The Two-Agent Claims-Truncated Proportional Rule Has No Consistent Extension: A Constructive Proof

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The two-agent claims-truncated proportional rule has no consistent extension: a constructive proof

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Abstract

We consider the problem of adjudicating conflicting claims. A rule to solve such problems is consistent if the choice it makes for each problem is always in agreement with the choice it makes for each "reduced problem" obtained by imagining that some claimants leave with their awards and reassessing the situation a that point. It says that each remaining claimant should receive what he received initially. We consider the version of the proportional rule that selects for each problem, the awards vector that is proportional to the vector of claims truncated at the amount to divide. We illustrate a geometric technique developed by Thomson (2001) by showing that the two-claimant truncated proportional rule has no consistent extension to general populations (Dagan and Volij, 1997).

Key-words: Conflicting claims. Division rules. Consistency. Consistent extension. Claims-truncated proportional rule.

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1 Introduction

A group of agents have claims on a resource adding up to more than what is available. How should the resource be divided? For instance, a firm goes bankrupt and its liquidation value has to be allocated among its creditors. How should it be done? A "rule" associates with each such "claims problem" a division of the amount available, an "awards vector" for the problem. The literature on the adjudication of conflicting claims, which originates in O'Neill (1982), is concerned with the identification of well-behaved rules.[1]

A central test of good behavior is "consistency". A rule is consistent if, for each problem, the awards vector it recommends is "in agreement" with the awards vector it recommends for each "reduced problem" obtained by imagining some claimants leaving with their awards and reassessing the situation at that point: in this lower-dimensionality problem, it should recommend for each of the remaining claimants what it recommended initially.[2]

In the various fields of game theory and the theory of resource allocation in which the idea of consistency has been applied, it has provided an appealing way of extending to arbitrary populations the choice of a rule for the two-agent case. Our intuition is stronger if there are only two agents. In particular, the difficult conceptual issue of how to treat coalitions does not arise. Also, less sophisticated mathematics often suffice. Hence the interest in so proceeding. Of course, not all two-agent rules have consistent extensions to arbitrary populations. In each case, the question is whether such an extension exists.

It is the question we ask here about an important variant of the proportional rule. The answer is negative. Dagan and Volij (1997) obtain it as follows. Given a two-claimant rule, they associate with each problem and each awards vector for it, a certain binary relation, and they establish transitivity of the relation as necessary and sufficient for a two-claimant rule satisfying some minor additional properties to have a consistent extension. This transitivity requirement can be understood as an algebraic description of the fact that the paths of awards of a consistent rules are related by projection. This fact is our point of departure. When no consistent extension exists, our technique provides a constructive way of identifying situations where the projection requirements implied by consistent are not met (and

[1]For as survey, see Thomson (2003).

[2]The idea of consistency has been the object of a considerable literature that now counts several hundred items. For a survey, see Thomson (2005).

equivalent, it helps identify awards vectors for which the Dagan and Volij necessary conditions are violated). When an extension exists, it allows its construction. When the point of departure is not a particular two-claimant rule, but a family of such rules, it allows to identify the selections that have to be made from that family so as to obtain consistency. The technique mostly relies on understanding the geometric properties of paths of awards, seen in their entirety (Thomson, 2001). (The path of awards of a rule for a claims vector is the locus of the awards vector it chooses as the amount to divide runs from 0 to the sum of the claims.)

The technique has been useful in other applications, not only in proving or disproving the existence of consistent extensions, but also in constructing these extensions when they exist. A first study identifies the generalizations of the Talmud rule (Aumann and Maschler, 1985) that do not necessarily satisfy "equal treatment of equals" (Hokari and Thomson, 2003a). A second study (Thomson, 2002a) offers an alternative characterization of a class of rules satisfying certain invariance properties, first obtained by Moulin (2000). A third study concerns a family of rules, the "ICI family", that provides a simultaneous generalization of several rules that are central to the literature, the proportional, constrained equal awards, constrained equal losses, and minimal overlap rules. The consistent members of this family can be completely described (Thomson, 2002b).

2 Adjudicating conflicting claims

There is a set of "potential" claimants indexed by the natural numbers, $\mathbb{N}$. Let $\mathcal{N}$ be the class of nonempty and finite subsets of $\mathbb{N}$. A claims problem, or simply a **problem**, is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$, where $N \in \mathcal{N}$, such that $\sum_N c_i \geq E$.[3] Let $\mathcal{C}^N$ be the class of all problems with claimant set N. A division rule, or simply a **rule**, is a function defined on $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$, which associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$ a point x of $\mathbb{R}^N_+$ such that $0 \leqq x \leqq c$ and $\sum x_i = E$ (this equality is the form taken by efficiency on this domain). Any such point is an **awards vector for (c, E)**. Let R be our generic notation for rules. For each $c \in \mathbb{R}^N_+$, the locus of $R(c, E)$ as E varies from 0 to $\sum c_i$ is the **path of awards of R for c**. The most prominent rule

[3]By the notation $\mathbb{R}^N$ we mean the Cartesian product of $|N|$ copies of $\mathbb{R}$ indexed by the members of N. Vector inequalities: $x \geqq y$, $x \geq y$, and $x > y$.

in practice as well as in the theoretical literature is the proportional rule, $\boldsymbol{P}$, which allocates the resource in proportion to claims.

A number of authors have argued that when a claim exceeds the amount to divide, it can be legitimately replaced by this amount. The difference between the claim and the amount to divide cannot be compensated anyway, so it can be judged irrelevant. Although one can certainly imagine counterarguments to this position, many interesting rules are invariant under truncation of claims at the amount to divide (constrained equal awards, Talmud, random arrival, minimal overlap), and it is certainly worth exploring. The idea leads naturally to associating to each rule R a new rule by truncating claims at the amount to divide before applying R. Due to the central role played by the proportional rule, its claims-truncated version is of particular interest (Dagan and Volij, 1997; Curiel, Maschler and Tijs, 1987):

Claims-truncated proportional rule, $\boldsymbol{P^t}$: For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, $P^t(c, E) \equiv \lambda(\min\{c_i, E\})_{i \in N}$, where $\lambda \in \mathbb{R}_+$ is chosen so as to achieve efficiency.

Many tests have been devised for the evaluation of rules. We focus on one that has been important not only in our current context, but in virtually all of the various models of game theory and the theory of resource allocation that have been the object of axiomatic analysis. Here, it says the following. Starting from some problem, apply the rule to obtain an awards vector for it. Now, imagine some claimants leaving the scene with their awards, and at that point, re-evaluate the situation. In the "reduced problem" involving the remaining claimants, the amount to divide is what is was initially minus what the claimants who left took with them. Apply the rule to this problem and check whether the rule assigns to each remaining claimant what it assigned to him initially. We require that this always be the case:[4]

Consistency: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $N' \subset N$, if $x \equiv R(c, E)$, then $x_{N'} = R(c_{N'}, \sum_{N'} x_i)$.[5]

[4]The first applications of the consistency idea to claims resolution are due to Aumann and Maschler (1985) and Young (1987). Young is the author of the most general theorems on the subject.

[5]Note that since we require rules to be such that for each $i \in N$, $0 \leq x_i \leq c_i$, then the sum of the claims of the remaining claimants is still at most as large as the amount that remains to divide, and therefore the problem $(c_{N'}, \sum_{N'} x_i)$ is well-defined.

Many rules are *consistent*, the proportional rule being an obvious example. However, the claims-truncated proportional rule is not.[6] To see this, let $N \equiv \{1,2,3\}$ and $(c,E) \in \mathcal{C}^N$ be defined by $(c,E) \equiv (1,1,3;3)$. Then $P^t(c,E) \equiv x = (\frac{3}{5}, \frac{3}{5}, \frac{9}{5})$ (note that no truncation is actually needed here), but after claimant 2 leaves with $\frac{3}{5}$, we obtain $P^t(c_1, c_3, x_1 + x_3) = P^t(1,3;\frac{12}{5}) = P(1, \frac{12}{5}; \frac{12}{5}) \neq (\frac{3}{5}, \frac{9}{5})$.

If a rule is *consistent* and coincides with a given two-claimant rule, it is its **consistent extension**. Suppose that a two-claimant rule is **resource monotonic**, that is, such that for each claims vector, it assigns to each claimant an amount that is a nowhere decreasing function of the amount to divide. Then, it has at most one *consistent* extension (Aumann and Maschler, 1985). Now, given a *resource monotonic* two-claimant rule, can one prove that it has a *consistent* extension if that is the case? Also, can one construct the extension?

A general technique to answer such questions is developed in Thomson (2001). It exploits the following projection property of the paths of awards of a *resource monotonic* and *consistent* rule: for each $N \in \mathcal{N}$, each $c \in \mathbb{R}^N_+$, and each $N' \subset N$, the path of awards of the rule for c, when projected onto $\mathbb{R}^{N'}$, is its path of awards for $c_{N'}$. Then, if a *resource-monotonic* two-claimant rule has a *consistent* extension, one can deduce, for each claims vector involving an arbitrary population, the path of awards of this extension from the paths of awards of the two-claimant rule for the projection of that claims vector onto two-dimensional subspaces. The result established here being negative, the following simplified presentation of the technique will suffice.

Let N be a three-claimant group. If a choice has been made of a rule R in the two-claimant case, the existence of a *consistent* extension of R implies the existence, for each $c \in \mathbb{R}^N_+$, of a path Π in $\mathbb{R}^N$ whose projections onto the two-dimensional subspaces of $\mathbb{R}^N$ coincide with the paths of awards of R for the projections of c onto these subspaces. In fact, if R is *strictly resource monotonic*, two of the three paths in the two-dimensional subspaces are sufficient to construct Π, as follows. To fix the ideas, let $N \equiv \{1,2,3\}$ and $c \in \mathbb{R}^N$. For each $t \in [0, c_1]$, let $H(t)$ be the plane of equation $x_1 = t$. By *strict monotonicity*, the plane intersects the paths Π_3 for $c_{\{1,2\}}$ and Π_2 for $c_{\{1,3\}}$ at exactly two points, denoted $x(t)$ and $y(t)$. *Consistency* tells us that Π contains the point $z(t) \in \mathbb{R}^N$ whose projections onto $\mathbb{R}^{\{1,2\}}$ and $\mathbb{R}^{\{1,3\}}$

[6] A study of which properties are preserved or not preserved by operators on the space of rules, including the "claims truncation operator", is due to Thomson and Yeh (2006).

are $x(t)$ and $y(t)$, namely $(t, x_2(t), y_3(t))$. In fact, Π is the locus of $z(t)$ as t varies in $[0, c_1]$. Once Π is constructed, we project it onto $\mathbb{R}^{\{2,3\}}$ and check whether this projection coincides with the path for $c_{\{2,3\}}$. It should. In our application, it does not.

3 The result

Theorem 1 *The two-claimant claims-truncated proportional rule has no consistent extension.*

The notation seg$[a, b]$ designates the segment connecting the points a and b.

Proof: We will derive a contradiction to the supposition that there exists a *consistent* rule R that coincides with P^t in the two-claimant case. Let $N \equiv \{1, 2, 3\}$ and $c \equiv (2, 4, 6) \in \mathbb{R}^N_+$.

Step 1: Constructing the paths of awards of P^t in the two-claimant case (Figure 1a). Let $i, j \in \mathbb{N}$ and $(c_i, c_j) \in \mathbb{R}^{\{i,j\}}_+$. Without loss of generality, suppose $c_i \leq c_j$. Then, the path of awards of P^t for (c_i, c_j) is as follows:

Case 1: $E \leq c_i$. Both claims are truncated. After truncation, they are both equal to E, so $P^t(c, E) = (\frac{E}{2}, \frac{E}{2})$. As E varies in $[0, c_i]$, $P^t(c, E)$ traces out seg$[(0, 0), (\frac{c_i}{2}, \frac{c_i}{2})]$.

Case 2: $c_i \leq E \leq c_j$. Only c_j is truncated, and $P^t(c, E)$ is the vector $x \in \mathbb{R}^{\{i,j\}}_+$ defined by (i) $x_i + x_j = E$, and (ii) for some λ, $x_i = \lambda c_i$ and $x_j = \lambda E$. As E varies in $[c_i, c_j]$, $P^t(c, E)$ traces out the part of the curve of equation $x_j = \frac{x_i^2}{c_i - x_i}$ (obtained by eliminating E and λ from (i) and (ii) above) that lies between the lines of equation $x_i + x_j = c_i$ and $x_i + x_j = c_j$. We call C the curve just identified when $c_i = 2$. In Figure 1a, it is the steep convex curve emanating from $(1, 1)$ and extending above the horizontal segment of ordinate 6. We designate by $(f(E), g(E))$ the point of C that lies on the line of equation $x_1 + x_2 = E$. The critical observation to make here is that C does not depend on c_j.

Case 3: $c_j \leq E$. No truncation takes place and $P^t(c, E) = P(c, E)$. As E varies in $[c_j, c_i + c_j]$, $P^t(c, E)$ traces out seg$[(f(c_j), g(c_j)), (c_i, c_j)]$.

Summarizing, for each $c_j \geq c_i = 2$, the path of P^t for $(2, c_j)$ consists of the union of seg$[(0, 0), (1, 1)]$ (Case 1), the part of C that lies between $(1, 1)$ and $(f(c_j), g(c_j))$ (Case 2), and seg$[(f(c_j), g(c_j)), (2, c_j)]$ (Case 3). Figure 1a illustrates these conclusions for $c_j = 3$, 4, and 6.

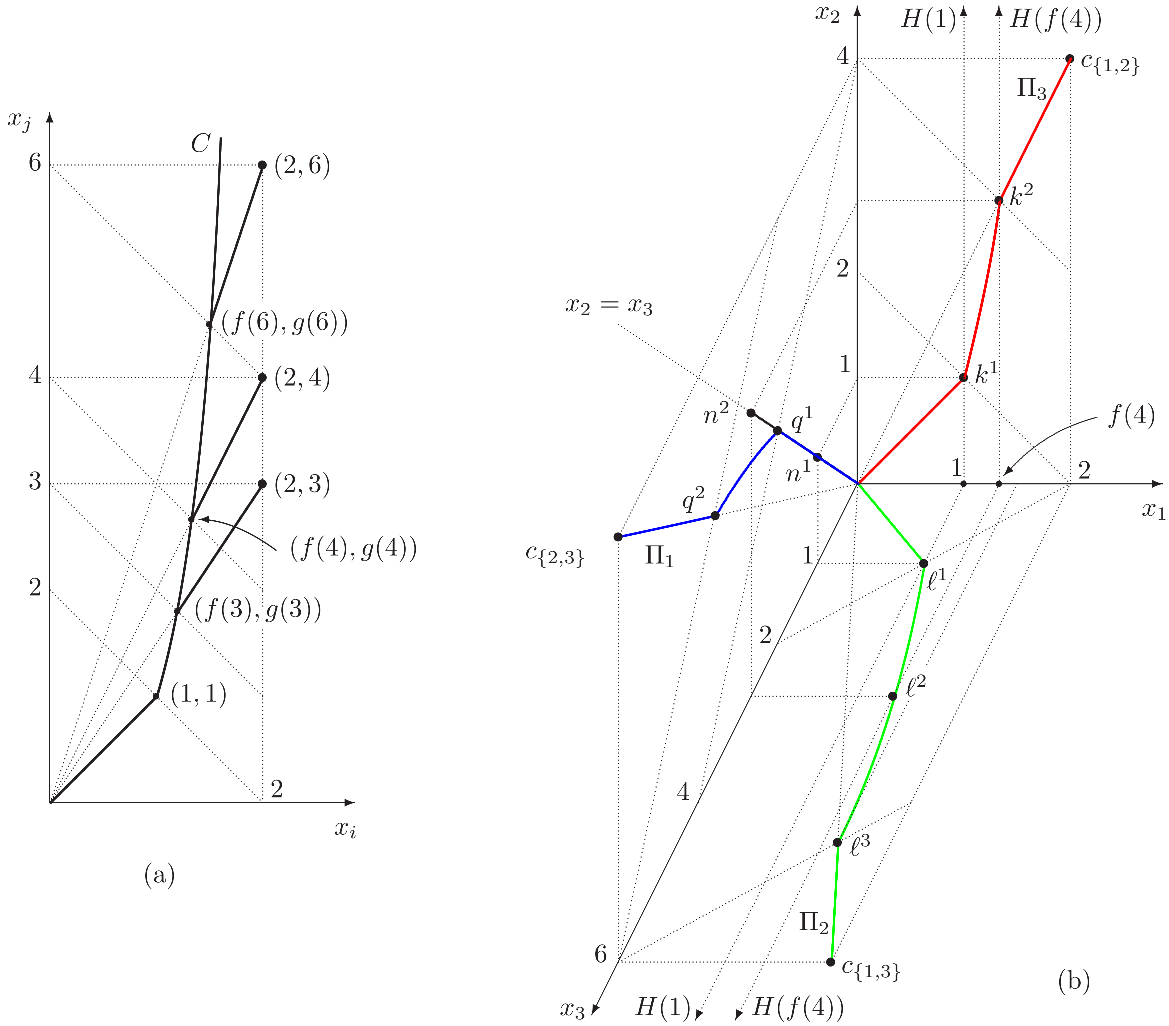

Figure 1: Proof of Theorem 1. (a) Step 1: Generating the paths of awards of P^t for $(2,3)$, $(2,4)$, and $(2,6)$. (b) Step 4: Deriving the contradiction. The path for $c_{\{2,3\}}$ in $\mathbb{R}_+^{\{2,3\}}$ consists of seg$[(0,0), q^1]$, the curvi-linear segment from q^1 to q^2, and seg$[q^2, c_{\{2,3\}}]$. The projection onto $\mathbb{R}_+^{\{2,3\}}$ of the path for $c \equiv (2,4,6)$ of the *consistent extension* of P^t, if such an extension exists, contains seg$[n^1, n^2]$.

Step 2: Representing in $\mathbb{R}^N$ the paths of awards of P^t for $(c_1, c_2) = (2, 4)$, $(c_1, c_3) = (2, 6)$, and $(c_2, c_3) = (4, 6)$ (Figure 1b). We designate these paths by Π_3, Π_2, and Π_1 respectively.

• The path Π_3 of panel (b) is a copy of the path for $(2, 4)$ of panel (a), with k^1 corresponding to $(1, 1)$ and k^2 to $(f(4), g(4))$.

• The path Π_2 of panel (b) is a copy of the path for $(2, 6)$ of panel (a), with ℓ^1 corresponding to $(1, 1)$ and ℓ^3 to $(f(6), g(6))$.

• The path Π_1 of panel (b) is obtained by a homothetic expansion of factor 2 of the path for $(2, 3)$ of panel (a). The points q^1 and q^2 correspond to the points obtained from $(1, 1)$ and $(f(3), g(3))$ by subjecting them to this expansion.

Step 3: Constructing the projection onto $\mathbb{R}^{\{2,3\}}$ of the path of awards of R for $(2, 4, 6)$, Π (Figure 1b). Since Π_3 and Π_2 are strictly monotonic in $\mathbb{R}^{\{1,2\}}$ and $\mathbb{R}^{\{1,3\}}$, Π can be uniquely deduced from them. It has four parts, separated by the planes parallel to $\mathbb{R}^{\{2,3\}}$ that contain either a kink of Π_3 or a kink of Π_2. For each $t \in [0, c_1]$, let $H(t)$ be the plane of equation $x_1 = t$. The critical values of t, those for which $H(t)$ contains a kink or an endpoint of Π_3 and Π_2, are 0, 1, $f(4)$, $f(6)$ and c_1 (for $t = 1$, the kink k^1 in Π_3 and the kink ℓ_1 in Π_2 are reached simultaneously). To avoid cluttering the figure, we do not represent Π itself and in fact, because this suffices for our purposes, we only identify in the next paragraph the part of Π that lies between $H(1)$ and $H(f(4))$.

Let ℓ^2 be the point of Π_2 whose first coordinate is $f(4)$. The part of Π_3 that lies between k^1 and k^2—let us call it C^3—and the part of Π_2 that lies between ℓ^1 and ℓ^2—let us call it C^2—are both copies of the part of C that lies between $(1, 1)$ and $(f(4), g(4))$, a strictly monotonic curvi-linear segment. Thus, the part of Π whose projection onto $\mathbb{R}^{\{1,2\}}$ is C^3 and whose projection onto $\mathbb{R}^{\{1,3\}}$ is C^2 is a strictly monotonic curvi-linear segment that lies in the plane of equation $x_2 = x_3$. Its endpoints are $(1, 1, 1)$ and $(f(4), g(4), g(4))$. Its projection onto $\mathbb{R}^{\{2,3\}}$ is a segment that lies in the 45° line of that space, namely seg$[n^1, n^2]$, where $n^1 \equiv (1, 1)$ and $n^2 \equiv (g(4), g(4))$.

Step 4: Deriving the contradiction. We have just determined that the projection of Π onto $\mathbb{R}^{\{2,3\}}$ is a segment contained in the 45° line in $\mathbb{R}^{\{2,3\}}$ that extends beyond the line of equation $x_2 + x_3 = c_2$. However, we had already seen that this is not how Π_1 extends beyond that line.[7]

□

[7]The four parts of Π are as follows. First is the segment in $\mathbb{R}^N$ whose projec-

4 Conclusion

Another operation on rules has been suggested in the literature. Given any problem, it consists in first assigning to each agent his "minimal right", namely the difference between the amount to divide and the sum of the claims of the others (if this difference is non-negative), and then applying the rule to divide the remainder, after revising claims down by the minimal rights. This operation is dual to the claims truncation operation (for the notion of duality, see Aumann and Maschler (1985), Herrero and Villar (2001), Moulin (2000), and Thomson and Yeh (2000)). *Consistency* being preserved under duality (this means that if a rule is *consistent*, so is its dual), a corollary of our main result is that the two-claimant version of the rule obtained from P by subjecting it to the "attribution of minimal rights" operator has no *consistent* extension. Curiel, Maschler, and Tijs (1987) suggest that the proportional rule be subjected to both operators. The rule they obtain is not *consistent* but in the two-claimant case, it coincides with the so-called contested-garment rule of the Talmud (Aumann and Maschler, 1985). The contested-garment rule does have a (unique) *consistent* extension, namely the Talmud rule, as Aumann and Maschler show. Thus, the two-claimant version of the rule Curiel, Maschler, and Tijs propose also does, although it is not the rule they suggest for more than two claimants. The technique we used here can also be applied to determine this extension (Thomson, 2001).

An alternative approach to the problem we address here is proposed by Dagan and Volij (1997). It involves calculating the "average consistent" extension of the two-claimant rule whose *consistent* extension is sought after, and if no such extension exists, showing that this extension is not *consistent*. As the authors note, the required calculations of this indirect approach may be prohibitive.

tion onto $\mathbb{R}^{\{1,2\}}$ is seg$[(0,0),k^1]$ and whose projection onto $\mathbb{R}^{\{1,3\}}$ is seg$[(0,0),\ell^1]$. It is seg$[(0,0,0),(1,1,1)]$. Its projection onto $\mathbb{R}^{\{2,3\}}$ is seg$[(0,0),n^1]$. Second is the part described in the body of the proof. Third is the curvi-linear segment whose projection onto $\mathbb{R}^{\{1,2\}}$ is seg$[k^2,(f(6),2f(6)]$ and whose projection onto $\mathbb{R}^{\{1,3\}}$ is the part of Π_2 that lies between ℓ^2 and $\ell^3 \equiv (f(6),g(6))$. Its projection onto $\mathbb{R}^{\{2,3\}}$ is a curvi-linear segment with endpoints n^2 and $(2f(6),g(6))$ (this segment is not represented). Fourth is the segment whose projection onto $\mathbb{R}^{\{1,2\}}$ is seg$[(f(6),g(6)),c_{\{1,2\}}]$ and whose projection onto $\mathbb{R}^{\{1,3\}}$ is seg$[(f(6),2f(6)),c_{\{1,3\}}]$. Its projection onto $\mathbb{R}^{\{2,3\}}$ is seg$[(2f(6),g(6)),c_{\{2,3\}}]$.

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