric equilibrium: in the first auction only $N-1$ players participate bidding competitively (the remaining player is called the excluded one) and in all remaining auctions there is competitive bidding among all players. This makes $\overline{w}$ just a little bit higher than w_{ce}, which is all that is needed for Proposition 2. The reason δ needs to be large is to provide some punishment in case the excluded player bids and wins in the first auction. ■

The results for First Price Auctions are more involved, as construction of asymmetric equilibria is more complicated. The following two lemmas provide us with sufficient asymmetric equilibria.

Lemma 1 *Consider an infinitely repeated First Price Auction. Denote by $x_{ce}(N)$ the expected payoff from a single auction when there is competitive bidding between N players. If the single period payoff from BRS, $\frac{1}{N}ev >$ $x_{ce}(N)$ then for large enough δ there exist asymmetric equilibria in repeated first price auctions, such that the strategies depend only on the history of wins and with expected payoffs $w_i = \underline{w}$ and $w_{-i} = \overline{w}$ satisfying the conditions of Proposition 1.*

Proof. (Sketch) A detailed proof is presented in the Appendix. We construct a class of asymmetric bid rotation strategies and we show that for large δ they form a SPNE. Then we show that in fact for any $v^h$and large δ we can choose such an equilibrium to satisfy $\delta(\overline{w} - \underline{w}) > v^h$.

We now describe asymmetric bid rotation strategies. Let T be a positive integer. Before each of the first $T(N-1)$ auctions one of the players achieving $\overline{w}$ is chosen to be the sole bidder. Then the player that is to obtain $\underline{w}$ is chosen to be the sole bidder in periods $T(N-1)+1$ to TN and the mechanism starts over. So every player wins on average the same number of auctions and the asymmetry is achieved by the fact that one player has to wait $T(N-1)$ before having chance to win. The extent of this asymmetry increases in T. Finally, to prevent cheating we use the threat of switching to competitive bidding forever. To make this a sufficient deterrent we need δ to be large enough. ■

Lemma 2 *Consider an infinitely repeated First Price Sealed Bid Auction* *with $N \geq 3$ players*[7]*. For large enough δ there exist asymmetric equilibria such that the strategies depend only on the history of wins and the equilibrium payoffs are $w_i = \underline{w}$ and $w_{-i} = \overline{w}$, s.t. $\overline{w} > w_{ce} > \underline{w}$.*

Proof. See Appendix for the detailed proof.

Along the equilibrium path the mechanism looks the same as the one in Corollary 3: In the first auction one player stays out while the remaining $N-1$ players bid competitively. From the second auction on all players bid competitively. The difficult part is to provide incentives for the one excluded player not to participate in the first auction. In the case of SPA we could use repeated asymmetric NE in which a deviant gets zero payoff forever but in FPA this equilibrium does not exist. In the previous lemma we used as a threat switching to competitive equilibrium forever. Here it is not sufficient as players switch to competitive bidding in the next period even if nobody deviates.

To find an equilibrium with payoffs lower than in competitive equilibrium we construct a stick and carrot "bidding war" equilibrium. However, this is not a conventional one. The stick phase (the bidding war) requires that players bid more aggressively than in a competitive equilibrium for a period of time. In the standard application of stick and carrot strategies such behavior is sustained by the threat of restarting the stick phase when somebody deviates. But in our setup all we observe is the identity of the winner, so players cannot tell if all are bidding as high as they are supposed to. To introduce the right incentives for high bidding we make this war a tournament for the carrot: Once the war is finished one player is chosen to get the carrot[8] and the probability of being the lucky one is proportional to the number of wins during the war phase. This makes players bid aggressively during the war not only for the sake of current gains but also to improve their chances of winning this tournament. Asymmetric values can be generated and one player can be punished with this bidding war by excluding him from the tournament. ■

We are now ready to state the corollaries:

[7]When $N = 2$ the presented mechanism does not work, because there are not enough players to "compete for the carrot". In that case exists a very similar mechanism in which the players condition on the history of wins and own bids. Details are available from the authors upon request.

[8]i.e. the players switch to an equilibrium in which this particular player has higher payoff than others - for example by allowing him to win a few auctions without competition.

Corollary 4 *In Repeated First Price Auctions with large enough $\delta < 1$, there exist a symmetric equilibrium with players publicly observing only the history of wins and with payoffs higher than in repeated competitive equilibrium*[9]*.*

Proof. The proof follows immediately from Proposition 2 and the above lemma. ■

Corollary 5 *In Repeated First Price Auctions with large enough $\delta < 1$, there exists a symmetric equilibrium with players publicly observing only the history of wins and with payoffs higher than in the repeated bid rotation scheme BRS.*

Proof. If $\frac{1}{N}ev < x_{ce}(N)$, then the BRS is dominated by the competitive equilibrium. If $\frac{1}{N}ev = x_{ce}(N)$, then BRS is dominated if and only if the competitive equilibrium is. As shown in the previous corollary, the competitive equilibrium is dominated for large enough δ. Finally if $\frac{1}{N}ev > x_{ce}(N)$ we can use Proposition 1 and Lemma 1 to show the existence of better equilibria than BRS. ■

This finishes our results on the existence of better equilibria than BRS and competitive bidding in repeated auctions.

3 Efficiency of the Cartel

In the previous section we have shown that if players are sufficiently patient, there exist SPNE that dominate both competitive bidding and bid rotation schemes even if the players do not use side transfers and observe only the identities of winners but not the submitted bids. This section considers the question of efficient collusion. We begin this section with a general proposition, that confirms a result of McAfee and McMillan (1992), that in any repeated standard auction any cartel cannot obtain efficiency unless it uses side payments. Then we show that there exist mechanisms that are asymptotically efficient as the number of players grows to infinity and the discount factor grows to one.

Total surplus of the cartel is maximized when in every auction the highest valuation bidder gets the object and the auctioneer gets paid the reserve price. If cartel members can engage in side payments, a mechanism such as holding an internal auction and then bidding the reserve price, can effectively

[9]For $N \geq 3$ it's sufficient that the players condition their strategies only on the public signal. For $N = 2$ they condition also on their own bids.

sustain perfect collusion. Such a mechanism requires pre-play communication and side payments. In absence of direct side payments, implicit transfers in the form of asymmetric continuation payoffs are essential to support such cooperation. The larger is the set of equilibrium payoffs, the better are the incentives that players can be given in any auction. Yet in providing incentives for cooperation in a stage game, a cost is borne in terms of future surplus. As shown below, this results in the impossibility of efficient collusion.

Proposition 3 *Consider any infinitely repeated standard auction with $N \geq 2$ players and with any discount factor $\delta < 1$, where players observe (and hence can condition upon) the bids from all previous auctions and can meet before each auction to coordinate the bids, but do not use side transfers. There does not exist a collusive mechanism that achieves efficient collusion.*

Proof. Suppose that such mechanism exists. Efficiency implies that the expected payoff from any single auction to any player after any history of the game is the same. Consider the first auction. By efficiency in auctions two to infinity the continuation payoffs are independent of the behavior in current auction. Hence the dynamic structure does not influence the incentives in the current auction. Consequently, the strategies are the same as in a one shot game and the only equilibrium in which the player with highest value obtains the good is the competitive equilibrium, as shown in McAfee McMillan (1992), contradicting full efficiency. ■

Remark Athey and Bagwell (1999) study a similar problem where players can communicate before each auction. A distinguishing feature of their setup is that players have only two possible valuations. As a consequence, different distributions of total surplus between the players along the Pareto frontier can be achieved by a choice of sharing rules in case of ties. This in turn allows for a scope of costless transfers and hence full efficiency can be supported for sufficiently large δ.

3.1 Asymptotic Full Efficiency

In this section we establish that as the number of players and the discount factor grow there exist collusive schemes that are asymptotically efficient. These mechanisms exist even if the players cannot communicate to coordinate bids and observe only the identity of winners. For convenience, we restrict our analysis to the case of second price auctions.

Definition 2 *A mechanism is asymptotically efficient if for any $p < 1$ and $\varepsilon > 0$ and $q > 0$ we can find N and $\delta < 1$ large enough such that in every auction:*

1. *The good is always obtained by some bidder.*
2. *The winner pays at most q.*
3. *With probability greater or equal to p, the winner has valuation at least equal to $v^{(1)} - \varepsilon$, where $v^{(1)}$ denotes the highest of the players' values (realization of the first order statistic from N players for the given auction).*[10]

We first describe informally the collusive scheme used. As seen above, in repeated Second Price Auctions, if the players can condition only on the history of wins, the bidding functions in most equilibria are of the form $b = v - c$ for $v \geq c$ and if otherwise, the bid is equal to 0 or the bidder does not participate.

The mechanism we propose is such that c is large and close to v^h. If a player bids a positive number then he has value close to v^h and the winner never pays more than $v^h - c$, which is small. Any given player quite seldom has a value above c, but if we allow for a large number of potentially bidding players, with high probability at least one of them does.

The mechanism works in the following way. At any given point in time, conditional on the history of the game, there is a set of players who are supposed to participate in the auction and a set of players that should not participate. Call them included and excluded players correspondingly. Out of the included players one is chosen randomly to be the sure bidder.

If an excluded player wins an auction, the continuation equilibrium is the worst possible for him (gives him payoff 0 in each future auction - the worst possible punishment in SPA). Any included player can submit any bid. The winner of the given auction becomes excluded for the next $T < N$ auctions. If there is no winner, the sure bidder becomes excluded.

[10] This is not the only possible definition of asymptotic efficiency. One could for example require that the probability that the player with the highest valuation wins to grow to 1. We do not know if such a mechanism can be constructed. To make this happen in our mechanism one would need the ratio of included to excluded players to grow to infinity, but in our proof it rather decreases to zero.

Another, often used criterion, is convergence of average payoffs to payoffs in efficient collusion.

To finish the description of the mechanism, we suggest the following *starting* procedure to make the mechanism symmetric and stationary:

Before the first auction, T players are chosen randomly in an order. The first of them is excluded in the first auction, the second is excluded in the first two auctions and the t one is excluded in auctions 1 through t. Hence, at any point in time there are $M = N - T$ included players. For asymptotic efficiency we need M to get large as N goes to infinity.

Proposition 4 *In repeated Second Price Auctions there exist asymptotically efficient mechanisms without communication and with no transfers in which players condition only on the history of wins.*

Proof. See Appendix ■

4 Concluding Remarks

In this paper we have shown that in the absence of side payments, some collusion is still possible through implicit transfers of equilibrium continuation payoffs. To keep bids low, bidders are *punished* for winning and *rewarded* for losing by lowering or increasing, respectively, their continuation payoffs. In this way, bidders are faced with an intertemporal trade-off: winning in the current period decreases their probability of winning in the future. This result suggests an empirical test for collusion where, accounting for fixed effects, the probability of a player winning an auction should decrease with previous wins.[11]

We have also established that in the absence of transfer payments, perfect collusion cannot be achieved. This leads to a question about an optimal collusive scheme. The set of all possible equilibrium payoffs (including the optimal ones) can be described recursively using the APS framework (Abreu, Pearce, Stacchetti, (1986) and (1990)).[12] If one focuses only on equilibria in which players condition their actions only on the history of wins, then this set is even relatively simple to compute, as the continuation payoffs are represented by finite number of points from the set of equilibrium payoffs. In repeated second price auctions the strategies of the players are easily found as functions of the continuation payoffs. In repeated first price auctions

[11] Recent empirical work on repeated auctions (see Pesendorfer (1996) and Porter and Zona (1999)) has focused exclusively on bid distributions.

[12] Athey and Bagwell (1999) follow this approach for a simplified model with two types.

the situation is more complicated, because in general we have to describe strategies when the players have different distributions of their values. This can be done however, as shown in Maskin and Riley (1998), and hence the computation of the set is also feasible for repeated first price auctions. Such analysis can provide answers to interesting questions such as the effect of different degrees of information release on collusive behavior.

A distinguishing feature of the repeated game considered in this paper is the existence of asymmetric information in the constituent game. This implies that even when bids are observed, the bidding functions used by players cannot be monitored. In consequence, the standard type of trigger strategies as support for efficient collusion are not applicable. Optimal punishments must rely on more complex intertemporal incentives. This structure is of interest beyond auctions, and applies generally to dynamic problems where there is lack of commitment and (temporary) private information. Examples of such problems include repeated oligopoly games and repeated partnerships with private information.

5 Appendix

Proof of Proposition 1.

Denote the expected payoffs from a repeated bid rotation by

$$w_{BRS} = \frac{1}{N}\frac{ev}{1-\delta} \tag{4}$$

If $\frac{1}{N}((N-1)\overline{w}+\underline{w}) > \frac{1}{N}\frac{ev}{1-\delta}$, then trivially, playing the asymmetric equilibria with equal probabilities already dominates the BRS. So let's look at a case when $\frac{1}{N}((N-1)\overline{w}+\underline{w}) = \frac{1}{N}\frac{ev}{1-\delta} = w_{BRS}$

Consider the following mechanism. Before the first auction one player is chosen randomly. Call this player a *sure bidder.* In the first auction all players can bid and the continuation payoffs depend on the identity of the winner. If there is a winner in the first auction (i.e. when at least one player submits a bid) the continuation payoffs for all players conditional on winning and losing respectively, are:

$$\begin{aligned} w^1 &= \alpha\underline{w} + (1-\alpha)w_{BRS} \\ w^2 &= \alpha\overline{w} + (1-\alpha)w_{BRS} \end{aligned} \tag{5}$$

If nobody wins, the sure bidder gets continuation payoff w^1 and all others get w^2. These continuation payoffs for $\alpha \in [0,1]$ are feasible because w_{BRS} can be obtained by randomizing between $\overline{w}$ and $\underline{w}$ with probabilities $\frac{N-1}{N}$ and $\frac{1}{N}$ respectively, and we allow for randomization among equilibria.

As we explained above, the repeated auction is equivalent to a single auction with values $v_i - c$ instead of v_i and with additional payoff δw^2 independent of the bid (and also independent of participation). First notice, that all players, except for the sure bidder, are better off not bidding at all than bidding any amount, when their value is below c. Second, the sure bidder's optimal choice is to bid 0 when he has value $v - c \leq 0$. This is optimal, because that bid leads to a win only if nobody else bids (or if somebody bids 0, what happens with zero probability in the equilibrium) and the sure bidder gets the same continuation payoff when he wins and when nobody wins, but the current payoff is higher in the first case. Hence he is better off bidding 0 than not bidding at all even if he has current value below c.

So players with values v below c (except for the sure bidder) do not bid at all. In our case $c = \delta\alpha(\overline{w} - \underline{w})$ hence by assumption, we can choose $\alpha \in (0,1]$, such that $c = v^h$. In that case in the first auction the sure bidder bids zero, and all others do not bid at all. So ex-ante (before the choice of the sure bidder) the behavior is as in BRS. Hence the probability of winning in the first auction is $\frac{1}{N}$ and the expected continuation payoff for any player is:

$$\delta\left(\frac{1}{N}w^1 + \frac{N-1}{N}w^2\right) = \delta w_{BRS} \tag{6}$$

So the mechanism with $c = v^h$ gives the same expected payoffs as the BRS.

Now we will show that decreasing c a little (by decreasing α) increases the payoff for any player.We can find that:

1. The expected utility for any player (but the sure bidder) with value v below c is:

$$U(v) = \delta w^2 \tag{7}$$

and for the sure bidder:

$$\begin{aligned} U(v) &= F(c)^{N-1}v + \delta\left[F(c)^{N-1}w^1 + (1 - F(c)^{N-1})w^2\right] && (8)\\ &= F(c)^{N-1}(v - c) + \delta w^2 && (9)\end{aligned}$$

So ex-ante the payoff for every player with value below c is:

$$U(v) = \frac{F(c)^{N-1}}{N}(v - c) + \delta w^2 \tag{10}$$

Considering the equivalent revelation game, the expected utility for any player with value v above c is:

$$U(v) = U(c) + \int_c^v F(x)^{N-1} dx \tag{11}$$

After integrating by parts, the total expected payoff is:

$$W(c) = -\frac{F(c)^{N-1}}{N} \int_0^c F(x)dx + \int_c^{v^h} F^{N-1}(x)(1 - F(x))dx + \delta w^2(c) \tag{12}$$

Take the derivative:

$$W'(c) = \frac{-f(c)(N-1)F^{N-2}(c)}{N} \int_0^c F(x)dx - \frac{F^N(c)}{N} - F^{N-1}(c)(1-F(c)) + \delta \frac{\partial w^2}{\partial c} \tag{13}$$

In our case we have:

$$\delta \frac{\partial w^2}{\partial c} = \frac{\overline{w} - w_{BRS}}{\overline{w} - \underline{w}} = \frac{1}{N} \tag{14}$$

Evaluate $W'(c)$ at $c = v^h$:

$$W'(c)|_{c=v^h} = \frac{-f(v^h)(N-1)}{N} \int_0^{v^h} F(x)dx < 0 \tag{15}$$

This implies that there exists a mechanism with $c < v^h$ that strictly dominates the BRS. ■

Proof of Proposition 2.

The mechanism is similar to the one above. Before the first auction one player is chosen randomly to be the sure bidder.

Consider the continuation payoffs:

$$\begin{aligned} w^1 &= \alpha \underline{w} + (1-\alpha) w_{ce} \\ w^2 &= \alpha \overline{w} + (1-\alpha) w_{ce} \end{aligned} \tag{16}$$

As before if there is a winner in the first auction, she gets continuation payoff w^1 and all others get w^2. If there is no winner the sure bidder gets w^1 and all others get w^2. Again, the sure bidder bids 0 if $v < c$ and all others do not bid at all when their value is below c. When $\alpha = 0$, we have $c = 0$ and this equilibrium is equivalent to the competitive equilibrium.

Now let's prove that increasing c a little (by increasing α) increases expected payoff. By the same reasoning as in the previous proof:

$$W'(c) = \frac{-f(c)(N-1)F^{N-2}(c)}{N} \int_0^c F(x)dx - \frac{F^N(c)}{N} - F^{N-1}(c)(1-F(c)) + \delta \frac{\partial w^2}{\partial c} \tag{17}$$

Now we have:

$$\delta \frac{\partial w^2}{\partial c} = \frac{\overline{w} - w_{ce}}{\overline{w} - \underline{w}} \tag{18}$$

Evaluate $W'(c)$ at $c = 0$:

$$W'(c)|_{c=0} = \frac{\overline{w} - w_{ce}}{\overline{w} - \underline{w}} > 0 \tag{19}$$

This implies that there exists a mechanism with $c > 0$ that strictly dominates the competitive bidding. ■

Proof of Corollary 3.

Consider the following mechanism.

Call *included players* the $(N-1)$ players that achieve $\overline{w}$ and the *excluded player* the one that obtains $\underline{w}$.

In the first auction the $N-1$ included players are supposed to bid competitively and the excluded player is supposed not to bid at all. If the winner turns out to be one of the included players, the continuation equilibrium is competitive bidding forever. If not, i.e. if it is observed that the excluded player won, the continuation payoff is $\underline{w} = 0$ for him and $\overline{w} = \frac{1}{1-\delta} \frac{ev}{N-1}$ for all others. To show that this is an equilibrium we need to show that the excluded player does not have incentives to deviate in the first auction. Depending on the number of bidders the gain from deviation can be smaller or larger. In any case it is bounded from above by v^h. Consequently a sufficient condition for this mechanism to constitute a SPNE is:

$$v^h \leq \frac{\delta}{1-\delta} x_{ce}(N) = \delta w_{ce} \tag{20}$$

where $x_{ce}(N)$ is one player's one-period expected profit from a competitive equilibrium with N players.

So for the given number of players there exists $\delta_0 < 1$ such that for all $\delta > \delta_0$ this mechanism constitutes a SPNE.

Finally as $x_{ce}(N)$ is decreasing in N, the equilibrium payoffs satisfy:

$$x_{ce}(N-1) + \frac{\delta}{1-\delta} x_{ce}(N) = \overline{w} > w_{ce}(N) > \underline{w} = \delta w_{ce}(N) \tag{21}$$

Thus we have constructed the sufficient asymmetric equilibria. ■

Proof of Lemma 1.

We construct a class of asymmetric bid rotation strategies and we show that for large δ they form a SPNE. Then we show that in fact for any $v^h$and large δ we can choose such an equilibrium to satisfy $\delta(\overline{w} - \underline{w}) > v^h$.

As usual, call the player obtaining $\underline{w}$ to be the excluded player and players obtaining $\overline{w}$ included players.

Let T be a positive integer. Before each of the first $T(N-1)$ auctions one of the included players is chosen randomly to be the sole bidder. He bids 0 in the given auction while all others do not bid at all. Then, from periods $T(N-1)+1$ to TN the excluded bidder is automatically chosen to be the winner by solely bidding 0.At this time the mechanism starts all over.

If at any point in time the designated winner does not win, all players switch to competitive bidding forever.

To verify that this is an equilibrium for large enough δ, it is sufficient to check that the excluded player doesn't have incentives to deviate even in the first auction when he gets value v^h.[13]

The expected ex-ante payoff for the excluded player in the mechanism is:

$$\underline{w} = ev \frac{\delta^{(N-1)T}(1-\delta^T)}{(1-\delta)(1-\delta^{NT})} \tag{22}$$

[13] Clearly, if the excluded player has no incentives to deviate, so does the included one, as she has higher cost of deviation in terms of continuation payoffs and the same bound on current gain.

The payoff from deviation is bounded from above by:

$$v^h + \frac{\delta}{1-\delta} x_{ce}(N) \tag{23}$$

So to show that this mechanism is incentive compatible it's sufficient to show that:

$$v^h \leq \frac{1}{1-\delta}\left[ev\frac{\delta^{(N-1)T}(1-\delta^T)}{1-\delta^{NT}} - \delta x_{ce}(N)\right] \tag{24}$$

Now notice that as $\delta \to 1$, the term in brackets converges to $ev\frac{1}{N} - x_{ce}(N) > 0$, so the whole RHS grows to infinity. By continuity, there exists $\delta_0 < 1$ s.t. for all $\delta \geq \delta_0$ this mechanism forms a SPNE.

Now let's look at $\overline{w}$. Any of the included players has equal probability of win in any of the first $(N-1)T$ auctions. Hence:

$$\overline{w} = ev\frac{1-\delta^{(N-1)T}}{(1-\delta)(1-\delta^{NT})}\left(\frac{1}{N-1}\right) \tag{25}$$

It can be easily verified that $\frac{1}{N}((N-1)\overline{w} + \underline{w}) = \frac{1}{N}\frac{ev}{1-\delta}$ for any δ, T and N.

Finally let's show that we can choose T in order to satisfy $\delta(\overline{w} - \underline{w}) > v^h$ without violating the incentive constraint.

$$\delta(\overline{w} - \underline{w}) = ev\delta\frac{1}{N-1}\left[\frac{1 - N\delta^{(N-1)T} + (N-1)\delta^{NT}}{(1-\delta)(1-\delta^{NT})}\right] \tag{26}$$

It can be found that:

$$\lim_{\delta \to 1} \delta(\overline{w} - \underline{w}) = ev\frac{T}{2} \tag{27}$$

Because δ_0 is increasing in T, we can find T large enough and δ_0 large enough that $\delta_0(\overline{w} - \underline{w}) > v^h$. ■

Proof of Lemma 2.

First we construct an equilibrium in which one player has higher payoffs than all others.

Consider the following collusive scheme consisting of two phases: a cooperative phase and a *bidding war* phase that plays a role of a punishment.

The recommended actions of the players in the cooperative phase are as follows. In the first T auctions there is one *favored player* who is supposed to bid 0 and all others not to bid at all. If it is observed that the favored player won in all T auctions, the continuation equilibrium is competitive bidding forever.

If at any of the T auctions somebody else wins this person is called the *deviant* and the mechanism goes immediately to a *bidding war* phase of length k (if nobody wins the favored player is the deviant).In this phase all other players are called *non-deviant*.

The bidding war is described as follows. During the k periods everybody can bid any amount. After period k one of the non-deviant players is chosen to become the new favored player and the probability of being chosen is proportional to the number of wins a player had in the bidding war phase. All other players as well as the deviant become the non-favored. (For $N = 2$, during the war the deviant is supposed to bid v^h and the non-deviant $v^h + \varepsilon$.)

Along the equilibrium path this mechanism generates payoffs:

$$\begin{aligned} \overline{w} &= \frac{1-\delta^T}{1-\delta} ev + \delta^T w_{ce} > w_{ce} \quad \text{for the favored player} \\ \underline{w} &= \delta^T w_{ce} < w_{ce} \quad \text{for all other players} \end{aligned} \tag{28}$$

Let's consider the bidding war. This phase is a variant of a stick and carrot equilibrium. We construct the mechanism in such a way that:

1) During the war the non-deviant players bid above v^h to try to obtain the carrot of being the new favored player after the war.

2) The deviant does not bid at all during the war.

3) The bidding war is a sufficiently severe for the deviant that he has no incentives to deviate in the first phase.

As before, at a given auction the behavior of a player depends on $v - c$ where $c = \delta(w^2 - w^1)$. Usually we have $w^1 < w^2$ to provide incentives for lower than competitive bidding. During the war phase however, a win increases the probability to become the favored player after the war, so the continuation payoff conditional on winning is higher than conditional on losing. So $c < 0$ and players bid more aggressively than in a competitive equilibrium. Hence the name *bidding war*.

As the bidding war continues, the players have different number of wins and hence differ in their current expected payoff. But as the difference $w^2 - w^1$ is independent of the number of past wins, the non-deviant players bid symmetrically in all k auctions (however the bidding functions differ over time, as explained below).

To make the deviant not want to participate in the auction, it is sufficient that in every auction all non-deviant players bid above v^h, as winning in that case would bring the deviant immediate loss and no gain in terms of the continuation payoff. So it is enough to show that $-c > v^h$ at every stage of the bidding war.

At the first period of the bidding war (given that the deviant has zero probability of winning which is consistent with the equilibrium strategy when $-c > v^h$) we have:

$$c = \delta(w^2 - w^1) = -\frac{1}{k}\delta^k(\overline{w} - \underline{w}) = -\frac{1}{k}\delta^k \frac{1-\delta^T}{1-\delta} ev \tag{29}$$

As there are less and less war stages left, c increases (making the firms bid more and more aggressively) so for the incentive for the deviant not to participate in the war is sufficient to find k and T s.t.:

$$v^h \leq \frac{1}{k}\delta^k \frac{1-\delta^T}{1-\delta} ev \tag{30}$$

Now we can check the incentive constraint for the non-favored player in phase 1 of the collusive scheme: It is sufficient to check that a player with value v^h does not have incentives to deviate in the first auction.

If he does deviate he gets:

$$v^h + \delta^{k+1}\underline{w} = v^h + \delta^{k+T+1} w_{ce} \tag{31}$$

If he does not deviate he gets:

$$\underline{w} = \delta^T w_{ce} \tag{32}$$

So it's sufficient to find δ, k, T to satisfy simultaneously:

$$v^h \leq \frac{1}{k}\delta^k \frac{1-\delta^T}{1-\delta} ev \tag{33}$$

$$v^h \leq \delta^T \frac{(1-\delta^{k+1})}{1-\delta} x_{ce} \tag{34}$$

These can be found as:

$$\lim_{\delta \to 1} \frac{1}{k}\delta^k \frac{1-\delta^T}{1-\delta} ev = \frac{T}{k} ev \tag{35}$$

and

$$\lim_{\delta\to 1} \delta^T \frac{(1-\delta^{k+1})}{1-\delta} x_{ce} = (k+1)x_{ce} \tag{36}$$

By continuity of these expressions in δ, there exists $\delta_0 < 1$ such that for all $\delta > \delta_0$ we can find T and k for which the mechanism constitutes a SPNE with expected payoffs:

$$\begin{aligned} \overline{w} &= \frac{1-\delta^T}{1-\delta} ev + \delta^T w_{ce} > w_{ce} \\ \underline{w} &= \delta^T w_{ce} < w_{ce} \end{aligned} \tag{37}$$

So we have constructed an asymmetric equilibrium with $w_i = \overline{w}$ and $w_{-i} = \underline{w}$ s.t. $\overline{w} > w_{ce} > \underline{w}$. But we need an equilibrium in which one player is "punished" and all others are "rewarded": $w_i = \underline{w}' < w_{ce} < w_{-i} = \overline{w}'$.

Unfortunately, simple randomizing between these equilibria s.t. $\underline{w}' = \underline{w}$ and $\overline{w}' = \frac{1}{N-1}\overline{w} + \frac{N-2}{N-1}\underline{w}$ does not necessarily satisfy the condition $w_{ce} < \overline{w}'$,as it is equivalent to $\frac{ev}{N-1} < x_{ce}(N)$ which in general does not have to hold.

However, using the bidding war phase, we can construct a different collusive scheme. Call the player that has to get the lower equilibrium payoff the *excluded* player and all others the *included* players. The mechanism is the following. In the first auction the excluded player is not supposed to bid at all. All included are supposed to bid competitively among themselves. If one of the included players or nobody wins, the continuation equilibrium is competitive with all players. If the excluded player wins, the continuation game is the *bidding war* mechanism constructed above: the excluded player becomes the deviant and all others non-deviant.

Along equilibrium path, this mechanism generates expected payoffs:

$$\underline{w}' = \frac{\delta}{1-\delta} x_{ce}(N) = \delta w_{ce} < w_{ce} \tag{38}$$

for the excluded player and

$$\overline{w}' = x_{ce}(N-1) + \frac{\delta}{1-\delta} x_{ce}(N) > w_{ce} \tag{39}$$

for the included players, where $x_{ce}(N)$ denotes expected payoff in one auction for one player in competitive equilibrium with N players.

In order to complete the proof we only need to show that this mechanism forms an equilibrium.

It's enough to check the incentives of the excluded player. The largest gain he can realize by deviating is:

$$v^h + \delta^{k+T+1} w_{ce} \tag{40}$$

so the sufficient incentive constraint is:

$$v^h \leq \delta(1-\delta^{k+T}) w_{ce} \tag{41}$$

As $\delta(1-\delta^{k+T}) \geq \delta^T(1-\delta^{k+1})$, every time the bidding war mechanism constitutes SPNE, this constraint is satisfied, what completes the proof. ■

Proof of Proposition 4.

The mechanism we use is the one described in the text. We already know that given the continuation payoffs, the sure bidder uses the bidding function $b_i = \max\{0, (v_i - c_i)\}$ and all other included players bid $b = (v - c)$ if $(v - c) \geq 0$ and do not bid at all otherwise.

We construct the proof in steps:

Step 1

Show that there exists M such that if $M - 1$ players are bidding $b_i = v_i - (v^h - r)$ and 1 *player (the sure bidder) bids $b_i = \max\{0, [v_i - (v^h - r)]\}$ then the mechanism satisfies the conditions of asymptotic efficiency.*

Set $r = min\{q, \varepsilon\}$. The $v^{(1)}$ from the whole set of players is bounded above by v^h. So for the third condition of efficiency it's sufficient that M satisfies

$$(1 - F^M(v^h - r)) \geq p \tag{42}$$

This clearly exists. The second condition is trivial because the largest bid consistent with the bidding function is equal to q.
Finally, the first condition is satisfied as the sure bidder always bids.

Step 2

Show that as M increases, the expected total (summed over all players) per-period payoff from the mechanism $b_i = \max\{0, (v_i - c)\}$ for the sure bidder and $b_i = v_i - c$ for all other included players converges to $c < v^h$ from above.

Introduce notation:

$v_M^{(1)}$ is the first order statistic from M draws and

$v_M^{(2)}$ is the second order statistic from M draws

Total (sum over all players) one-period profit from the mechanism is:

$$x_{Total} = (1 - F^M(c))E_{v_M^{(1)}}\left[\left(v_M^{(1)} - Prob(v_M^{(2)} > c)E_{v_M^{(2)}}(v_M^{(2)} - c|v_M^{(2)} > c)\right)|v_M^{(1)} > c\right]$$

$$+F^M(c)E(v|v \leq c) \tag{43}$$

This expression is derived by the fact that with probability $(1 - F^M(c))$ at least one player has value above c. In that case the winner is the one with the highest value $v_M^{(1)}$. His payment is 0 when the second highest valuation is below c and is $v_M^{(2)} - c$ when the second highest valuation is above c. Finally, with probability $F^M(c)$ all players have value below c and in that case the winner is the sure bidder. He pays 0 and has expected value $E(v|v \leq c)$.

To see that this value converges to c is trivial as $F^M(c)$ converges to 1, and the highest and the second highest valuations converge to v^h.

Now let's show that it converges from above.

The expected current auction value of the winner is at least equal to the expected first order statistic given it is larger than c times the probability of $v_M^{(1)}$ being greater than c.

Expected payment is equal to:

$$M\int_c^{v^h}(v-c)\left(f(v)+F(v)-1\right)F^{M-1}(v)dv \tag{44}$$

This leads to:

$$x_{Total} \geq M\int_c^{v^h}\left[vf(v)F^{M-1}(v) - (v-c)\left(f(v)+F(v)-1\right)F^{M-1}(v)\right]dv$$

$$= c(1 - F^{M-1}(c)) + M\int_c^{v^h}(1 - F(v))F^{M-1}(v)dv \tag{45}$$

Therefore to show that the convergence is from above, it is sufficient to show that for large M:

$$cF^{M-1}(c) < M\int_c^{v^h}(1 - F(v))F^{M-1}(v)dv \tag{46}$$

This is done by noting that:

$$M\int_{c}^{v^h}(1-F(v))F^{M-1}(v)dv \geq MF^{M-1}(c)\int_{c}^{v^h}(1-F(v))dv \tag{47}$$

Because $\int_{c}^{v^h}(1-F(v))dv$ is positive and independent of M, there clearly exists M_0 large enough such that for all $M > M_0$, $x_{Total} > c$.

Step 3

Show that for any $c < v^h$ and M_0 *there exist* δ, N *and* T *such that the mechanism implements these bidding functions in every auction and* $N - T = M \geq M_0$.

Start with finding M large enough that $x_{Total} > c$ and $M \geq M_0$. Fix c and M. Now show that we can find δ and T such that the mechanism indeed implements c :

We have a total of N players. Due to the stationarity of the mechanism it's easy to find their expected payoffs given x_{Total}. We have $M = N-T$ included players that are ex ante (before choosing the sure bidder) symmetric. Denote their average per-period expected payoff by x_0. There are also T excluded players that have distinct waiting times to become included. For a player that cannot bid in the next t auctions (including the current) the average per-period expected payoff is $\delta^t x_0$. The sum of all individual average expected payoffs has to be equal to x_{Total}:

$$Mx_0 + \sum_{t=1}^{T}\delta^t x_0 = x_{Total} \tag{48}$$

Therefore:

$$x_0 = \frac{x_{Total}}{M + \delta\frac{1-\delta^T}{1-\delta}} \tag{49}$$

If a player wins he is not allowed to bid in the next T auctions and if he doesn't win he is allowed. So the continuation payoffs are :

$$\begin{aligned} w^1 &= \frac{\delta^{T+1}}{1-\delta}x_0 \\ w^2 &= \frac{\delta}{1-\delta}x_0 \end{aligned} \tag{50}$$

This gives us $c = w^2 - w^1 = \delta\frac{1-\delta^T}{1-\delta}x_0$. Substituting the expression for x_0 we get:

$$c = \frac{z}{M+z}x_{Total} \tag{51}$$

where $z = \delta\frac{1-\delta^T}{1-\delta}$. Because $x_{Total} > c$ and $\lim\limits_{\delta\to 1} z = T$ and $\lim\limits_{T\to\infty} z = \frac{\delta}{1-\delta}$, we can clearly find such a combination of these two to make $\frac{z}{M+z}$ arbitrarily close to 1 and hence in fact implement c.

Step 4

Show that no player has ever incentives to deviate

It's clear that given M and T and δ coming from the previous steps, the included players do not have incentives to deviate from the prescribed bidding functions. The question is if the excluded players do as well.

It's sufficient to consider incentives of a player that is excluded for the next T auctions (i.e. a player who won in the previous auction) and got a draw v^h in the current one.

If he bids he can gain at most v^h in the current auction and his continuation payoff is set to zero if he does. If he doesn't bid, his current profit is zero and his continuation payoff is

$$\frac{\delta^{T+1}}{1-\delta}x_0 = \frac{\delta^{T+1}}{1-\delta}\frac{x_{Total}}{M+z} \tag{52}$$

It's sufficient to show that we can choose T and δ such that both $z \to \infty$ and $\frac{\delta^{T+1}}{1-\delta}/z \to \infty$, because M is fixed and incentives require:

$$v^h \leq \frac{\delta^{T+1}}{1-\delta}x_0 \tag{53}$$

This is done by making δ to grow to 1 and then increasing (slowly) T, as:

$$\lim_{\delta\to 1} z = T \quad \text{and} \quad \lim_{\delta\to 1}\frac{\delta^{T+1}}{1-\delta}/z = \lim_{\delta\to 1}\frac{\delta^{T+1}}{\delta-\delta^{T+1}} = \infty. \tag{54}$$

Now look at the interception of the conditions obtained so far. Clearly the maximum of M's found in steps 1 and 2 satisfies conditions in both of them. Then, fix c and M what gives x_{Total}. Next, find $\frac{\delta^{T+1}}{1-\delta}/z$ large enough to satisfy incentive constraint in Step 4. Finally given this value we find T and δ that satisfy jointly conditions in Steps 3 and 4 (they clearly exist as we have shown in Step 4). Of course throughout the construction we have to take care of the integer problems, but as δ can change smoothly this is possible. ■

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