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1. *Introduction*. *Bargaining theory* is concerned with the formulation of rules to select, for each *bargaining problem* (S,d)--where S is a set of feasible alternatives and d is the disagreement point, both being given in the utility space--a point of S, suggested as a compromise among the agents' conflicting interests. Various such rules, or *solutions*, have been proposed, starting with Nash's solution (1950), defined in his seminal article. Other solutions were introduced later, the solutions proposed by Kalai-Smorodinsky (1975) and Perles-Maschler (1981) being important alternative ones.

Various properties of solutions have been extensively studied. Of particular interest are monotonicity properties, an example of which is the property which we will call "strong monotonicity:" the solution F is strongly *monotonic* if, given two problems (S,d) and (S',d') with S' containing S and d=d', the compromise F(S',d') specified for (S',d') dominates the compromise F(S,d) specified for (S,d). It is known that none of the three solutions mentioned earlier satisfy this property. We are interested here in finding out whether such negative results are preserved when the solutions are applied to economics.

First, we should say a few words on how solutions can be applied to economics. An economic problem of fair division is given by specifying a list of agents equipped with utility functions, together with an aggregate endowment. The agents have "equal rights" on the resources and the aim is to determine allocations at which all these rights have been properly recognized. A solution F to the bargaining problem can be used to solve an economic problem of fair division by simply taking S to be the image in the utility space of the set of feasible allocations and d to be the image of the zero allocation, solving (S,d) according to F, (this of course requires that some regularity assumptions be satisfied by the economy under consideration), and finally selecting the allocations whose images in utility space coincide with F(S,d).

In such a context, the typical circumstance that would cause expansions of opportunities as described in the formulation of the property of strong monotonicity is an increase in the aggregate endowment, and the property of interest is whether a given solution would respond to such a change by recommending that all agents benefit from it. Of course the monotonicity properties of solutions when applied to bargaining problems as compared to economic problems need not be the same, and our object is to determine the extent to which properties will carry over.

Billera and Bixby (1973) establish conditions under which a bargaining problem can be seen as the image in the utility space of some economic problem, but their results are not relevant here, and this for two reasons. First of all, one of their conditions is that the number of commodities available be not too small in relation to the number of agents, while we would like to analyze economies with no a priori restrictions on the numbers of commodities and agents. Second, and more importantly, what we would really need are results on pairs of problems: taking as an example the property of strong monotonicity discussed above, in order to be able to deduce whether a given solution satisfies it when applied to economics, from the knowledge that it does or does not satisfy it on the domain of abstract bargaining problems, we would have to be able to answer the following question: given any two bargaining problems with one containing the other, under what conditions can these problems be seen as the images in the utility space of two economic problems with the same agents but with two different aggregate endowments, one dominating the other. Questions of this type have, to our knowledge, not been analyzed in the literature. One can guess that conditions on the number of commodities would appear here just as they did above. For these reasons, we will address the question that interests us directly.

This issue concerning the number of commodities is of great interest. Indeed, the most widely discussed problem of fair division is the so-called "divide the dollar game" (see Luce and Raiffa (1957)) in which two agents can receive a dollar if they agree on a way to divide it and nothing otherwise. It is important to know whether the insights that have been gained by the study of this canonical problem extend to more general situations and in particular to the case of more than 1 good and 2 agents.

As it turns out, as shown here, the divide the dollar game is indeed exceptional: bargaining solutions tend to behave quite well in that case; however, as soon as the number of commodities is two, solutions are much less well behaved. In particular, the Nash and Kalai-Smorodinsky solutions, although they are strongly monotonic in the one-commodity, two-person case lose this property as soon as the number of commodities increases to two. The Perles-Maschler solution fares the worst; it is not strongly monotonic even in the one-commodity case. Similar results are obtained for two related but weaker properties that we also consider: "individual monotonicity" and "weak monotonicity." These properties pertain to situations in which expansions in the feasible set are constrained, as would result when agents are initially satiated. A final property we study is "population monotonicity," i.e. whether an increase in population size, unaccompanied by an increase in resources, causes all agents originally present to share the burden of supporting the newcomers. There again the Nash solution behaves well in the one-commodity case but not as soon as the number of commodities is greater than one. (The Kalai-Smorodinsky solution behaves well in the general class of bargaining problems, as discussed in Thomson (1983).)

These results should help clarify the usefulness of bargaining solutions in solving economic problems of fair division. They show the fallacy of relying on the "divide the dollar game" as the main paradigm and identify $\ell = 1$ as the exact bound on the number of commodities beyond which solutions start behaving in the way they do for abstract bargaining problems. Since the Egalitarian solution trivially satisfies all of the properties, our result should reinforce its appeal.

2. *Definitions*. *Notation*. $E(\ell,n)$ is the class of *exchange economies with* $\ell \in \mathbb{N}$[1] *commodities and* $n \in \mathbb{N}$ *agents*, in which each agent is characterized by his consumption set $\mathbb{R}^{\ell}_{+}$ and his utility function $u_i: \mathbb{R}^{\ell}_{+} \to \mathbb{R}_{+}$, which is continuous, concave, non-constant and non-decreasing and satisfies $u_i(0) = 0$. $\Omega \in \mathbb{R}^{\ell}_{+}$ is the economy's aggregate endowment. A typical element of $E(\ell,n)$ is denoted (u,Ω) with $u = (u_1,\ldots,u_n)$.

An *n-person bargaining problem* is a pair (S,d) of a convex, compact subset S of $\mathbb{R}^n$ and of a point $d \in S$, strictly dominated by at least one point of S. Each point of S is a utility vector attainable by the n agents through some joint action, and d, the "disagreement point", is the utility vector that would result if the agents failed to reach a compromise among the points of S. An *n-person bargaining solution* or simply a *solution* is a function defined on the class Σ^n of these problems, which associates with every $(S,d) \in \Sigma^n$ a point of S, interpreted as the recommended compromise.

Solutions can be applied to the problem of fairly dividing a bundle Ω among n agents with utility functions $u_1,\ldots,u_n$ as follows. Given $(u,\Omega) \in E(\ell,n)$, let $Y(u,\Omega) \equiv \{x = (x_1,\ldots,x_n) \in \mathbb{R}^{\ell n}_{+} | \Sigma x_i \leq \Omega\}$[2] be the set of feasible allocations of (u,Ω) and let $S \subseteq \mathbb{R}^n_{+}$ and $d \in \mathbb{R}^n_{+}$ be defined by

$$\begin{cases} S \equiv \{\bar{u} \in \mathbb{R}^n_+ \mid \exists\, x \in Y(u,\Omega) \text{ s.t. } \forall i,\ u_i(x_i) = \bar{u}_i \\ d = 0 \end{cases}$$

Note that (S,d) is an element of Σ^n if $\Omega > 0$ so that given a solution F, $F(S,d)$ can be determined. The allocations $x \in Y(u,\Omega)$ such that $u(x) = F(S,d)$ are called the *F-allocations* of (u,Ω). They are the allocations recommended by F for (u,Ω). With a slight abuse of notation, the set they constitute is denoted $F(u,\Omega)$, just as the utility vector was denoted $F(S,d)$.

Under the assumptions imposed above on economies, more can actually be said about the set S associated with any economy $(u,\Omega) \in E(\ell,n)$: S is comprehensive (i.e. for all $\bar{u}$, $\bar{u}' \in \mathbb{R}^n_+$, if $\bar{u} \in S$ and $\bar{u} \geq \bar{u}'$, then $\bar{u}' \in S$). Since we are interested in problems obtained in this way, we explicitly introduce the class $\Sigma^n_e \subset \Sigma^n$ of "economic" problems: such a problem is a subset S of $\mathbb{R}^n_+$ which is convex, compact and comprehensive and contains at least one point strictly dominating the origin. Note that the disagreement point is omitted from the notation. It should be understood to be the origin.

The set of Pareto-optimal points of S is denoted $PO(S)$; it is defined by $PO(S) \equiv \{\bar{u} \in S \mid \nexists\, \bar{u}' \in S$ with $\bar{u}'_i \geqq \bar{u}_i$ for all i and $\bar{u}'_i > \bar{u}_i$ for some $i\}$.

The domain Σ^n_e is of particular interest not only because it arises naturally from an examination of economic problems but also because solutions are often better behaved on it than on Σ^n. In particular, all the solutions that we will consider select weakly Pareto-optimal, and usually Pareto-optimal, points on this domain. The other properties of solutions that will be central to our analysis are:

strong monotonicity (S.M.): For all $S, S' \in \Sigma_e^n$ with $S' \supset S$, $F(S') \geqq F(S)$.

individual monotonicity (I.M.): For all $S, S' \in \Sigma_e^n$ with $S' \supset S$, for all i, if $\{\bar{u} \in S \mid \bar{u}_i = 0\} = \{\bar{u} \in S' \mid \bar{u}_i = 0\}$, then $F_i(S') \geqq F_i(S)$.

weak monotonicity (W.M.): For all $S, S' \in \Sigma_e^n$ with $S' \supset S$, if for all i, $\max\{u_i \mid u \in S\} = \max\{u_i \mid u \in S'\}$, then $F(S') \geqq F(S)$.

Note that S.M. implies I.M. and W.M. and that if $n=2$, I.M. implies W.M.

The last property concerns solutions that are defined on $\bigcup_{n \in \mathbb{N}} \Sigma_e^n$.

population monotonicity (P.M.): For all $n, n' \in \mathbb{N}$ with $n > n'$, for all $S \in \Sigma_e^n$, $S' \in \Sigma_e^{n'}$, if $S' = \{\bar{u}' \in \mathbb{R}_+^{n'} \mid \exists \bar{u} \in S$ with $\bar{u}_i' = \bar{u}_i \ \forall i \leq n'\}$, then $F_i(S) \leqq F_i(S')$ for all $i \leq n'$.

These four properties were respectively introduced by Luce and Raiffa (1957), Kalai and Smorodinsky (1975), Roth (1979) and Thomson (1983).

Next, we define the solutions on which our analysis will focus.

Given $S \in \Sigma_e^n$, for the Nash (1950) solution, $N(S)$ is the unique maximizer of Πx_i for $x \in S$; for the Kalai-Smorodinsky (1975) solution, $K(S)$ is the maximal point of S on the segment connecting the origin to the point $a(S)$ where $a_i(S) \equiv \max\{\bar{u}_i \mid \bar{u} \in S\}$; for the Egalitarian solution, $E(S)$ is the maximal point of S with equal coordinates; for a Utilitarian solution, $U(S)$ is a maximizer of Σx_i for $x \in S$; finally, for the Perles-Maschler (1981) 2-person solution, and assuming that S is polygonal, PM(S) is the first point in common of the sequences $\{A_k\}$ and $\{B_k\}$ constructed as follows: A_0 and B_0 are the points of PO(S) the closest to the 1^{st} and 2^{nd} axes respectively; A_1 and B_1 are the points of PO(S) such that (i) the segments $[A_0,A_1]$ and $[B_0,B_1]$ are contained in PO(S), (ii) they do not overlap, although their endpoints A_1 and B_1 may coincide, (iii) the areas of the right-angle triangles with hypothenuses $[A_0,A_1]$ and $[B_0,B_1]$ are equal and maximal; ...; A_k and B_k for k=2,..., are defined from A_{k-1} and B_{k-1} in a similar way. This construction is illustrated in Figure 1 with an example for which the limits of the sequences are obtained in three steps. (The definition of PM(S) when S is not a polygonal problem is given by a limiting argument involving approximations of S by polygonal problems; here we will need only polygonal problems.)

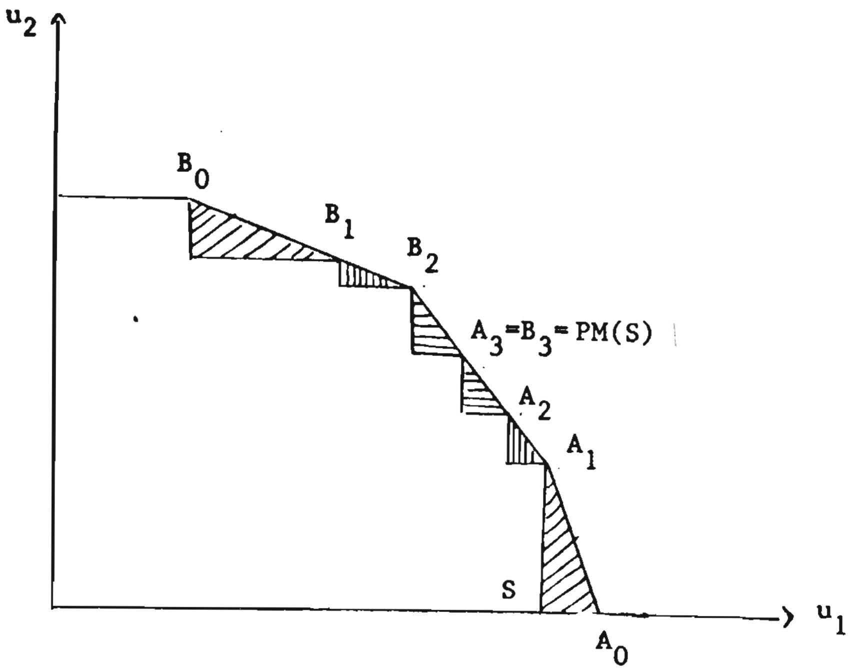

Figure 1

In the economic context in which we are placing our analysis of monotonicity properties of solutions, the parameter changes that are of interest are increases and decreases in Ω, keeping the number of agents constant for the first three, and increases and decreases in the number of agents keeping Ω constant for the last one. An increase in Ω leads to an expansion of the feasible set as described in the hypotheses of S.M. If each agent i satiated at Ω, the hypotheses of W.M. will be satisfied. Finally, an increase in the number of agents accompanied by no change in Ω and no external effects will lead to expansion as described in the hypothesis of P.M.

3. *The results*. First, we consider the Nash solution. It is known from Kalai and Smorodinsky (1975) that if n=2 the Nash solution is not weakly monotonic and therefore not individually monotonic and *a fortiori* not strongly monotonic. However, when applied to economic division problems with one commodity, it is strongly monotonic, and this independently of the number of agents. Unfortunately, this result does not extend to the case of more than one commodity, and even weak monotonicity is violated as soon as there are two commodities. These negative results hold as soon as $n \geqq 2$.

Theorem 1. The Nash solution is strongly monotonic on $\mathcal{E}(1,n)$ for all n.

Proof. For simplicity we consider an economy $(u,\Omega) \in \mathcal{E}(1,n)$ for which all utility functions are differentiable and increasing. (The other cases can be dealt with by an approximation argument

making use of the fact that the Nash solution is continuous).
$N(u,\Omega)$ is obtained by maximizing $\Pi u_i(x_i)$ in $x \in \mathbb{R}_+^n$ subject to $\Sigma x_i = \Omega$. This problem is solved by requiring that

$$(1) \qquad \frac{u_i'(x_i)}{u_i(x_i)} = \frac{u_j'(x_j)}{u_j(x_j)} \text{ for all } i,j \text{ and } \Sigma x_i = \Omega$$

Since each u_i is concave and increasing, each function $g_i: \mathbb{R}_+ \to \mathbb{R}$ defined by $g_i(x_i) = \frac{u_i'(x_i)}{u_i(x_i)}$ is decreasing. If Ω increases, (1) can be preserved only if all x_i increase, which implies that all agents gain.

QED

Theorem 2: The Nash solution is not strongly monotonic on $E(\ell,n)$ as soon as $(\ell,n) \geqq (2,2)$.

Proof. (i) The main step of the proof consists of an example with $(\ell,n) = (2,2)$, illustrated in Figure 2. The other values of (ℓ,n) are dealt with later.

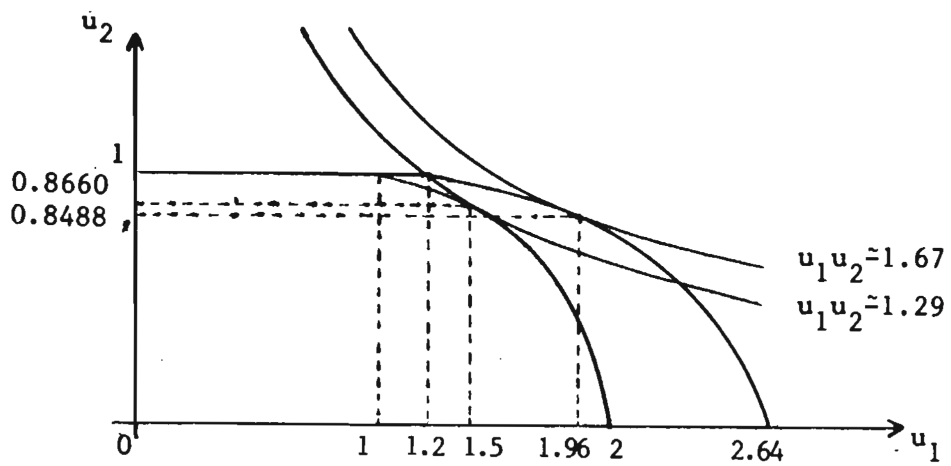

Figure 2

Let $u_1,u_2: \mathbb{R}_+^2 \to \mathbb{R}_+$ be defined by

$$\begin{cases} u_1(x_1) = x_{11}^{1/2}\, x_{12}^{1/2} + x_{12}^{1/4} \\ u_2(x_2) = x_{21}^{1/2} \end{cases}$$

At first, we assume that $\Omega = (1,1)$. $N(u,\Omega)$ is obtained by solving

$$\max_{x_1, x_2} u_1(x_1)u_2(x_2) \text{ s.t. } x_1 + x_2 \leqq (1,1).$$

Since u_2 is independent of x_{22} and u_1 is increasing in x_{12}, solving this problem requires setting $x_{22} = 0$ and $x_{12} = 1$. We are led to solving

$$\max_{x_{11}} (x_{11}^{1/2} + 1)(1 - x_{11})^{1/2} \text{ s.t. } 0 \leqq x_{11} \leqq 1.$$

After differentiating, we obtain the equation in x_{11}

$$2x_{11} + x_{11}^{1/2} - 1 = 0$$

whose unique positive solution is

$$x_{11} = 1/4.$$

Then $x_{21} = 3/4$ and the agents' utilities are $u_1 = 1.5$, $u_2 = \frac{\sqrt{3}}{2} \simeq 0.8660$.

Next, we assume that Ω increases to $\Omega' = (1,(1.2)^4)$. An analogous reasoning leads to the problem

$$\max_{x_{11}} (1.44\, x_{11}^{1/2} + 1.2)(1 - x_{11})^{1/2} \text{ s.t. } 0 \leqq x_{11} \leqq 1,$$

which yields the equation in x_{11}

$$2.88\, x_{11} + 1.2\, x_{11}^{1/2} - 1.44 = 0$$

whose unique positive solution is

$$x_{11} \simeq 0.2796.$$

The utilities are then $u_1 = 1.9614$ and $u_2 \simeq 0.8488$. Agent 2 has lost in spite of the increase in the social endowment. This concludes the proof for case (i).

(ii) Next, we take care of the case $n > 2$. Let S and S' be the problems obtained in (i) for Ω and Ω' respectively. We now introduce $n-2$ additional agents with utility functions $u_k: \mathbb{R}^2_+ \to \mathbb{R}$ for $2 < k \leqq n$, defined by

$$u_k^\epsilon(x_k) = \begin{cases} \frac{1}{\epsilon}(x_{k1} + x_{k2}) & \text{if } x_{k1} + x_{k2} < \epsilon \\ 1 & \text{otherwise} \end{cases}$$

where $\epsilon > 0$. (The motivation for introducing such agents is that they become satiated quickly if ϵ is small so that small amounts of resources will ever be allocated to them by any solution satisfying Pareto-optimality, such as the Nash solution. Thus their presence will hardly disturb the problem faced by the agents originally present).

First, let $T \in \mathbb{R}^n_+$ be defined by $T \equiv \{\bar{u} \in \mathbb{R}^n_+ | (\bar{u}_1, \bar{u}_2) \in S, u_k \leqq 1 \ \forall k > 2\}$. T is a truncated cylinder with base S; it is represented on Figure 3 in the case $n = 3$, for which k takes only the value $k = 3$.

Now it is easily checked (see Thomson (1984) for a study of related properties of solutions) that $N(T)$ projects onto the coordinate plane pertaining to the original two agents at $N(S)$: in fact $N_1(T) = N_1(S)$, $N_2(T) = N_2(S)$ and $N_k(T) = 1$ for all $k > 2$. Similarly, if T' is constructed from S' in the same way T was constructed from S, we have that $N_1(T') = N_1(S')$, $N_2(T') = N_2(S')$ and $N_k(T) = 1$ for all $k > 2$.

Let $T^\epsilon, T'^\epsilon \in \Sigma^n_e$ be the problems derived from the economies with utility functions u_1, u_2 and u_k^ϵ for $k > 2$, and aggregate endowments $\Omega = (1,1)$ and $\Omega' = (1,(1.2)^4)$ respectively. Note that

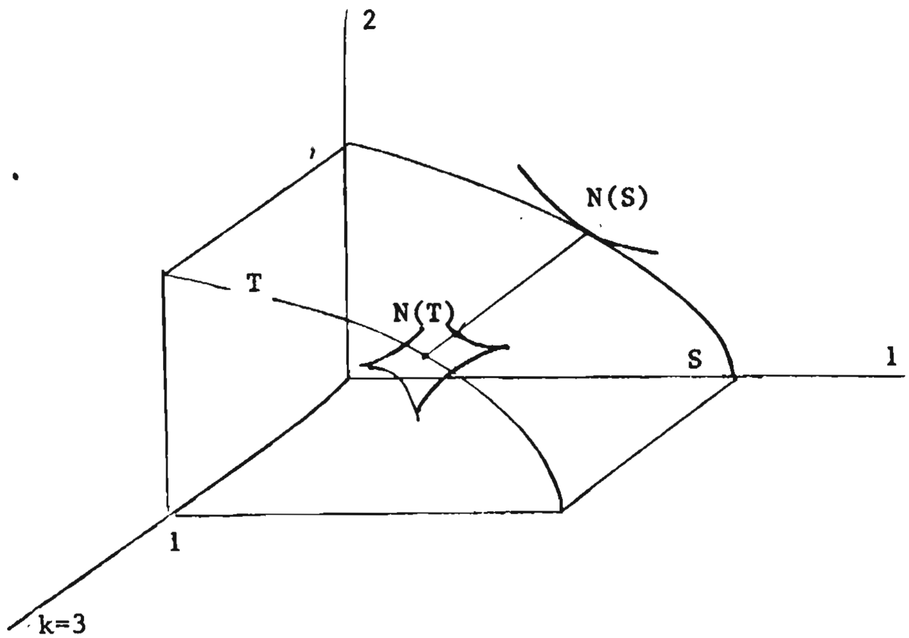

Figure 3

as $\epsilon \to 0$, $T^{\epsilon} \to T$ and $T'^{\epsilon} \to T'$. Since the Nash solution is continuous, we have that $N(T^{\epsilon}) \to N(T)$ and $N(T'^{\epsilon}) \to N(T')$. Given that $N_2(T) = N_2(S) > N_2(T') = N_2(S')$, we obtain that for ϵ small enough, $N_2(T^{\epsilon}) > N_2(T'^{\epsilon})$, the desired conclusion.

(iii) We conclude by examining the case $\ell > 2$. To that effect, it suffices to modify the example of (ii) by introducing utility functions $v_i\colon \mathcal{R}_+^{\ell} \to \mathcal{R}_+$ related to the u_i defined earlier by

$$v_i(x_i) = u_i(x_{i1}, x_{i2}) \text{ for all } x_i \in \mathcal{R}_+^{\ell} \text{ and for all } i,$$

since this choice will leave unaffected the problems T and T', T^{ϵ} and T'^{ϵ}.

QED

Remark 1. The example of (i) can be modified to show that the Nash solution does not satisfy weak monotonicity either on $E(2,n)$ for all $n \geq 2$; it suffices to replace u_1 by w_1 defined by:

$$w_1(x_1) = \min\{u_1(x_1), 2\} \text{ for all } x_1 \in \mathcal{R}_+^2.$$

The derived $\tilde{S}$ and $\tilde{S}'$ are such that $\tilde{S} = S$ while $\tilde{S}'$ is obtained from S' by a truncation by a vertical line of abscissa $a_1(S)$. (Recall that $a_i(S) \equiv \max\{\bar{u}_i | u \in S\}$.) Obviously $N(\tilde{S}) = N(S)$ and since $N_1(S') < a_1(S)$, $N(\tilde{S}') = N(S')$. The claim is proved by noting that $a(\tilde{S}) = a(\tilde{S}')$.

Remark 2. It is conceivable that the imposition of additional conditions on the utility functions would permit to extend the positive result of Theorem 1. We have determined that if the utility functions are separable additive, the Nash solution is strongly monotonic for $(\ell,n) = (2,2)$ and $(\ell,n) = (3,2)$. It is an open question whether this result extends to other values of (ℓ,n).

Remark 3. Theorem 1 can easily be generalized to the case of solutions defined by the maximization of a function of the form

$$\Sigma f_i(u_i) \text{ for } u \in S,$$

where each f_i is concave. This family of solutions include in particular all CES solutions. (Solutions obtained by maximizing a function with constant elasticity of substitution.) The Utilitarian solution is covered by such a generalization.

Next, we turn to the Kalai-Smorodinsky solution. Although this solution satisfies individual monotonicity (it is principally on this fact that Kalai and Smorodinsky based their characterization of the solution for n=2) and weak monotonicity, it violates strong monotonicity. We show here that it does satisfy strong monotonicity on $\mathcal{E}(1,2)$ but not on $\mathcal{E}(\ell,n)$ for any other pair (ℓ,n). This is in unfavorable contrast to the Nash solution.

Theorem 3. The Kalai-Smorodinsky solution is strongly monotonic on $\mathbb{E}(1,2)$.

Proof. For simplicity, we consider an economy $(u,\Omega) \in \mathbb{E}(1,2)$ for which the utility functions are twice differentiable and increasing (the other cases are dealt with by making use of the continuity of the KS solution). Then $K(u,\Omega)$ is obtained by finding the unique $x \in \mathbb{R}^2_+$ such that

$$(1) \qquad \frac{u_1(x_1)}{u_1(\Omega)} = \frac{u_2(x_2)}{u_2(\Omega)} \text{ s.t. } x_1 + x_2 = \Omega$$

Let $(x_1(\Omega), x_2(\Omega))$ be the solution to this equation. By substituting into (1) we obtain an identity both sides of which can be differentiated with respect to Ω, yielding a second identity

$$(2) \qquad f_1(\Omega) \equiv f_2(\Omega) \text{ where}$$

$$f_i(\Omega) = \frac{u_i'(x_i(\Omega))}{u_i(\Omega)} x_i'(\Omega) - \frac{u_i(x_i(\Omega))}{(u_i(\Omega))^2} u_i'(\Omega) \text{ for } i = 1,2.$$

Suppose now that $x_1'(\Omega) < 0$ for some Ω. Since $x_1(\Omega) + x_2(\Omega) = 1$ for all Ω, we conclude that $x_1'(\Omega) + x_2'(\Omega) = 1$ for all Ω. Therefore $x_2'(\Omega) > 1$. Since u_2 is concave and

$$x_2(\Omega) \leqq \Omega, \quad \frac{u_2'(x_2(\Omega))}{u_2(x_2(\Omega))} \geqq \frac{u_2'(\Omega)}{u_2(\Omega)}.$$

This, in conjunction with $x_2'(\Omega) > 1$ yields that $f_2(\Omega) > 0$. But if $x_1'(\Omega) < 0$, $f_1(\Omega) < 0$. These two statements on the signs of $f_1(\Omega)$ and $f_2(\Omega)$ are incompatible with (2).

QED

Theorem 4. The KS solution is not strongly monotonic on $E(\ell,n)$ whenever $(\ell,n) \neq (1,2)$.

Proof. (i) We first prove that the KS solution is not strongly monotonic on $(1,3)$. The proof is by way of an example. Let $u_1,u_2,u_3: \mathbb{R}_+ \to \mathbb{R}_+$ be defined by

$$u_1(x_1) = \begin{cases} 4\ x_1 & \text{if} \quad 0 \leqq x_1 \leqq 5 \\ (1/2)\ x_1 + 35/2 & \text{if} \quad 5 \leqq x_1 \leqq 25 \\ 30 & \text{if} \quad 25 \leqq x_1 \end{cases}$$

$$u_1(x_2) = \begin{cases} 2\ x_2 & \text{if} \quad 0 \leqq x_2 \leqq 10 \\ (2/3)\ x_2 + 40/3 & \text{if} \quad 10 \leqq x_2 \end{cases}$$

$$u_3(x_3) = \begin{cases} 2\ x_3 & \text{if} \quad 0 \leqq x_3 \leqq 10 \\ (2/3)\ x_3 + 40/3 & \text{if} \quad 10 \leqq x_3 \end{cases}$$

First, we assume that $\Omega = 25$. We claim that the KS allocation is $(5,10,10)$. Indeed,

$$\frac{u_1(5)}{u_1(25)} = \frac{u_2(10)}{u_2(25)} = \frac{u_3(10)}{u_3(25)} = 20/30$$

and feasibility holds.

Next, we increase Ω to $\Omega' = 40$. Then $u_1(40) = 30$, $u_2(40) = u_3(40) = 40$. The new KS allocation $x = (x_1,x_2,x_3)$ has to satisfy

$$\frac{u_1(x_1)}{30} = \frac{u_2(x_2)}{40} = \frac{u_3(x_3)}{40} \text{ and } x_1 + x_2 + x_3 = 40.$$

Since agents 2 and 3 are identical and the KS solution treats identical agents identically, $x_2 = x_3$ and therefore $x_1 = 40 - 2x_2$. We can determine the KS allocation by solving the following equation in x_2:

$$\frac{4(40-2x_2)}{30} = \frac{(2/3)x_2+40/3}{40}.$$

The solution is $x_2 = x_3 \simeq 17.647$ and $x_1 \simeq 4.706$. This is the desired result; since agent 1 has lost as resources increased from Ω to Ω'.

(ii) Next, we take care of the case $(1,n)$ for $n > 3$. Starting from the 3-person economy (u,Ω) defined in step (i), let S be its utility feasibility set S; and let us add n-3 new agents with utility functions $u_k^\epsilon: \mathbb{R}_+ \to \mathbb{R}_+$ for $4 \leqq k \leqq n$, defined by

$$u_k^\epsilon(x_k) = \begin{cases} \frac{1}{\epsilon}x_k & \text{if } 0 \leqq x_k \leqq \epsilon \\ 1 & \text{if } \epsilon \leqq x_k \end{cases}$$

As $\epsilon \to 0$, we find that the enlarged economy yields a utility feasibility set T^ϵ which converges to $T \equiv \{x \in \mathbb{R}^n | (x_1,x_2,x_3) \in S, x_k = 1 \text{ for all } k > 3\}$. Also $K(T^\epsilon) \to K(T)$, since the KS solution is continuous. The other steps of the argument are analogous to those in (ii) of the proof of Theorem 2.

(iii) The case of arbitrary ℓ and $n > 3$ is treated as in (iii) of the proof of Theorem 2.

(iv) Finally, we discuss the case $(\ell,n) = (2,2)$. The proof is by way of an example, the same as the example used in step (i) of the proof of Theorem 2. First, we assume that $\Omega = (1,1)$. Then $u_1(\Omega) = 2$ and $u_2(\Omega) = 1$. The KS allocation satisfies

$$\frac{x_{11}^{1/2} + 1}{2} = (1-x_{11})^{1/2}$$

since u_2 is independent of x_{22} and therefore x_{22} should be set equal to 0, and x_{12} to 1. Solving this equation yields $x_{11} = 0.36$, $x_{21} = 0.64$, and corresponding utilities $u_1 = 1.6$ and $u_2 = 0.8$.

Next, we assume that Ω increases to $\Omega' = (1, (1.2)^4)$. An analogous calculation leads to the KS allocation $x_{11} \simeq 0.3769$, $x_{21} \simeq 0.6231$, $x_{12} = (1.2)^4 = 2.0736$ and $x_{22} = 0$, and corresponding utilities $u_1 \simeq 2.0840$, $u_2 \simeq 0.7894$. Agent 2 has lost as resources increased from Ω to Ω'.-

QED

Finally, we consider the Perles-Maschler solution. This solution satisfies very few of the properties that have been advocated in bargaining theory. The unsatisfactory behavior of the solution is confirmed here since, even if $(\ell, n) = (1,2)$, it does not satisfy W.M. We do not consider the case of more agents since the generalization of the solution to that case has not appeared in print.

Theorem 5. The Perles-Maschler solution is not weakly monotonic on $E(\ell,2)$ for all ℓ.

Proof. The proof is by way of an example, illustrated in Figure 4. Let $u_1, u_2 \colon \mathbb{R}_+ \to \mathbb{R}_+$ be given by

$$u_1(x_1) = \begin{cases} (9/5)\ x_1 & \text{if} \quad 0 \leqq x_1 \leqq 5 \\ (1/5)\ x_1 + 8 & \text{if} \quad 5 \leqq x_1 \leqq 15 \\ 11 & \text{if} \quad 15 \leqq x_1 \end{cases}$$

and

$$u_2(x_2) = \begin{cases} (1/5)\ x_2 & \text{if} \quad 0 \leq x_2 \leq 15 \\ 3 & \text{if} \quad 15 \leq x_2 \end{cases}$$

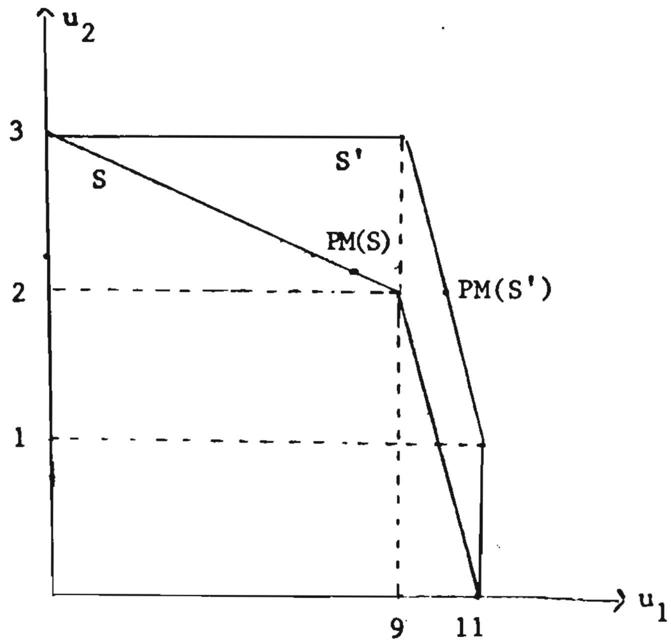

Figure 4

We assume first that $\Omega = 15$. Then, the utility feasibility set is given by $S \equiv \text{cch}\{(11,0),\ (9,2),\ (0,3)\}$, these three points being the images of the allocations $(15,0)$, $(5,10)$ and $(0,15)$ respectively. The PM solution outcome of S is $(\frac{15}{2}, \frac{13}{6})$; it is obtained by noting that the areas of the triangles with vertices $(11,0)$, $(9,2)$ and $(9,0)$ on the one hand and $(0,3)$, $(0,\frac{7}{3})$, $(6,\frac{7}{3})$ on the other are equal and that $(\frac{15}{2}, \frac{13}{6})$ is the midpoint of the segment $[(9,2),\ (6,\frac{7}{3})]$.

For $\Omega' = 20$, the utility feasibility set is given by $S' \equiv \text{cch}\{(11,1),\ (9,3)\}$, these two points being the images of the allocations (15,5) and (5,15) respectively. The PM solution outcome of S' is (10,2); it is obtained by noting that (10,2) is the midpoint of the segment [(11,1), (9,3)].

We conclude by observing that as Ω increases to Ω', agent 2's utility decreases from $\frac{13}{6}$ to 2, in violation of weak monotonicity which applies since $S' \supset S$, and $a(S) = a(S')$.

This takes care of the case $\ell = 1$. The case of arbitrary ℓ is dealt with as in the previous theorems.

QED

The last property we consider is population monotonicity. Since KS satisfies the property in general (Thomson, 1983), and as pointed out earlier, the PM solution is not defined for more than two persons, we limit our attention to the Nash solution. It is known that the Nash solution does not satisfy the property in general (Thomson, 1983). However, we have:

Theorem 6. The Nash solution is population monotonic on $E(1,n)$ for all n.

Proof. For simplicity we consider an economy $(u,\Omega) \in E(1,n)$ for which utility functions are differentiable and increasing. Then, we recall that the Nash allocation is given by $x_1,\dots,x_n \in \mathbb{R}_+$ such that $\Sigma x_i = \Omega$ and

$$\frac{u_i'(x_i)}{u_i(x_i)} = \frac{u_j'(x_j)}{u_j(x_j)} \quad \text{for all } i,j.$$

Also recall that the function

$$\frac{u_i'(x_i)}{u_i(x_i)}$$

is non-increasing for all i, and finally, note that the increase in the number of agents yields the same effect as the decrease in the social endowment on the original agents.

QED

Theorem 7. The Nash solution is not population monotonic on $E(\ell,n)$ if $\ell \geq 2$ and $n \geq 2$.

Proof. (i) We first consider the case $\ell = 2$ and we show that the Nash solution is not population monotonic when n increases from 2 to 3. The proof is by way of an example. Let u_1, u_2 be as in Theorem 2 and $\Omega = (1,1)$. There, we already computed that at the Nash allocation x, $x_{21} = \frac{3}{4}$ and $x_{22} = 0$.

Next, we assume that a third agent enters the scene with $u_3\colon \mathbb{R}^2_+ \to \mathbb{R}_+$ defined by $u_3(x_3) = x_{32}^{1/2}$. The Nash solution is obtained by solving

$$\max_{x_{11},x_{12}} \; (x_{11}^{1/2}x_{12}^{1/2} + x_{12}^{1/4})\,(1-x_{11})^{1/2}(1-x_{12})^{1/2},$$

s.t. $0 \leqq x_{11} \leqq 1$, $0 \leqq x_{12} \leqq 1$

using the fact that u_2 is independent of x_{22} and u_3 of x_{31}.

After differentiating, we obtain the equation in x_{11}, x_{12},

$$2x_{11} + x_{11}^{1/2}\, x_{12}^{-1/4} = 1.$$ [3]

From the feasibility condition, $x_{12} \leqq 1$. At the Nash allocation, $x_{12} < 1$. This implies that $x_{11} < 1/4$. Therefore $x_{21} > 3/4$ and $x_{22} = 0$. Agent 2 gains upon the arrival of agent 3.

(ii) In order to take care of the case when the economy enlarges from an element of $E(2,n)$ to an element of $E(2,m+n)$ we introduce $m+n-3$ additional agents with utilities $u_k\colon \mathbb{R}^2_+ \to \mathbb{R}_+$ defined by

$$u_k^{\epsilon}(x_k) = \begin{cases} \frac{1}{\epsilon} x_k & \text{for} \quad 0 \leq x_{k1} + x_{k2} \leq \epsilon \\ 1 & \text{for} \quad \epsilon \leq x_{k1} + x_{k2} \end{cases}$$

The argument concludes as in the proof of Theorem 2.

(iii) The case of arbitrary ℓ can be treated as it was in the proof of Theorem 2.

QED

Remark. As discussed in a previous remark concerning the strong monotonicity of the Nash solution, the positive result of Theorem 6 can be extended to solutions defined by the maximization of a function of the form $\Sigma f_i(u_i)$ for $u \in S$.

Footnotes

1. $\mathbb{N}$ designates the positive integers and $\mathbb{R}$ the real numbers.
2. Vector inequalities : $x \geqq y$, $x \geq y$, $x > y$.
3. Given a list $x^1,\ldots,x^k$ of points in $\mathbb{R}^n_+$, $cch\{x^1,\ldots,x^k\}$ denotes the convex and comprehensive hull of these points, i.e. the smallest subset of $\mathbb{R}^n_+$ containing these points which is convex and comprehensive (for all $x,y \in \mathbb{R}^n_+$, if $x \in S$ and $x \geq y$, then $y \in S$).
4. In order to obtain the inequality "$x_{11} < 1/4$", which is all that we need, it suffices to use this first-order condition. The precise caluclation of x_{11} requires the use of the second first-order condition too. We omit these calculations.

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