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**Abstract**

We construct "simple" games implementing in Nash equilibria several solutions to the problem of fair division. These solutions are the no-envy solution, which selects the allocations such that no agent would prefer someone else's bundle to his own, and several variants of this solution. Components of strategies can be interpreted as allocations, consumption bundles, permutations, points in simplices of dimensionalities equal to the number of goods or to the number of agents, and integers. We also propose a simple game implementing the Pareto solution and games implementing the intersections of the Pareto solution with each of these solutions.

**Key words.** Nash implementation. No-envy. Divide-and-permute.

# 1 Introduction

We are concerned here with the implementation by "simple" games of solutions to the problem of fair division.

An informal description of this objective is as follows. A bundle of goods has to be divided among a group of agents with equal claims on them. Given a domain of such **problems of fair division**, defined by a space of possible preferences for the agents involved, a **solution** is a mapping associating with each problem in the domain a subset of its set of feasible allocations. Suppose that a solution $\varphi$ has been chosen as being the most desirable. Given some admissible problem, computing the allocations selected by $\varphi$ for it requires that preferences be known. Unfortunately, agents may well benefit from not reporting their true preferences. In fact, it is well-known that on sufficiently wide domains of preferences there is no well-behaved "strategy-proof" solution: for such a solution, each agent finds it in his best interest to reveal his true preferences, independently of what they are and independently of the preferences announced by the others.[1] In the two-person case, this was shown for classical economies under the requirements of efficiency and individual rationality (Hurwicz, 1972), and under the requirement of efficiency and one of several alternative requirements of fairness (Thomson, 1987). Considerably more general results hold since strategy-proofness and efficiency together imply dictatorship (Zhou, 1991). The dictatorship conclusion holds even on very narrow domains of homothetic or linear preferences (Schummer, 1997). The program of identifying in the $n$-person case the implications of strategy-proofness alone, that is, even in the absence of efficiency, has been pushed much further (Barberà and Jackson, 1995).

Unless preferences belong to some special domains,[2] obtaining truthful information as a dominant strategy is therefore incompatible with efficiency and minimal distributional objectives. We will then weaken our incentive requirement as follows. Given a solution, we will only ask whether there is a game form such that for each admissible problem, its set of Nash equilibrium allocations coincides with the set of allocations that the solution would have selected for this problem. If such a game form exists, the solution is said to be "implementable". The properties that a solution should satisfy in

[1] Then, truth-telling is a "dominant" strategy in the direct revelation game associated with the solution.

[2] For instance, in a one-commodity economy with single-peaked preferences, the solution known as the uniform rule is implementable in dominant strategies (Sprumont, 1991).

order to be implementable are now well-understood, and general algorithms have been proposed to construct game forms achieving the implementation when it is possible. However, these algorithms, being designed to solve the problem in very general situations, produce game forms that have the disadvantage of involving complex strategy spaces. Indeed, strategies include either whole preference profiles or whole indifference sets for several agents. In the economic applications we have in mind, these are infinite-dimensional objects. Fortunately, when it comes to implementing specific solutions, the particular features of the space of feasible outcomes and of the space of admissible preferences can be used to achieve considerable simplifications. For instance, a variety of game forms whose strategy spaces are subsets of finite dimensional Euclidean spaces have been constructed to implement the Walrasian and Lindahl solutions. (Hurwicz, 1979, and Walker, 1981; Tian, 1989; Corchón and Wilkie, 1996.) Our objective here is the construction of such simple game forms to implement several solutions to the problem of fair division.

We first consider (i) the solution that associates with each problem its set of "envy-free" allocations: an allocation is envy-free if each agent finds his bundle at least as desirable as the bundle assigned to anyone else. Alternatively, we require that each agent finds his bundle at least as desirable as: (ii) the average bundle received by the other agents; (iii) the average bundle received by any group to which he does not belong; (iv) any bundle in the convex hull of the bundles of all the agents.[3] In each case, we show that implementation is possible by means of a game form whose strategy spaces are the cross-product of a subset of a finite dimensional Euclidean space with some finite set. We also propose a simple game form (v) implementing the Pareto solution, the solution that associates with each economy its set of Pareto-efficient allocations. Here, strategy spaces are slightly more complicated but they are still finite dimensional Euclidean spaces. Finally, (vi) we show how the distributional objectives embodied in the various solutions listed above can be reconciled with efficiency by "combining" the game forms implementing them with the game form implementing the Pareto solution. We achieve this by identifying a class of solutions that contains each of the above-mentioned examples, and constructing a general game form for the class. However, we also devise a separate game form for the no-envy solution. We name this game form "Divide-and-Permute".

[3]The relations between these various notions are explained below.

For each of the game forms we construct, the components of strategies have a straightforward economic interpretation as allocations, consumptions bundles, or prices, and (except for Divide-and-Permute which does not have it), one component is an integer that is used as a device to select an agent who is granted the right to choose in a certain set. This set is determined as a function of the other components of the strategies. The use of such "integer" constructions has been criticized (Jackson, 1992), but not having access to them severely limits the range of solutions that can be implemented (again, see Jackson, 1992). An appealing aspect of our game forms is that at equilibrium, agents receive the bundles that have been announced. (This feature essentially corresponds to what Dutta, Sen, and Vohra, 1995, calls "truth-telling".) Without it, there would be little meaning to evaluating how complicated a game form is by the dimensionality of its strategy spaces, since information can be "smuggled" by means of such mathematical devices as Peano's space filling curve. Hurwicz (1977) was the first to recognize this possibility in this context (see also Chakravorti, 1991), and avoided it by placing restrictions on the game forms.

A contribution with a similar aim as ours, namely the investigation of the implications of requiring game forms to be "simple", is the paper just mentioned by Dutta, Sen, and Vohra. These authors characterize the class of solutions that can be implemented by what they call "elementary" game forms, that is, game forms such that at equilibrium, the set of bundles that each agent can attain by varying his own strategy, given the strategies chosen by the others, is a subset of the half-space bounded by a hyperplane of support to his upper contour set at his assigned bundle. The Pareto solution is one of the solutions covered by their theorem, but none of the solutions to the problem of fair division that we discuss here can be handled because they all fail to satisfy a condition that is critical to their approach, namely that the desirability of an allocation be verifiable on the basis of local information only (in the case of smooth preferences, the marginal rates of substitution).[4] Therefore, our results throw some light on the strength of the requirement that the game form be elementary. Similarly, Sjöström (1991a) considers implementation by "demand game forms", that is, game forms in which strategies are points in the commodity space that can be interpreted

[4]Although one should note that if preferences are smooth, the smallest of the correspondences that we consider coincides then with the Walrasian solution, to which the condition does apply.

as agents' desired bundles. He shows that the no-envy solution can be so implemented but that neither the Pareto solution nor its intersection with the no-envy solution can, indicating that limiting oneself to demand game forms may also be too restrictive. We finally comment on the results independently obtained by Saijo, Tatamitani, and Yamato (1996). These authors identify the conditions that a solution has to satisfy in order to be implemented by a game form in which strategies are required to be interpretable as one of the following: a consumption bundle, a pair of bundles, an allocation, a bundle and a price vector, and finally a pair of bundles and a price vector.

They apply their result to the no-envy solution and its intersection with the Pareto solution. They show that the former can be implemented by a game form where each agent announces two consumption bundles, and its intersection with the Pareto solution by a game form in which each agent announces a bundle and a price. In the game forms they consider, all agents have the same strategy space, whereas here, two agents are made to play a special role. The implications in terms of dimensionality of the requirement that strategy spaces be the same for all agents may be a question worthy of further study. For a more complete comparison of the results of Saijo, Tatamitani, and Yamato with those of the current paper, one would also need to verify whether the other examples of solutions examined here satisfy the conditions they derive. We will also leave this question to future research.

Finally, we note that although our main objective was to construct game forms with strategy spaces of low dimensionality, we did not address the issue of identifying the dimensionality that is minimal for implementation. A contribution on this issue is Reichelstein and Reiter (1988). Dutta, Sen and Vohra (1995), Sjöström (1991a), and Saijo, Tatamitani, and Yamato (1996) provide useful additional information on this matter.

# 2 The Model

There are $\ell$ private goods and $n$ agents indexed by $i \in N = \{1, \ldots, n\}$. Each agent $i \in N$ is equipped with a continuous, convex, and strictly monotone preference relation on $\mathbb{R}^{\ell}_{+}$, denoted by $R_i$. Let $P_i$ be the associated strict preference relation and $I_i$ the corresponding indifference relation. There is a bundle $\Omega \in \mathbb{R}^{\ell}_{++}$ of goods to be distributed. This bundle is fixed and known. Thus, a problem of fair division, or an **economy**, is simply a list $R = (R_1, \ldots, R_n)$ of preference relations. Let $Z = \{z \in \mathbb{R}^{\ell n}_{+} \colon \sum z_i = \Omega\}$ be the set of feasible allocations and $Z_0 = \{z_0 \in \mathbb{R}^\ell_+ : z_0 \leqq \Omega\}$ be the set of possible consumption bundles for any agent.

Given a domain of economies $\mathcal{R}^n$, a **solution** is a correspondence $\varphi: \mathcal{R}^n \to Z$ associating with each economy in the domain a non-empty subset of the set of feasible allocations, each point of which is interpreted as a recommendation.

A **game form** is a pair $\Gamma = (S, h)$, where $S = S_1 \times \cdots \times S_n$ is the cross-product of **strategy spaces**, and $h: S \to \mathbb{R}^{\ell n}_+$ is the **outcome function**. Given $R \in \mathcal{R}^n$, let $E(\Gamma, R) \subseteq S$ be the set of (Nash)-**equilibria of $\Gamma$ when played in $R$**, and $E_Z(\Gamma, R)$ be the corresponding set of **equilibrium allocations**: $z \in E_Z(\Gamma, R)$ if there is $s \in E(\Gamma, R)$ such that $z = h(s)$. The game form **$\Gamma$ implements the solution $\varphi: \mathcal{R}^n \to Z$** if for each $R \in \mathcal{R}^n$, $E_Z(\Gamma, R) = \varphi(R)$. A solution is **implementable** if there is a game form that implements it.

Consider now an abstract set of alternatives $A$, a domain $\mathcal{R}$ of preference relations defined over $A$, and a correspondence $\varphi: \mathcal{R}^n \to A$. Given $i \in N$, $R_i \in \mathcal{R}$ and $a \in A$, let $L(R_i, a) = \{b \in A: a \ R_i \ b\}$ be the **lower contour set of $R_i$ at a**. Maskin (1999) shows that if a correspondence $\varphi: \mathcal{R}^n \to A$ is implementable, then it satisfies the following condition:

**Monotonicity:** For each $\{R, R'\} \subseteq \mathcal{R}^n$, and each $a \in \varphi(R)$, if for each $i \in N$, $L(R'_i, a) \supseteq L(R_i, a)$, then $a \in \varphi(R')$.

A correspondence satisfies "no veto power" if, when an alternative is at the top of the preferences of all agents but possibly one, then it is selected by the solution. Maskin shows that if there are at least three agents, and the correspondence is *monotonic* and satisfies *no veto power*, then it is implementable. In private good economies, as soon as there is one good with respect to which preferences are strictly monotone, *no veto power* is trivially satisfied since then its hypothesis is never met. Having assumed that all preferences are strictly monotone, *monotonicity* is then a necessary and sufficient property that a correspondence need to satisfy in order to be implementable, and we will make no more mention of *no veto power*.

Maskin's proof is constructive: he exhibits an algorithm producing, for each implementable correspondence, a game form implementing it. Although he restricts his attention to the case when the set of feasible alternatives is finite, his result can be extended to general domains (Repullo, 1987; Saijo, 1988). Unfortunately, the game forms used by all of these authors have the drawback of involving complex strategy spaces, as explained in the introduction. Our objective is to construct game forms with simple strategy spaces. Implementation of the Walras and Lindahl solutions by simple game forms has been achieved by several authors, but few attempts have been made to implement by simple game forms solutions to the problem of fair division. The exceptions of which we are aware are the following: Crawford (1979) proposes a game form implementing a selection from Pazner and Schmeidler (1978)'s "egalitarian-equivalent" solution (see also Demange, 1984). Since this selection is not *monotonic*, the implementation is of course not achieved in Nash equilibrium; instead, it involves stage game forms and the concept of perfect equilibrium. Palfrey and Srivastava (1987) verify conditions for the implementation of the no-envy solution but their focus is on Bayesian implementation. Here, we consider normal form games and implementation in Nash equilibrium. Another exception is Tadenuma and Thomson (1995), who offer an implementation of the no-envy solution for a class of economies with one indivisible object and one infinitely divisible good by means of a game form in which each strategy space is the real line[5,6]. A final reference is Sjöström (1991a), already discussed.

# 3 Divide-and-Permute : an implementation of the no-envy solution

A requirement that plays a central important role in the literature on fair allocation is that each agent should find his bundle at least as desirable as anyone else's bundle (Foley, 1967):

**The no-envy solution, $\boldsymbol{F}$,** selects, for each $R \in \mathcal{R}^n$, all the allocations $z \in Z$ such that for each $\{i, j\} \subseteq N$, $z_i \; R_i \; z_j$. These allocations are called **envy-free for $\boldsymbol{R}$**.

It is easy to see that the no-envy solution is *monotonic*. Thus, it can be implemented by the Maskin-Repullo-Saijo game forms. The following is

[5] Whether this game form can be generalized to handle the multiple object case remains to be determined however.

[6] The implementation of solutions to the problem of fair division in economies with one good when preferences are single-peaked is considered by Thomson (1990), Sjöström (1991b), and Yamato (1992,1993), but these papers do not address the issue of implementation by simple game forms.

a simple game form achieving the implementation. Let $\Pi^n$ be the class of permutations on $N$ and $\pi_0$ be the identity permutation.

**Divide-and-Permute, $\boldsymbol{\Gamma^F}$:** $S_1 = S_2 = Z \times \Pi^n$ and $S_3 = \cdots = S_n = \Pi^n$.

Given $s = ((z^1, \pi_1),\ (z^2, \pi_2), \pi_3, \ldots, \pi_n) \in S = S_1 \times \cdots \times S_n$, let

$$h(s) = \begin{cases} (0, 0, ..., 0) & \text{if } z^1 \neq z^2; \\ \pi_n \circ \pi_{n-1} \circ \ldots \circ \pi_1(z^1) & \text{if } z^1 = z^2. \end{cases}$$

The game form can be informally described as follows. The first two agents are dividers (each proposes an allocation), and everyone proposes a permutation; if the dividers disagree, they are penalized. If they agree on an allocation, each agent (including them) can reach any of its components by appropriately choosing his permutation, and this, independently of the permutations chosen by the others.

The name "Divide-and-Permute" is meant to bring to mind the well-known two-person "Divide-and-Choose" procedure (one agent divides and the other chooses). However, a number of important distinctions should be noted. In Divide-and-Choose, only one agent proposes a division; here, two agents do so. Divide-and-Choose is a stage game form; here, we consider normal form games. Divide-and-Choose provides a **partial** implementation of the no-envy solution since only the allocation in the envy-free set that is the most favorable to the divider is obtained at equilibrium (Kolm, 1972; Crawford, 1977); here, we obtain **full** implementation (in addition, each envy-free allocation is obtained at some equilibrium.) Finally, a number of complications arise in extending Divide-and-Choose to the $n$-person case (See Brams and Taylor, 1996, for a presentation of the literature devoted to such extensions); here, the $n$-person case poses no special problem. In fact, for $n \geq 3$, additional desirable properties can be imposed on the game form (See Remark 3 below).

We use the following additional notation. For each $s \in S$ and each $i \in N$, $\mathbf{att}_{\boldsymbol{i}}\boldsymbol{(s)}$ is the set of bundles attainable by agent $i$ at $s$, namely $\{z_i \in \mathbb{R}^\ell_+ \colon z_i = h_i(s'_i, s_{-i})$ for some $s'_i \in S_i\}$.[7]

**Theorem 1** *Divide-and-Permute implements the no-envy solution.*

**Proof: Step 1: If $\boldsymbol{z \in E_Z(\Gamma^F; R)}$, then $\boldsymbol{z \in F(R)}$.** Let $s = ((z^1, \pi_1), (z^2, \pi_2), \pi_3, \ldots, \pi_n)$ be an equilibrium supporting $z$. We have

[7]For convenience of notation, we write $\text{att}_i(s)$ instead of $\text{att}_i(s_{-i})$.

$\text{att}_1(s) = \{0, z_1^2, z_2^2, ..., z_n^2\}$; the first bundle results from choosing any $s_1' = (z', \pi) \in S_1$ such that $z' \neq z^2$; each of the remaining ones is obtained by choosing a pair $s_1' = (z^2, \pi)$ for some appropriate $\pi$. Since at least one of the components of $z^2$ contains a positive amount of at least one good, and agent 1 has strictly monotone preferences, $s_1 = (z^1, \pi_1)$ is a best response to $s_{-1}$ only if $z^1 = z^2$. Thus, $z = \pi_n \circ \pi_{n-1} \circ \ldots \pi_2 \circ \pi_1(z^1)$. (Similarly, $\text{att}_2(s) = \{0, z_1^1, z_2^1, \ldots, z_n^1\}$). Now, given that $z^1 = z^2$, if follows that for each $i \in N \backslash \{1, 2\}$, $\text{att}_i(s) = \{z_1, z_2, \ldots, z_n\}$. Thus, equilibrium occurs only because the $\pi_i$'s are such that for each $i \in N$, the $i^{th}$ component of $\pi_n \circ \pi_{n-1} \circ \ldots \pi_2 \circ \pi_1(z^1)$ maximizes $R_i$ over $\{z_1, z_2, \ldots, z_n\}$, which means that $z \in F(R)$.

**Step 2: If $\boldsymbol{z \in F(R)}$, then $\boldsymbol{z \in E_Z(\Gamma^F; R)}$.** Indeed, let $s = ((z, \pi_0), (z, \pi_0), \pi_0, \ldots, \pi_0)$. Then, $h(s) = z$. Here, $\text{att}_1(s) = \{0, z_1, \ldots, z_n\}$, and since for each $i \in N$, $z_1 \; R_1 \; z_i$, and $z_1 \; R_1 \; 0$, it follows that $(z, \pi_0)$ is a best response to $s_{-1}$ for agent 1. Similarly, $(z, \pi_0)$ is a best response to $s_{-2}$ for agent 2. Finally, for each $i \in N \backslash \{1, 2\}$, $\text{att}_i(s) = \{z_1, \ldots, z_n\}$. Since for each $j \in N$, $z_i \; R_i \; z_j$, it follows that $\pi_0$ is a best response for agent $i$ to $s_{-i}$.

□

**Remark 1.** It would suffice to have each agent announce a transposition (a permutation exchanging only two components), instead of an arbitrary permutation.

**Remark 2.** Implementation occurs even for $n = 2$. This is worth noting since, as is well-known (see Moore, 1992, or Corchón, 1996, for surveys of the relevant results), implementation for $n = 2$ is often more difficult to achieve than for $n > 2$.

**Remark 3.** If $n > 2$, the outcome function can be re-specified so that no resource is ever thrown away, that is, so that for each $s \in S$, $\sum h_i(s) = \Omega$ (not just at equilibrium). When $z^1 \neq z^2$, set $h(s) = (0, 0, \Omega, 0 \ldots, 0)$ for instance (any distribution of the social endowment $\Omega$ between agents 3 to $n$ would do).[8]

**Remark 4.** Divide-and-Permute can be used to implement the no-envy solution on domains of economies with heterogeneous goods such as land or time, the resource to be divided being modelled as a measurable subset of a finite-dimensional Euclidean space, with preferences defined over its measurable subsets. (The issue of existence of envy-free and efficient allocations in this

[8] I owe this remark to B. Dutta.

context is addressed by Weller, 1985, Stromquist, 1980, and Berliant, Dunz, and Thomson, 1992). For time (the one-dimensional case), it is often natural to assume that agents have preferences defined over intervals and that of two intervals ordered by inclusion, the larger one is preferred to the smaller one. In this case, an envy-free allocation is necessarily efficient (Berliant, Dunz and Thomson, 1992), so that Divide-and-Permute achieves both efficiency and the distributional objective of no-envy.

# 4 Implementation of other solutions

In this section, we consider a variety of other solutions to the problem of fair division, but instead of dealing with each of them separately, we offer a general procedure. The no-envy solution is covered by this procedure, However, Divide-and-Permute seems particularly natural for that solution, which is why we felt that there would be some advantage to giving it separately.

We begin by listing the examples of solutions to which the general procedure applies. First, we require that each agent should find his bundle at least as desirable as the average of the bundles of the others. This definition can be found in Thomson (1979, 1982), Baumol (1986) and Kolpin (1991), to which we refer the reader for motivation.

**The average no-envy solution, $\boldsymbol{A}$,** selects, for each $R \in \mathcal{R}^n$, all the allocations $z \in Z$ such that for each $i \in N$, $z_i \; R_i \; a_i(z)$, where $a_i(z) = (\sum_{j \in N \setminus \{i\}} z_j)/(n-1)$. These allocations are called **average envy-free for $\boldsymbol{R}$**.

A stronger definition is that each agent should find his bundle at least as desirable as the average bundle of any group not containing him. This definition is due to Zhou (1992) to which we refer the reader for motivation and discussion. For each $i \in N$, let $\mathcal{G}_i = \{G \subseteq N \colon i \notin G\}$ denote the set of groups not containing agent $i$:

**The strict no-envy solution, $\boldsymbol{C}$,** selects, for each $R \in \mathcal{R}^n$, all the allocations $z \in Z$ such that for each $i \in N$ and each $G \in \mathcal{G}_i$, $z_i \; R_i \; \sum_{j \in G} z_j/|G|$. These allocations are called **strictly envy-free for $\boldsymbol{R}$**.

Finally, we require that each agent should find his bundle at least as desirable as any bundle in the convex hull of the $n$ bundles it comprises. Given $z_1, \ldots, z_n \in \mathbb{R}^{\ell}_{+}$, $H\{z_1, \ldots, z_n\}$ denotes their convex hull (Kolm, 1973):

**The super no-envy solution, $\boldsymbol{K}$,** selects all the allocations $z \in Z$ such that for each $i \in N$ and each $z'_i \in H\{z_1, \ldots, z_n\}$, $z_i \; R_i \; z'_i$. These allocations are called **super envy-free for $\boldsymbol{R \in \mathcal{R}^n}$.**

Note that if $n = 2$, no-envy and average no-envy coincide. However, if $n > 2$, there is no logical relation between the concepts (Thomson, 1982). The strict no-envy solution is a subsolution of both the no-envy and the average no-envy solutions.[9] The super no-envy solution is the smallest of the distributional criteria discussed in this paper. It is a subsolution of the strict no-envy solution, and if preferences are smooth, it actually coincides with the Walrasian solution operated from equal division (Zhou, 1992).

The three solutions listed above are *monotonic.* Next, we present a game that implements any solution in the following broad class, which includes all three. The definition is based on the simple observation that most equity notions involve comparing what each agent receives to the bundles in some appropriately defined set. The set appears in the definition of the game form, each player being indeed given the choice to maximize his preference relation in it:

**Definition** A solution $\varphi$ belongs to the **family $\boldsymbol{\Phi}$** if there are a list $(V_i)_{i\in N}$ of sets and a list $(\tau_i)_{i\in N}$ of functions $\tau_i \colon V_i \times Z \to Z_0$ such that for each $R \in \mathcal{R}^N$, each $z \in \varphi(R)$, and for each $i \in N$,

1. $z_i \in \tau_i(V_i, z)$, and for each $z'_i \in \tau_i(V_i, z)$, $z_i \; R_i \; z'_i$;
2. There is $z'_i \in \tau_i(V_i, z)$ such that $z'_i \neq 0$.

Although this definition may appear somewhat technical, it has the advantage of being quite general. In particular, the three examples given above belong to $\Phi$, as now demonstrated:[10] for the average no-envy solution, let $V_i = \{0, 1\}$ with generic element denoted by $k_i$, and $\tau_i(k_i, z) = k_i z_i + (1 - k_i) a_i(z_i)$; for the strict no-envy solution, let $V_i = \{\mathcal{G}_i, \{i\}\}$ with generic element denoted by $G_i$, and $\tau_i(G_i, z) = \sum_{j\in G_i} z_j / |G_i|$; for

[9] It can also be related by means of a consistency property to the average no-envy solution. It is the "largest" consistent solution contained in it (Thomson, 1994).

[10] For the examples considered in the paper, it would actually suffice to choose the sets $V_i$ to be subsets of the simplex. At the price of a somewhat greater complexity, our formulation covers a wider family of solutions.

the super no-envy solution, let $V_i = \Delta^{n-1}$ with generic element $\lambda_i$, and $\tau_i(\lambda_i, z) = \sum_{j \in N} \lambda_{ij} z_j$.

Note that all the members of $\Phi$ are *monotonic*. We now present a game form implementing any one of them. In the specification of the outcome function, when reference is made to agent $i$, the only non-zero component of an allocation such as $(0, \dots, z_i, \dots, 0)$ should be understood to appear in the $i^{th}$ place. To implement a given $\varphi \in \Phi$, we use its associated sets $(V_i)_{i \in N}$, so that the simplicity of the implementation will obviously be directly related to that of the $(V_i)_{i \in N}$.

**Game form $\Gamma^\varphi$:** $S_1 = Z \times N \times V_1$, $S_2 = Z \times N \times V_2$, and for each $i \in N \backslash \{1, 2\}$, $S_i = N \times V_i$.

Given $s = ((z^1, t_1, v_1), (z^2, t_2, v_2), (t_3, v_3), \dots, (t_n, v_n)) \in S$, let $i(s) = \sum t_i$ (mod $n$) and

$$h(s) = \begin{cases} (0, \dots, 0) & \text{if } z^1 \neq z^2; \\ (0, \dots, \tau_{i(s)}(v_{i(s)}, z^1), \dots, 0) & \text{if } z^1 = z^2 \text{ and } \tau_{i(s)}(v_{i(s)}, z^1) \neq z^1_{i(s)}; \\ z^1 & \text{if } z^1 = z^2 \text{ and } \tau_{i(s)}(v_{i(s)}, z^1) = z^1_{i(s)}. \end{cases}$$

Just as in the previous game form, this outcome function is designed so as to ensure that at equilibrium, the first two agents announce the same allocation. Once they agree on some $z \in Z$, then for each $i \in N$, agent $i$ has the opportunity to choose any of the points of $\tau_i(V_i, z^1)$ (a set which includes $z^1_i$) by appropriately choosing in $N$. Each point of $\tau_i(V_i, z^1)$ is obtained by an appropriate choice of $v_i$.

**Theorem 2** *Given any $\varphi \in \Phi$, the game form $\Gamma^\varphi$ implements the solution $\varphi$.*

The proof is relegated to the appendix.

**Remark 5.** As claimed earlier, the no-envy solution is covered by the theorem. Take $V_i = N$ with generic element $k_i$ and $\tau_i(k_i, z) = z_{k_i}$.

# 5 Implementation of the Pareto solution

The solution considered next is central to economics: it associates with each economy its set of Pareto efficient allocations.

**The Pareto solution** selects, for each $R \in \mathcal{R}^n$, all the allocations $z \in Z$ such that there is no $z' \in Z$ such that for each $i \in N$, $z'_i \; R_i \; z_i$, and for at least one $i \in N$, $z'_i \; P_i \; z_i$.

Under strict monotonicity of preferences, the Pareto solution is *monotonic.* (For an example showing that if preferences are only weakly monotone, the property may not hold, see Thomson, 1985, 1999). The solution that associates with each economy its set of efficient allocations such that at some supporting prices the value of each agent's bundle is positive is also *monotonic.* We find it convenient to work with this variant of the Pareto solution, which we refer to as the "strong" Pareto solution. Note that under strict monotonicity of preferences, the two differ only in that the allocations at which some agent receives nothing are excluded by the strong Pareto solution. These allocations are also excluded by all of the distributional criteria considered above.

**The strong Pareto solution, $\boldsymbol{P}$,** selects, for each $R \in \mathcal{R}^n$, all the allocations $z \in Z$ that are Pareto-efficient for $R$ and in addition have supporting prices $p \in \Delta^{\ell-1}$ such that $pz_i > 0$ for each $i \in N$. These allocations are called **strongly Pareto-efficient for $\boldsymbol{R}$.**

Consider the following game form, where $D = \{(z,p) \in Z \times \Delta^{\ell-1}\colon$ for each $i \in N$, $pz_i > 0\}$, and given $(z_i, p) \in \mathbb{R}^{\ell}_{+} \times \Delta^{\ell-1}$, $B(z_i,p) = \{z'_i \in Z_0\colon pz'_i \leq pz_i\}$.

**Game form $\boldsymbol{\Gamma^P}$:** $S_1 = S_2 = D \times N \times Z_0$ and $S_3 = \cdots = S_n = N \times Z_0$. Given $s = ((z^1, p_1, t_1, z_1), (z^2, p_2, t_2, z_2), (t_3, z_3), \ldots, (t_n, z_n)) \in S$, let $i(s) = \sum t_i \pmod{n}$ and

$$h(s) = \begin{cases} (0,\ldots,0) & \text{if } \begin{cases} (z^1,p_1) \neq (z^2,p_2); \\ \text{or} \\ (z^1,p_1) = (z^2,p_2) \text{ and } z_{i(s)} \notin B(z^1_{i(s)}, p_1); \end{cases} \\ (0,\ldots,z_{i(s)},\ldots,0) & \text{if } \begin{cases} (z^1,p_1) = (z^2,p_2), z_{i(s)} \in B(z^1_{i(s)}, p_1); \\ \text{and } z_{i(s)} \neq z^1_{i(s)}; \end{cases} \\ z^1 & \text{if } (z^1,p_1) = (z^2,p_2) \text{ and } z_{i(s)} = z^1_{i(s)}. \end{cases}$$

The inspiration for this game form is the Second Fundamental Welfare theorem. Indeed, it works as follows: the first two agents, the dividers, announce an allocation-price pair; the allocations can be interpreted as recommendations for the entire economy and the prices as supporting prices; the outcome function is specified so as to guarantee that the dividers agree on some allocation-price pair; in addition, each agent announces (i) an integer, which is used to determine who is granted the right to object to the dividers' recommendation, and (ii) a bundle that can be interpreted as the bundle that he feels he should receive; to be accepted, an objection should be "reasonable" in that its value at the common prices announced by the dividers should not exceed the value of the bundle the dividers intended for him. If the objection is not reasonable, every one ends up with nothing. If it is, the objector receives what he requested and the others receive nothing.

**Theorem 3** *The game form $\Gamma^P$ implements the strong Pareto solution.*

The proof is relegated to the appendix.

# 6 Implementation of equitable and efficient solutions

In the final section, we show how to take care of both distributional and efficiency objectives. Essentially, we combine the game form just proposed to implement the strong Pareto solution, first with Divide-and-Permute—this is the game form $\Gamma^{F\cap P}$ below—and then with the game form designed for the implementation of any solution $\varphi$ in the family $\Phi$, whenever this intersection is well-defined—this is the game form $\Gamma^{\varphi\cap P}$. Note that the intersection of two *monotonic* solutions is also *monotonic*, provided it is a well-defined solution. This proviso is met for each of the examples in which we are interested since they all contain the Walrasian solution operated from equal division.

Given $(z_i, p) \in \mathbb{R}^\ell_+ \times \Delta^{n-1}$, the sets $D$ and $B(z_i, p)$ are defined as for the game form $\Gamma^P$.

**Game form $\boldsymbol{\Gamma^{F\cap P}}$:** $S_1 = S_2 = D \times N \times Z_0 \times \Pi^n$ and $S_3 = \cdots = S_n = N \times Z_0 \times \Pi^n$.
Given $s = ((z^1, p_1, t_1, z_1, \pi_1), (z^2, p_2, t_2, z_2, \pi_2), (t_3, z_3, \pi_3), \ldots, (t_n, z_n, \pi_n)) \in S$, let $i(s) = \sum t_i \pmod n$ and

$$h(s) = \begin{cases} (0, \ldots, 0) & \text{if } \begin{cases} (z^1, p_1) \neq (z^2, p_2); \\ \text{or} \\ (z^1, p_1) = (z^2, p_2) \text{ and } z_{i(s)} \notin B(z^1_{i(s)}, p_1); \end{cases} \\ (0, \ldots, z_{i(s)}, \ldots, 0) & \text{if } \begin{cases} (z^1, p_1) = (z^2, p_2), z_{i(s)} \in B(z^1_{i(s)}, p_1), \\ \text{and } z_{i(s)} \neq z^1_{i(s)}; \end{cases} \\ \pi_n \circ \ldots \circ \pi_1(z^1) & \text{if } (z^1, p_1) = (z^2, p_2) \text{ and } z_{i(s)} = z^1_{i(s)}. \end{cases}$$

**Game form $\Gamma^{\varphi\cap P}$:** $S_1 = S_2 = D \times N \times Z_0 \times V_i$ and for each $i \in N\backslash\{1,2\}$, $S_i = N \times Z_0 \times V_i$.
Given $s = ((z^1, p_1, t_1, z_1, v_1), (z^2, p_2, t_2, z_2, v_2), (t_3, z_3, v_3), \ldots, (t_n, z_n, v_n)) \in S$, let $i(s) = \sum t_i \pmod n$ and $h(s)$ be as in the above game form for the first two cases. Otherwise

$$h(s) = \begin{cases} (0, \ldots, \tau_{i(s)}(v_{i(s)}, z^1), \ldots, 0) & \text{if } \begin{cases} (z^1, p_1) = (z^2, p_2), z_{i(s)} = z^1_{i(s)}, \\ \text{and } \tau_{i(s)}(v_{i(s)}, z^1) \neq z^1_{i(s)}; \end{cases} \\ z^1 & \text{if } \begin{cases} (z^1, p_1) = (z^2, p_2), z_{i(s)} = z^1_{i(s)}, \\ \text{and } \tau_{i(s)}(v_{i(s)}, z^1) = z_{i(s)}. \end{cases} \end{cases}$$

**Theorem 4** *The game form $\Gamma^{F\cap P}$ implements the solution $F \cap P$.*

**Theorem 5** *For each $\varphi \in \Phi$ such that $\varphi \cap P$ is well-defined, the game form $\Gamma^{\varphi\cap P}$ implements the solution $\varphi \cap P$.*

We omit the proofs of Theorems 4 and 5, which are similar to the proofs of Theorems 2 and 3, limiting ourselves to noting that as before, they involve showing that at equilibrium, agents 1 and 2 announce the same allocation-price pair, and that at the announced prices, the value of the bundle that the agent designated by the function $i(.)$ announces for himself is equal to the value of his component of that common allocation. Given agents 1 and 2's common announcement $z$, each agent's attainable set contains all the feasible bundles whose value is less than the value of his component of $z$, and it may contain additional bundles. These bundles are the components of $z$ for $\Gamma^{F\cap P}$ and the elements of $\psi_{i(s)}(z)$ for $\Gamma^{\varphi\cap P}$. By feasibility, no agent can end up above his budget set. This implies the existence of parallel lines of support to the indifference curves at each of the equilibrium bundles, and therefore efficiency. The attainability of the additional bundles guarantees that the relevant equity criterion is met.
**Remark 6.** Implementation occurs even if $n = 2$.
**Remark 7.** If $n > 2$, the outcome function can be modified so that no resource is ever thrown away.
**Remark 8.** The spaces of feasible allocations in models with heterogeneous goods such as land or time do not have a convex structure and apart from the no-envy solution, the examples of solutions examined above cannot be applied to them.
**Remark 9.** The game forms we have proposed do not have continuous outcomes functions. It is probably the case that techniques such as the ones

developed by Postlewaite and Wettstein (1989) and others could be applied to obtain this property.

## 7 An open question

Given two solutions $\varphi$ and $\varphi'$ whose intersection is well-defined, and two game forms $\Gamma$ and $\Gamma'$ implementing them, is there a simple way of "connecting" the game forms so as to obtain a game form $c(\Gamma, \Gamma')$ that implements the intersection $\varphi \cap \varphi'$? The present paper started out with the objective of developing such a general procedure $c$. In applications, one game form would typically be chosen so as to implement a solution meeting some participation or distributional criterion, another to implement the Pareto solution, and the combined game form would implement their intersection. We presented separately Divide-and-Permute and a game form implementing the Pareto solution, and then combined them into a game form implementing their intersection in order to give the flavor of what such an operation may look like.[11] We did not reach our initial objective with the generality that we hoped but preliminary results are described in Thomson (1995).

[11] Of course, the question could be asked for each equilibrium concept, not just for Nash equilibrium, on which we focused here. Of course, for the operation to be successful, we would need the implementability conditions to be closed under intersection, but most conditions are.

# 8 Appendix

In this appendix, we give the proofs of Theorems 2 and 3.

**Proof of Theorem 2:**

**Step 1: If $z \in E_Z(\Gamma^{\varphi}; R)$, then $z \in \varphi(R)$.** Let $s = ((z^1, t_1, v_1), (z^2, t_2, v_2), (t_3, v_3), \ldots, (t_n, v_n))$ be an equilibrium supporting $z$. First, we claim that $z^1 = z^2$. Indeed, $\mathrm{att}_1(s) = \{0\} \cup \tau_1(V_1, z^2)$, the bundle 0 being obtained for any $s'_1 = (z'^1, t'_1, v'_1)$ such that $z'^1 \neq z^2$, and each bundle $z'_1 \in \tau_1(V_1, z^2)$ being obtained for $s'_1 = (z^2, t'_1, v'_1)$, such that $t'_1 + \sum_{i \neq 1} t_i = 1$, (the expression on the left of the equality sign being calculated mod $n$), and $\tau_1(v'_1, z^2) = z'_1$. By strict monotonicity of preferences, and the fact that at least one bundle in $\tau_1(V_1, z^2)$ is not equal to 0, then necessarily for some strategy available to agent 1, he prefers the bundle he receives to 0. Thus, since $s_1 = (z^1, t_1, v_1)$ is a best response to $s_{-1}$ for agent 1, we have $z^1 = z^2$. Now, we cannot have $\tau_{i(s)}(v_{i(s)}, z^1) \neq z^1_{i(s)}$. Indeed, for any $j \in N \backslash \{i(s)\}$, $\mathrm{att}_j(s) = \{0\} \cup \tau_j(V_j, z^1)$, where the bundle 0 is obtained by playing $s_j$, and each point $z'_j \in \tau_j(V_j, z^1)$ is obtained by switching to $t'_j$ such that $t'_j + \sum_{k \in N \backslash \{j\}} t_k = j$, (the expression on the left being calculated mod $n$), and $v'_j$ such that $\tau_j(v'_j, z^1) = z'_j$. Since, again by strict monotonicity of preferences, necessarily one of these attainable bundles is preferred to 0, the claim is proved.

**Step 2: If $z \in \varphi(R)$, then $z \in E_Z(\Gamma^{\varphi}; R)$.** Indeed, let $s = ((z, 1, v_1), (z, 1, v_2), (1, v_3), \ldots, (1, v_n))$, where $v_i$ is such that for each $i \in N$, $\tau_i(v_i, z) = z_i$. Then, $h(s) = z$. We have $\mathrm{att}_1(s) = \{0\} \cup \tau_1(V_i, z)$, where the bundle 0 is obtained for any $s'_1 = (z', t', v'_1)$ with $z' \neq z$, and each bundle $z'_1 \in \tau_1(V_i, z)$ is obtained for $s'_1 = (z, 2, v'_1)$ such that $\tau_1(v'_1, z) = z'_1$. Since $z_1 \ R_1 \ z'_1$ for each $z'_1 \in \psi_1(z)$ and $z_1 \ R_1 \ 0$, $s_1$ is a best response to $s_{-1}$ for agent 1. Similarly, $s_2$ is a best response to $s_{-2}$ for agent 2. Finally, for any $i \in N \backslash \{1, 2\}$, we also have $\mathrm{att}_i(s) = \tau_i(V_i, z)$, where each bundle $z'_i \in \tau_i(V_i, z)$ is obtained for $s'_i = (i + 1, v'_i)$, ($i + 1$ being calculated mod $n$), and $v'_i$ is such that $\tau_i(v'_i, z) = z'_i$. Again, since $z_i \ R_i \ z'_i$ for each $z'_i \in \tau_i(V_i, z)$, $s_i$ is a best response to $s_{-i}$ for agent $i$. □

**Proof of Theorem 3**

**Step 1: If $z \in E_Z(\Gamma^P; R)$, then $z \in P(R)$.** First, if $s = ((z^1, p_1, t_1, z_1), (z^2, p_2, t_2, z_2), (t_3, z_3), \ldots, (t_n, z_n)) \in E_Z(\Gamma^P; R)$, then $(z^1, p_1) = (z^2, p_2)$. Indeed, $\mathrm{att}_1(s) = B(z^2_1, p_2)$ : agent 1 receives 0 by announcing $(z'^1, p'_1) \neq (z^2, p_2)$; he receives any point $z'_1 \in B(z^2_1, p_2)$ by announcing $s_1' = (z^2, p_2, t_1', z_1')$ such that $t_1' + \sum_{i \in N \setminus \{1\}} t_i \pmod{n} = 1$. Since preferences are strictly monotonic and $B(z_1^2, p_2)$ contains positive points, it follows that at equilibrium, $(z^1, p_1) = (z^2, p_2)$. Similarly, $\text{att}_2(s) = B(z_2^1, p_2)$. Next, if the equality $(z^1, p_1) = (z^2, p_2)$ holds, then for each $i \in N$, $\text{att}_i(s) = B(z_i^1, p_1)$ : indeed, agent $i$ can reach any point $z_i'$ in this set by announcing $s_i' = (t_i', z_i')$ such that $t_i' + \sum_{j \neq i} t_j = i$ (the expression on the left being calculated mod $n$). Equilibrium requires that he obtains his preferred point in the set. This is possible only if for each $i \in N$, $z_i' = z_i$. And then, $z \in P(R)$.

**Step 2: If $z \in P(R)$, then $z \in E_Z(\Gamma^P; R)$.** If $z \in P(R)$, then by the second fundamental welfare theorem, there is $p \in \Delta^{\ell-1}$ such that for each $i \in N$ and each $z_i' \in \mathbb{R}_+^\ell$ such that $pz_i' \leq pz_i$, $z_i \; R_i \; z_i'$. Now, let $s = ((z, p, 1, z_1), (z, p, 1, z_2), (1, z_3), \ldots, (1, z_n))$. Then, for each $i \in N$, $\text{att}_i(s) = B(z_i, p)$. We omit the straightforward proof that $s$ is indeed an equilibrium and $z$ the corresponding equilibrium allocation. □

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