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Operators for the Adjudication of Conflicting Claims

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for the adjudication of conflicting claims

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Abstract

We consider the problem of dividing some amount of an infinitely divisible and homogeneous resource among agents having claims on this resource that cannot be jointly honored. A "rule" associates with each such problem a feasible division. Our goal is to uncover the structure of the space of rules. For that purpose, we study "operators" on the space, that is, mappings that associate to each rule another one. Duality, claims truncation, attribution of minimal rights, and convex combinations are the four operators we consider. We first establish a number of results linking these operators, such as idempotence, commutativity, and distributivity. Then, we determine which properties of rules are preserved under each of these operators, and which are not.

Key-words: Conflicting claims. Division rules. Operators. Minimal rights. Maximal claims. Duality. Convexity.

JEL Classification numbers: C79-D63-D74.

1 Introduction

We address the problem of dividing some amount of an infinitely divisible and homogeneous resource among agents having claims on this resource that cannot be jointly honored. A primary example is when the liquidation value of a bankrupt firm has to be allocated among its creditors. A "division rule" is a function that associates with each situation of this kind, which we call a "claims problem", a division of the amount available. We call this division an "awards vector". It is interpreted as the choice that a judge or arbitrator could make. In the search for the most desirable rules, the literature[1], initiated by O'Neill (1982), has proceeded on several fronts, much recent progress having been made on the axiomatic front.

We will consider the issue from a higher perspective than is standard however, and examine the space of rules itself. Our goal is to uncover its structure. When surveying the literature, one is struck by the richness of the inventory of rules that have been proposed. This richness is also confusing and one feels the need to put some order in the inventory, to organize it in some fashion. Several approaches can be taken for this purpose. The first approach simply consists in searching for resemblances between rules, in their formulas and in the geometry of their graphs. Rules can be usefully organized in families exploiting these resemblances. The parametric family introduced by Young (1987), as well as certain families defined by Thomson (2000), collect a number of important rules that can be described in a common way. The identification of these families allows us to relate rules to one another and also to understand what is unique to each of them. A second approach is to organize rules by means of the properties they share. Axiomatic analysis is the principal methodology here. Of course, these two approaches are related. The general formulas that one writes down to gather rules among which one has recognized patterns will often cause all members of the family to share certain properties.

The approach we follow here is based on a third way of "connecting" rules. It exploits and generalizes a phenomenon one quickly notices, namely that one can often pass from a rule to another by means of a simple algebraic or geometric operation. Let us define an "operator" on the space of rules as a mapping that associates with each rule another one. We propose to undertake a systematic study of such objects. We consider four of them. First

[1] For a survey, see Thomson (2003).

is a duality operator. When looking at a claims problem, two perspectives can be taken: we can think of the issue as dividing what is available; or, as dividing this deficit (the difference between the sum of the claims and the amount to divide). Let S be a rule. The rule associated with S by the duality operator, its "dual", treats what is available in the same way as S treats what is missing. The second operator associates with S the rule defined for each problem by first truncating claims at the amount to divide and then applying S to the problem so revised. The rule associated with S by the third operator calculates the awards vector for each problem in two steps: first, each claimant is attributed the difference between the amount to divide and the sum of the claims of the other agents (or 0 if this difference is negative); this difference is an obvious minimum to which he is entitled; second, S is applied to allocate what remains, the part that is truly contested, claims being adjusted down by the "minimal rights" of the first step. The last operator differs from the others as its arguments are lists of rules and weights for the rules: it produces the weighted average of the rules.[2]

We establish a number of results linking the four operators. Obviously the duality operator composed with itself is the identity; also the claims truncation operator composed with itself is equivalent to itself; somewhat less obvious is that a similar statement holds for the attribution of minimal rights operator. We then show that if two rules are dual, then the version of one obtained by subjecting it to the attribution of minimal rights operator is dual to the version of the other obtained by subjecting it to the claims truncation operator. Next, we study the composition of the claims truncation and attribution of minimal rights operators (a composition on which a rule suggested by Curiel, Maschler, and Tijs, 1987, is based). We show that the order in which they are composed does not matter: the rule that results is independent of the order. Second, in the two-claimant case, starting from any two rules satisfying the basic property that claimants having equal claims should receive equal amounts, subjecting them to the composition of the two operators always produces the same rule. Third, this rule is not just any rule, but it is one that has been central in the literature. We refer to it as "concede-and-divide" because it emerges from the following natural two-step scenario: each claimant first concedes to the other the difference between the amount to divide and his own claim (or 0 if this difference is negative); what remains, the part that we described earlier as being truly contested, is divided

[2]There is one operator for each choice of a weight vector.

equally (Aumann and Maschler, 1985). Finally, we show that the convexity operator is distributive with respect to each of the other three operators.

Given a property that a rule may have, a natural question is whether the property is also enjoyed by the rule obtained by subjecting it to a certain operator. The fact that a property is preserved under an operator is an interesting and very useful feature it may have. We show that, of the properties that have been frequently discussed in the literature, most are preserved under the duality operator, but our main results concerning this operator pertain to two basic monotonicity properties, which somewhat surprisingly, are not. One is "claims monotonicity": if an agent's claim increases, his award should be at least as large as it was initially. The other is "population monotonicity": upon the arrival of additional claimants, the award to each claimant initially present should be at most as large as it was initially.

Next, we turn to the claims truncation and attribution of minimal rights operators. These operators tend to be more disruptive, but they are disruptive in "symmetric" ways. We also study their composition and find that the central property of "self-duality"—invariance under the duality operator—which is preserved by neither operator, is preserved under their composition.

The convexity operator preserves most properties, but not all, and we give two important examples of properties that are not preserved. One is "consistency", which says that the choice made for each problem should always be in agreement with the choice made for the problem derived from it by imagining that some claimants leave with their awards, and reevaluating the situation from the viewpoint of the remaining claimants. The second property is "converse consistency", which says that a certain awards vector should be chosen for a problem if for each two-claimant subgroup of the claimants it involves, its restriction to that subgroup is chosen for the problem these claimants face when the complementary group of claimants leave with their awards.

Our results have a number of benefits. First, as was our goal, they allow us to structure the existing inventory of rules available to solve claims problems, and to help ensure that no important rule has been missed. The structural relations between the operators we uncover also allow us to provide easy proofs that certain properties hold for particular rules (examples are the properties established by Curiel, Maschler, and Tijs, 1987, for the rule they define), and they should also be useful in identifying which properties each newly constructed rule may or may not satisfy. Finally, the operators—

the duality operator is particularly useful in this regard—allow us to derive new characterizations from existing ones. (For an earlier example of such a derivation, see Herrero and Villar, 2001). Altogether, we believe that they help clarify the existing literature, and that they will provide useful tools to keep it organized as it develops further.

2 Model

There is a finite set of **claimants**, N. Each agent $i \in N$ has a **claim** $c_i \in \mathbb{R}_+$ over an **amount to divide** $E \in \mathbb{R}_+$. This amount is insufficient to honor all the claims. Altogether, a **claims problem** is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum_N c_i \geq E$.[3] Let $\boldsymbol{\mathcal{C}^N}$ denote the class of all claims problems. An **awards vector of** $\boldsymbol{(c, E)}$ is a point of $\mathbb{R}^N_+$ bounded above by c and whose coordinates add up to E, a condition we call "efficiency". Let $X(c, E)$ be the set of awards vectors of (c, E). A **rule** is a function defined on $\mathcal{C}^N$ that associates with each $(c, E) \in \mathcal{C}^N$ an awards vector of (c, E). Let S be our generic notation for rules. For the two-claimant case, a rule can be conveniently described in a two-dimensional space by representing, for each claims vector, the path followed by the awards vector as the amount to divide increases from 0 to the sum of the claims. We refer to this path as the **path of awards of the rule for this claims vector**. We denote by $p(S, c)$ the path of awards of S for c.

We also consider a variable-population version of the model. There is a population of "potential" claimants, either $\mathbb{N}$, the set of natural numbers, or some subset of it. However, only a finite number of claimants are present at any given time. Let $\mathcal{N}$ be the class of finite subsets of the set of potential claimants. To specify a claims problem, we first choose $N \in \mathcal{N}$, then $(c, E) \in \mathcal{C}^N$. A rule is a function defined over $\cup_{N \in \mathcal{N}} \mathcal{C}^N$, which associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, an awards vector of (c, E).

3 Operators

Next, we define the four operators with which we are concerned. They are the duality operator, $\boldsymbol{O^d}$, the claims truncation operator, $\boldsymbol{O^t}$, the attribution

[3] By the notation $\mathbb{R}^N$ we mean the Cartesian product of $|N|$ copies of $\mathbb{R}$ indexed by the members of N. Vector inequalities: $x \geqq y$, $x \geq y$, and $x > y$.

of minimal rights operator, $\boldsymbol{O^m}$, and the convexity operator, $\boldsymbol{O^c}$. Then, we illustrate by means of several examples. Given any rule S, the rule obtained by subjecting it to operator O^p is denoted S^p.

1. Duality. The dual of a rule S treats what is available for division in the same way as S treats what is missing. Formally, given $(c, E) \in \mathcal{C}^N$, we replace E by $\sum c_i - E$; we use S to divide this difference, and then subtract the result from c. The idea is suggested by Aumann and Maschler (1985), who provide motivation for it, as well as note passages in the Talmud to support their thesis that its seed was already there:

Dual of $\boldsymbol{S}$, $\boldsymbol{S^d}$: For each $(c, E) \in \mathcal{C}^N$, $S^d(c, E) \equiv c - S(c, \sum c_i - E)$.

It is easy to check that the pair $(c, \sum c_i - E)$ is a well-defined problem and that S^d is a well-defined rule. The operator O^d has a geometric interpretation that is particularly convenient: for each $c \in \mathbb{R}^N_+$, $p(S, c)$ and $p(S^d, c)$ are symmetric of each other with respect to $\frac{c}{2}$. Also, since, as formally stated below (Theorem 1), $(S^d)^d = S$, we can speak of rules being "dual" of each other. Examples of dual rules are the **constrained equal awards rule**, $\boldsymbol{CEA}$, which equates the amounts received by all claimants subject to no one receiving more than his claim, and the **constrained equal losses rule**, $\boldsymbol{CEL}$, which equates the losses experienced by all claimants subject to no one receiving a negative amount: formally, $CEA(c, E) \equiv (\min\{c_i, \alpha\})_{i \in N}$, and $CEL(c, E) \equiv (\max\{0, c_i - \alpha\})_{i \in N}$, where in each case, $\alpha \in \mathbb{R}_+$ is chosen so as to achieve efficiency. Figure 1a illustrates the definitions, and this duality, for $|N| = 2$.[4]

A rule is **self-dual** if it treats the problem of dividing what is available symmetrically to the problem of dividing what is missing (Aumann and Maschler, 1985).[5] To say that a rule is *self-dual* is to say that it is invariant under O^d. For such a rule S, and for each $c \in \mathbb{R}^N_+$, $p(S, c)$ is symmetric with respect to $\frac{c}{2}$. A number of rules are *self-dual*. An obvious example is the **proportional rule**, $\boldsymbol{P}$, which chooses awards proportional to claims: formally, for each $(c, E) \in \mathcal{C}^N$, $P(c, E) \equiv \alpha c$, where $\alpha \in \mathbb{R}_+$ is chosen so as to achieve efficiency. However, other important rules share this property. One of them is the **Talmud rule**, $\boldsymbol{Tal}$, (Aumann and Maschler, 1985), which can

[4] We write the formal definitions of rules for the fixed-population case. To obtain their variable-population versions, it suffices to add a universal quantification over N.

[5] Aumann and Maschler (1985) note a number of passages in the Talmud where the idea that the two perspectives should be equivalent is implicit.

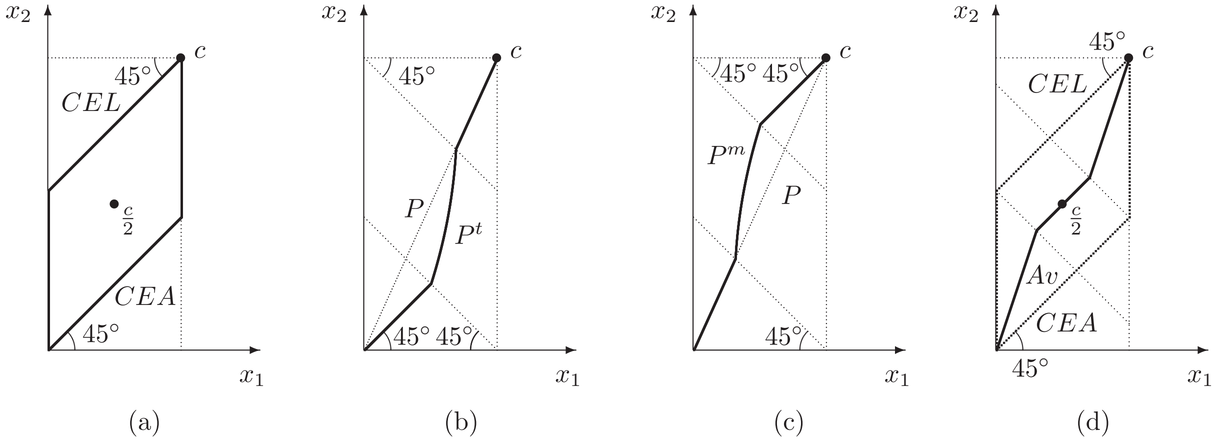

Figure 1: Illustrating the four operators. (a) The operator O^d applied to CEA produces CEL: for each $c \in \mathbb{R}^N_+$, $p(CEA, c)$ and $p(CEL, c)$ are symmetric of each other with respect to $\frac{c}{2}$. (b) The operator O^t applied to P. (c) The operator O^m applied to P. (d) The operator O^c with equal weights, applied to CEA and CEL, produces the simple average of these rules, Av.

be described as a hybrid of CEA and CEL. The former is used for $E \leq \frac{\sum c_i}{2}$, each claim being first divided by two; the latter is used for the remaining cases, here too, $\frac{c}{2}$ being used in the formula. Another is the **random arrival rule**, ***RA*** (O'Neill, 1982; see Thomson, 1998, for a proof), which assigns to each claimant the expected value of what he would obtain on a first-come first-serve basis, assuming that all orders of arrival of claimants occur with equal probabilities.[6]

2. Claims truncation. The second operator truncates claims: given $(c, E) \in \mathcal{C}^N$, each claim that is greater than E is replaced by E. The operator O^t is critical for the study of claims problems as "games with transferable utility" (O'Neill, 1982). Indeed, if a rule is such that for each problem, the awards vector it recommends is the payoff vector chosen by a solution to TU games for the game associated with the problem in the manner first suggested by O'Neill (1982)[7], then it is invariant under O^t (Curiel,

[6]For references to the relevant ancient literature, see O'Neill (1982), Aumann and Maschler (1985), Young (1987), and Dagan (1996). Both CEA and CEL are discussed by Maimonides. Proportionality is explicitly advocated by Aristotle as the basis for "just" distribution. The Talmud rule is defined by Aumann and Maschler (1985) to rationalize numerical examples given in the Talmud. We should also mention the "minimal overlap rule", MO, (O'Neill, 1982), which calculates awards by arranging claims over the amount to divide so as to minimize in a lexicographic way the extent to which they conflict, and then dividing each unit equally among all agents claiming it. Remarkably, RA, MO, and Tal all coincide for $|N| = 2$; moreover, they coincide with "concede-and-divide", defined below. For further discussion of these relationships, see Thomson (2003). We will see below that many other rules share this feature.

[7]Given $(c, E) \in \mathcal{C}^N$, and $S \subseteq N$, the "worth of S" is defined to be $\max\{E - \sum_{i \in N \setminus S} c_i, 0\}$. "Correspondences" between rules and solutions to coalitional games have proved to be very useful tools in the literature on the problem of claims resolution.

Maschler and Tijs, 1987). Formally, for each $(c, E) \in \mathcal{C}^N$ and each $i \in N$, let $t_i(c, E) \equiv \min\{c_i, E\}$ denote **agent *i*'s truncated claim at the amount to divide**, and $t(c, E) \equiv (t_i(c, E))_{i \in N}$ the vector of truncated claims. Figure 1b illustrates O^t applied to P for $|N| = 2$.

Rule *S* operated from truncated claims, S^t: For each $(c, E) \in \mathcal{C}^N$, $S^t(c, E) \equiv S\left(t(c, E), E\right)$.

The inequality between $\sum c_i$ and E is not reversed by the truncation: after carrying it out, we still have a well-defined claims problem.

If a rule is invariant under O^t, we say that it is **invariant under claims truncation**: then, for each $(c, E) \in \mathcal{C}^N$, one can equivalently calculate the awards vector (i) directly, or (ii) after truncating claims at E (Dagan, 1996).

3. Attribution of minimal rights. Given $(c, E) \in \mathcal{C}^N$ and $i \in N$, it is natural to think of the difference $E - \sum_{N \setminus \{i\}} c_j$, (or 0 if this difference is negative), as a minimal amount that he can reasonably expect. There should be no dispute about this payment. Given any rule S, a version of it can be defined by first attributing to each claimant his minimal amount; then after adjusting all claims down by these "first-round awards", applying S to divide the remainder. This remainder is what is truly disputed. Formally, for each $(c, E) \in \mathcal{C}^N$ and each $i \in N$, let $m_i(c, E) \equiv \max\{E - \sum_{N \setminus \{i\}} c_j, 0\}$ denote **claimant *i*'s minimal right** and $m(c, E) \equiv (m_i(c, E))_{i \in N}$ the vector of minimal rights. Figure 1c illustrates the operator O^m applied to P for $|N| = 2$.

Rule *S* operated from minimal rights, S^m: For each $(c, E) \in \mathcal{C}^N$, $S^m(c, E) \equiv m(c, E) + S\left(c - m(c, E), E - \sum m_i(c, E)\right)$.

Since $E - \sum m_i(c, E) \geq 0$ (Curiel, Maschler and Tijs, 1987), and $\sum(c_i - m_i(c, E)) \geq E - \sum m_i(c, E)$, here too, at the second round, we obtain a well-defined claims problem.

If a rule is invariant under O^m, we say that it satisfies **minimal rights first**: then, for each problem, one can equivalently calculate the awards vector (i) directly, or (ii) in two steps, first attributing to each claimant his "minimal right", and after adjusting down each agent's claim by his minimal right, dividing what remains (Curiel, Maschler, and Tijs, 1987).

4. Convexity. Our final operator takes several arguments, but we will refer to it in the singular. When two rules express opposite viewpoints on how to solve a claims problem, it is natural to compromise between them by averaging. More generally, we consider a flexible formulation that allows arbitrary convex combinations. Let K be a finite index set and Δ^K the unit simplex in the $|K|$-dimensional Euclidean space. Let $(S^k)_{k\in K}$ be a list of rules, and $(\lambda^k)_{k\in K}$ a vector of weights in Δ^K. Figure 1d illustrates the operator O^c with equal weights applied to CEA and CEL for $|N| = 2$.

Weighted average of rules $(S^k)_{k\in K}$ with weights $(\lambda^k)_{k\in K} \in \Delta^K$, $w((S^k)_{k\in K}, (\lambda^k)_{k\in K})$: For each $(c, E) \in \mathcal{C}^N$, $w((S^k)_{k\in K}, (\lambda^k)_{k\in K})(c, E) \equiv \sum_{k\in K} \lambda^k S^k(c, E)$.

That O^c is well-defined is a direct consequence of the fact that the set of awards vectors is a convex set.

4 Relating the operators

The following theorem describes the result of composing each of the first three operators with itself. It uses the following notation, which appears repeatedly in the sections to follow. Given any rule S, the rule obtained by subjecting it to the operator O^p and then to the operator $O^{p'}$ is denoted $S^{p'\circ p}$.[8]

Theorem 1 *For each rule S, we have $S^{d\circ d} = S$, $S^{t\circ t} = S^t$, and $S^{m\circ m} = S^m$.*

Proof: The statement concerning O^d is obtained by straightforward manipulation of the definitions.[9] We omit the obvious proof of the second statement. To prove the last statement, let $(c, E) \in \mathcal{C}^N$. We need to show that, in the problem obtained from (c, E) by attributing minimal rights, namely $(c - m(c, E), E - \sum m_j(c, E))$, minimal rights are all 0. Let $i \in N$, and note that claimant i's minimal right in this revised problem is equal to

$$\max\{E - \sum m_j(c, E) - \sum_{N\setminus\{i\}} (c_j - m_j(c, E)), 0\}.$$

After canceling out terms, we obtain the expression

[8] We find this notation a little easier in formulas than $O^{p'} \circ O^p(S)$.

[9] The standard proof of this fact can be found in Herrero and Villar (2002).

$$\max\{E - \sum_{N\setminus\{i\}} c_j - m_i(c,E), 0\},$$

which is easily seen to be equal to 0, by using the definition of $m_i(c,E)$. $\square$

For $|N| = 2$, straightforward calculations reveal that CEA subjected to O^m and CEL subjected to O^t are dual. Indeed, they both coincide with **concede-and-divide**, ***CD***. This rule is defined only for $|N| = 2$ but it is very important because a large number of ways of looking at the problem of adjudicating conflicting claims lead to it.[10] Formally, setting $N \equiv \{i,j\}$, $CD(c,E) \equiv (\max\{E-c_j,0\} + \frac{E-\sum_N \max\{E-c_k,0\}}{2}, \max\{E-c_i,0\} + \frac{E-\sum_N \max\{E-c_k,0\}}{2})$. This duality result is not an accident. It is a consequence of the following theorem:

Theorem 2 *Let S and R be two dual rules. Then S^m and R^t are dual too.*

Proof: We need to show that for each $(c,E) \in \mathcal{C}^N$, $S^m(c,E) = c - R^t(c, \sum_N c_i - E)$, or equivalently that

$$(*) \quad m(c,E) + S(c - m(c,E), E - \sum_N m_i(c,E)) = c - R(t(c, \sum_N c_i - E), \sum_N c_i - E).$$

Since S is dual to R,

$$S(c - m(c,E), E - \sum_N m_i(c,E)) =$$

$$c - m(c,E) - R\left(c - m(c,E), \sum_N (c_i - m_i(c,E)) - \left(E - \sum_N m_i(c,E)\right)\right),$$

[10] The following scenario, which provided the reason for the name we chose for the rule, is one of them (Aumann and Maschler, 1985): agent i, by claiming c_i, is implicitly conceding to claimant j the difference $E - c_i$, or 0 if this difference is negative, namely $\max\{E-c_i, 0\}$. Similarly, by claiming c_j, agent j can be understood as conceding $\max\{E - c_j, 0\}$ to agent i. Let us first attribute to each claimant the amount conceded to him by the other (this can be done because the sum of these concessions is at most as large as the amount to divide), and in a second step let us divide the remainder, the "contested part", equally (no agent ends up with more than his claim).

and substituting in $(*)$, we obtain

$$R(c - m(c,E), \sum_N c_i - E) = R(t(c, \sum_N c_i - E), \sum_N c_i - E).$$

We prove this equality by showing that for each $i \in N$, $c_i - m_i(c,E) = t(c, \sum_N c_i - E)$, or equivalently that

$$(**) \quad c_i - \max\{E - \sum_{N\backslash\{i\}} c_j, 0\} = \min\{c_i, \sum_N c_j - E\}.$$

If $E \leq \sum_{N\backslash\{i\}} c_j$, then $\max\{E - \sum_{N\backslash\{i\}} c_j, 0\} = 0$ and the left-hand side of $(**)$ is c_i; the right-hand side is also c_i. If $\sum_{N\backslash\{i\}} c_j < E$, the left-hand side of $(**)$ is $c_i - E + \sum_{N\backslash\{i\}} c_j = -E + \sum_N c_j$, and so is the right-hand side.

□

We give two other illustrations of Theorem 2 for $|N| = 2$. First, an implication of this theorem is that if a rule S is such that S^m is *self-dual* and $R \equiv S^d$, then $R^t = S^m$. This is what occurs for **Piniles' rule**, **Pin**, which is defined, for each $(c,E) \in \mathcal{C}^N$, by applying CEA when $E \leq \frac{\sum c_i}{2}$, but using in the formula $\frac{c}{2}$ instead of c itself; then, doing so again when $\frac{\sum c_i}{2} < E \leq \sum c_i$. Then, each award is in two "installments".[11] Formally, if $\frac{c_i}{2} \geq E$, $Pin(c,E) \equiv (\min\{\frac{c_i}{2}, \alpha\})_{i\in N}$, and otherwise, $Pin(c,E) \equiv (\frac{c_i}{2} + \min\{\frac{c_i}{2}, \alpha\})_{i\in N}$, where in each case, $\alpha \in \mathbb{R}_+$ is chosen so as to achieve efficiency. The rule is represented for $|N| = 2$ in Figure 2b and its dual in Figure 2e. It is easy to calculate that for $|N| = 2$, $Pin^m = CD$ (Figure 2a). Since CD is *self-dual*, Theorem 2 implies that $Pin^{tod} = CD$, as is also easily verified (Figure 2d).

As a final illustration of Theorem 2, once again we consider Pin for $|N| = 2$, but this time we subject it to O^t. The resulting rule is shown in Figure 2c. The rule obtained by subjecting Pin^d to O^m is shown in Figure 2f. For each $c \in \mathbb{R}^N_+$ the symmetry of $p(Pin^t, c)$ and $p(Pin^{mod}, c)$ announced by Theorem 2 can be verified on panels (c) and (f).

When a rule is subjected to both O^t and O^m, the question arises whether the order in which these operators are applied matters. It is an important

[11] Piniles' (1861) rule, defined below, is an only partially successful attempt to explain the recommendations made in the Talmud for the numerical examples given there. On the other hand, the rule obtained from Pin by subjecting it to O^t is not *self-dual*. We return to this example to illustrate a later theorem.

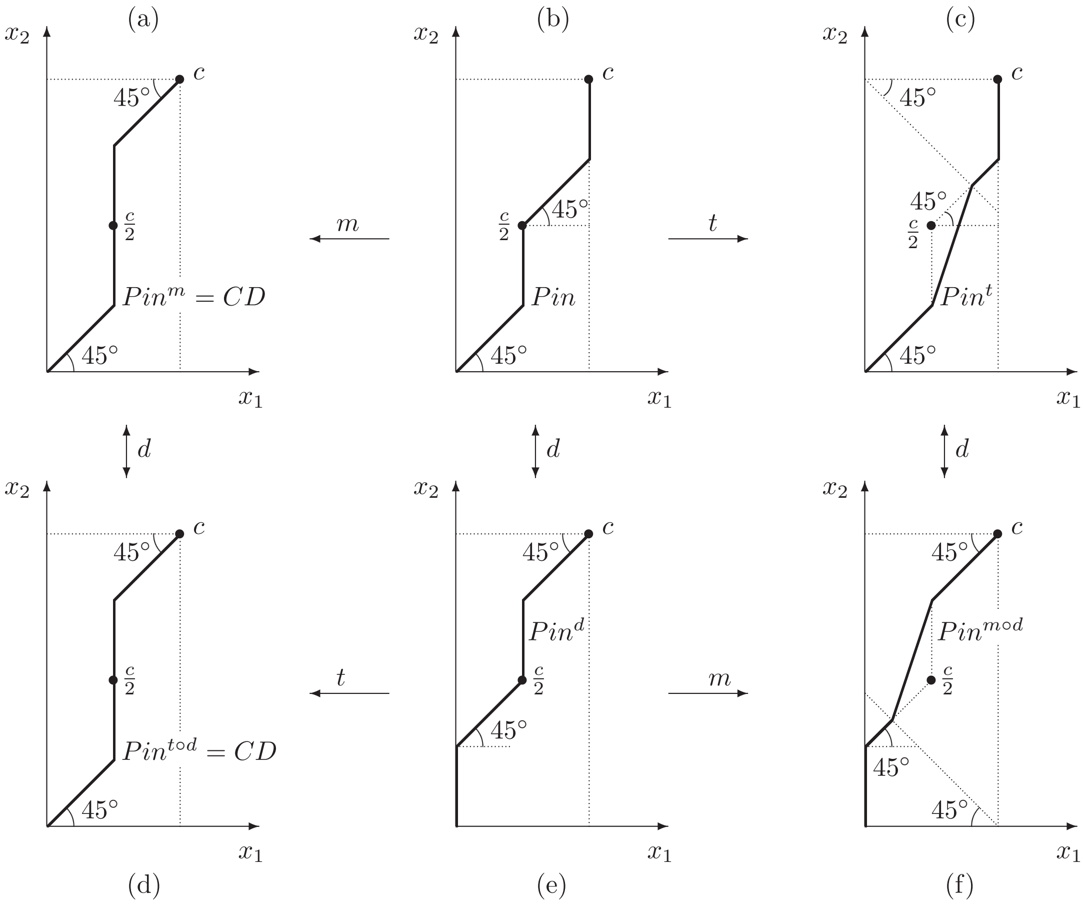

Figure 2: Illustrating Theorem 2. (a) Piniles' rule subjected to O^m. (b) Piniles' rule. (c) Piniles' rule subjected to O^t. (d) The dual of Piniles' rule subjected to O^t. (e) The dual of Piniles' rule. (f) The dual of Piniles' rule subjected to O^m. The dual of each rule represented in the top row is the rule represented just underneath.

question since neither order appears more compelling than the other. Fortunately, the answer is no. We give the proof of this invariance first for $|N| = 2$, as it is very transparent, and also because then, not only is the resulting rule independent of the order, but it is also independent of which rule is taken as a starting point, provided the rule assigns equal awards to agents with equal claims. This is the property of **equal treatment of equals**: for each $(c, E) \in \mathcal{C}^N$ and each pair $\{i, j\} \subseteq N$, if $c_i = c_j$, then $S_i(c, E) = S_j(c, E)$. Moreover, the end-result is CD. This feature of CD is in fact one of the reasons why we feel that this rule is so important.

A preliminary observation is worth making: after being subjected to O^t, any rule satisfying *equal treatment of equals* chooses equal division if the amount to divide is at most as large as the smallest claim. Also, under O^m, for a rule satisfying *equal treatment of equals*, if the amount to divide is at least as large as the sum of the $n - 1$ largest claims, all claimants experience equal losses.

Theorem 3 *For $|N| = 2$. For each rule S satisfying equal treatment of equals, $S^{tom} = S^{mot} = CD$.*

Proof: We assume, without loss of generality, that $c_1 \leq c_2$.
Case 1: $E \leq c_1$. The amount conceded to each claimant (also his minimal right) is 0. First-round awards are all 0, and no adjustment of claims is needed. Truncation of claims at E yields revised claims both equal to E. By *equal treatment of equals*, equal division prevails.

Case 2: $c_1 < E \leq c_2$. Claimant 1 concedes to claimant 2 the amount $E - c_1$, and claimant 2 concedes nothing to claimant 1. Claims are adjusted down to c_1 and $c_2 - (E - c_1)$. After these first-round awards, what remains to divide is c_1. Truncating claims at c_1 yields new claims both equal to c_1. In this second round, by *equal treatment of equals*, each claimant receives half of the amount that remains available, namely $\frac{c_1}{2}$. Altogether, claimant 1 receives $\frac{c_1}{2}$ and claimant 2 whatever is left.

Case 3: $c_2 < E$. The amounts conceded are $E - c_2$ and $E - c_1$. Claims are adjusted down to $c_1 - (E - c_2)$ and $c_2 - (E - c_1)$, and the amount that remains available is $E - \sum(E - c_i)$. After truncation at this revised amount, claims are equal (and in fact, equal to the revised amount to divide $c_1 + c_2 - E$). Then, in this second round, by *equal treatment of equals*, equal division of what remains prevails.

It is easy to see that the awards made in each of the three cases are those specified by CD, and that reversing the order in which the two operators are composed also yields CD. $\square$

If in Theorem 3, *equal treatment of equals* is dropped, order independence still holds but now a family of rules is obtained (defined in Hokari and Thomson, 2003). If the rule is such that multiplying all data of a problem by some positive number results in a problem whose awards vector is obtained from the awards vector of the original problem by the same multiplication (see below for a more formal statement of this property of "homogeneity"), a one-parameter subfamily is obtained. In the general n-claimant case, we lose uniqueness also, but not order independence.

Theorem 4 *For each rule S, we have $S^{mot} = S^{tom}$.*

Proof: The proof is in three steps. Let $(c, E) \in \mathcal{C}^N$ be given.

Step 1: $m(c, E) = m(t(c, E), E)$.

Let $i \in N$. If there is $j \in N \backslash \{i\}$ such that $c_j \geq E$, then $m_i(c, E) = 0$. Also, $t_j(c, E) = E$, so $m_i(t(c, E), E) = 0$. Thus, $m_i(c, E) = m_i(t(c, E), E)$. If for each $j \in N \backslash \{i\}$, $c_j < E$, then for each $j \in N \backslash \{i\}$, $t_j(c, E) = c_j$, and thus, $m_i(t(c, E), E) \equiv \max\{E - \sum_{N \backslash \{i\}} t_j(c, E), 0\} = \max\{E - \sum_{N \backslash \{i\}} c_j, 0\} \equiv m_i(c, E)$.

Step 2: $t\left(c - m(c, E), E - \sum m_i(c, E)\right) = t(c, E) - m(t(c, E), E)$.

By Step 1, we only need to show that for each $i \in N$,

$$t_i\left(c - m(c, E), E - \sum m_k(c, E)\right) = t_i(c, E) - m_i(c, E) \quad \textit{(Relation Ri.)}$$

Using the definitions of $t(\cdot, \cdot)$ and $m(\cdot, \cdot)$, Relation Ri reads:

$\min\{c_i - \max\{E - \sum_{N \backslash \{i\}} c_j, 0\}, E - \sum_{h \in N} \max\{E - \sum_{N \backslash \{h\}} c_j, 0\}\} = \min\{c_i, E\} - \max\{E - \sum_{N \backslash \{i\}} c_j, 0\}$.

Adding $\max\{E - \sum_{N \backslash \{i\}} c_j, 0\}$ to both sides, we have to prove that

$$(*) \quad \min\{c_i, E - \sum_{h \in N \backslash \{i\}} \max\{E - \sum_{N \backslash \{h\}} c_j, 0\}\} = \min\{c_i, E\}.$$

If, for each $h \in N \setminus \{i\}$, $E - \sum_{N\setminus\{h\}} c_j \leq 0$, the desired conclusion follows directly. Otherwise, there is $h^* \in N \setminus \{i\}$ such that $E - \sum_{N\setminus\{h^*\}} c_j > 0$. Then, obviously $c_i < E$ and

$$\begin{aligned}
E - \sum_{h\in N\setminus\{i\}} \max\{E - \sum_{N\setminus\{h\}} c_j, 0\} &= E - (E - \sum_{N\setminus\{h^*\}} c_j) - \sum_{h\in N\setminus\{i,h^*\}} \max\{E - \sum_{N\setminus\{h\}} c_j, 0\} \\
&= c_i + \sum_{j\in N\setminus\{i,h^*\}} c_j - \sum_{h\in N\setminus\{i,h^*\}} \max\{E - \sum_{N\setminus\{h\}} c_j, 0\} \\
&= c_i + \sum_{h\in N\setminus\{i,h^*\}} \min\{\sum_N c_j - E, c_h\} \\
&\geq c_i.
\end{aligned}$$

Hence, both left and right hand sides of $(*)$ are equal to c_i.

Step 3: Conclusion. Using Step 2 and Step 1 in turn, we obtain,

$$\begin{aligned}
S^{tom}(c, E) &\equiv m(c, E) + S\Big(t\Big(c - m(c, E), E - \sum m_i(c, E)\Big), E - \sum m_i(c, E)\Big) \\
&= m(c, E) + S\Big(t(c, E) - m(t(c, E), E), E - \sum m_i(c, E)\Big). \\
&= m(t(c, E), E) + S\Big(t(c, E) - m(t(c, E), E), E - \sum m_i(t(c, E), E)\Big) \\
&\equiv S^{mot}(c, E).
\end{aligned}$$

□

It is natural to ask what would happen if the operators O^t and O^m were reapplied. The answer is: nothing. We have already noted that once minimal rights are attributed, claims adjusted down by the minimal rights, and the amount to divide adjusted down by the sum of the minimal rights, the minimal rights of the problem that results are all 0. In other words, the minimal rights in $(c - m(c, E), E - \sum m_j(c, E))$ are all 0 (Theorem 1). But consider the problem obtained from the above by truncating claims at the amount to divide. In this new problem,

$$(c', E') \equiv \Big(t\Big(c - m(c, E), E - \sum m_j(c, E)\Big), E - \sum m_j(c, E)\Big),$$

we assert that minimal rights are still all 0. Formally:

Proposition 1 *For each $(c, E) \in \mathcal{C}^N$, consider the problem obtained from it by attributing minimal rights, revising claims down by these minimal rights and the amount to divide down by the sum of the minimal rights. Then, after claims truncation, minimal rights are all 0.*

Proof: We need to show that for each $i \in N$, $m_i(c', E') \equiv \max\{E' - \sum_{N\backslash\{i\}} c'_j, 0\} = \max\{E' - \sum_{N\backslash\{i\}} t_j(c - m(c, E), E'), 0\} = 0$. Replacing E' by its value, this is equivalent to showing that

$$E - \sum m_j(c, E) \leq \sum_{N\backslash\{i\}} t_j(c - m(c, E), E - \sum m_k(c, E)),$$

and using the equality established in the proof of Theorem 4 (Relation Ri),

$$t_j(c - m(c, E), E - \sum m_k(c, E)) = t_j(c, E) - m_j(c, E),$$

showing that $E - m_i(c, E) \leq \sum_{N\backslash\{i\}} t_j(c, E)$, and equivalently that

$$(*) \quad E - \sum_{N\backslash\{i\}} t_j(c, E) \leq m_i(c, E).$$

To prove $(*)$, we distinguish two cases.

Case 1: there is $j \in N\backslash\{i\}$ such that $c_j \geq E$. Then the left-hand side of $(*)$ is at most equal to 0, whereas the right-hand side is 0. The desired inequality is satisfied.

Case 2: there is no such j. Then the left-hand side of $(*)$ is equal to $E - \sum_{N\backslash\{i\}} c_j$ and the right-hand side is the maximum of that same expression and 0. Once again, the desired inequality is satisfied.

□

Thanks to Theorem 4, we conclude that parallel statements can be made when O^t and O^m are applied in reverse order.

Our final result in this section relates O^c to the other three operators. We omit its proof, which follows directly from the definitions.

Theorem 5 *The convexity operator is distributive with respect to each of the claims truncation, attribution of minimal rights, and duality operators:*

For each $(S^k)_{k \in K}$ and each $(\lambda^k)_{k \in K}$,

	t	m	d
t	t (Thm 1)	$t \circ m$ (Thm 4)	$m \circ d$ (Thm 2)
m	$m \circ t$ (Thm 4)	m (Thm 1)	$t \circ d$ (Thm 2)
d	$d \circ m$ (Thm 2)	$d \circ t$ (Thm 2)	id (Thm 1)

Table 1: Summary table relating operators. The result of applying the operator indexing a row and then the operator indexing a column is indicated at the intersection of the row and the column. To illustrate, the composition $m \circ t$, in the (t, m) cell, is shown in that cell to be the equivalent to the $t \circ m$ composition. The notation id refers to the identity operator.

$$[w((S^k)_{k\in K}, (\lambda^k)_{k\in K})]^t = \sum_{k\in K} \lambda^k [S^k]^t.$$

Similar formulas hold with either the minimal rights or duality operators replacing the claims truncation operator.[12]

Table 1 here.

5 Preservation of properties under operators

In this section we undertake a systematic investigation of which properties are preserved under the operators defined in the previous section. The properties we consider have a straightforward interpretation, and to save space we refer readers to earlier literature for motivation and formal definitions. For the same reason, we do not consider properties that have been less frequently discussed.[13] We apologize for the enumeration, which nevertheless has the advantage of gathering all the material we need. Formal definitions can be found in Thomson (2003). The proofs are available from the authors upon request.

Order preservation (Aumann and Maschler, 1985): if agent i's claim is at least as large as agent j's claim, his award should be at least as large as agent j's award; also, his loss should be at least as large as agent j's

[12] We use superscripts either to indicate an operator or a number, but the context always makes it clear which is intended.

[13] Additional results are listed in Thomson (2005b).

loss; **group order preservation** (Thomson, 1998; Chambers and Thomson, 2002): given two groups of claimants, if the aggregate claim of the first group is at least as large as the aggregate claim of the second group, similar inequalities should hold between the aggregate award of the two groups, as well as between the aggregate loss incurred by the two groups; **anonymity**: any "renaming" of claimants should be accompanied by a parallel reassignment of awards; **homogeneity**: if claims and amount to divide are multiplied by the same positive number, so should all awards; and **continuity**: the awards vector should be a continuous function of the data of the problem.

Next are monotonicity properties. They are **claims monotonicity**: if an agent's claim increases, his award should be at least as large as it was initially; **resource monotonicity**: if the amount to divide increases, each claimant's award should be at least as large as it was initially.[14]

Our next group consists of invariance properties. First is **no advantageous transfer** (Moulin, 1987; Ju, Miyagawa, and Sakai, 2004): no group of claimants should receive more in the aggregate as a result of redistributing their claims among themselves. Two "composition" properties follow. If the amount to divide decreases from some initial value, this decrease can be dealt with in either one of two ways: (i) by canceling the initial division and recalculating the awards for the final amount; (ii) by taking the awards calculated on the basis of the initial amount as claims in dividing the final amount. **Composition down** (Moulin, 2000a) says that (i) and (ii) should result in the same awards vector. Now, suppose that instead, the amount to divide increases from some initial value. Here too, we can handle this increase in either one of two ways: (i) by canceling the initial division and simply recalculating the awards for the final amount; (ii) by letting claimants keep their initial awards, revising their claims down by these awards, and reapplying the rule to divide the incremental amount (the difference between the final and initial amounts). **Composition up** (Young, 1987) says that (i) and (ii) should give the same awards vector.

We close with several properties pertaining to the variable-population version of the model. **Population monotonicity** (Thomson, 1983): if new claimants arrive, the award to each of the claimants initially present should be at most as large as it was initially;[15] **replication invariance** (for a study of

[14]For the "inequality conditions", a "strict" version is obtained by requiring that the conclusion should be strict if the inequality appearing in the hypothesis is strict.

[15]For a survey of the literature on *population monotonicity*, see Thomson (1995).

the idea in this context, see Chun and Thomson, 2005): the awards vector of a replica problem should be the replica of the awards vector of the problem that is replicated (in the k-replica of a problem, each claimant has $k-1$ clones with claims equal to his and the amount to divide is multiplied by k); **consistency** (Young, 1987, is the author of the most general results on the subject[16]): if some claimants leave with their awards and the problem of dividing among the remaining claimants what is left is considered, these claimants should receive the same awards as was decided initially; **converse consistency** (see Chun, 1999, and Thomson, 2005a, for discussions of the property in this context): suppose that an awards vector x is such that its restriction to each two-claimant group is chosen for the problem of dividing between them the sum of their components of x; then, x should be chosen.

We say that **a property is preserved under an operator** if whenever a rule satisfies it, so does the rule obtained by subjecting to the operator. In the following pages, we discuss which properties are preserved under our operators, and which are not. The results are summarized in Table 2. We omit most proofs. The appendix contains the proofs of those results that are more difficult or from which a lesson can be drawn. For the remaining properties, the proofs are available from the authors upon request.

1. **Duality operator.** The properties that are preserved under the duality operator are numerous. We say that **two properties are dual** if whenever a rule satisfies one of them, its dual satisfies the other. A simple example of a pair of dual properties are the two parts of *order preservation.* This is most easily seen for $|N| = 2$, thanks to the convenient geometric interpretation of *self-duality.* Let $N \equiv \{1, 2\}$ and $c \in \mathbb{R}^N_+$ be such that $c_1 \leq c_2$, say. Then, $p(S, c)$ is above the 45° line (the first part of *order preservation*) if and only if $p(S^d, c)$ lies below the line of slope 1 emanating from c (the second part of *order preservation*). **A property is self-dual** if it is preserved under O^d.

Two basic monotonicity properties are not preserved under the duality operator, and we state their duals, to show how one performs this operation. First is *claims monotonicity.* Its dual says that if an agent's claim and the amount to divide increase by the same amount γ, this claimant's award should not increase by more than γ.[17] Indeed, let $i \in N$, and note that $S_i(c_i+\gamma, c_{-i}, E) \geq S_i(c, E)$ is equivalent to $c_i + \gamma - S^d_i(c_i + \gamma, c_{-i}, \sum c_j + \gamma - E) \geq$

[16]For a survey of the literature on *consistency* and its *converse*, see Thomson (2005a).

[17]This property is independently formulated by Moulin (2000b) for a discrete version of the model of claims resolution.

$c_i - S_i^d(c, \sum c_j - E)$.[18] After canceling out c_i from both sides of this inequality and replacing $\sum c_j - E$ by E', we obtain $\gamma \geq S_i^d(c_i+\gamma, c_{-i}, E'+\gamma) - S_i^d(c, E')$, as announced.

Second is *population monotonicity*. Its dual says that if new claimants arrive and the amount to divide increases by an amount equal to the sum of their claims, then the award to none of the claimants initially present should decrease. Indeed, for each $N' \subset N$, and each $i \in N'$, the inequality $S_i(c_{N'}, E) \geq S_i(c, E)$ is equivalent to the inequality $c_i - S_i^d(c_{N'}, \sum_{N'} c_j - E) \geq c_i - S_i^d(c, \sum_N c_j - E)$. After canceling out c_i from both sides of the inequality sign and introducing $E' \equiv \sum_{N'} c_j - E$, we obtain that for each $i \in N'$, $S_i^d(c_{N'}, E') \leq S_i^d(c, E' + \sum_{N \setminus N'} c_j)$.

2. Claims truncation and minimal rights operators. Many properties are preserved under O^t. Having at hand such a list, the concept of duality of properties, together with the following theorem, allows us to easily determine which properties are preserved under O^m. The only properties whose case cannot be settled by invoking these theorems are *claims monotonicity* and *population monotonicity*, and direct proofs are needed (see the Appendix).

Theorem 6 *A property is preserved under O^t if and only if its dual is preserved under O^m.*

Proof: Let A be a property that is preserved under O^t, A^d its dual, and let S be a rule satisfying A^d. We need to show that S^m satisfies A^d. Since A is dual to A^d, S^d satisfies A. Since A is preserved under O^t, S^{tod} satisfies A. Since A^d is dual to A, S^{dotod} satisfies A^d. We will show that $S^{dotod} = S^m$.

Recall that Theorem 2 asserts that if R is the dual of S, then R^t is the dual of S^m. Thus, $R^{dot} = S^m$. Since $R = S^d$, then $S^{dotod} = S^m$.

We have therefore shown that $S^{dotod} = S^m$, and since S^{dotod} was assumed to satisfy A^d, so does S^m. This completes the proof of the theorem in one direction.

We omit the "dual" proof for the other direction. □

Theorem 6 suggests an additional definition: **two operators are dual** if whenever a property is preserved under the first one, the dual property is preserved under the second one. According to this definition, O^t and O^m are dual.

[18]The notation c_{-i} designates the vector c from which the i-th coordinate has been deleted and (c_i', c_{-i}) the vector c in which the i-th coordinate has been replaced by c_i'.

3. Composition of the claims truncation and attribution of minimal rights operators. The next theorem says that the composition of O^t and O^m preserves duality of rules (according to Theorem 4, the operators can be composed in either order):

Theorem 7 *If two rules are dual, the versions of the rules obtained by truncating claims first and then attributing minimal rights are also dual.*

Proof: Let S and R be a pair of dual rules. By Theorem 2, S^t and R^m are dual too. Applying Theorem 2 to this second pair, we deduce that $S^{m\circ t}$ and $R^{t\circ m}$ are dual. By Theorem 4, $R^{t\circ m} = R^{m\circ t}$. Thus, $S^{m\circ t}$ and $R^{m\circ t}$ are dual (and of course, so are $S^{t\circ m}$ and $R^{t\circ m}$). □

If a property is preserved under both O^t and O^m separately, then clearly it is preserved under their composition. However, a property may be preserved under neither of these operators and yet be preserved under their composition. An example is *self-duality*, for which we obtain the following result, which is a corollary of Theorem 7.

Corollary 1 *If a rule is self-dual, the version of the rule obtained by truncating claims first and then attributing minimal rights is also self-dual.*

Curiel, Maschler and Tijs (1987) take P as a starting point and define a new rule from it by first attributing minimal rights and then truncating claims. They show that their rule is *self-dual* by invoking a game-theoretic argument. Since P is *self-dual*, this result can be obtained as an application of Corollary 1.

Typically however, when a property is preserved by neither O^t nor O^m, it is not recovered by their composition. To illustrate, consider *resource monotonicity.* The rule CEA is *resource monotonic* but CEA^m is not. Neither is $CEA^{t\circ m}$, which coincides with it: indeed, since this rule is *invariant under claims truncation*, $CEA^{m\circ t} = CEA^m$.

Consistency is another example of a property that is not preserved under the composition of O^t and O^m. To see this, recall that this property is not preserved under O^m (Table 2). This can be proved by means of P, but CEA could have been used to make the point.

This reasoning can be applied more generally. Consider a property that can be shown not to be preserved under O^m by means of a rule that satisfies *invariance under claims truncation.* Then the property is not preserved under the composition of O^t and O^m. Also, consider a property that can be shown not to be preserved under O^t by means of a rule that satisfies *minimal rights first*. Then, the property is not preserved under the composition of O^t and O^m. These observations are direct consequences of Theorem 6.

4. Convexity operator. We conclude with the convexity operator, for which no particular remarks need be made. It preserves quite a few properties, but not several important ones. We simply refer to Table 2 for the complete list.

6 Conclusion

Concerning the extent to which the operators preserve properties, we can offer no strict ranking, but only an informal observation that of the four operators, the convexity operator tends to be the least disruptive. Also, due to the duality between the claims truncation and attribution of minimal rights operators, these two operators are equivalent in that regard. Switching our focus from operators to properties, here no easy generalization can be made: punctual properties (properties of rules that apply to each point in their domain separately, such as *order preservation*) do not seem to be preserved more frequently than relational properties (properties of rules that relate the choices they make for problems that are related in some way, such as *resource monotonicity*). A similar statement can be made about the fixed-population properties as compared to the variable-population properties.

Prop \operators	convexity	duality	truncation	min rights	trunc ∘ min rights
Equal treat of equals	+	+	+	+	+
Order pres	+	+	+	+	+
Anonymity	+	+	+	+	+
Group order pres	+	+	− (P)	− (P)	− (P)
Continuity	+	+	+	+	+
Claims mon	+	− (Prop 3)	+	− (Prop 4)	− (Prop 7)
Resource mon	+	+	− (CEL) (Prop 6)	− (CEA)	− (CEA)
Homogeneity	+	+	+	+	+
Claims trunc inv	+	− (CEA)	+	+	+
Min rights first	+	−(CEL)	+	+	+
Comp down	− (Av)	− (ES^u) (Prop 2)	− (P)	− (P)	− (CEA)
Comp up	− (Av)	−(ES^u) (Prop 2)	− (P)	− (P)	− (CEA)
Self-duality	+	+	− (P)	− (P)	+ (Cor 1)
No adv trans	+	+	− (P)	− (P)	− (P)
Pop mon	+	− (Prop 5)	+	− (CEA)	− (CEA)
Repli inv	+	+	− (P)	− (P)	− (P)
Consistency	− (Av)	+	− (P)	− (P)	− (P)
Conv cons	− (Av)	+	− (P)	− (P)	− (P)

Table 2: Showing which properties are preserved under the operators. In each cell for which a negative result holds, we indicate in parenthesis a rule allowing to prove the assertion. For instance, the notation (P) at the intersection of the row labelled "*group order preservation*" and column labelled "truncation" means that P satisfies the property but that P^t does not.

APPENDIX

In this appendix, we provide the proofs of selected results presented in Table 2. We use the following additional notation. Given $x^1, x^2, \ldots, x^k \in \mathbb{R}^N$, $\text{seg}[x^1, x^2]$ denotes the segment connecting them and $\text{seg}]x^1, x^2] \equiv \text{seg}[x^1, x^2] \setminus \{x^1\}$; also $\text{bro.seg}[x^1, x^2, \ldots, x^k] \equiv \text{seg}[x^1, x^2] \cup \text{seg}[x^2, x^3] \cup \ldots \cup \text{seg}[x^{k-1}, x^k]$.

Proposition 2 *The following properties are not preserved under the duality operator but they come in dual pairs: invariance under claims truncation and minimal rights first; composition down and composition up.*

For the proof of the second part of this proposition, we use the following "equal sacrifice rule" (Young, 1988; Moulin, 2000a). Let $u\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $u(x) \equiv \frac{-1}{x}$ and ES^u the rule that selects for each $(c, E) \in \mathcal{C}^N$ the vector $x \in X(c, E)$ such that for each pair $\{i, j\} \subseteq N$, $u(c_i) - u(x_i) = u(c_j) - u(x_j)$. (This is equivalent to setting $x_i = \frac{c_i}{1+\beta c_i}$, where $\beta \in \mathbb{R}_+$ is chosen so as to achieve efficiency.)

Proof: The duality between *invariance under claims truncation* and *minimal rights first* is proved by Herrero (1998) (Dagan, 1996, proves a related result).

Minimal rights first: CEL can be used to make the point. One can also appeal to the example used to prove that *invariance under claims truncation* is not preserved and to the fact that this property and *minimal rights first* are dual properties.

Composition down: We assert first that ES^u satisfies *composition down*. To see this, let $(c, E) \in \mathcal{C}^N$ be given and $E' < E$. Let $x \equiv ES^u(c, E)$, $x' \equiv ES^u(c, E')$, and $y \equiv ES^u(x, E')$. We show that $x' = y$. Let $i \in N$. Let $\beta_x \in \mathbb{R}_+$ be such that $\sum \frac{c_i}{1+\beta_x c_i} = E$. Let $\beta_{x'}$ and β_y be similarly defined. By definition of ES^u, $x_i = \frac{c_i}{1+\beta_x c_i}$ and $y_i = \frac{x_i}{1+\beta_y x_i}$. Thus, $y_i = \frac{c_i}{1+(\beta_y+\beta_x)c_i}$. Since $\sum y_i = \sum x'_i = E'$ and $\beta_{x'}$ is uniquely determined, $\beta_{x'} = \beta_y + \beta_x$. Thus, $x'_i = y_i$, as announced.

Next, we assert that $(ES^u)^d$ violates *composition down*. Let $N \equiv \{1, 2\}$, $(c, E) \in \mathcal{C}^N$ be defined by $(c, E) \equiv (1, 3; \frac{68}{21})$, and $E' = \frac{11}{4}$. Then, $(ES^u)^d(c, E) = (\frac{2}{3}, \frac{18}{7})$ and $(ES^u)^d(c, E') = (\frac{1}{2}, \frac{9}{4})$. Let $c' \equiv (\frac{2}{3}, \frac{18}{7})$ and $x \equiv (ES^u)^d(c', E')$. We claim that $x \neq (\frac{1}{2}, \frac{9}{4})$. Suppose that $x = (\frac{1}{2}, \frac{9}{4})$. Since $x \equiv c' - (ES^u)(c', \sum c'_i - E')$, then $(ES^u)(c', \sum c'_i - E') = (\frac{1}{6}, \frac{9}{28})$. Let $\beta \in \mathbb{R}_+$ be such that $\sum (ES^u)_i(c', \sum c'_i - E') = \sum \frac{c'_i}{1+\beta c'_i} = \sum c'_i - E'$. We obtain $\beta = \frac{9}{2}$ but also $\frac{49}{18}$. This is impossible since β is uniquely determined.

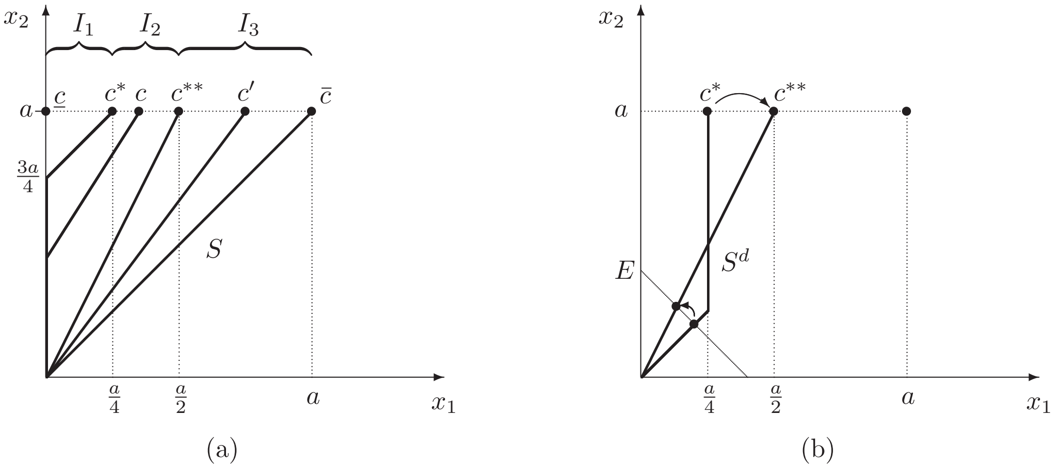

Figure 3: Claims-monotonicity is not preserved under the duality operator (Proposition 3). (a) Sample paths of awards of the rule S defined in the proof. (b) Paths $p(S^d, c^*)$ and $p(S^d, c^{**})$. If the amount to divide is E, as agent 1's claim increases from $c_1^* = \frac{a}{4}$ to $c_1^{**} = \frac{a}{2}$, he receives less (follow the arrows).

The duality between the two *composition* properties is proved by Moulin (2000a). $\square$

Our next result concerns *claims monotonicity*, a property that is satisfied by every rule encountered in the literature.[19] Unfortunately we have:

Proposition 3 *Claims monotonicity is not preserved under the duality operator.*

The proof is by means of an example. It is of interest that the example is *anonymous*, *order-preserving*, *homogeneous*, and *resource monotonic* (and therefore *resource continuous*; it is in fact fully *continuous*, that is, jointly continuous with respect to the claims and the amount to divide). This shows that these properties do not help preserve *claims monotonicity*.

Proof: We define a rule S on $\mathcal{C}^N$, where $N \equiv \{1, 2\}$. The rule is depicted in Figures 3 and 4. We show that S is *claims monotonic* whereas S^d is not.

Let $a > 0$, $\underline{c} \equiv (0, a)$, $c^* \equiv (\frac{a}{4}, a)$, $c^{**} \equiv (\frac{a}{2}, a)$, and $\bar{c} \equiv (a, a)$. We first specify $p(S, c)$ for each $c \in \text{seg}[\underline{c}, \bar{c}]$. We then choose $p(S, c)$ for each $c \in \text{seg}[(a, 0), \bar{c}]$ as the symmetric image with respect to the 45° line of $p(S, (c_2, c_1))$. Finally, we choose $p(S, c)$ for each other $c \in \mathbb{R}^2_+$ by first calculating μ such that $\mu c \in \text{bro.seg}[(a, 0), \bar{c}, (0, a)]$ and subjecting $p(S, \mu c)$ to a

[19] Here too, few of the standard rules satisfy the stronger requirement that an agent whose claim increases should receive more, unless $E = 0$ of course (equality is not permitted any more). The rule P is a rare example that does. However, it is easy to construct rules that do. Most "parametric rules" (Young, 1987) do.

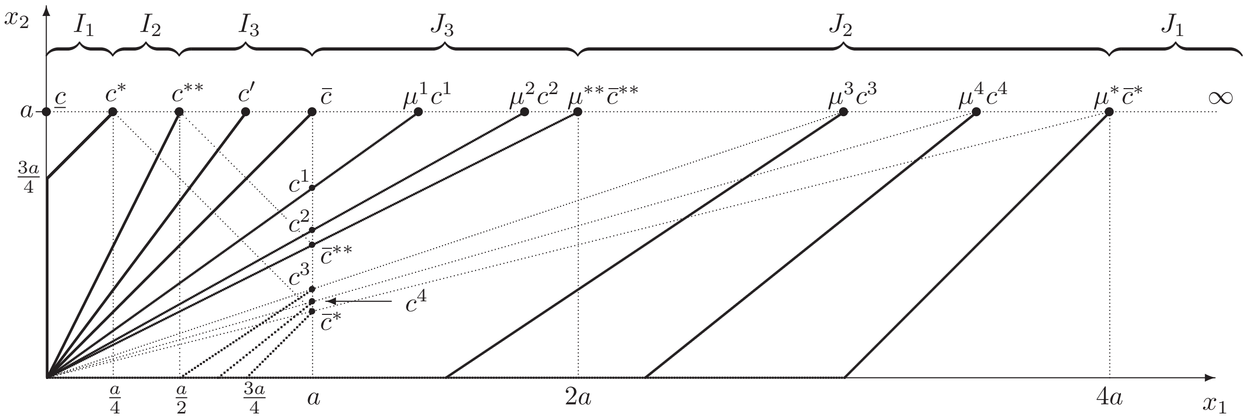

Figure 4: The rule S of Proposition 3 is claims monotonic. Sample paths $p(S,c)$ for $c \equiv (c_1, a)$ when $c_1 \in [0, \infty[$. The path $p(S,c)$ for each $c \in \text{seg}[(a,0), \bar{c}]$, where $\bar{c} \equiv (a,a)$, is obtained by symmetry from the path for the symmetric image of c with respect to the 45° line. The paths for two critical claims vectors, $\bar{c}^*$ and $\bar{c}^{**}$, the symmetric images of c^* and c^{**}, are represented. The path $p(S,c)$ for each $c \in J_3 \cup J_2 \cup J_1$ is obtained by homothetic expansion of the path for the homothetic image of c that belongs to $\text{seg}[(a,0), \bar{c}]$. For each amount to divide, as agent 1's claim increases, he receives at least as much as he did initially.

homothetic transformation of ratio $\frac{1}{\mu}$. This construction guarantees that S is *anonymous* and *homogeneous*.

For each $c \in I_1 \equiv \text{seg}[\underline{c}, c^*]$ (see Figure 3a for illustrations of I_1 and I_2 and I_3 defined below), $p(S,c) = p(CEL, c)$. For each $c \in I_2 \equiv \text{seg}]c^*, c^{**}]$, $p(S,c)$ is piecewise linear in two pieces: given $0 \leq \lambda \leq 1$, $p(S, \lambda c^* + (1-\lambda)c^{**}) = \text{bro.seg}[(0,0), \lambda(0, \frac{3}{4}a), \lambda c^* + (1-\lambda)c^{**}]$. (Note that for $\lambda = 0$, the path is that of P.) For each $c \in I_3 \equiv \text{seg}]c^{**}, \bar{c}]$, $p(S,c) = p(P,c)$.

Figure 4 illustrates that when agent 2's claim is fixed at a, and as agent 1's claim increases from 0 to ∞, agent 1's award does not decrease. The *claims monotonicity* of S is a consequence of this fact and of its being *anonymous* and *homogeneous*. The figure indicates some paths $p(S,c)$ for $c \in \mathbb{R}^N_+$ with $c_2 = a$. We show that these paths never cross. Given any claims vector c of the form $c \equiv (c_1, a)$ for $c_1 \in [a, \infty[$, there is a claims vector on $\text{seg}[(a,0), \bar{c}]$ that is proportional to it. We call $\mu \geq 1$ the expansion factor required to pass from the latter to the former, using the same superscript to keep track of this pairing, $\bar{c}^*$ and $\mu^*\bar{c}^*$ being an example of a pair so defined.

1. For each $c \in J_3 \equiv \text{seg}]\bar{c}, \mu^{**}\bar{c}^{**}]$ (again, see Figure 4), where $\bar{c}^{**}$ is

the symmetric image of c^{**}, $p(S,c) = p(P,c)$ (examples are $p(S,\mu^1c^1)$ and $p(S,\mu^2c^2)$).

2. For each $c \in J_2 \equiv \text{seg}]\mu^{**}\bar{c}^{**}, \mu^*\bar{c}^*]$, $p(S,c)$ is obtained by a homothetic expansion of the path for the reduced image of c that belongs to $\text{seg}[(a,0),\bar{c}]$. For example, consider two points in J_2, such as μ^3c^3 and μ^4c^4 in the figure, where μ^4c^4 is to the right of μ^3c^3. Then, the paths for these points are obtained by homothetic expansions of the paths for c^3 and c^4, with c^4 below c^3. The slope of the oblique segment in $p(S,c^4)$ is greater than the slope of the oblique segment in $p(S,c^3)$. Therefore the same statement can be made about the slopes of the oblique segments in $p(S,\mu^4c^4)$ and $p(S,\mu^3c^3)$, which imply that they do not cross.

3. Finally, for each $c \in J_1 \equiv \{(c_1,a) : c_1 \in]4a,\infty[\}$ (J_1 is the open half-line $\{\mu^*\bar{c}^* + t(1,0) : t > 0\}$ in the figure), $p(S,c)$ consists of a horizontal segment from the origin and a segment of slope 1.

The fact that S^d violates *claims monotonicity* can be seen by considering $p(S^d,c^*)$ and $p(S^d,c^{**})$. These paths are obtained by symmetry of $p(S,c^*)$ and $p(S,c^{**})$. Inspection of Figure 3b reveals that the paths cross: in fact, for each amount to divide in the interval $]0,\frac{3}{4}a[$, agent 1 loses as his claim increases from $c_1^* = \frac{a}{4}$ to $c_1^{**} = \frac{a}{2}$, agent 2's claim being kept fixed at a. □

The strengthening of *claims monotonicity* obtained by requiring that if an agent's claim increases, he should receive more, is not preserved under the duality operator either. To see this, it suffices to modify the example used to prove Proposition 3. Informally, for each $c \in I_1 \cup I_2$, replace the vertical segment of $p(S,c)$ by a very steep segment whose slope varies continuously and monotonically between ∞ and 2 as c varies in $I_1 \cup I_2$ from $\underline{c}$ to c^{**}.

Proposition 4 *Claims monotonicity is not preserved under the attribution of minimal rights operator.*

We prove this result by exhibiting a rule that is *claims monotonic*, but once subjected to O^m, it is not. Although the assertion can be proved by means of the example used to prove Proposition 7, we exhibit here a rule that is *order preserving*, *anonymous*, *resource monotonic*, and *continuous*.

Proof: The proof is by means of an example of a rule S defined on $\mathcal{C}^N$ where $N \equiv \{1,2\}$. It is depicted in Figure 5a.

Step 1: Construction of S. We first consider $c \in \mathbb{R}^N_+$ with $c_1 \leq c_2$. If $c_2 \leq 2$, then $p(S,c) = p(P,c)$. If $c_2 \geq 4$, then $p(S,c) = p(CEL,c)$. If

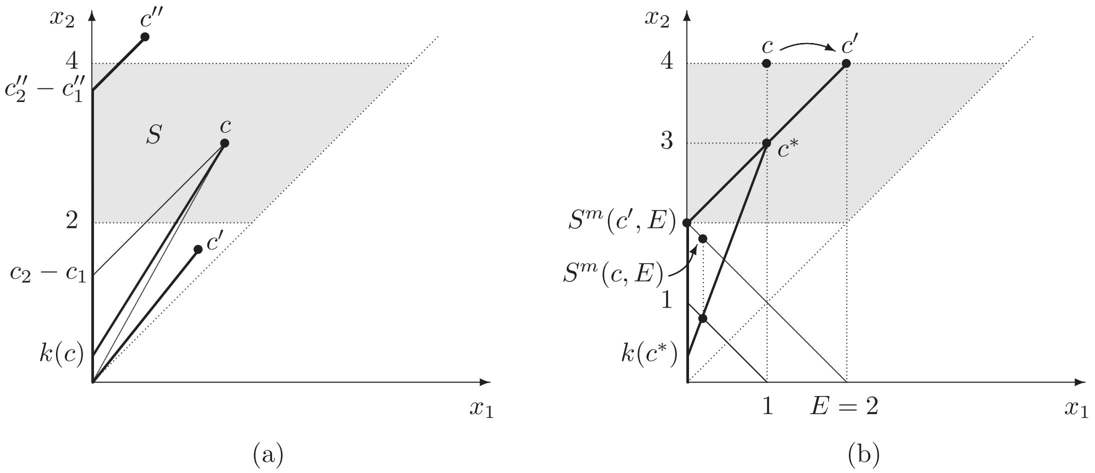

Figure 5: Claims monotonicity is not preserved under the attribution of minimal rights operator (Proposition 4.) (a) Sample paths of awards of the rule S defined in the proof. The path for c in the shaded region is a linear combination of $p(P, c)$ and $p(CEL, c)$. (b) Paths $p(S^m, c)$ and $p(S^m, c')$. If $E = 2$, as agent 1's claim increases from 1 to 2, he receives less.

$2 < c_2 < 4$ (the shaded region), then $p(S, c)$ is a linear combination of $p(P, c)$ and $p(CEL, c)$. The construction uses an arbitrary continuous and monotone function $g\colon [0,1] \to [0,1]$ such that for each $t \in [0,1]$, $g(t) \leq t$, and $g(0) = 0$, $g(\frac{1}{2}) = \frac{1}{4}$, and $g(1) = 1$. Now, let $k(c) \equiv g(\frac{c_2-2}{2})(c_2 - c_1)$, and $p(S, c) \equiv \text{bro.seg}[(0,0), (0, k(c)), c]$.

We then choose $p(S, c)$ for each $c \in \mathbb{R}^N_+$ with $c_1 > c_2$ as the symmetric image with respect to the 45° line of $p(S, (c_2, c_1))$. This guarantees that S is *anonymous*.

Step 2: S is *claims monotonic*. Since S is *anonymous*, it is enough to examine the rule in the region $\{c \in \mathbb{R}^N_+ \colon c_1 \leq c_2\}$. First, let $c_2 > 0$ and let $c_1, c'_1 \in [0, c_2]$ be such that $c'_1 < c_1$. Let $c'_2 \equiv c_2$ and $c' \equiv (c'_1, c'_2)$. There are three subcases. If $c_2 \leq 2$, then $p(S, c) = p(P, c)$ and $p(S, c') = p(P, c')$, and since P satisfies *claims monotonicity*, we are done. If $c_2 \geq 4$, then $p(S, c) = p(CEL, c)$ and $p(S, c') = p(CEL, c')$, and since CEL satisfies *claims monotonicity*, we are done. If $2 < c_2 < 4$, then $p(S, c) = \text{bro.seg}[(0,0), (0, k(c)), c]$. Also, $p(S, c') = \text{bro.seg}[(0,0), (0, k(c'), c']$. The conclusion follows from the fact that $c_2 - c_1 < c'_2 - c'_1$, and since $c'_2 = c_2$, $g(\frac{c_2-2}{2}) = g(\frac{c'_2-2}{2})$, so that altogether $k(c) \equiv g(\frac{c_2-2}{2})(c_2 - c_1) < g(\frac{c'_2-2}{2})(c'_2 - c'_1) \equiv k(c')$.

Next, let $c_1 > 0$ and let $c_2, c'_2 \in [c_1, \infty[$ be such that $c_2 < c'_2$. Let $c'_1 \equiv c_1$ and $c' \equiv (c'_1, c'_2)$. We have $p(S, c) = \text{bro.seg}[(0,0), (0, k(c)), c]$; also, $p(S, c') = \text{bro.seg}[(0,0), (0, k(c')), c']$. The conclusion follows from the fact that since $c'_2 > c_2$ and $c'_1 = c_1$, then $c_2 - c_1 < c'_2 - c'_1$, and since g is increasing, $g(\frac{c_2-2}{2}) \leq g(\frac{c'_2-2}{2})$, so that altogether $k(c) \equiv g(\frac{c_2-2}{2})(c_2 - c_1) < g(\frac{c'_2-2}{2})(c'_2 - c'_1) \equiv k(c')$.

Step 3: S^m is not *claims monotonic* (Figure 5b). To see this, let $c \equiv (1,4)$, $c' \equiv (2,4)$, and $E = 2$. Note that $m(c,E) = (0,1)$ and $m(c',E) = (0,0)$. We have $S^m(c,E) = m(c,E) + S(c - m(c,E), E - \sum m_i(c,E))$. Let $c^* \equiv c - m(c,E)$. To calculate the second term in this sum, we note that $c^* = (1,3)$ and $E - \sum m_i(c,E) = 1$. Then, $p(S,c^*)$ is $\text{seg}[(0,0), k(c^*), c^*]$, where $k(c^*) \equiv g(\frac{c_2^* - 2}{2})(c_2^* - c_1^*)$. Since $g(\frac{c_2^*-2}{2}) = g(\frac{1}{2}) = \frac{1}{4}$, we have $k(c^*) = \frac{1}{4}(c_2^* - c_1^*) = \frac{1}{2} < 1 = E - \sum m_i(c,E)$. So, $S_1(c^*, E - \sum m_i(c,E)) > 0$. This implies $S_1^m(c,E) > 0$. Also, $S_1^m(c',E) = S_1(c',E) = CEL_1(c',E) = 0$. Thus, as agent 1's claim increases from $c_1 = 1$ to $c_1' = 2$, he receives less, in violation of *claims monotonicity.* □

Next, we turn to *population monotonicity* for which a negative result also holds. We prove this fact by exhibiting a rule S that is *anonymous, homogeneous,* and *resource monotonic*, and *population monotonic* but S^d is not *population monotonic.* (Since *resource monotonicity* implies *resource continuity, S* is also *resource continuous*):

Proposition 5 *Population monotonicity is not preserved under the duality operator.*

Proof: The proof is by means of an example of a rule S defined on $\bigcup_{N' \subseteq N} \mathcal{C}^{N'}$ where $N \equiv \{1,2,3\}$. Sample paths of awards of S are plotted in Figures 6a and 6b. We show that S is *population monotonic* but S^d is not.

Step 1: Construction of S. On the subdomain of two-claimant problems, $S \equiv P$. Let Q be the unit cube in $\mathbb{R}^N_+$, and for each $t \in \{1,2,3\}$, let F_t be the face of Q consisting of all $c \in \mathbb{R}^N_+$ such that $c_t = 1$. Given $c \equiv (c_1, 1, c_3) \neq (1,1,1)$, a typical claims vector in F_2, let L be the line passing through c and $e \equiv (1,1,1)$. Also, let $x \equiv L \cap \text{seg}[(\frac{2}{3},1,1),(1,1,\frac{2}{3})]$ and $y \equiv L \cap \text{seg}[(0,1,1),(1,1,0)]$ (we use the notation L', x', y' for the claims vector c'). If $0 < c_1 + c_3 \leq 1$, then $p(S,c) = p(P,c)$ (c in Figure 6a). If $1 < c_1 + c_3 \leq 1 + \frac{2}{3}$, then $p(S,c) = \text{bro.seg}[(0,0,0), y, c]$ (c' in Figure 6a). If $1 + \frac{2}{3} < c_1 + c_3 \leq 2$, then $p(S,c)$ is piecewise linear in two pieces defined as follows: let $0 \leq \lambda \leq 1$ be such that $c = \lambda x + (1-\lambda)e$. Then $p(S,c) = \text{seg}[(0,0,0), d, c]$ where $d \equiv \lambda y + (1-\lambda)\frac{2}{3}e$ (Figure 6b). Finally, if $c = e$, then $p(S,c) = p(P,c)$. We deduce $p(S,c)$ for each $c \in F_1$ by symmetry with respect to the plane of equation $x_1 = x_2$ of $p(S,c')$ where c' is symmetric image of c with respect to that plane; similarly we deduce $p(S,c)$ for each $c \in F_3$ by symmetry with respect to the plane of equation $x_2 = x_3$ of $p(S, c')$ where c' is the symmetric image of c with respect to that plane. If c is not in any of the faces F_1, F_2, and F_3, let $\mu \in \mathbb{R}_+$ be such that μc does belong to such a face. Then, $p(S, c)$ is obtained from $p(S, \mu c)$ by the homothetic transformation of ratio $\frac{1}{\mu}$. This construction guarantees that S is *anonymous* and *homogeneous.*

Step 2: S is *population monotonic*. Let $E > 0$ and $c \equiv (c_1, 1, c_3)$ be an arbitrary point in F_2. We distinguish three cases.

Case 1: $0 < c_1 + c_3 \leq 1$. Then, $S(c, E) \equiv P(c, E)$. Since $S(c_{N'}, E) \equiv P(c_{N'}, E)$ for each N' with $|N'| = 2$ and P is *population monotonic*, the *population-monotonicity* inequalities hold.

Case 2: $1 < c_1 + c_3 \leq 1 + \frac{2}{3}$. We imagine the departure of each agent in turn (Figure 6c).

Subcase 2.1: Claimant 1 leaves. We have to compare $z \equiv S(c, E)$ and $z' \equiv S\left(c_{\{2,3\}}, E\right)$. We assume that $E \leq 1 + c_3$ since otherwise there is nothing to check. Since $y = L \cap \text{seg}[(0, 1, 1), (1, 1, 0)]$, then $y_1 + 1 + y_3 = 2$. Note that y belongs to the simplex in the plane of equation $\sum v_i = 2$. Thus $z = \left(\frac{y_1 E}{2}, \frac{E}{2}, \frac{y_3 E}{2}\right)$. Also $z' = \left(\frac{E}{1+c_3}, \frac{c_3 E}{1+c_3}\right)$. Then $z'_3 - z_3 = \frac{c_3 E}{1+c_3} - \frac{y_3 E}{2}$. Since $c_3 \geq y_3$ and $1 + c_3 \leq 2$, then $z'_3 - z_3 \geq 0$.

Also $z'_2 - z_2 = \frac{E}{1+c_3} - \frac{E}{2}$. Since $1 + c_3 \leq 2$, then $z'_2 - z_2 \geq 0$.

Subcase 2.2: Claimant 2 leaves. We have to compare $z \equiv S(c, E)$ and $z' \equiv S\left(c_{\{1,3\}}, E\right)$. We assume that $E \leq c_1 + c_3$ since otherwise there is nothing to check. Since $y = L \cap \text{seg}[(0, 1, 1), (1, 1, 0)]$, as already calculated, $y_1 + 1 + y_3 = 2$. Thus $z = \left(\frac{y_1 E}{2}, \frac{E}{2}, \frac{y_3 E}{2}\right)$. Also $z' = \left(\frac{c_1 E}{c_1+c_3}, \frac{c_3 E}{c_1+c_3}\right)$. Thus $z'_1 - z_1 = \frac{c_1 E}{c_1+c_3} - \frac{y_1 E}{2}$. Since $c_1 \geq y_1$ and $c_1 + c_3 \leq 2$, then $z'_1 - z_1 \geq 0$.

Also, $z'_3 - z_3 = \frac{c_3 E}{c_1+c_3} - \frac{y_3 E}{2}$. Since $c_3 \geq y_3$ and $c_1 + c_3 \leq 2$, then $z'_3 - z_3 \geq 0$.

Subcase 2.3: Claimant 3 leaves. We apply the same argument as in Subcase 2.1.

Case 3: $1 + \frac{2}{3} < c_1 + c_3 \leq 2$. Let λ be such that $c = \lambda x + (1 - \lambda)e$ and $d \equiv \lambda y + (1 - \lambda)\frac{2}{3}e$. Since $x \geq y$ and $e > \frac{2}{3}e$, then $c \geq d$.

Subcase 3.1: Claimant 1 leaves. We have to compare $z \equiv S(c, E)$ and $z' \equiv S\left(c_{\{2,3\}}, E\right)$. We assume that $E \leq 1 + c_3$ since otherwise there is nothing to check. Note that $d_1 + d_2 + d_3 = 2$ and $E \leq 1 + c_3 \leq d_1 + d_2 + d_3$, so $z = \left(\frac{d_1 E}{2}, \frac{d_2 E}{2}, \frac{d_3 E}{2}\right)$ and $z' = \left(\frac{E}{1+c_3}, \frac{c_3 E}{1+c_3}\right)$. Thus $z'_3 - z_3 = \frac{c_3 E}{1+c_3} - \frac{d_3 E}{2}$. Since $c_3 \geq d_3$ and $1 + c_3 \leq 2$, then $z'_3 - z_3 \geq 0$.

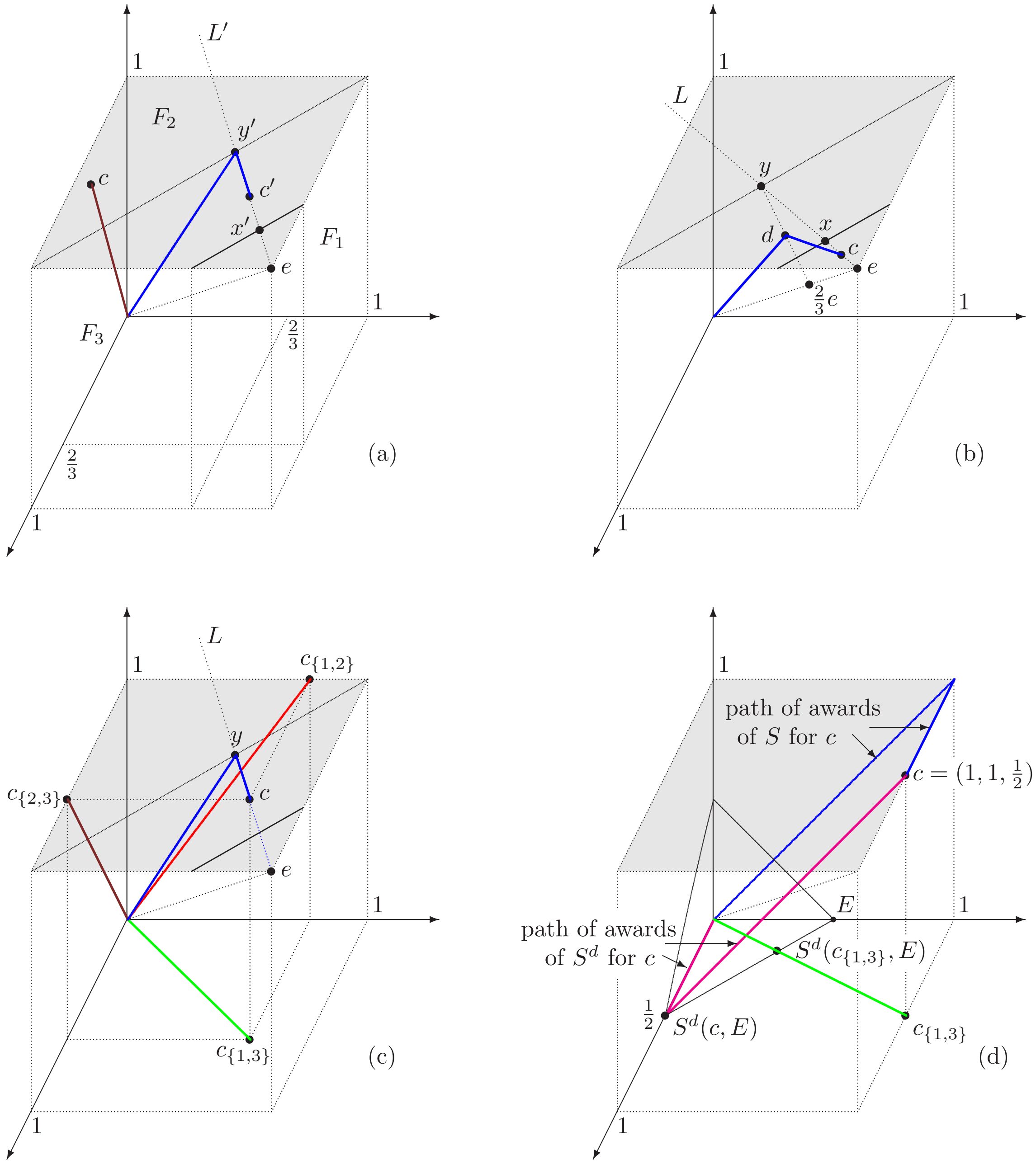

Figure 6: Population monotonicity is not preserved under the duality operator (Proposition 5). (a) Construction of $p(S, c)$ for c such that $c_2 = 1$ and $0 \leq c_1+c_3 \leq 1$, and for c' such that $c'_2 = 1$ and $1 < c'_1+c'_3 \leq 1+\frac{2}{3}$. (b) Construction of $p(S, c)$ for c such that $c_2 = 1$ and $1+\frac{2}{3} < c_1+c_3 \leq 2$. (c) The rule S is *population monotonic*. Given c in F_2, we determine $p(S, c)$ (it consists of two line segments), and the paths of awards of S for each of the projections of c onto the three two-dimensional coordinates subspaces (these paths are segments connecting the origin to these projections). Then, given E, we calculate the awards vectors selected by S for the resulting problems. (d) The rule S^d is not *population monotonic*. For $(c, E) \equiv (1, 1, \frac{1}{2}; \frac{1}{2})$, it selects $(0, 0, \frac{1}{2})$, but for the problem that results from the departure of claimant 2, it selects $(\frac{1}{3}, \frac{1}{6})$. Claimant 3 loses.

Also, $z'_2 - z_2 = \frac{E}{1+c_3} - \frac{d_2 E}{2}$. Since $1 \geq d_2$ and $1 + c_3 \leq 2$, $z'_2 - z_2 \geq 0$.

Subcase 3.2: Claimant 2 leaves. We have to compare $z \equiv S(c, E)$ and $z' \equiv S(c_{\{1,3\}}, E)$. We assume that $E \leq c_1 + c_3$ since otherwise there is nothing to check. Note that $d_1 + d_2 + d_3 = 2$ and $E \leq c_1 + c_3 \leq d_1 + d_2 + d_3$, so $z = \left(\frac{d_1 E}{2}, \frac{d_2 E}{2}, \frac{d_3 E}{2}\right)$ and $z' = \left(\frac{c_1 E}{c_1+c_3}, \frac{c_3 E}{c_1+c_3}\right)$. Thus $z'_1 - z_1 = \frac{c_1 E}{c_1+c_3} - \frac{d_1 E}{2}$. Since $c_1 \geq d_1$ and $c_1 + c_3 \leq 2$, then $z'_1 - z_1 \geq 0$.

Also, $z'_3 - z_3 = \frac{c_3 E}{c_1+c_3} - \frac{d_3 E}{2}$. Since $c_3 \geq d_3$ and $c_1 + c_3 \leq 2$, then $z'_3 - z_3 \geq 0$.

Subcase 3.3: Claimant 3 leaves. We apply the same argument as in Subcase 3.1.

Step 3: S^d is not *population monotonic* (Figure 6d). Let $(c, E) \equiv \left(1, 1, \frac{1}{2}; \frac{1}{2}\right)$. We have $p(S, c) = \text{bro.seg}[(0,0,0), (1,1,0), (1,1,\frac{1}{2})]$. The path $p(S^d, c)$ is obtained from $p(S, c)$ by symmetry with respect to $\frac{c}{2}$. Thus, $p(S^d, c) = \text{bro.seg}[(0,0,0), (0,0,\frac{1}{2}), (1,1,\frac{1}{2})]$. Then, $S^d\left(1, 1, \frac{1}{2}; \frac{1}{2}\right) = (0, 0, \frac{1}{2})$.

Let claimant 2 leave. Then $c_{\{1,3\}} \equiv \left(1, \frac{1}{2}\right)$. By definition of S, $S\left(1, \frac{1}{2}; \frac{1}{2}\right) = P\left(1, \frac{1}{2}; \frac{1}{2}\right) = \left(\frac{1}{3}, \frac{1}{6}\right)$. Since P is *self-dual*, $S^d\left(1, \frac{1}{2}; \frac{1}{2}\right) = \left(\frac{1}{3}, \frac{1}{6}\right)$. Since claimant 3 receives less in the two-claimant problem than in the three-claimant problem, S^d violates *population monotonicity.* □

Proposition 6 *Resource monotonicity is not preserved under the claims truncation operator.*

Proof: The rule CEL satisfies the property but CEL^t does not. To see this, let $N \equiv \{1, 2, 3\}$ and $(c, E) \in \mathcal{C}^N$ be defined by $(c, E) \equiv (10, 20, 30; 10)$. Then $CEL^t(c, E) = (\frac{10}{3}, \frac{10}{3}, \frac{10}{3})$. However, for $E' \equiv 20$, we obtain $CEL^t(c, E') = (0, 10, 10)$. Claimant 1 loses when the amount to divide increases from E to E'.

Since for $|N| = 2$, CEL^t coincides with CD, which is *resource monotonic*, this negative result can be proved by means of CEL only with an example involving at least three claimants. However, rules can be constructed to make the point that the property is not preserved under the claims truncation operator for $|N| = 2$. Any such rule has to fail *claims monotonicity*, a property that CEL satisfies. The proof is by means of an example S. Let $c \equiv (4, 7)$, $c' \equiv (4, 6)$, $p(S, c) \equiv \text{bro.seg}[(0,0), (3.5, 3.5), c]$, and $p(S, c') \equiv \text{bro.seg}[(0,0), (2,4), c']$. Both of these paths are monotone, and to obtain a *resource monotonic* rule it suffices to choose $p(S, \tilde{c})$ to be a monotone path for any other $\tilde{c}$. Let $E = 7$ and $E' = 6$. Now, note that $S^t(c, E) = (3.5, 3.5)$

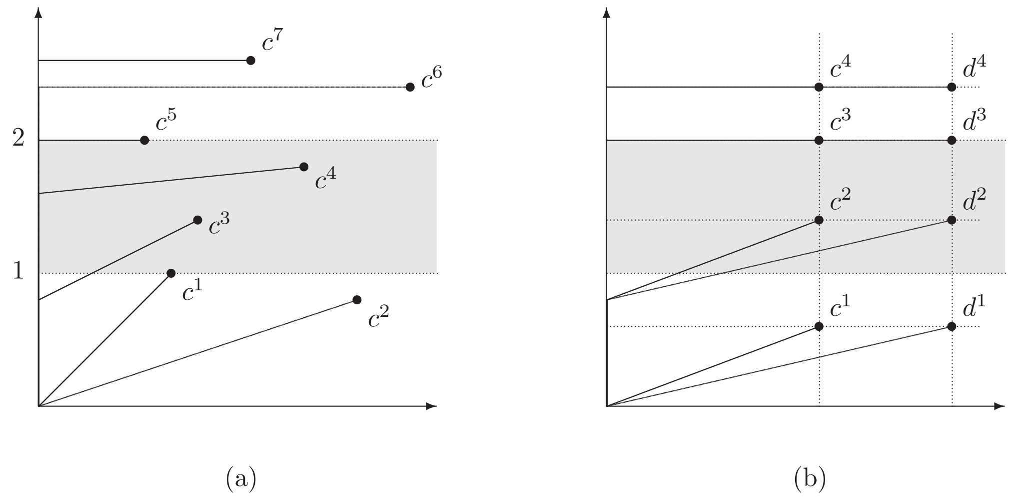

Figure 7: Claims monotonicity is not preserved under the composition of the claims truncation and attribution of minimal rights operator (Proposition 7). (a) Sample paths of awards of the rule S defined in the proof. (b) Showing that S is *claims monotonic*.

but $S^t(c, E') = S(t(c, E'), E') = S(c', E') = (2, 4)$. Claimant 2 loses when the amount to divide increases from E' to E.

□

Proposition 7 *Claims monotonicity is not preserved under the composition of the claims truncation and attribution of minimal rights operators.*

The proof is by means of an example of a rule, called S. The rule is *claims continuous* and *resource monotonic*. It violates both *equal treatment of equals* and *homogeneity* but this is unavoidable. Indeed, as shown in Theorem 3, if a rule R defined on $\mathcal{C}^N$ for $|N| = 2$ satisfies *equal treatment of equals*, then $R^{tom} = R^{mot} = CD$. Since CD is *claims monotonic*, then S violates *equal treatment of equals*. In addition, still for $|N| = 2$, the "weighted concede-and-divide rules" are the only rules satisfying *homogeneity*, *invariance under claims truncation*, and *minimal rights first* (Hokari and Thomson, 2003). Since these rules are *claims monotonic*, then S violates *homogeneity*. However, we design S to be *claims continuous* and *resource monotonic*. An ingredient of our construction is the "sequential priority rule associated with the order $2 \prec 1$," denoted $D^{2\prec 1}$, which, as the amount to divide increases from 0, assigns all of it to claimant 2 until he is fully compensated, and only then starts compensating claimant 1. *Claims monotonicity* of S means that the paths of awards associated with two claims vectors that differ in only one coordinate do not cross.

Proof: We define a rule S on $\mathcal{C}^N$, where $N \equiv \{1, 2\}$. We show that S is *claims monotonic* whereas S^{tom} is not.

Step 1: Construction of S (Figure 7a). Let $c \in \mathbb{R}^N_+$. If $c_2 \leq 1$, then $p(S,c) = p(P,c)$. If $c_2 \geq 2$, $p(S,c) = p(D^{2\prec 1},c)$. If $1 < c_2 < 2$, then $p(S,c) = \text{bro.seg}[(0,0),(0,2(c_2-1)),c]$. (Thus, for each $c_1 \in \mathbb{R}_+$, and as c_2 increases from 1 to 2, $p(S,c)$ changes continuously from $p(P,(c_1,1))$ to $p(D^{2\prec 1},(c_1,2))$.) The first case is illustrated by c^1 and c^2, the second case by c^5, c^6, and c^7, and the third case by c^3 and c^4.

Step 2: S is *claims monotonic* (Figure 7b). Let $c, c' \in \mathbb{R}^N_+$ be such that $c'_2, c_2 \leq 1$. Then the paths of awards of S for c and c' are that of the proportional rule, and since this rule satisfies *claims monotonicity*, we are done. If $c'_2, c_2 \geq 2$, then the paths of awards of S for c and c' are that of $D^{2\prec 1}$, and since this rule satisfies *claims monotonicity*, we are done. If $c'_1 > c_1$, $c'_2 = c_2$, and $1 < c_2 < 2$, then $p(S,c) = \text{bro.seg}[(0,0),(0,2(c_2-1)),c]$. Also, $p(S,c') = \text{bro.seg}[(0,0),(0,2(c'_2-1)),c']$. Since $c'_1 > c_1$ and $c'_2 = c_2$, the two paths begin with the same vertical segment. The second segment of $p(S,c')$ is flatter than the second segment of $p(S,c)$. Thus, the paths do not cross.

Next, let $c, c' \in \mathbb{R}^N_+$ be such that $c'_1 = c_1$ and $c'_2 > c_2$. When $1 \leq c_2 < c'_2 \leq 2$, then $p(S,c) = \text{bro.seg}[(0,0),(0,2(c_2-1)),c]$ and $p(S,c') = \text{bro.seg}[(0,0),(0,2(c'_2-1)),c']$. Since $2(c_2-1) < 2(c'_2-1)$, these paths do not cross. The desired conclusion for the other cases follows directly from the facts that the proportional and constrained equal losses rules are *claims monotonic*, and that S is *continuous*.

Step 3: S^{mot} is not *claims monotonic*. To see this, let $c \equiv (1,4)$, $c' \equiv (3,4)$, and $E = 2$. By Step 1 in Theorem 4, $m(t(c,E),E) = m(c,E)$. Thus, $S^{mot}(c,E) = m(c,E) + S(t(c,E) - m(c,E), E - \sum_{i\in N} m_i(c,E))$. Note that $m(c,E) = (0,1)$, $t(c,E) = (1,2)$, $m(c',E) = (0,0)$, and $t(c',E) = (2,2)$. It follows that $S^{mot}(c,E) = (\frac{1}{2}, \frac{3}{2})$ and $S^{mot}(c',E) = (0,2)$. Thus, as agent 1's claim increases from $c_1 = 1$ to $c'_1 = 3$, he receives less, in violation of *claims monotonicity*. □

The rule defined in this proof can also be used to show that *claims monotonicity* is not preserved under O^m (to see this, set $c \equiv (1,4)$, $E \equiv 4$, and $c' \equiv (2,4)$), but examples can be constructed to prove this fact that are *anonymous*.

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