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Abstract

We define two families of rules to adjudicate conflicting claims. The first family contains the constrained equal awards, constrained equal losses, Talmud, and minimal overlap rules. The second family, which also contains the constrained equal awards and constrained equal losses rules, is obtained from the first one by exchanging, for each problem, how well agents with relatively larger claims are treated as compared to agents with relatively smaller claims. In each case, we identify the subfamily of consistent rules.

JEL classification number: C79; D63; D74

Key-words: claims problems; constrained equal awards rule; constrained equal losses rule; Talmud rule; minimal overlap rule; ICI rules; CIC rules; consistency.

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1 Introduction

When a firm goes bankrupt, how should its liquidation value be divided among its creditors? More generally, when several agents have claims on a resource adding up to more than is available, how should that amount be divided? A "division rule" is a function that associates with each such problem a recommendation for it, namely a division of the resource.[1]

We propose and study two simple and yet surprisingly rich families of rules.[2] When surveying the literature, one cannot help but notice resemblances between rules and our families provide formal connections between several central ones. The first family contains the following four: the "constrained equal awards rule" and the "constrained equal losses rule", both already familiar to Maimonides (12th Century); the "Talmud rule", introduced by Aumann and Maschler (1985) to rationalize certain examples found in the Talmud; and the "minimal overlap rule", proposed by O'Neill (1982) as an extension of an incompletely specified rule appearing in Rabad (12th Century) (Section 3). The second family is a counterpart in which relatively small claims and relatively large claims are treated in a reverse way to the way they are treated by the first family. It contains the constrained equal awards and constrained equal losses rules. (These two rules are the only ones the two families have in common.)

The main ingredient in the definitions of the families is the basic idea of equality, applied either to the amounts claimants receive or to the losses they incur (the differences between their claims and their awards). For the first family, keeping the claims vector fixed, let us describe how awards evolve as the resource endowment increases from 0 to the sum of the claims. Initially, all claimants share each increment equally; they are excluded in succession and for a while, in the order of increasing claims; for each increment, all claimants who are not excluded yet share it equally; this goes on until the agent with the largest claim is the only one left, and for a while, he receives the totality of each increment; then claimants come back in the order of decreasing claims; here too, for each increment, all claimants who are present share it equally. This goes on until all claimants have returned and until they are fully compensated (Section 4).

For the second family of rules, the reverse occurs: initially, only the agent

[1] For a survey of the literature devoted to this subject, see Thomson (2003).
[2] These families are introduced by Thomson (2000).

with the largest claim is present and claimants are introduced to partake in the distribution in the order of decreasing claims; for each increment, all claimants who are present share it equally; claimants leave in the order of increasing claims, each claimant leaving when he is fully compensated. The agent with the largest claim is the last one to leave (Section 5).

The richness of the families comes from the freedom in choosing where claimants are dropped off and where they are invited back in (or conversely, for the second family), and in particular, from the fact that these drop-off and pick-up points are allowed to depend on the claims vector.

We identify the basic properties of the two families. We then turn to "duality" notions. Two rules are "dual" if one divides the amount available as the other divides the deficit (the difference between the sum of the claims and the endowment). Both families are closed under duality. A rule is "self-dual" if it is invariant under duality. The only self-dual member of the first family is the Talmud rule and the only self-dual member of the second family is a "reverse" of the Talmud rule (Section 6).

Finally, we investigate the existence of "consistent" members of the families. Consistency relates the recommendations made for problems involving different sets of claimants. It says that the awards vector selected for some problem need not be revised after some claimants have left with their awards and the situation is reevaluated at that point. The constrained equal awards, constrained equal losses, and Talmud rules are consistent members of the first family. Are there others? The answer is yes and we offer a characterization of the subfamily they constitute (Section 7). If we also require that when claims and endowment are multiplied by the same positive number, so is the chosen awards vector—this is the property called "homogeneity"—we obtain a one-parameter subfamily, introduced by Moreno-Ternero and Villar (2006a). (The parameter is a point in the unit interval.) We also characterize a similarly defined one-parameter subfamily of the second family on the basis of homogeneity and consistency.

2 The model of adjudication of conflicting claims

A group N of agents have claims on a resource, $c_i \in \mathbb{R}_+$ being the **claim** of agent $i \in N$ and $c \equiv (c_i)_{i \in N}$ the vector of claims. Initially, N is some finite subset of the set of natural numbers $\mathbb{N}$. Using the notation $\mathbb{R}^N_+$ for the cross-product of $|N|$ copies of $\mathbb{R}_+$ indexed by the members of N,[3] the claims vector is therefore an element of $\mathbb{R}^N_+$. There is an **endowment $\boldsymbol{E}$** of the resource, and this endowment is insufficient to honor all of the claims. Altogether, a claims problem, or simply a **problem**, is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum c_i \geq E$. Let $\boldsymbol{\mathcal{C}^N}$ denote the domain of all problems.

A division rule, or simply a **rule**, is a function that associates with each problem $(c, E) \in \mathcal{C}^N$ a vector $x \in \mathbb{R}^N$ such that $0 \leqq x \leqq c$ and satisfying the efficiency condition $\sum x_i = E$. Such an x is an **awards vector for $(\boldsymbol{c}, \boldsymbol{E})$**. Let $\boldsymbol{X(c, E)}$ be the set of these vectors.

We use two kinds of graphical representations for a rule. First, for each claims vector, we plot in a Euclidean space of dimension equal to the number of claimants the awards vector chosen by the rule as a function of the endowment. The locus of this vector is the **path of awards of the rule for the claims vector**. This representation is limited to the two- and three-claimant cases but nevertheless it is very useful for proofs. Alternatively, we plot the award to each claimant as a function of the endowment. The graphs of the resulting functions are the **schedules of awards of the rule for the claims vector**. This representation accommodates any number of claimants.

Other notation: Given $N \in \mathcal{N}$, we write n for $|N|$. Let $\boldsymbol{e}\colon \mathbb{R}_+ \to \mathbb{R}_+$ be the identity function: for each $c_0 \in \mathbb{R}_+$, $e(c_0) = c_0$.

3 Four important rules

The following are important rules in the literature. They will be central to our analysis as well. The first two implement the idea of equality, of awards on the one hand, subject to claims boundedness, and of losses on the other hand, subject to non-negativity:

Constrained equal awards rule, $\boldsymbol{CEA}$: For each $(c, E) \in \mathcal{C}^N$ and each $i \in N$, $CEA_i(c, E) \equiv \min\{c_i, \lambda\}$, where λ is chosen so as to achieve efficiency.

[3] The superscript N indicates a cross-product of objects indexed by the members of the set N, or more generally it refers to a set pertaining to the agents in N. Which interpretation is intended should be unambiguous from the context.

Constrained equal losses rule, *CEL*: For each $(c,E) \in \mathcal{C}^N$ and each $i \in N$, $CEL_i(c,E) \equiv \max\{0, c_i - \lambda\}$, where λ is chosen so as to achieve efficiency.

The next rule was critical in leading to the formulation of the families we introduce here. It was initially defined so as to make sense of the resolutions proposed in the Talmud for certain problems. The story has been told many times (O'Neill, 1982; Aumann and Maschler, 1985) but it bears repeating. In the **contested garment problem**, two men disagree over the ownership of a garment, worth 200, say. The first man claims half of it (100) and the other claims it all (200). The Talmud recommends the division $g \equiv (50, 150)$.[4] For the **estate division problem**, a man has three wives whose marriage contracts specify that upon his death, they should receive 100, 200, and 300 respectively. The man dies and his estate is found to be worth only 100. The Talmud recommends $e \equiv (33\frac{1}{3}, 33\frac{1}{3}, 33\frac{1}{3})$. If the estate is worth 200, it recommends $k \equiv (50, 75, 75)$, and if it is worth 300, it recommends $p \equiv (50, 100, 150)$.[5] The rule, proposed by Aumann and Maschler, can be thought of as a hybrid of the constrained equal awards and constrained equal losses rules. It is defined as follows (Figure 1):

Talmud rule, *T*: For each $(c,E) \in \mathcal{C}^N$ and each $i \in N$,

$$T_i(c,E) \equiv \begin{cases} \min\{\frac{c_i}{2}, \lambda\} & \text{if } \sum \frac{c_i}{2} \geq E, \\ c_i - \min\{\frac{c_i}{2}, \lambda\} & \text{otherwise,} \end{cases}$$

where in each case, λ is chosen so as to achieve efficiency.

The following historical precedent, called Ibn Ezra's problem (12th Century)[6] is not as well-known, but it is the main rationale for the rule defined next, as it was for an incompletely specified rule due to Rabad[7]. It too has been important in providing motivation for the introduction of our families. A man dies whose estate is worth 120. He has four sons, to whom he

[4] See Baba Metzia 2a. References to the relevant passages of the Talmud are taken from O'Neill (1982), Aumann and Maschler (1985), and Dagan (1996).

[5] Kethubot 93a; the author of this Mishna is Rabbi Nathan.

[6] This attribution is by O'Neill (1982). O'Neill notes several of the properties of the rule defined below. Another rule that also delivers Ibn Ezra's numbers is proposed by Bergantiños and Méndez-Naya (2001) and Alcalde, Marco, and Silva (2005).

[7] Rabad's rule is defined only for claims problem in which the endowment is no greater than the smallest claim.

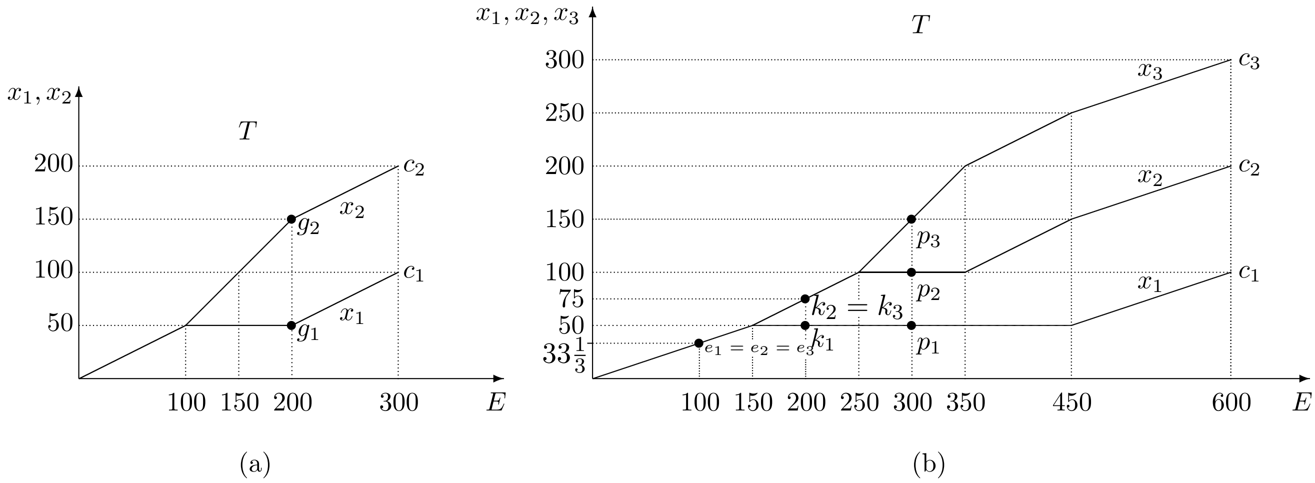


Figure 1: Schedules of awards for the Talmud rule, proposed by Aumann and Maschler to rationalize the recommendations made in the Talmud for the contested garment and estate division problems. The award vectors found in the Talmud are indicated as dots. (a) For the contested garment problem, claims are $(c_1, c_2) \equiv (100, 200)$. (b) For the estate division problem, claims are $(c_1, c_2, c_3) \equiv (100, 200, 300)$.

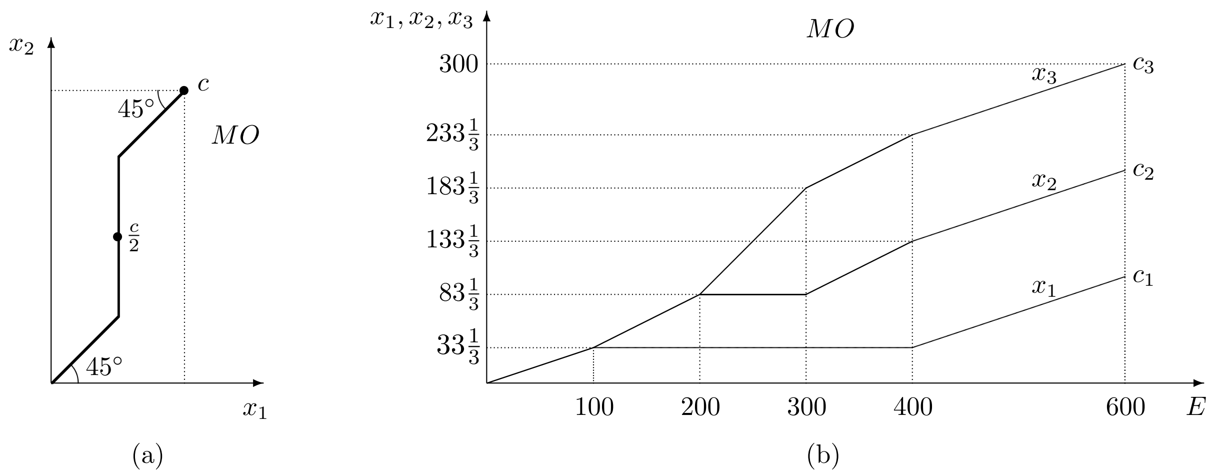


Figure 2: Schedules of awards for the minimal overlap rule, proposed by O'Neil to rationalize Ibn Ezra's recommendation for an estate division problem. (a) Two-claimant problem. In that case, the minimal overlap rule coincides with the Talmud rule. (b) The estate division problem of the Talmud. The general shape of each schedule of awards is the same as for the Talmud rule, but the breakpoints are different.

bequeathed 30, 40, 60, and 120. Ibn Ezra recommends the awards vector $(\frac{30}{4}, \frac{30}{4} + \frac{10}{3}, \frac{30}{4} + \frac{10}{3} + \frac{20}{2}, \frac{30}{4} + \frac{10}{3} + \frac{20}{2} + \frac{60}{1})$. O'Neill (1982) proposed the following rule to rationalize these numbers (Figure 2):

Minimal overlap rule, MO: Claims on specific parts of the endowment are arranged so that the part that is claimed by exactly one claimant (whoever he is; different subparts can be claimed by different claimants) is maximized, and for each $k = 2, \ldots, n-1$ successively, subject to the previous k maximizations being solved, the part that is claimed by exactly $k+1$ claimants (whoever they are; each subpart can be claimed by different sets of $k+1$ claimants) is maximized. Once claims are arranged in this way, for each part of the endowment, equal division prevails among all agents claiming it. Each claimant receives the sum of the partial compensations coming from the various parts that he claimed.

Formulae for the rule are available (O'Neill, 1982; Chun and Thomson, 2005; Alcalde, Marco, and Silva, 2007).

It is of interest that the Talmud and minimal overlap rules coincide in the two-claimant case, and that they also coincide with the rule based on the following scenario: each claimant concedes to the other the difference between the endowment and his claim (or 0 if this difference is negative), and whatever remains is divided equally between them (Aumann and Maschler, 1985). We refer to it as **concede-and-divide**.

4 A new family of rules, the ICI family

Although the constrained equal awards and constrained equal losses rules clearly enter as ingredients into the definition of the Talmud rule, the considerations underlying the definition of the minimal overlap rule seem to be far removed from those that underlie these three rules. Nevertheless, we will see that a very simple general formula exists that delivers all four as special cases.

We noted that in the two-claimant case, the minimal overlap and Talmud rules both coincide with concede-and-divide, but in fact many other rules do too. Without giving formal definitions, let us only mention (i) the random arrival rule (O'Neill, 1982), defined by imagining that claimants arrive in random order to get compensated, and fully compensating each of them until money runs out, (ii) the version of the constrained equal awards rule obtained by first assigning to each claimant his "minimal right" (the difference between the endowment and the sum of the claims of the other agents if this difference is non-negative, and 0 otherwise), and (iii) the version of the constrained equal losses rule obtained by first truncating claims at the endowment. If there are more than two claimants, these various rules usually make different recommendations.

Figure 1b, which illustrates the Talmud rule in the three-claimant case, and Figure 2b, which illustrates the minimal overlap rule, also in the three-claimant case, reveal a striking resemblance between them. What is similar is not just the fact that their schedules of awards are piece-wise linear—many other rules exhibit this feature, including all of the rules just enumerated—but mainly the pattern of increases in awards as the endowment increases. For each claims vector, the award to each claimant increases initially, then it remains constant, then it increases again until the claimant is fully compensated; moreover, any two awards that are increasing at any given moment do so at the same rate. For each of the two rules, each claims vector, and each claimant, the interval of constancy of his award depends on where his claim stands in the ordered list of claims.

Let us then consider all rules exhibiting these features. We designate the family they constitute by the name of "Increasing-Constant-Increasing" family, or **ICI family** for short (the interval in which a claimant's award is increasing can be subdivided into subintervals in which the rate of increase is constant; for the agent with the largest claim, the interval of constancy is actually degenerate), and let us refer to each member of the family as an **ICI rule**. Proceeding in this way is justified as follows. If the endowment is very small, differences in claims can be judged irrelevant, and equal shares make sense, as "consolation prizes". As the endowment increases, at some point, it is felt that the agent with the smallest claim starts receiving too large a percentage of his claim, so he is excluded from receiving a share of new increments. Differences in the claims of the others might still be judged irrelevant, at least for a while, and we continue with equal division for them until it is felt that the agent with the second smallest claim starts receiving too large a percentage of his claim. Then, he is dropped off too and we continue with equal division for the remaining $n-2$ claimants, and so on. We proceed until agent $n-1$ has been dropped off and for a while, agent n is the only recipient. Then claimants return in the order of decreasing claims. The richness of the family comes from the considerable freedom one has in choosing the various points at which claimants are dropped off and picked up again, and in particular from the fact that these drop-off and pick-up points may depend on the claims vector.

Here is the general definition of the family. Its members are indexed by a list $H \equiv (F_k, G_k)_{k=1}^{n-1}$ of pairs of functions from $\mathbb{R}_+^N$ to $\mathbb{R}_+$ such that for each $c \in \mathbb{R}_+^N$, the sequence $(F_k(c))_{k=1}^{n-1}$ is nowhere decreasing, the sequence $(G_k(c))_{k=1}^{n-1}$ is nowhere increasing, $G_1(c) \leq \sum c_i$, and the following relations hold.

$$\begin{array}{ccccccccc}
 & & \frac{F_1(c)}{n} & + & \frac{\sum c_i - G_1(c)}{n} & = & c_1 \\
c_1 & + & \frac{F_2(c)-F_1(c)}{n-1} & + & \frac{G_1(c)-G_2(c)}{n-1} & = & c_2 \\
\vdots & + & \vdots & + & \vdots & = & \vdots \\
c_{k-1} & + & \frac{F_k(c)-F_{k-1}(c)}{n-k+1} & + & \frac{G_{k-1}(c)-G_k(c)}{n-k+1} & = & c_k \\
\vdots & + & \vdots & + & \vdots & = & \vdots \\
c_{n-1} & + & \frac{-F_{n-1}(c)}{1} & + & \frac{G_{n-1}(c)}{1} & = & c_n
\end{array}$$

As we will see, the reason for these relations is that, when the endowment reaches $\sum c_i$, each claimant should be fully compensated. Let us refer to them as the **ICI relations**, and let us denote by $\boldsymbol{\mathcal{H}^N}$ the family of lists $H \equiv (F_k, G_k)_{k=1}^{n-1}$ of pairs of functions satisfying them. The ICI relations are not independent; multiplying the first one through by n, the second one by $n-1$, ..., and the last one by 1, gives new relations whose sum is an identity.

When agents have equal claims, the ICI relations imply that successive F_k's and the corresponding G_k's are equal. Then, these agents drop out and come back together.

For each $k = 1, \dots, n-1$, let $\boldsymbol{\delta_k(c)} \equiv G_k(c) - F_k(c)$. Interestingly, the sequence $(\delta_k(c))_{k=1}^{n-1}$ is the same for all functions $H \in \mathcal{H}^N$, as the following calculations show:

$$\begin{array}{rclcccccccccc}
\delta_1(c) & = & -(n-1)c_1 & + & c_2 & + & c_3 & + & \cdots & + & c_{n-1} & + & c_n \\
\delta_2(c) & = & & & -(n-2)c_2 & + & c_3 & + & \cdots & + & c_{n-1} & + & c_n \\
\cdots & = & & & & + & \cdots & + & \cdots & + & c_{n-1} & + & c_n \\
\delta_{n-2}(c) & = & & & & & & & -2c_{n-2} & + & c_{n-1} & + & c_n \\
\delta_{n-1}(c) & = & & & & & & & & & -c_{n-1} & + & c_n
\end{array}$$

To describe an ICI rule, it suffices to specify, for each $c \in \mathbb{R}_+^N$, a nowhere decreasing sequence $(F_k(c))_{k=1}^{n-1}$ such that $F_1(c) \geq 0$. The sequence $(G_k(c))_{k=1}^{n-1}$ can then be recovered from $(F_k(c))_{k=1}^{n-1}$ by adding $(\delta_k(c))_{k=1}^{n-1}$. It should be nowhere increasing and such that $G_1(c) \leq \sum c_i$. Alternatively, we can start from the sequence $(G_k(c))_{k=1}^{n-1}$ and derive the sequence $(F_k(c))_{k=1}^{n-1}$ from it by subtracting the sequence $(\delta_k(c))_{k=1}^{n-1}$.

ICI rule associated with $\boldsymbol{H \equiv (F_k, G_k)_{k=1}^{k=n-1} \in \mathcal{H}^N}$, $\boldsymbol{R^H}$: For each $c \in \mathbb{R}_+^N$ with $c_1 \leq c_2 \leq \cdots \leq c_n$ say, the awards vector chosen by R^H is given by the following algorithm. As the endowment first increases from 0 to $F_1(c)$, equal division prevails. As it increases from $F_1(c)$ to $F_2(c)$, claimant 1's award remains constant, and equal division of each increment prevails among the others. As it increases from $F_2(c)$ to $F_3(c)$, claimants 1 and 2's awards remain constant, and equal division of each increment prevails among the others. ... This process goes on until the endowment reaches $F_{n-1}(c)$. As it increases from $F_{n-1}(c)$ to $G_{n-1}(c)$, each increment goes entirely to claimant n. As it increases from $G_{n-1}(c)$ to $G_{n-2}(c)$, equal division of each increment prevails between claimants n and $n-1$. ... This process goes on until the endowment reaches $G_1(c)$, after which each increment is divided equally among all claimants, until all are fully compensated.

Next, we show that the process just described does deliver a well-defined rule, referring to the stages into which it is divided as "steps". There are $2n-1$ of them. Claimant 1's award increases on two occasions, at Step 1 by

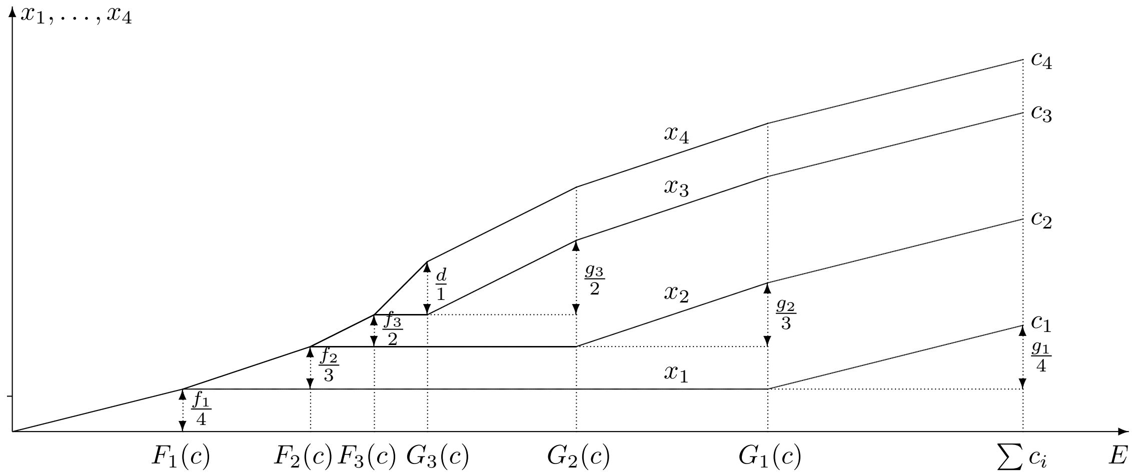


Figure 3: Schedule of awards of a four-claimant ICI rule for a particular claims vector. Let $f_1 \equiv F_1(c)$ and for $k = 2, 3$, let $f_k \equiv F_k(c) - F_{k-1}(c)$. Let $g_1 \equiv \sum c_i - G_1(c)$, and for $k = 2, 3$, let $g_k \equiv G_k(c) - G_{k-1}(c)$. Finally, let $d \equiv G_3(c) - F_3(c)$.

$\frac{F_1(c)}{n}$, and at Step $2n - 1$ (the last step) by $\frac{\sum c_i - G_1(c)}{n}$, for a total of c_1 (by the first ICI relation). Claimant 2's award increases along with claimant 1's award on both occasions and at the same rate, also for a total of c_1; in addition, it increases at Step 2 by $\frac{F_2(c)-F_1(c)}{n-1}$ and at Step $2n-2$ by $\frac{G_1(c)-G_2(c)}{n-1}$. Altogether then, he ends up with c_2 (by the second ICI relation). Similar statements can be made about the remaining claimants.

The schedules of awards of a four-claimant ICI rule for a particular claims vector are illustrated in Figure 3.

The family $\mathcal{H}^N$ is convex: a convex combination of members of the family is also a member of the family. However, the ICI family itself is not a convex family.[8]

Next, we show that the four rules defined in Section 3 are ICI rules. We will need the following lemma, which concerns the minimal overlap rule:

Lemma 1 (O'Neill, 1982) *Up to relabelling of parts of the endowment, there is a unique arrangement of claims achieving minimal overlap. It is obtained*

[8] In fact, a non-trivial convex combination of two members of the family is never a member of the family. We could say that the family is "nowhere convex".

as follows:

Case 1: *There is $j \in N$ such that $c_j \geq E$. Then, each agent $i \in N$ such that $c_i \geq E$ claims $[0, E]$ and each other agent i claims $[0, c_i]$ (nesting of claims occurs).*

Case 2: $\max\{c_j\} < E$. *Let $t(E)$ be the solution to the equation in t, $\sum_{i \in N :\, c_i > t}(c_i - t) = E - t$. Then,*

(a) *each agent $i \in N$ such that $c_i \leq t(E)$ claims $[0, c_i]$;*

(b) *each agent $i \in N$ such that $c_i \geq t(E)$ claims $[0, t(E)]$ as well as a part of $[t(E), E]$ of size $c_i - t(E)$, with no overlap among the subsets of $[t(E), E]$ claimed by any two of these agents.*

Lemma 2 *The constrained equal awards, constrained equal losses, Talmud, and minimal overlap rules are ICI rules.*

Proof: Without loss of generality, suppose $c_1 \leq c_2 \leq \cdots \leq c_n$.

(a) Constrained equal awards rule: For each $c \in \mathbb{R}^N_+$, set $F(c) \equiv (nc_1, c_1 + (n-1)c_2, \ldots, c_1 + c_2 + \cdots + c_{k-1} + (n-k+1)c_k, \ldots, c_1 + c_2 + \cdots + c_{n-2} + 2c_{n-1})$.

(b) Constrained equal losses rule: For each $c \in \mathbb{R}^N_+$, set $F(c) \equiv (0, \ldots, 0)$.

(c) Talmud rule: For each $c \in \mathbb{R}^N_+$, set $F(c) \equiv (n\frac{c_1}{2}, \frac{c_1}{2} + (n-1)\frac{c_2}{2}, \ldots, \frac{c_1}{2} + \frac{c_2}{2} + \cdots + \frac{c_{k-1}}{2} + (n-k+1)\frac{c_k}{2}, \ldots, \frac{c_1}{2} + \frac{c_2}{2} + \cdots + \frac{c_{n-2}}{2} + 2\frac{c_{n-1}}{2})$ (then, the sequence $(G_k(c))_{k=1}^{k=n-1}$ is symmetric of the sequence $(F_k(c))_{k=1}^{k=n-1}$ with respect to $\sum \frac{c_i}{2}$).

(d) Minimal overlap rule: We show that the schedules of payments follow the pattern defining the ICI family, in the process identifying the function $H \in \mathcal{H}^N$ associated with the rule. As long as $E \leq c_{n-1}$, Case 1 of Lemma 1 applies. Indeed, as E increases from 0 to c_1, equal division among all agents prevails. As E increases from c_1 to c_2, equal division of each increment prevails among agents 2,…, n. Thus, $F_1(c) = c_1$. As E increases from c_2 to c_3, equal division of each increment prevails among agents 3,…, n. Thus, $F_2(c) = c_2$. This goes on until $E = c_{n-1}$. Thus, $F_{n-1}(c) = c_{n-1}$. As E increases beyond c_{n-1}, and until $E = c_n$, agent n receives each increment.

As E increases from c_n to $\sum c_i$, Case 2 of Lemma 1 applies. When $E = c_n$, agent $n-1$ returns. Thus, $G_{n-1}(c) = c_n$. As E increases from c_n, the equation to solve is $\sum_{i \in \{n-1, n\}}(c_i - t) = E - t$. Its solution $t(E)$ satisfies $c_{n-2} < t(E) < c_{n-1}$. Each increment is divided equally between claimants $n-1$ and n. When E increases further, the equation to solve is $\sum_{i \in \{n-2, n-1, n\}}(c_i - t) = E - t$.

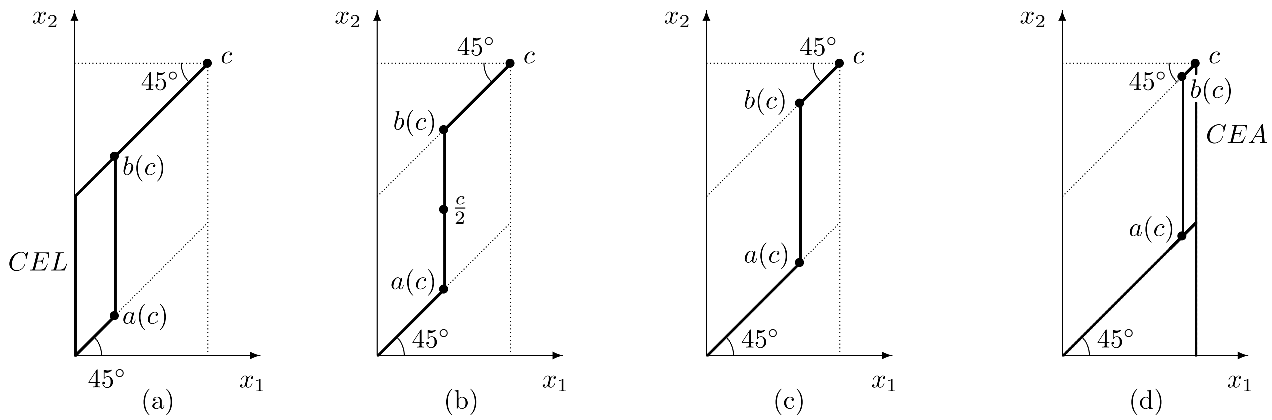


Figure 4: Paths of awards of two-claimant ICI rules for a particular claims vector. The claimant set is $N \equiv \{1, 2\}$ and $c_1 < c_2$. The four panels represent paths of awards of ICI rules associated with progressively greater and greater values of the endowment at which the kinks in the paths, $a(c)$ and $b(c)$, occur. At one extreme (panel (a)), when they lie on the vertical axis, we obtain the constrained equal losses rule. At the other extreme (panel (d)), when they lie on the vertical line passing through the claims vector, we obtain the constrained equal awards rule. Panel (b) shows the Talmud rule.

Its solution $t(E)$ is such that $c_{n-3} < t(E) < c_{n-2}$. Each increment is divided equally between claimants n, $n-1$, and $n-2$. At the $(n-k)$-th step, as E increases further, the equation to solve is $\sum_{i \in \{k,\ldots,n-1,n\}}(c_i - t) = E - t$. Its solution $t(E)$ is such that $c_{k-1} < t(E) < c_k$. Each increment is divided equally between claimants n, $n-1$, $\ldots$, and $n-k$. At the $(n-2)$-th step, the equation to solve is $\sum_{i \in \{2,\ldots,n-1,n\}}(c_i - t) = E - t$. At the last step, the equation to solve is $\sum_N (c_i - t) = E - t$. Altogether then, in each of the intervals into which we just divided the range of variation of the endowment, all agents who are included share each increment equally.

Thus, the minimal overlap rule is the ICI rule associated with the function $H = (F, G) \in \mathcal{H}^N$ given by $F(c) = (c_1, c_2, \ldots, c_{n-1})$ and $G(c) = (-(n-2)c_1 + c_2 + \cdots + c_{n-1} + c_n, -(n-3)c_2 + c_3 + \cdots + c_{n-1} + c_n, \ldots, -(k-1)c_{n-k} + c_{n-k+1} + \cdots + c_n, \ldots, -c_{n-2} + c_{n-1} + c_n, c_n)$. □

If $|N| = 2$, the list H has only one term (F_1, G_1), with for each $c \in \mathbb{R}^N_+$, $F_1(c)$ and $G_1(c)$ satisfying the ICI relations—there are two of them then—but given the dependence between these relations, we obtain a one-parameter family, the parameter being a function of c. This family "connects" the constrained equal awards and constrained equal losses rules, passing through concede-and-divide, (which, in the two-claimant case, is also the minimal overlap rule). The paths of awards are easily determined (Figure 4). Let $N \equiv \{1,2\}$. If $c_1 = c_2$, the path is seg$[(0,0), c]$. If $c_1 < c_2$, the path consists of the segment connecting the origin to the point $a(c) \equiv (\frac{F_1(c)}{2}, \frac{F_1(c)}{2})$ (this segment is degenerate if $F_1(c) = 0$), the vertical segment with endpoints $a(c)$ and $b(c) \equiv a(c) + (0, c_2 - c_1)$, and the segment connecting $b(c)$ to c (this segment has slope 1; it is degenerate if $G_1(c) = \sum c_i$). We noted earlier that the family $\mathcal{H}^N$ is convex. Now, let S and S' be two ICI rules, associated with the functions H and $H' \in \mathcal{H}^N$. Then, for each $c \in \mathbb{R}^N_+$, the path of awards of the rule associated with a convex combination of H and H' is a convex combination of the paths of S and S' parallel to the 45° line. Each line of slope 1 that intersects the path of S also intersects the path of S'. Then, we take the convex combination of these points of intersection (sometimes, segments of intersection).

Moreno-Ternero and Villar (2006a) propose a family of rules indexed by a point in the unit interval. This family happens to be a subfamily of the ICI family. Let $\theta \in [0,1]$. Assuming $c_1 \leq c_2 \leq \cdots \leq c_n$, simply choose $F(c) \equiv \theta(nc_1, c_1 + (n-1)c_2, \cdots, c_1 + c_2 + \cdots + c_{k-1} + (n-k+1)c_k, \ldots, c_1 + c_2 + \cdots + c_{n-2} + 2c_{n-1})$. Let $\boldsymbol{T^\theta}$ denote the rule associated with θ in this manner, and $\boldsymbol{\{T^\theta\}_{\theta \in [0,1]}}$ be the resulting family. Note that $T^0 = CEL$, $T^1 = CEA$, and $T^{\frac{1}{2}} = T$.

The paths of awards of two-claimant ICI rules are depicted in Figure 4. For each $\theta \in [0,1]$, the rule T^θ is obtained when $a(c)$ takes the simple form $a(c) = (2\theta \min\{c_i\}, 2\theta \min\{c_i\})$. Thus, in the two-claimant case, the path of T^θ has the same shape as for any ICI rule, but each T^θ can be recovered from only one of its paths, whereas for a general ICI rule, the kinks in paths of awards can depend in some arbitrary manner on the claims vector.

5 Reverse family: the CIC family

A reverse algorithm to the one underlying the definition of the ICI family suggests itself: instead of beginning with equal division and dropping claimants in succession, starting with the ones with the smallest claims, we start by giving everything to claimant n and progressively enlarge the set of recipients, adding agents in the order of decreasing claims. More precisely, claimant n is the only one in until the endowment reaches a first critical value, at which point claimant $n-1$ enters the scene. Then, claimants n and $n-1$ share equally each increment until the endowment reaches a second critical value.

Then, claimants n, $n-1$, and $n-2$ share equally each increment ... This process goes on until claimant 1 enters the scene, at which point all claimants share equally each increment. At some point, claimant 1 is fully compensated and is dropped off. Then, equal division of each increment prevails among claimants 2 through n until claimant 2 is fully compensated, and is dropped off and so on. During the last step, claimant n is the only one left. We refer to a rule so defined as an **CIC rule**, this acronym reflecting the fact that each claimant's award is first Constant, then Increasing, then Constant. The family so obtained is the **CIC family**.

The CIC rules are indexed by pairs of functions $H \equiv (F_k, G_k)_{k=1}^{n-1}$ from $\mathbb{R}_+^N$ to $\mathbb{R}_+$, where $n \equiv |N|$, satisfying properties parallel to those imposed on the pairs of functions indexing the ICI rules. Specifically, for each $c \in \mathbb{R}_+^N$, $(F_k(c))_{k=1}^{n-1}$ is nowhere decreasing, $(G_k(c))_{k=1}^{n-1}$ is nowhere increasing, $G_1(c) \leq \sum c_i$, and the following relations, which we call the **CIC relations**, hold:

$$
\begin{array}{ccccccc}
 & & \frac{-F_{n-1}(c)}{n} & + & \frac{G_{n-1}(c)}{n} & = & c_1 \\
c_1 & + & \frac{F_{n-1}(c)-F_{n-2}(c)}{n-1} & + & \frac{G_{n-2}(c)-G_{n-1}(c)}{n-1} & = & c_2 \\
\dots & + & \dots & + & \dots & = & \dots \\
c_{k-1} & + & \frac{F_{n-k+1}(c)-F_{n-k}(c)}{n-k+1} & + & \frac{G_{n-k}(c)-G_{n-k+1}(c)}{n-k+1} & = & c_k \\
\dots & + & \dots & + & \dots & = & \dots \\
c_{n-1} & + & \frac{F_1(c)}{1} & + & \frac{\sum c_i - G_1(c)}{1} & = & c_n
\end{array}
$$

Here too, the relations are not independent: multiplying the first one through by n, the second one by $n-1$, ..., and the last one by 1, gives new relations whose sum is an identity. Let $\bar{\mathcal{H}}^N$ be the family of pairs of functions satisfying these relations.

If agents have equal claims, the CIC relations imply that these agents come in and drop out together.

CIC rule associated with $H \equiv (F_k, G_k)_{k=1}^{k=n-1} \in \bar{\mathcal{H}}^N$, R^H: For each $c \in \mathbb{R}_+^N$ with $c_1 \leq c_2 \leq \cdots \leq c_n$ say, the awards vector chosen by R^H is given by the following algorithm. As the endowment first increases from 0 to $F_1(c)$, everything goes to claimant n. As it increases from $F_1(c)$ to $F_2(c)$, equal division of each increment prevails between claimants n and $n-1$. As it increases from $F_2(c)$ to $F_3(c)$, equal division of each increment prevails among claimants n, $n-1$, and $n-2$... This process goes on until claimant 1 enters the scene, at which point equal division of each increment prevails among all claimants until, when the endowment is equal to $G_{n-1}(c)$, he is

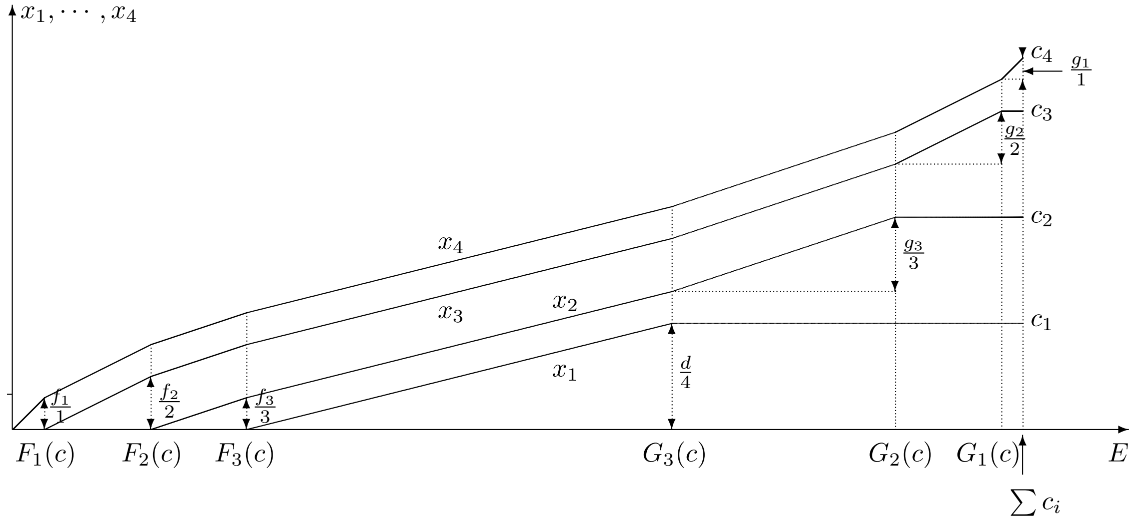


Figure 5: Schedules of awards of a four-claimant CIC rule for a particular claims vector. Let $f_1 \equiv F_1(c)$ and for $k = 2, 3$, let $f_k \equiv F_k(c) - F_{k-1}(c)$. Let $g_1 \equiv \sum c_i - G_1(c)$ and for $k = 2, 3$, let $g_k \equiv G_k(c) - G_{k-1}(c)$. Finally, let $d \equiv G_3(c) - F_3(c)$.

fully compensated and is dropped off. Then equal division of each increment prevails among claimants 2 through n until, when the endowment is equal to $G_{n-2}(c)$, claimant 2 is fully compensated and is dropped off, and so on. At the end of the process, claimant n is the only one left, and he receives each increment until he is fully compensated.

The schedules of awards of a four-claimant CIC rule for a particular claims vector are illustrated in Figure 5.

For each $k = 1, \ldots, n-1$, let $\boldsymbol{\epsilon_k(c)} \equiv G_k(c) - F_k(c)$. The sequence $(\epsilon_k(c))_{k=1}^{n-1}$ is the same for all members of the ICI family, as the following calculations show:[9]

$$\begin{array}{rcccccccccccc}
\epsilon_1(c) & = & c_1 & + & c_2 & + & \cdots & + & c_{n-3} & + & c_{n-2} & + & 2c_{n-1} \\
\epsilon_2(c) & = & c_1 & + & c_2 & + & \cdots & + & c_{n-3} & + & 3c_{n-2} & & \\
\cdots & = & \cdots & + & \cdots & + & \cdots & + & \cdots & & & & \\
\epsilon_{n-2}(c) & = & c_1 & + & (n-1)c_2 & & & & & & & & \\
\epsilon_{n-1}(c) & = & nc_1 & & & & & & & & & &
\end{array}$$

An alternative way to describe a member of the CIC family is to specify for each $c \in \mathbb{R}^N_+$ a sequence $(F_k(c))_{k=1}^{n-1}$. The sequence $(G_k(c))_{k=1}^{n-1}$ can then be recovered from $(F_k(c))_{k=1}^{n-1}$ by adding $(\epsilon_k(c))_{k=1}^{n-1}$. The sequence $(F_k(c))_{k=1}^{n-1}$ should be nowhere decreasing and such that $F_1(0) \geq 0$ and the sequence $(G_k(c))_{k=1}^{n-1}$ should be nowhere increasing and such that $G_1(c) \leq \sum c_i$.

[9] We have $\delta_1 + \epsilon_{n-1} = \delta_2 + \epsilon_{n-2} = \cdots = \delta_{n-1} + \epsilon_1 = \sum c_i$.

The next lemma identifies three rules as CIC rules. Interestingly, two of them are the constrained equal awards and constrained equal losses rules, which as we saw, also belong to the ICI family. The third one is defined like the Talmud rule, which we described as a hybrid of the constrained equal awards and constrained equal losses rules, by exchanging the order in which these component rules are applied. For that reason, we name it the **reverse Talmud** rule.[10]

Lemma 3 *The constrained equal awards, constrained equal losses, and reverse Talmud rules are CIC rules.*

Proof: Without loss of generality, suppose $c_1 \leq c_2 \leq \cdots \leq c_n$.

(a) Constrained equal awards rule: For each $c \in \mathbb{R}^N_+$, set $F(c) \equiv (0, 0, \ldots, 0)$.

(b) Constrained equal losses rule: For each $c \in \mathbb{R}^N_+$, set $F(c) \equiv (c_n - c_{n-1}, c_n + c_{n-1} - 2c_{n-2}, \ldots, c_n + c_{n-1} + \cdots + c_2 - (n-1)c_1)$.

(c) Reverse Talmud rule: For each $c \in \mathbb{R}^N_+$, set $F(c) \equiv (-\frac{c_{n-1}}{2} + \frac{c_n}{2}, -2\frac{c_{n-2}}{2} + \frac{c_{n-1}}{2} + \frac{c_n}{2}, \cdots, -k\frac{c_{n-k}}{2} + \frac{c_{n-k+1}}{2} + \cdots + \frac{c_n}{2}, -(n-1)\frac{c_1}{2} + \frac{c_2}{2} + \cdots + \frac{c_n}{2})$ (then, the sequence $(G_k(c))_{k=1}^{k=n-1}$ is symmetric of the sequence $(F_k(c))_{k=1}^{k=n-1}$ with respect to $\sum \frac{c_i}{2}$). □

A subfamily of CIC rules parallel to $\{T^\theta\}_{\theta\in[0,1]}$ can be defined. Let $\theta \in [0,1]$. Assuming $c_1 \leq c_2 \leq \cdots \leq c_n$, let us choose $F(c) \equiv \theta(-c_{n-1} + c_n, -2c_{n-2} + c_{n-1} + c_n, \cdots, -kc_{n-k} + c_{n-k+1} + \cdots + c_n + \cdots, -(n-1)c_1 + c_2 + \cdots + c_n)$. Let $\boldsymbol{U^\theta}$ denote the rule associated with θ in this manner, and $\boldsymbol{\{U^\theta\}_{\theta\in[0,1]}}$ be the resulting family. Note that $U^0 = CEA$, $U^1 = CEL$, and $U^{\frac{1}{2}} = T^r$.

The paths of awards of two-claimant CIC rules are depicted in Figure 6. Let $N \equiv \{1,2\}$ and $c \in \mathbb{R}^N$ with $c_1 < c_2$. Then, for each $\theta \in [0,1]$, the rule U^θ is obtained when $a(c) = (0, \theta(c_2 - c_1))$. A comment similar to one we made for the ICI rules applies: in the two-claimant case, the path of U^θ has the same shape as for any CIC rule, but each U^θ can be recovered from only one of its paths, whereas for a general CIC rule, the kinks in paths of awards can depend in some arbitrary manner on the claims vector.

[10]It is discussed under that name by Chun, Schummer, and Thomson (2001) and Hokari and Thomson (2003).

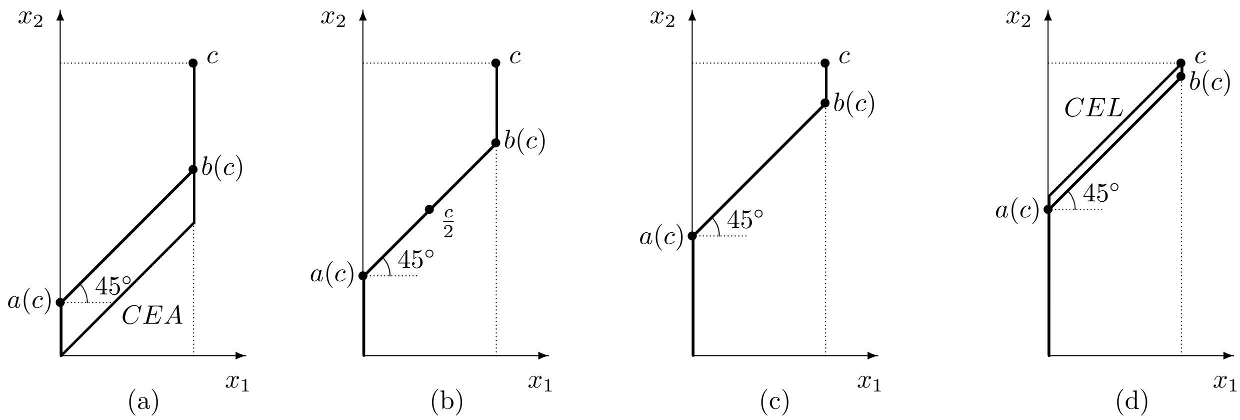


Figure 6: Paths of awards of two-claimant CIC rules for a particular claims vector. The claimant set is $N \equiv \{1,2\}$ and $c_1 < c_2$. The four panels represent paths of awards of CIC rules associated with progressively greater and greater values of the endowment at which the kinks in the paths, $a(c)$ and $b(c)$, occur. At one extreme (panel (a)), when the kinks belong to the 45° line, we obtain the constrained equal awards rule. At the other extreme (panel (d)), when they lie on the line of slope 1 passing through the claims vector, we obtain the constrained equal losses rule. Panel (b) shows the reverse Talmud rule.

6 Basic properties of the ICI and CIC rules

In this section, we identify basic properties of the ICI and CIC rules. Several are obviously satisfied by each of them. Other properties are met if the functions with which a rule is associated (a pair $H = (F, G) \in \mathcal{H}^N$ for an ICI rule and a pair $H = (F, G) \in \bar{\mathcal{H}}^N$ for a CIC rule) satisfy certain restrictions that we identify.

Obviously, each ICI rule awards equal amounts to agents with equal claims, that is, satisfies **equal treatment of equals**. In fact, each of them satisfies **order preservation of awards**: if agent i's claim is at least as large as agent j's claim, he receives at least as much as agent j does; and each satisfies **order preservation of losses**, which says that under the same hypotheses, agent i's loss is at least as large as agent j's loss (Aumann and Maschler, 1985). Also, for each of them, the names of claimants do not matter, a property called **anonymity**. Each is also such that if the endowment increases, each claimant receives at least as much as he did initially: they are **resource monotonic.** Each is **resource continuous**: a small change in the endowment does not lead to a large change in the recommended awards. An ICI rule associated with a continuous H is **continuous**: small changes in the data of the problem (claims and endowment) do not produce large changes in the recommended awards. **Homogeneity** says that if claims and endowment are multiplied by the same positive number, then so should all awards. An ICI rule is *homogenous* if it is associated with a pair $H = (F, G) \in \mathcal{H}^N$ that is homogeneous of degree 1 (for each $c \in \mathbb{R}^N_+$ and each $\alpha \in \mathbb{R}_+$, $H(\alpha c) = \alpha H(c)$. It suffices to check homogeneity of F; homogeneity of G follows from the ICI relations.)

Similar observations hold for the CIC family.

Next, we consider the requirement that what is available (the endowment) should be divided symmetricly to what is missing (the deficit):[11]

Self-duality: For each $(c, E) \in \mathcal{C}^N$, $S(c, E) = c - S(c, \sum c_i - E)$.

Two rules ***S* and *R* are dual** if for each $(c, E) \in \mathcal{C}^N$, $S(c, E) = c - R(c, \sum c_i - E)$. A family of rules is **closed under duality** if whenever it contains a rule, it also contains its dual.

The following proposition collects a number of observations pertaining to duality about the ICI and CIC families. (Parallel statements concerning the family $\{T^\theta\}_{\theta \in [0,1]}$ are established by Moreno-Ternero and Villar, 2006a.)

Proposition 1 *(a) The only self-dual ICI rule is the Talmud rule.*
(b) The only self-dual CIC rule is the reverse Talmud rule.
(c) The ICI family is closed under duality.
(d) The CIC family is closed under duality.

Proof: (a) For a rule to be *self-dual*, for each $c \in \mathbb{R}^N_+$, the graph of the function giving each claimant i's award as a function of the endowment should be symmetric with respect to the point $(\sum \frac{c_i}{2}, \frac{c_i}{2})$. For the ICI rule associated with $H \equiv (F, G) \in \mathcal{H}^N$, this is equivalent to requiring that, starting with the agent with the smallest claim, $F_1(c)$ and $\sum c_i - G_1(c)$ should be equal. Thus, the sum $F_1(c) + G_1(c)$ is known. Since the difference $F_1(c) - G_1(c)$ is given by the first ICI relation, the numbers $F_1(c)$ and $G_1(c)$ are uniquely determined. Turning to the agent with the second smallest claim, and using the second ICI relation, we deduce that $F_2(c)$ and $G_2(c)$ are also uniquely determined. Proceeding by induction, we establish uniqueness of each pair $(F_k(c), G_k(c))$ for $k = 1, \ldots, n-1$. But then, we have obtained the Talmud rule.

[11]The condition is formulated by Aumann and Maschler (1985), who note a number of passages in the Talmud where the idea is implicit.

(b) The reasoning is similar. Again, we start with the agent with the smallest claim, but this time, we first obtain the symmetry with respect to $\frac{1}{2}\sum c_i$ of $F_{n-1}(c)$ and $G_{n-1}(c)$. By turning to the agent with the second smallest claim, we then obtain the symmetry of $F_{n-2}(c)$ and $G_{n-2}(c)$, and so on. Altogether, we derive the reverse Talmud rule.

(c) Let S be the ICI rule associated with $H \equiv (F, G) \in \mathcal{H}^N$. For each $c \in \mathbb{R}^N_+$, and each $k \in \{1, \ldots, n-1\}$, let $G^d_k(c) \equiv \sum c_i - F_k(c)$ and $F^d_k(c) \equiv \sum c_i - G_k(c)$. It is easy to see that the function $H^d \equiv (G^d, F^d)$ so defined satisfies the ICI relations, and that the ICI rule associated with G^d is indeed the dual of S.

(d) The proof is similar to that of (c) and we omit it. □

7 Consistency

Next we consider a variable-population version of the model. There is an infinite set of "potential" claimants, indexed by the natural numbers $\mathbb{N}$. In each given problem, however, only a finite number of them are present. Let $\mathcal{N}$ be the class of finite subsets of $\mathbb{N}$. Here, a **problem** is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$, where $N \in \mathcal{N}$, such that $\sum_N c_i \geq E$. A **rule** is a function defined over the domain $\mathcal{C} \equiv \bigcup_{N\in\mathcal{N}} \mathcal{C}^N$ consisting of all problems involving some population in $\mathcal{N}$, which associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$ a vector in $X(c, E)$. Given $N \in \mathcal{N}$, we refer to the restriction of a rule to the subdomain $\mathcal{C}^N$ as its "N-component". What we call an "ICI rule" or a "CIC rule" is a rule whose components are rules as defined in Sections 4 and 5 respectively.

Our central property of a rule here is designed to relate the recommendations it makes as the population of claimants changes. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. Apply the rule to (c, E) and let x be the awards vector that it selects. Now, imagine that some claimants receive their awards and leave. Let N' denote the set of remaining claimants. The situation, re-evaluated at that point, can be seen as the problem whose agent set is N' and whose endowment is the difference between the initial endowment and the sum of the amounts assigned to the departing claimants, namely $E - \sum_{N'} x_i$. Equivalently, the revised endowment is the sum $\sum_{N'} x_i$ of the amounts initially assigned to the remaining claimants. The pair $(c_{N'}, \sum_{N'} x_i)$ is the **reduced problem of (c, E) with respect to N' and x**. By definition of a rule, for each $i \in N'$, $x_i \leq c_i$. Thus, $\sum_{N'} x_i \leq \sum_{N'} c_i$, and we do obtain a well-defined claims problem. We require that for this problem, the rule should award to each remaining claimant the same amount as it did initially.[12]

Consistency: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $N' \subset N$, if $x \equiv S(c, E)$, then $x_{N'} = S(c_{N'}, \sum_{N'} x_i)$.

Our next theorem describes the subfamily of the ICI family of *consistent* rules. It is well-known that the constrained equal awards, constrained equal losses, and Talmud rules are *consistent* (Young, 1987). Thus, they are examples of *consistent* ICI rules (the minimal overlap rule is not *consistent* however.[13]) To obtain a complete description, let $\mathbf{\Gamma}$ be the class of nowhere decreasing functions $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\gamma(0) = 0$ and the function $e - \gamma$ is also nowhere decreasing. (These requirements imply that γ is continuous.) To each $\gamma \in \Gamma$ we associate a rule as follows. Let us denote by $\tilde{c}$ the vector in $\mathbb{R}^n_+$ obtained from c by writing its coordinates in increasing order.

ICI* rule associated with $\boldsymbol{\gamma} \in \mathbf{\Gamma}$, $\boldsymbol{S^\gamma}$: For each $N \in \mathcal{N}$ and each $c \in \mathbb{R}^N_+$, S^γ is the ICI rule whose N-component has breakpoints $(F_k(c), G_k(c))_{k=1}^{n-1}$ given by $(F_k(c))_{k=1}^{n-1} = (n\gamma(\tilde{c}_1), \gamma(\tilde{c}_1) + (n-1)\gamma(\tilde{c}_2), \ldots, \gamma(\tilde{c}_1) + \gamma(\tilde{c}_2) + \cdots + \gamma(\tilde{c}_{n-2}) + 2\gamma(\tilde{c}_{n-1}))$, $(G_k(c))_{k=1}^{n-1}$ being obtained from $(F_k(c))_{k=1}^{n-1}$ by means of the ICI relations.

[12]The first application of consistency ideas in the context of the current model is due to Aumann and Maschler (1985). For a survey of the consistency principle, see Thomson (1999).

[13]It does not satisfy the self-explanatory property of *replication invariance*, as noted by Chun and Thomson (2005). This can be proved in two ways. First, since *equal treatment of equals* and *consistency* imply *replication invariance* and the rule satisfies *equal treatment of equals* and violates *replication invariance*, it violates *consistency.* To see a violation of *replication invariance*, let $N \equiv \{1, 2\}$ and $(c, E) \in \mathcal{C}^N$ be defined by $c \equiv (2, 4)$ and $E \equiv 2$. Then, $MO(c, E) = (1, 1)$. Now, let $N' \equiv \{1, \ldots, 4\}$ and $(c', E') \in \mathcal{C}^{N'}$ be defined by $c' \equiv (2, 4, 2, 4)$ and $E' \equiv 4$. We have $MO(c', E') = (\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{3}{2})$. Claimants 1 and 2 are not getting the same amounts in the initial problem and in the problem obtained by introducing two new claimants with claims equal to theirs, and doubling the endowment. Violations of *consistency* can also be established directly, starting from a three-claimant example. Let $N \equiv \{1, 2, 3\}$ and $(c, E) \in \mathcal{C}^N$ be defined by $c \equiv (3, 6, 6)$ and $E \equiv 7$. Then $MO(c, E) = (1, 3, 3)$. If claimant 3 leaves with his award, the set of remaining claimants is $\{1, 2\}$, their claims are still $(3, 6)$ and the endowment is $7 - 3$. For this reduced problem, the minimal overlap rule recommends $(1.5, 2.5)$. (In this problem, agent 2's claim is greater than the endowment, and its minimal overlap awards vector is obtained by nesting claims.)

If $\gamma = e$, $S^\gamma = CEA$; if $\gamma = 0$, $S^\gamma = CEL$; and if $\gamma = \frac{e}{2}$, $S^\gamma = T$.

A **parametric rule** (Young, 1987) S is such that there is a continuous function $f\colon \mathbb{R}_+ \times \Lambda \to \mathbb{R}$, where $\Lambda = [\underline{\lambda}, \bar{\lambda}]$ is an interval in $\mathbb{R}$, with the property that for each $c_0 \in \mathbb{R}_+$, $f(c_0, \underline{\lambda}) = 0$ and $f(c_0, \bar{\lambda}) = c_0$, and $f(c_0, \cdot)$ is nowhere decreasing; then, for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, $S(c, E) = (f(c_i, \lambda))_{i \in N}$, where $\lambda \in \Lambda$ is such that $\sum f(c_i, \lambda) = E$. Any such λ is called an **equilibrium $\boldsymbol{\lambda}$**. The function f is a **parametric representation of $\boldsymbol{S}$**. Parametric representations of a parametric rule are not unique. All parametric rules are *consistent*. Of particular interest for us is the following:

Lemma 4 *The ICI* rules are parametric rules.*

Proof: Let $\gamma \in \Gamma$. We simply exhibit a representation f of S^γ (Figure 7a). Let $\underline{\lambda} = 0$, $\bar{\lambda} > 0$, and $\lambda^* \in]\underline{\lambda}, \bar{\lambda}[$. Let A be the graph of a continuous and increasing function g defined on $[0, \lambda^*[$ such that $g(0) = 0$ and $g(\lambda) \to \infty$ as $\lambda \to \lambda^*$. Let $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and increasing function such that $h(0) = 0$ and $h(c_0) \to \bar{\lambda} - \lambda^*$ as $c_0 \to \infty$. Let $B^\gamma \equiv \{(\bar{\lambda} - h(c_0), \gamma(c_0))\colon c_0 \in \mathbb{R}_+\}$. Let $C^\gamma \equiv \{(\bar{\lambda} - h(c_0), \gamma(c_0) - c_0)\colon c_0 \in \mathbb{R}_+\}$. Now, for each $c_0 \in \mathbb{R}_+$, let $f(c_0, \cdot)\colon [\underline{\lambda}, \bar{\lambda}] \to \mathbb{R}_+$ be the function whose graph follows A from $(0, 0)$ until the point of ordinate $\gamma(c_0)$, continues horizontally until the point of coordinates $(\bar{\lambda} - h(c_0), \gamma(c_0))$, and concludes with the part of $C^\gamma + \{(0, c_0)\}$ that lies to the right of the vertical line of abscissa $\bar{\lambda} - h(c_0)$. It is easy to see that the function f is well-defined and satisfies the properties listed in the definition of a parametric rule.

To see that indeed f is a parametric representation of S^γ, it is enough to consider the two-claimant case, the logic being the same for the general case. Let $N \equiv \{1, 2\}$ and suppose that agent 1's claim is c_0 and agent 2's claim c'_0. The graphs of $f(c_0, \cdot)$ and $f(c'_0, \cdot)$ are plotted in Figure 7b. As the endowment increases from 0 to $c_0 + c'_0$, the equilibrium λ increases from $\underline{\lambda}$ to $\bar{\lambda}$. Initially, as both graphs follow A, we obtain equal division. At $\lambda = \lambda^1$, the graph of $f(c_0, \cdot)$ becomes horizontal whereas that of $f(c'_0, \cdot)$ is still going up, so claimant 2 receives the totality of each increment. At $\lambda = \lambda^2$, claimant 2's graph also becomes horizontal. We have a "dead interval" of values of λ in which both awards are stationary. At $\lambda = \bar{\lambda} - h(c'_0)$, the graph of $f(c'_0, \cdot)$ starts going up again whereas that of $f(c_0, \cdot)$ is still horizontal. Thus, claimant 2 receives the totality of each increment. At $\lambda = \bar{\lambda} - h(c_0)$, claimant 1's graph starts going up too. Since the parts of the two graphs that we now follow are vertical translates of the same curve (C^γ), both claimants receive equal

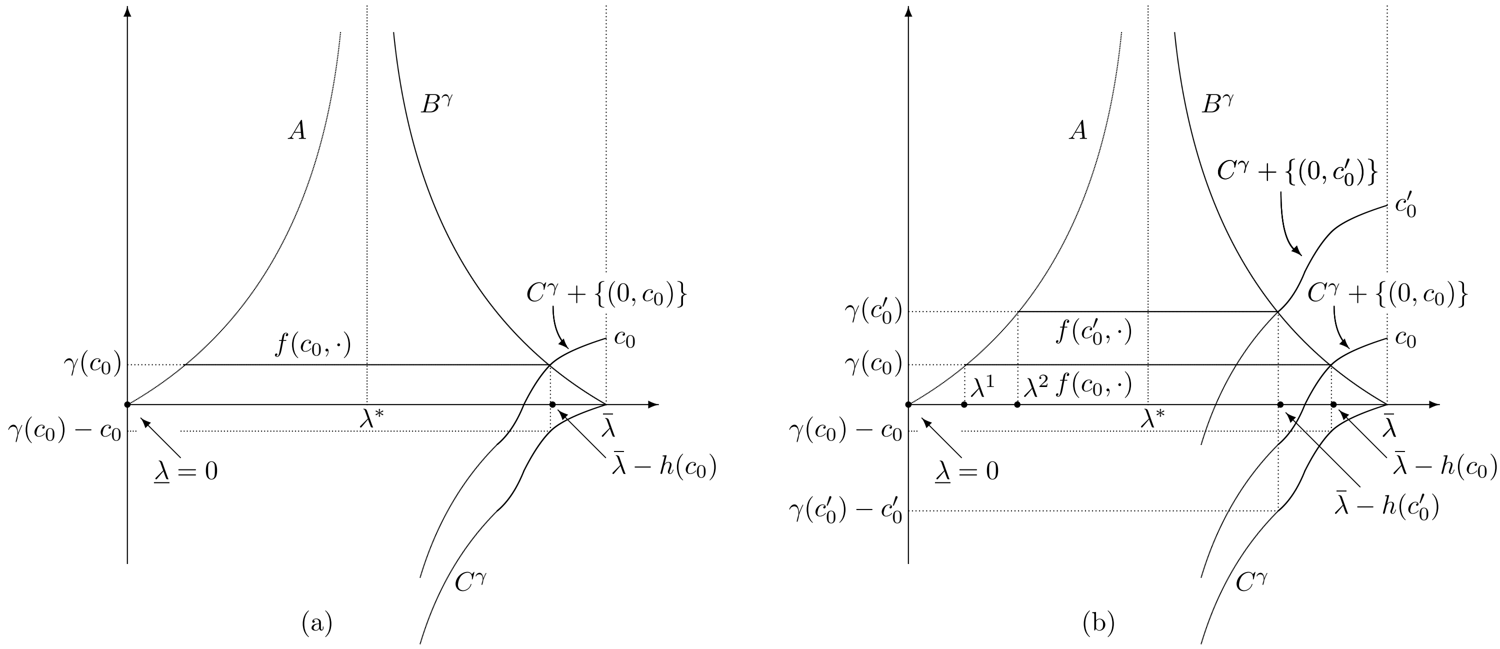


Figure 7: Parametric representations of ICI* rules. (a) Schedule $f(c_0, \cdot)$ of a parametric representation of the ICI* rule associated with some $\gamma \in \Gamma$, S^γ. (b) The schedules for claims c_0 and c_0' used to determine the path for (c_0, c_0').

shares of each increment. This goes on until both are fully compensated. Altogether, we have obtained the path of S^γ for (c_0, c_0'). $\square$

If $\gamma = e$, B^γ is a copy of the graph of h turned counterclockwise 90 degrees and C^γ is the horizontal segment $]\lambda^*, \bar{\lambda}] \subset \mathbb{R}_+$; then, $S^\gamma = CEA$. If $\gamma = 0$, B^γ is what C^γ is when $\gamma = e$ and C^γ is the symmetric image with respect to the horizontal axis of what B^γ is when $\gamma = e$; then $S^\gamma = CEL$ (in this case, A is not used). If $\gamma = \frac{e}{2}$, B^γ and C^γ are symmetric of each other with respect to the horizontal axis; then, $S^\gamma = T$. For each $\theta \in [0, 1]$, if $\gamma \equiv \theta e$, then B^γ and C^γ are symmetric images with respect to the horizontal axis of two curves that are vertical homothetic images of each other; then $S^\gamma = T^\theta$.

The proofs of Theorems 1 and 2 repeatedly exploit Fact 1 below, which is nothing other than a geometric restatement of *consistency*, the claims vector being kept fixed, and the endowment varying from 0 to the sum of the claims:[14]

[14] The line of reasoning is developed by Thomson (2007b), and illustrated there with a series of examples. It is also applied in Thomson (2001, 2007a), Hokari and Thomson (2003), and Dominguez and Thomson (2006).

Fact 1: Let S be a *consistent* rule. Then, for each $N \in \mathcal{N}$ and each $c \in \mathbb{R}^N_+$, the projection of the path of awards of S for c onto the subspace pertaining to each $N' \subset N$ is a subset of its path for the projection of c onto $\mathbb{R}^{N'}_+$. Moreover, if S is *resource continuous*, coincidence occurs. (This is because in that case, a path of awards is a continuous curve; by projection, we also obtain a continuous curve.)

Fact 1 applies to our search for *consistent* ICI and CIC rules because these rules are *resource continuous*, as noted in Section 6.

Another important logical relation involves *anonymity*, which has to be restated for our variable population framework. In this context, this is the requirement that not only within each group $N \in \mathcal{N}$, the names of claimants should not matter, but also that two problems involving distinct groups $N, N' \in \mathcal{N}$ of the same size in which the ordered lists of claims are equal and the endowments are equal too should be handled in the same way: a member of N and a member of N' whose claims are equal should receive equal amounts.[15]

Lemma 5 (Chambers and Thomson, 2002) *Equal treatment of equals and consistency imply anonymity.*

The following property of a rule will be useful in proving our next theorem. Let $N \in \mathcal{N}$, $(c, E) \in \mathcal{C}^N$, and $x \in X(c, E)$. If x is such that for each two-claimant subgroup of N, its restriction to the subgroup is chosen by the rule for the reduced problem with respect to the subgroup and x, then x should be chosen by the rule for (c, E):

Converse consistency: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $x \in X(c, E)$, if for each $N' \subset N$ with $|N'| = 2$, we have $x_{N'} = S(c_{N'}, \sum_{N'} x_i)$, then $x = S(c, E)$.

The next lemma, whose proof is straightforward, is an important model-free structural result that has appeared in numerous studies:[16]

[15] To illustrate the distinction, the rule that agrees with the constrained equal awards rule for any problem involving claimant 1 and with the constrained equal losses rule for any other problem satisfies what could be called *intra-group anonymity* but not *inter-group anonymity*.

[16] It is presented in this form by Thomson (1999).

Lemma 6 (Elevator Lemma) *If a rule is consistent and coincides with a conversely consistent rule in the two-claimant case, then it coincides with it for any number of claimants.*

Here is our main result on *consistency* for the ICI rules.

Theorem 1 *An ICI rule is consistent if and only if it is an ICI* rule.*

Proof: Each ICI* rule is by definition an ICI rule. It is *consistent* because it is a parametric rule (Lemma 4) and parametric rules satisfy this property.[17]

Conversely, let S be a *consistent* ICI rule. Step 1 says that the middle segments[18] of the paths of the two-claimant components of S are related in a special way.

Step 1: There is a function $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that the following holds. Let $N \in \mathcal{N}$ with $|N| = 2$ and $c \in \mathbb{R}^N_{++}$ be a claims vector of unequal coordinates. Then, the common value of the coordinates of the lowest endpoint of the middle segment in the path of S for c is $\gamma(\min c_i)$.

Let $N \equiv \{1, 2, 3\}$ and $c \in \mathbb{R}^N_+$ be such that $0 < c_1 < c_2 < c_3$. Let Π_1, Π_2, and Π_3 be the paths of awards of S for (c_2, c_3), (c_1, c_3), and (c_1, c_2) respectively, and Π be the path of S for c. By Fact 1, the projections of Π on each of the two-dimensional subspaces $\mathbb{R}^{\{1,2\}}$, $\mathbb{R}^{\{1,3\}}$, and $\mathbb{R}^{\{2,3\}}$, are Π_3, Π_2, and Π_1 respectively.

Since $0 < c_1 < c_2$, by definition of an ICI rule, Π_3 contains a segment parallel to $\mathbb{R}^{\{2\}}$—let us denote its endpoints k^1 and k^2, with $k^1 \leq k^2$, and since $0 < c_1 < c_3$, Π_2 contains a segment parallel to $\mathbb{R}^{\{3\}}$ (Figure 8a). Let a_1 and b_1 be the first coordinates of these segments. We will show that $a_1 = b_1$.

Suppose by contradiction that $a_1 \neq b_1$. Consider the plane in $\mathbb{R}^N$ of equation $x_1 = a_1$. Its intersection with Π_3 is $\text{seg}[k^1, k^2]$ (Figure 8a) and its

[17] A direct proof of the *consistency* of the ICI* rules is as follows. Let $\gamma \in \Gamma$ and consider S^γ. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. Suppose first that $E \leq \sum_N \gamma(c_i)$. Then, there is $k \in \{0, \ldots, |N|\}$ such that each claimant $i \in N$ whose claim is among the k smallest claims in c receives $\gamma(c_i)$—let $\tilde{N}$ be the set they constitute—and all others receive equal amounts. Let $N' \subset N$. Let $\tilde{N}' \equiv N' \cap \tilde{N}$. The members of $\tilde{N}'$ have the smallest claims in N'. Let us assign to each $i \in \tilde{N}'$ the amount $\gamma(c_i)$ and to the others equal shares of what is left. The claimants in $N' \setminus \tilde{N}$ are the claimants in $N \setminus \tilde{N}$ and this common amount is what they had been assigned initially. The case $E > \sum \gamma(c_i)$ can be analyzed similarly.

[18] The expression "middle segment" of a path is an abuse of language when either of the two segments of slope 1 in the path are degenerate.

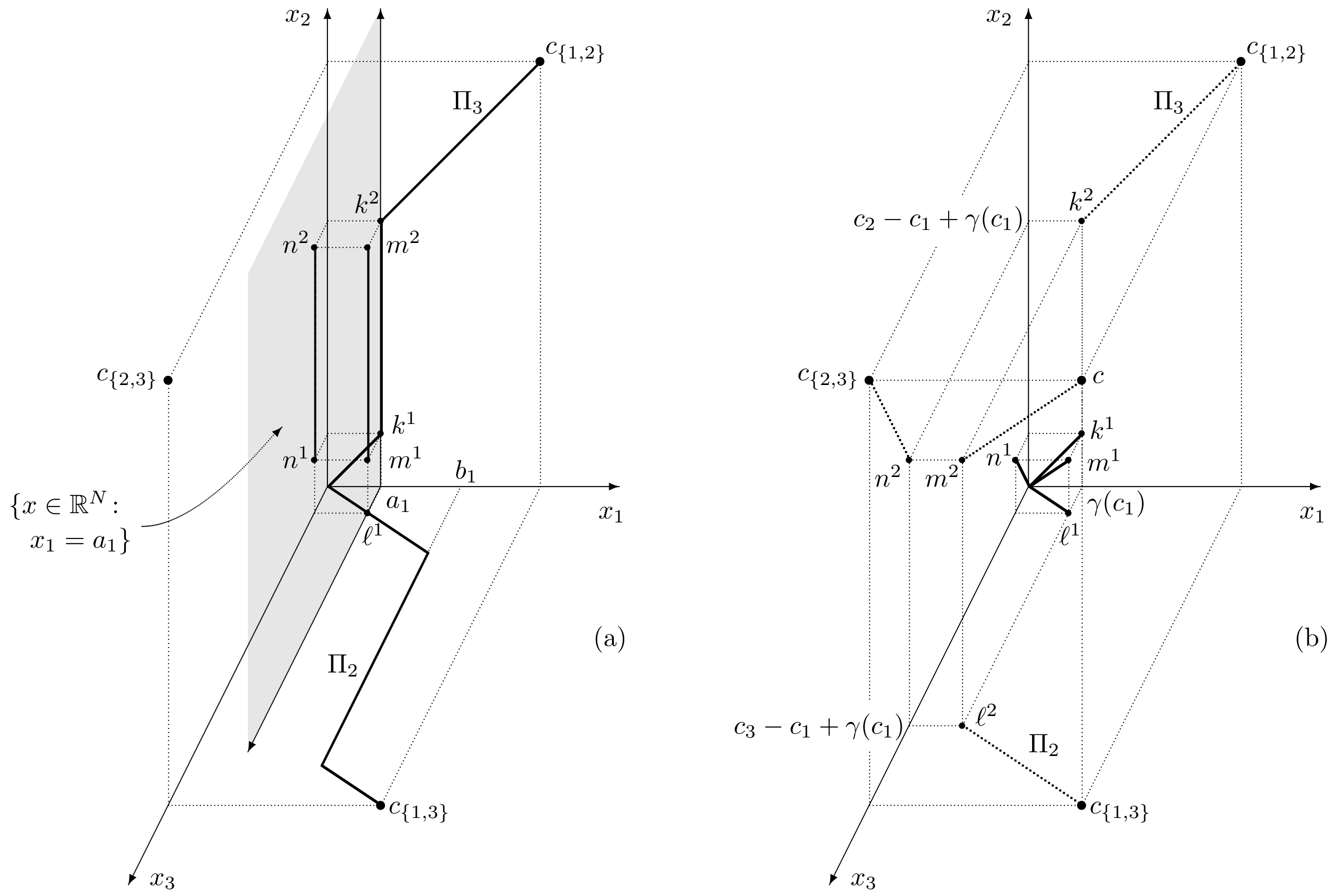


Figure 8: Proof of Theorem 1. (a) Step 1. Here, we show by contradiction that the middle segments of the paths for $c_{\{1,2\}}$ and $c_{\{1,3\}}$ have to be in the same plane parallel to the 2-3-coordinate subspace. This allow us to derive the existence of the function $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$. (b) Step 3. To show that γ is nowhere decreasing, we use seg$[0, k^1]$ in the path for $c_{\{1,2\}}$ and seg$[0, \ell^1]$ in the path for $c_{\{1,3\}}$ to deduce seg$[0, m^1]$ in the path for c, from which we deduce seg$[0, n^1]$ in the path for $c_{\{2,3\}}$. To show that $e - \gamma$ is nowhere decreasing, we use seg$[k^2, c_{\{1,2\}}]$ in the path for $c_{\{1,2\}}$ and seg$[\ell^2, c_{\{1,3\}}]$ in the path for $c_{\{1,3\}}$ to deduce seg$[m^2, c]$ in the path for c, from which we deduce seg$[n^2, c_{\{2,3\}}]$ in the path for $c_{\{2,3\}}$.

intersection with Π_2 is a singleton, denoted ℓ^1. (Figure 8a shows the case $a_1 < b_1$ but the argument is independent of whether $a_1 < b_1$ or $a_1 > b_1$.) Thus, by Fact 1, Π contains the segment whose projection onto $\mathbb{R}^{\{1,2\}}$ is $\text{seg}[k^1, k^2]$ and whose projection onto $\mathbb{R}^{\{1,3\}}$ is ℓ^1. It is $\text{seg}[(a_1, k_2^1, \ell_3^1), (a_1, k_2^2, \ell_3^1)]$ (= $\text{seg}[m^1, m^2]$ in the figure). The projection of this segment onto $\mathbb{R}^{\{2,3\}}$ is a segment parallel to $\mathbb{R}^{\{2\}}$, $\text{seg}[(0, k_2^1, \ell_3^1), (0, k_2^2, \ell_3^1)]$ (= $\text{seg}[n^1, n^2]$). By Fact 1, this segment belongs to the path of S for (c_2, c_3). However, since $0 < c_2 < c_3$, the path of an ICI rule for (c_2, c_3) does not contain a segment parallel to $\mathbb{R}^{\{2\}}$. Thus, $a_1 = b_1$.

Appealing to the *anonymity* of S, which follows from Lemma 5 (this lemma applies because the ICI rules satisfy *equal treatment of equals*), we conclude the following. Let $N \in \mathcal{N}$ with $|N| = 2$ and $c \in \mathbb{R}^N_{++}$ of unequal coordinates. Then, the common value of the coordinates of the lowest end-point of the middle segment in the path of S for c is independent of who the owner of the larger claim is (above, the owner of the larger claim in (c_1, c_2) is claimant 2, and the owner of the larger claim in (c_1, c_3) is claimant 3). Also, this common value is independent of who the owner of the smaller claim is (there, for both (c_1, c_2) and (c_1, c_3), this owner is claimant 1). Thus, there is $\gamma\colon \mathbb{R}_{++} \to \mathbb{R}_+$ as in the statement of Step 1.

If one or both coordinates of c are 0, the path of S for c is degenerate. To cover this case, it suffices to extend the domain of definition of γ to $\mathbb{R}_+$ by setting $\gamma(0) = 0$.

Step 2: The functions γ and $e - \gamma$ are nowhere decreasing. First, we show that γ is nowhere decreasing. Let $a, b \in \mathbb{R}_{++}$ be such $a < b$. Let $c \in \mathbb{R}^N_+$ be such that $c_1 = a$, $c_2 = b$, and $c_3 > c_2$. To show that $\gamma(a) \le \gamma(b)$, we consider the claims vector $c \equiv (c_1, c_2, c_3)$ (Figure 8b). By Step 1, the path of S for (c_1, c_2) contains $\text{seg}[(0,0), (\gamma(c_1), \gamma(c_1))]$ (= $\text{seg}[(0,0), k^1]$ on the figure) and its path for (c_1, c_3) contains $\text{seg}[(0,0), (\gamma(c_1), \gamma(c_1))]$ (= $\text{seg}[(0,0), \ell^1]$). Thus, by Fact 1, its path for c contains $\text{seg}[(0,0,0), (\gamma(c_1), \gamma(c_1), \gamma(c_1))]$ (= $\text{seg}[(0,0,0), m^1]$), and thus by Fact 1 again, projecting onto $\mathbb{R}^{\{2,3\}}$, its path for (c_2, c_3) contains $\text{seg}[(0,0), (\gamma(c_1), \gamma(c_1))]$ (= $\text{seg}[(0,0), n^1]$). By Step 1, and since $c_2 < c_3$, the path of S for (c_2, c_3) contains $\text{seg}[(0,0), (\gamma(c_2), \gamma(c_2))]$ and it has a kink at $(\gamma(c_2), \gamma(c_2))$. These two conclusions can hold together only if $\gamma(c_2) \ge \gamma(c_1)$.

Next, we show that $e - \gamma$ is nowhere decreasing. By Step 1, the path of S for (c_1, c_2) contains $\text{seg}[(\gamma(c_1), c_2 - c_1 + \gamma(c_1)), (c_1, c_2)]$ (= $\text{seg}[k^2, c_{\{1,2\}}]$ on the figure) and its path for (c_1, c_3) contains $\text{seg}[(\gamma(c_1), c_3 - c_1 + \gamma(c_1)), (c_1, c_3)]$

$(= \text{seg}[\ell^2, c_{\{1,3\}}])$. Thus, by Fact 1, its path for c contains $\text{seg}[(\gamma(c_1), c_2 - c_1 + \gamma(c_1), c_3 - c_1 + \gamma(c_1)), c]$ $(= \text{seg}[m^2, c])$, and by Fact 1 again, projecting onto $\mathbb{R}^{\{2,3\}}$, its path for (c_2, c_3) contains $\text{seg}[(c_2 - c_1 + \gamma(c_1), c_3 - c_1 + \gamma(c_1)), (c_2, c_3)]$ $(= \text{seg}[n^2, c_{\{2,3\}}])$. By Step 1, and since $c_2 < c_3$, the path of S for (c_2, c_3) contains $\text{seg}[(\gamma(c_2), c_3 - c_2 + \gamma(c_2)), (c_2, c_3)]$ and it has a kink at $(\gamma(c_2), c_3 - c_2 + \gamma(c_2))$. These two conclusions can hold together only if $c_2 - \gamma(c_2) \geq c_1 - \gamma(c_1)$.

Step 3: Concluding. Steps 1 and 2 together imply that there is $\gamma \in \Gamma$ such that on the domain of two-claimant problems in which the two claimants have unequal claims, $S = S^\gamma$. Since the ICI rules satisfy *equal treatment of equals*, this equality also holds, trivially, for two-claimant problems in which the two claimants have equal claims. By hypothesis, S is *consistent* and because S^γ is *consistent* and *resource monotonic*, it is *conversely consistent* (for a proof of this implication, see Chun, 1999). Thus, by the Elevator Lemma, $S = S^\gamma$ for any number of claimants. □

Next, we search for the *consistent* CIC rules. The constrained equal awards and constrained equal losses rules are among them, and so is the reverse Talmud rule. We will show that the *consistent* CIC rules can also be indexed by the elements of Γ. They are defined as follows:

CIC* rule associated with $\boldsymbol{\gamma \in \Gamma}$, $\boldsymbol{R^\gamma}$: For each $N \in \mathcal{N}$ and each $c \in \mathbb{R}^N_+$, R^γ is the CIC rule whose N-component has breakpoints $(F_k(c), G_k(c))_{k=1}^{n-1}$ given by $(F_k(c))_{k=1}^{n-1} = (\gamma(\tilde{c}_n) - \gamma(\tilde{c}_{n-1}),\ \gamma(\tilde{c}_n) + \gamma(\tilde{c}_{n-1}) - 2\gamma(\tilde{c}_{n-2}),\ \ldots,$ $\gamma(\tilde{c}_n) + \gamma(\tilde{c}_{n-1}) + \cdots + \gamma(\tilde{c}_2) - (n-1)\gamma(\tilde{c}_1))$, $(G_k(c))_{k=1}^{n-1}$ being obtained from $(F_k(c))_{k=1}^{n-1}$ by means of the CIC relations.

If $\gamma = 0$, $R^\gamma = CEA$; if $\gamma = e$, $R^\gamma = CEL$; and if $\gamma = \frac{e}{2}$, $R^\gamma = T^r$ (the reverse Talmud rule).

Lemma 7 *The CIC* rules are parametric rules.*

Proof: Let $\gamma \in \Gamma$. We simply exhibit a representation f of R^γ (Figure 9a). Let $\underline{\lambda} = -\infty$ and $\bar{\lambda} = \infty$. Let $B^\gamma \equiv \{(c_0 - \gamma(c_0), c_0) \colon c_0 \in \mathbb{R}_+\}$. For each $c_0 \in \mathbb{R}_+$, let $f(c_0, \cdot) \colon \mathbb{R} \to \mathbb{R}_+$ whose graph consists of the horizonal half-line $\{(t, 0) \colon t \in]-\infty, -\gamma(c_0)]\}$, the segment of slope 1 whose endpoints are $(-\gamma(c_0), 0)$ and $(c_0 - \gamma(c_0), c_0)$ (this second point belongs to B^γ), and the horizontal half-line $\{(t, c_0) \colon t \in [c_0 - \gamma(c_0), \infty[\}$. It is easy to see that the function f is well-defined and satisfies the properties listed in the definition of a parametric rule.

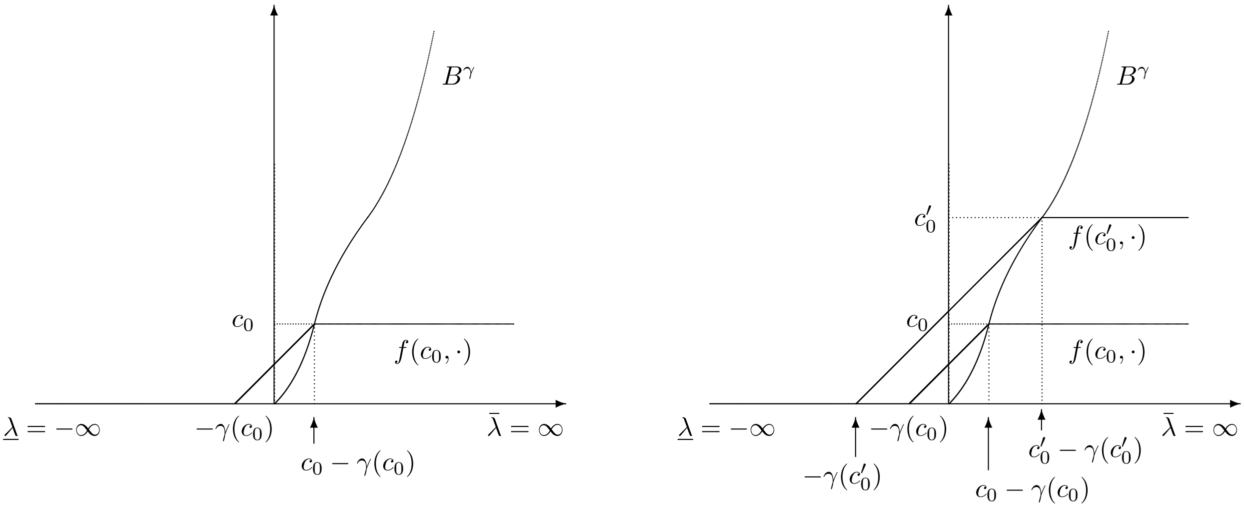
To see that indeed f is a parametric representation of R^γ, it is enough to consider the two-claimant case, as the logic is the same for the general case. Let $N \equiv \{1, 2\}$ and suppose that agent 1's claim is c_0 and agent 2's claim $c'_0 > c_0$. The functions $f(c_0, \cdot)$ and $f(c'_0, \cdot)$ are plotted in Figure 9b. As the endowment increases from 0 to the sum of the claims, the equilibrium λ increases from $-\infty$ to $-\gamma(c'_0)$, the graphs of both $f(c_0, \cdot)$ and $f(c'_0, \cdot)$ follow the horizontal axis, and neither claimant gets anything. (We have what we called in the proof of Lemma 4 a "dead interval".) As λ increases from $-\gamma(c'_0)$ to $-\gamma(c_0)$, the graph of $f(c_0, \cdot)$ remains horizontal but the graph of $f(c'_0, \cdot)$ starts going up. Thus, claimant 2 receives each increment of the endowment. As λ increases from $-\gamma(c_0)$ to $c_0 - \gamma(c_0)$, the graphs of both $f(c_0, \cdot)$ and $f(c'_0, \cdot)$ are parallel lines, so equal division of each increment prevails. At $\lambda = c_0 - \gamma(c_0)$, claimant 1 is fully compensated. As λ increases from that point, the graph of $f(c_0, \cdot)$ is horizontal, but that of $f(c'_0, \cdot)$ is still going up, so claimant 2 receives the totality of each increment. At $\lambda = c'_0 - \gamma(c'_0)$, claimant 2 is also fully compensated and his graph also becomes horizontal. We have a dead interval of values of λ in which both awards remain stationary and equal to claims. Altogether, we have obtained the path of R^γ for (c_0, c'_0). $\square$

If $\gamma = 0$, B^γ is the 45° line; then, $R^\gamma = CEA$. If $\gamma = e$, B^γ is the vertical axis, then, $R^\gamma = CEL$. If $\gamma = \frac{e}{2}$, B^γ is the ray of slope 2; then, $R^\gamma = T^r$. For each $\theta \in [0, 1[$, if $\gamma = \theta e$, B^γ is the ray of slope $\frac{1}{1-\theta}$ (for $\theta = 1$, it is the vertical axis), and $R^\gamma = U^\theta$.

Theorem 2 *A CIC rule is consistent if and only if it is a CIC* rule.*

Proof: Each CIC* rule is by definition a CIC rule. It is *consistent* because it is a parametric rule (Lemma 7) and parametric rules satisfy this property.[19]

[19] The *consistency* of the CIC* rules can also be seen directly as follows. Let $\gamma \in \Gamma$ and consider R^γ. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. Suppose first that $E \leq \sum \gamma(c_i)$. Then, there is $k \in \{0, \ldots, |N|\}$ such that each claimant $i \in N$ whose claim is among the k largest claims in c receives $\gamma(c_i)$ and all others receive 0. Let $\tilde{N}$ be the first of these two sets. Let $N' \subset N$. Let $\tilde{N}' \equiv N' \cap \tilde{N}$. The members of $\tilde{N}'$ have the largest claims in N'. Let us assign to each $i \in \tilde{N}'$ the amount $\gamma(c_i)$ and to the others 0. Also, $N' \setminus \tilde{N} = N \setminus \tilde{N}$. The case when $E \geq \sum \gamma(c_i)$ can be analyzed similarly.



(a) (b)

Figure 9: Parametric representation of CIC* rules. (a) Schedule $f(c_0, \cdot)$ of a parametric representation of the CIC* rule associated with some $\gamma \in \Gamma$, R^γ. (b) Two schedules, for claims c_0 and c_0', used to determine the path for (c_0, c_0').

Conversely, let S be a *consistent* CIC rule.

Step 1: For each $c_0 > 0$, there is a function $\gamma\colon\ ]c_0, \infty[\to \mathbb{R}_+$ such that the following holds. Let $N \in \mathcal{N}$ with $|N| = 2$ and $c \in \mathbb{R}^N_{++}$ of unequal coordinates, both of which larger than c_0. Then, the greater coordinate of the lowest endpoint of the segment of slope 1 contained in the path of S for c is $\gamma(\max\{c_i\}) - \gamma(\min\{c_i\})$. Also, γ is nowhere decreasing.

Let $c_0 > 0$. Let $N \equiv \{1, 2, 3\}$ and $c \in \mathbb{R}^N_+$ be such that $c_1 \equiv c_0$, and $c_0 < c_2 < c_3$. Let Π_1, Π_2, and Π_3 be the paths of awards of S for (c_2, c_3), (c_1, c_3), and (c_1, c_2).

By definition of a CIC rule, since $c_1 < c_2$, Π_3 contains a segment seg$[k^1, k^2]$ with $k^1 \in \mathbb{R}^{\{1,2\}}$ such that $k^1_1 = 0$ and $k^2 \equiv k^1 + (c_1, c_1)$, and since $c_1 < c_3$, Π_3 contains a segment seg$[\ell^1, \ell^2]$ with $\ell^1 \in \mathbb{R}^{\{1,3\}}$ such that $\ell^1_1 = 0$ and $\ell^2 \equiv \ell^1 + (c_1, c_1)$ (Figure 10a). By *anonymity* of S, which holds by Lemma 5, the larger coordinate of k^1 and the larger coordinate of ℓ^1 are independent of the fact that Π_3 and Π_2 are paths of S for problems involving the groups $\{1, 2\}$ and $\{1, 3\}$: there is a function $\gamma\colon\ ]c_0, \infty[\to \mathbb{R}_+$ such that $k^1_2 = \gamma(c_2)$ and $\ell^1_3 = \gamma(c_3)$.

Next, we show that γ is nowhere decreasing. By Fact 1, Π contains the segment whose projection onto $\mathbb{R}^{\{1,2\}}$ is seg$[k^1, k^2]$ and whose projection onto $\mathbb{R}^{\{1,3\}}$ is $[\ell^1, \ell^2]$. It is seg$[(0, k^1_2, \ell^1_3)(c_1, k^2_2, \ell^2_3)]$, ($=$ seg$[m^1, m^2]$ in Figure 10a). The projection of this segment onto $\mathbb{R}^{\{2,3\}}$ is a segment of slope 1, seg$[(0, k^1_2, \ell^1_3), (0, k^2_2, \ell^2_3)]$ ($=$ seg$[n^1, n^2]$ in the figure). By Fact 1, this segment is contained in the path of S for (c_2, c_3). If $\gamma(c_2) > \gamma(c_3)$, the line containing

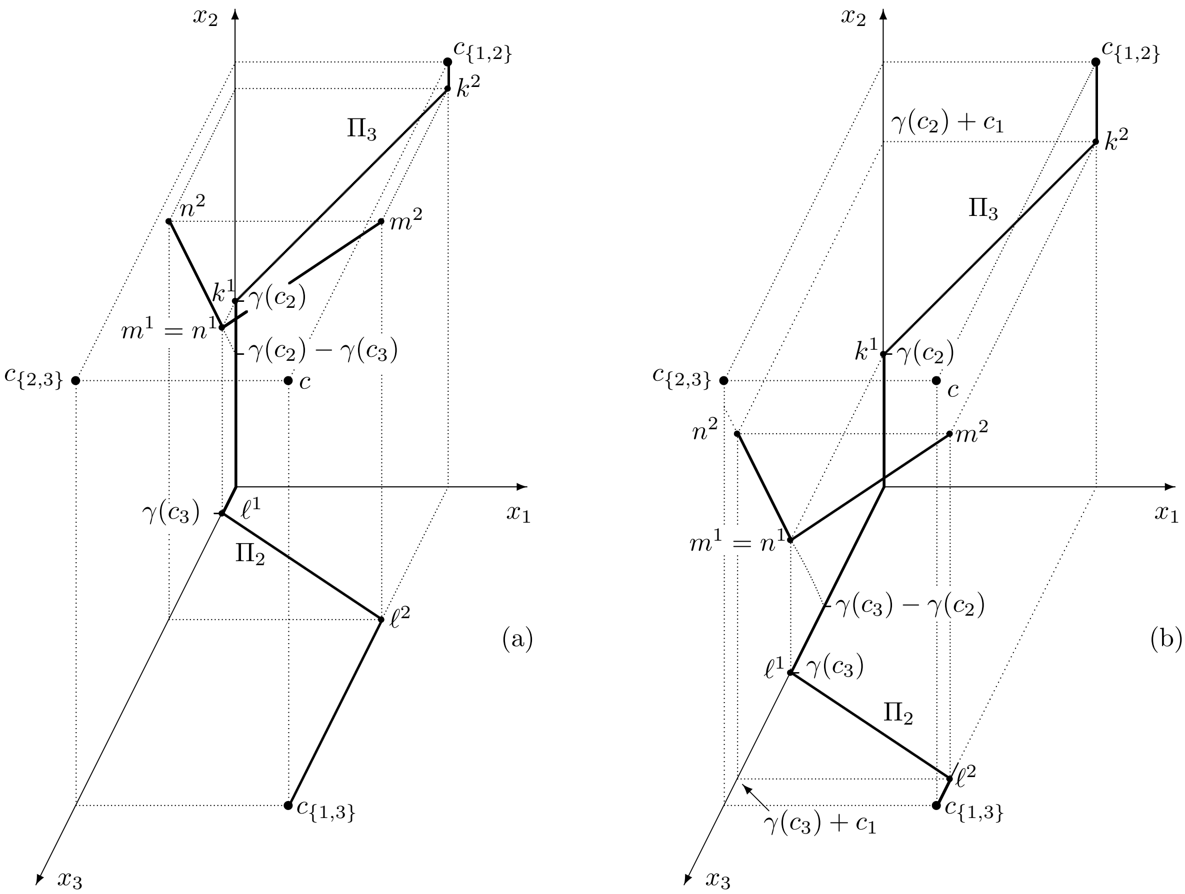


Figure 10: Proof of Theorem 2. (a) Step 1: deriving the existence of the function $\gamma\colon]c_0, \infty[\to \mathbb{R}_+$. Also, showing that γ is nowhere decreasing. (b) Step 2: showing that $e - \gamma$ is nowhere decreasing.

this segment contains the point $(\gamma(c_3) - \gamma(c_2), 0)$, which belongs to one of the shorter sides of the rectangle with vertices $(0, 0)$ and (c_2, c_3) (Figure 10a). Since $c_2 < c_3$, this contradicts what we know of the path of an CIC rule for (c_2, c_3): the middle segment contained in such a path should meet both of the longer sides of this rectangle. Thus, $\gamma(c_2) \geq \gamma(c_3)$. Since this holds for each pair $\{c_2, c_3\}$ such that $c_1 \equiv c_0 < c_2 < c_3$, γ is nowhere decreasing on its domain $]c_0, \infty[$.

Step 2: The function $e - \gamma$ is nowhere decreasing. Suppose by contradiction that for some $a, b \in \mathbb{R}_+$ with $c_0 < a < b$, $a - \gamma(a) > b - \gamma(b)$. Let $N \equiv \{1, 2, 3\}$, $c_1 \equiv c_0$, $c_2 \equiv a$, and $c_3 \equiv b$ (Figure 10b). The segment of slope 1 contained in the path of S for (c_2, c_3) contains the point $(c_3, \gamma(c_2) + c_3 - \gamma(c_3))$, which belongs to one of the shorter sides of the rectangle with vertices $(0, 0)$ and (c_2, c_3). Since $c_2 < c_3$, this contradicts what we know of the path of S for (c_2, c_3).

Step 3: Extending the domain of definition of γ to $\mathbb{R}_+$. The function γ obtained in Step 2 is defined over $]c_0, \infty[$ for a particular $c_0 > 0$. We now extend its domain of definition to the whole of $\mathbb{R}_+$. Let $0 < a \leq c_0$. We will define $\gamma(a)$. Given $0 < \bar{c}_0 < a$, we can construct a function $\bar{\gamma}\colon]\bar{c}_0, \infty[\to \mathbb{R}_+$ as we constructed γ. We will show that over the common part of their domains of definition, namely $]c_0, \infty[$, the functions γ and $\bar{\gamma}$ are equal up to an additive constant. Indeed, let $c_2, c_3 \in]c_0, \infty[$ be such that $c_2 < c_3$. By the definition of γ, the middle segment in the path of S for (c_2, c_3) intersects $\mathbb{R}^{\{3\}}$ at $(0, \gamma(c_3) - \gamma(c_2))$, and by the definition of $\bar{\gamma}$, this segment intersects $\mathbb{R}^{\{3\}}$ at $(0, \bar{\gamma}(c_3) - \bar{\gamma}(c_2))$. Thus, $\gamma(c_3) - \gamma(c_2) = \bar{\gamma}(c_3) - \bar{\gamma}(c_2)$. Keeping c_2 fixed, we conclude that over $]c_2, \infty[$, γ and $\bar{\gamma}$ are equal up to an additive constant. Since c_2 was chosen arbitrarily subject to $c_0 < c_2$, it follows that over $]c_0, \infty[$, the two functions are equal up to an additive constant.

Now, let $b > c_0$, and set $\gamma(a) \equiv \bar{\gamma}(a) + \gamma(b) - \bar{\gamma}(b)$. This value of $\gamma(a)$ is independent of the choice of $\bar{c}_0$. Thus, γ is well-defined. Also, by its construction, γ inherits the two monotonicity properties established in Steps 1 and 2. They imply that $\lim_{t \to 0} \gamma(t)$ exists. Let g designate this limit. To obtain a function in Γ, it now suffices to add $-g$ to γ.

Step 4: Concluding. Steps 1, 2, and 3 together imply that there is $\gamma \in \Gamma$ such that on the domain of two-claimant problems with unequal claims, $S = R^\gamma$. Since the CIC rules satisfy *equal treatment of equals*, this conclusion extends, trivially, to the case of two claimants with equal claims. By hypothesis, S is *consistent* and since R^γ is *consistent* and *resource mono-*

tonic, it is *conversely consistent* (implication already noted in the proof of Theorem 1). Thus, by the Elevator Lemma, $S = R^\gamma$ for any number of claimants. □

Remark 1: Given a two-claimant rule S, an interesting question is whether there is a rule R defined for all populations that coincides with S in the two-claimant case and is *consistent*. If such a rule exists, it is the **consistent extension of** $\boldsymbol{S}$. Instead of asking whether one can select, for each $N \in \mathcal{N}$, an ICI rule defined on $\mathcal{C}^N$, so that the rule on $\bigcup_{N\in\mathcal{N}} \mathcal{C}^N$ so defined is *consistent* (the question answered by Theorem 1), one can ask whether a given two-claimant ICI rule has a *consistent* extension R. This is a more general question because now the components of R relative to groups with more than two claimants are not required to be fixed-population ICI rules themselves.

We have all the elements for the answer. Since the ICI rules satisfy *equal treatment of equals*, if R exists, it does too in the two-claimant case. If a rule satisfies this property in the two-claimant case and is *consistent*, it satisfies the property in general. (This is an example of a Lifting Theorem, of the kind established by Hokari and Thomson, 2000.) Thus, Lemma 5 applies. The proof can then proceed as before. We derive the function γ as we did. The result can now be stated as follows: if the two-claimant components of a rule R on $\bigcup_{N\in\mathcal{N}} \mathcal{C}^N$ are two-claimant ICI rules and R is *consistent*, then there is $\gamma \in \Gamma$ such that the two-claimant components of R are the two-claimant components of the ICI* rule associated with γ; moreover, its components relative to greater populations are also the components of an ICI* rule, and they too are associated with γ.

The same comment applies to the CIC rules.

Remark 2: Suppose that instead of specifying for each $N \in \mathcal{N}$, an arbitrary ICI rule, one specifies a member of the Moreno-Ternero–Villar family (2006a) associated with some parameter $\theta^N \in [0, 1]$. One can then inquire about what is needed for the resulting rule $\{T^{\theta^N}\}_{N\in\mathcal{N}}$ to be *consistent.* Under these more restrictive assumptions, the proof of Theorem 1 simplifies. The answer is that there is $\theta \in [0, 1]$ such that for each $N \in \mathcal{N}$, $\theta^N = \theta$. A similar comment applies to the CIC rules.

Remark 3: The subfamily of *homogeneous* ICI* rules is $\{T^\theta\}_{\theta\in[0,1]}$. This is the Moreno-Ternero–Villar family (2006a). Also, the subfamily of *homogeneous* CIC* rules is $\{U^\theta\}_{\theta\in[0,1]}$.

Remark 4: In a companion paper (Thomson, 2007c), we identify conditions relating the functions (F, G) and (F', G') with which two ICI rules are associated for them to be comparable in the Lorenz order. We derive parallel conditions for two CIC rules.

References

Alcalde, J., M. Marco, and J. A. Silva, "Bankruptcy problems and the Ibn Ezra's proposal", *Economic Theory* 26 (2005), 103-114.

——, ——, and ——, "The minimal overlap rule revisited", *Social Choice and Welfare*, forthcoming, 2007.

Aumann, R. and M. Maschler, "Game theoretic analysis of a bankruptcy problem from the Talmud," *Journal of Economic Theory* 36 (1985), 195-213.

Bergantiños, G. and L. Méndez-Naya, "Additivity in bankruptcy problems and in allocation problems", *Spanish Economic Review* 3 (2001), 223-229.

Chambers, C. and W. Thomson, "Group order preservation and the proportional rule for bankruptcy problems", *Mathematical Social Sciences* 44 (2002), 235-252.

Chun, Y., "Equivalence of axioms for bankruptcy problems", *International Journal of Game Theory 28* (1999), 511-520.

Chun, Y., J. Schummer, and W. Thomson, "Constrained egalitarianism: A new solution to bankruptcy problems", *Seoul Journal of Economics* 14 (2001), 269-297.

Chun, Y. and W. Thomson, "Convergence under replication of rules to adjudicate conflicting claims", *Games and Economic Behavior* 50 (2005), 129-142.

Dagan, N., "New characterizations of old bankruptcy rules", *Social Choice and Welfare* 13 (1996), 51-59.

Dominguez, D. and W. Thomson, "A new solution to the problem of adjudicating conflicting claims", *Economic Theory* 28 (2006), 283-307.

Hokari, T. and W. Thomson, "Claims problems and weighted generalizations of the Talmud rule", *Economic Theory* 21 (2003), 241-261.

—— and ——, "On properties of division rules lifted by bilateral consistency", mimeo 2000, revised 2006, *Journal of Mathematical Economics*, forthcoming.

Moreno-Ternero, J. and A. Villar, "The TAL-family of rules for bankruptcy problems", *Social Choice and Welfare* 27 (2006a), 231-249.

—— and ——, "On the relative equitability of a family of taxation rules", *Journal of Public Economic Theory* 8 (2006b), 283-291.

O'Neill, B., "A problem of rights arbitration from the Talmud," *Mathematical Social Sciences* 2 (1982), 345-371.

Thomson, W., "Consistent allocation rules", mimeo, 1999.

——, Condorcet Lecture, Fourth International Meeting of the Society for Social Choice and Welfare, Alicante, 2000.

——, "A characterization of a family of rules for the resolution of conflicting claims", mimeo, 2001.

——, "Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey", *Mathematical Social Sciences* 45 (2003), 249-297.

——, "On the existence of consistent rules to adjudicate conflicting claims: a geometric approach", *Review of Economic Design* 11 (2007a), 225-251.

——, "The two-agent claims-truncated proportional rule has no consistent extension: a constructive proof", *Economics Letters*, forthcoming, 2007b.

——, "Lorenz comparisons of rules for the adjudication of conflicting claims", mimeo, 2007c.

——, *How to Divide When there Isn't Enough*, mimeo, 2007d.

Young, P., "On dividing an amount according to individual claims or liabilities," *Mathematics of Operations Research* 12 (1987), 398-414.

Appendix

The remaining figures give parametric representations of selected members of the ICI* and CIC* families. To read Figures 11 and 12, follow the instructions given in Lemma 4, and to read Figures 13 and 14, follow the instructions given in Lemma 7.

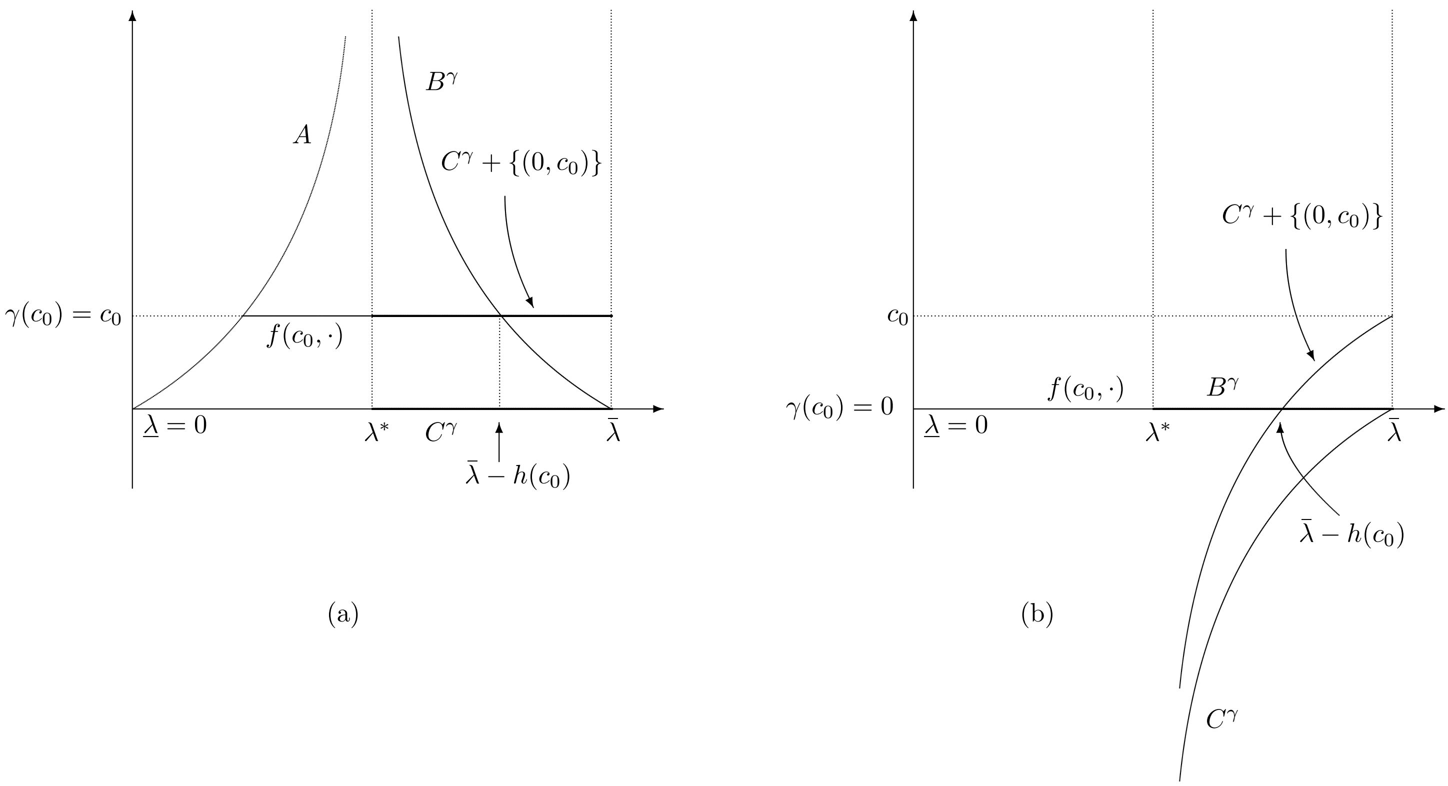


Figure 11: Two special members of the ICI* family. (a) Constrained equal awards rule (obtained for $\gamma = 0$). (b) Constrained equal losses rule (obtained for $\gamma = e$. The curve A is not used then, so we have omitted it).

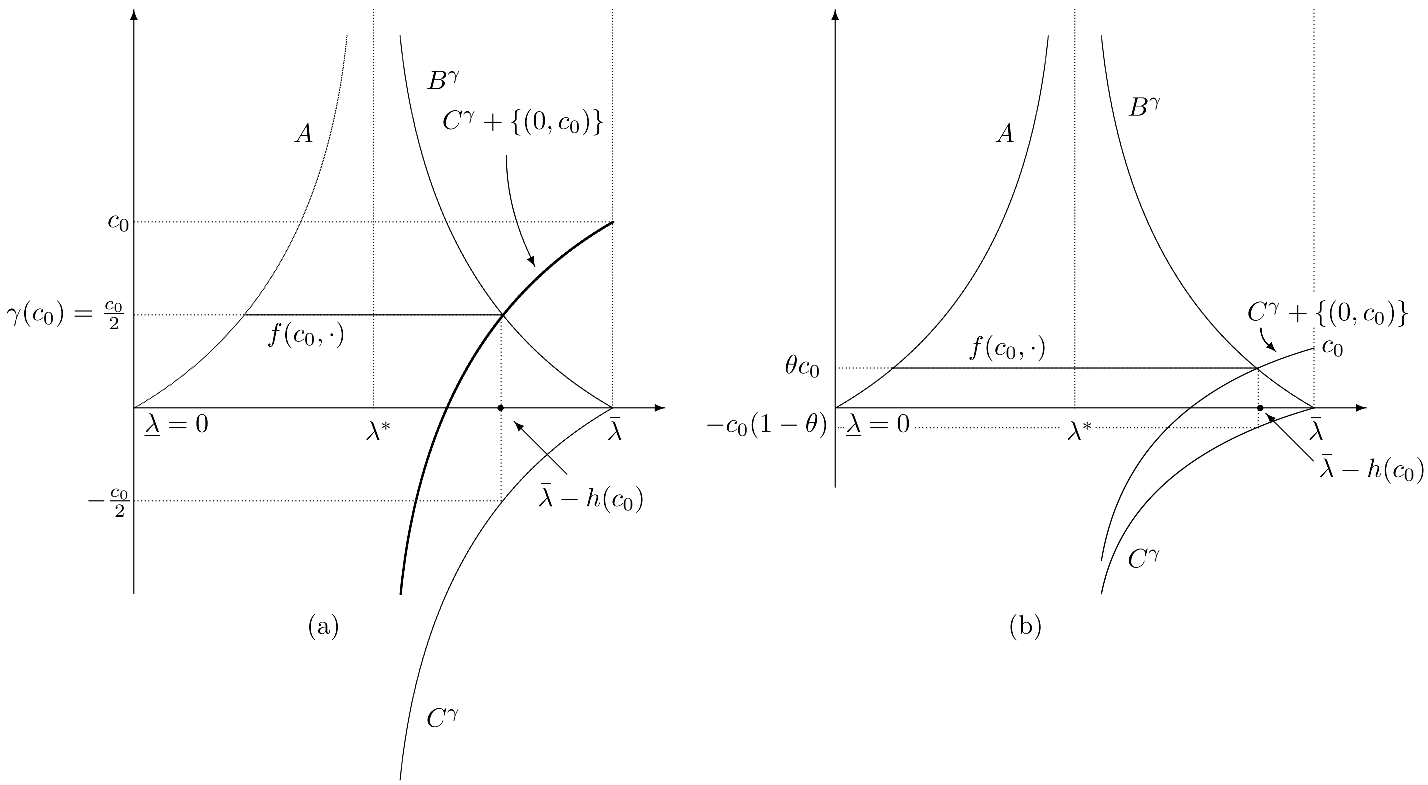


Figure 12: Two other special member of the ICI* family. (a) Talmud rule (obtained for $\gamma = \frac{c}{2}$). (b) Member of the subfamily $\{T^\theta\}_{\theta \in [0,1]}$ (obtained for $\gamma = \theta e$).

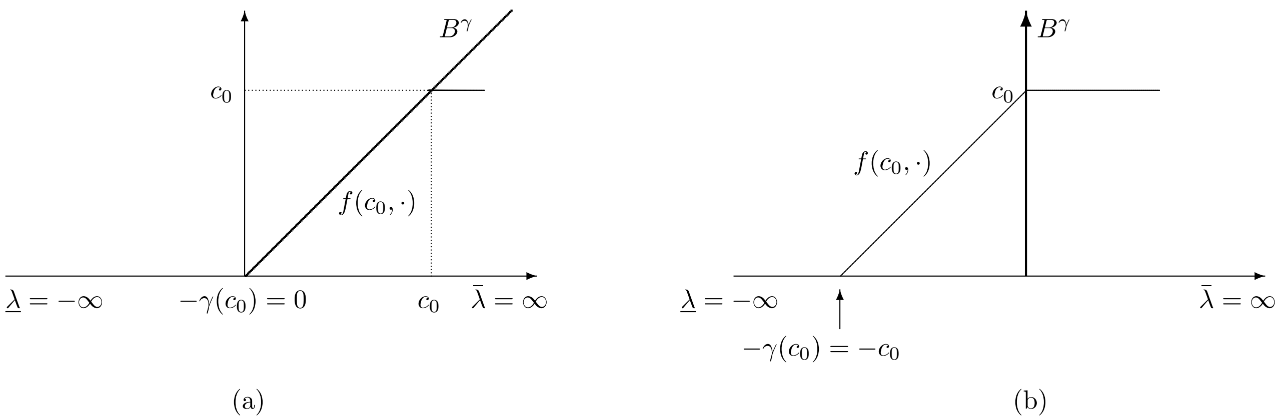


Figure 13: Two special members of the CIC* family. (a) Constrained equal awards rule (obtained for $\gamma = 0$). (b) Constrained equal losses rule (obtained for $\gamma = e$).

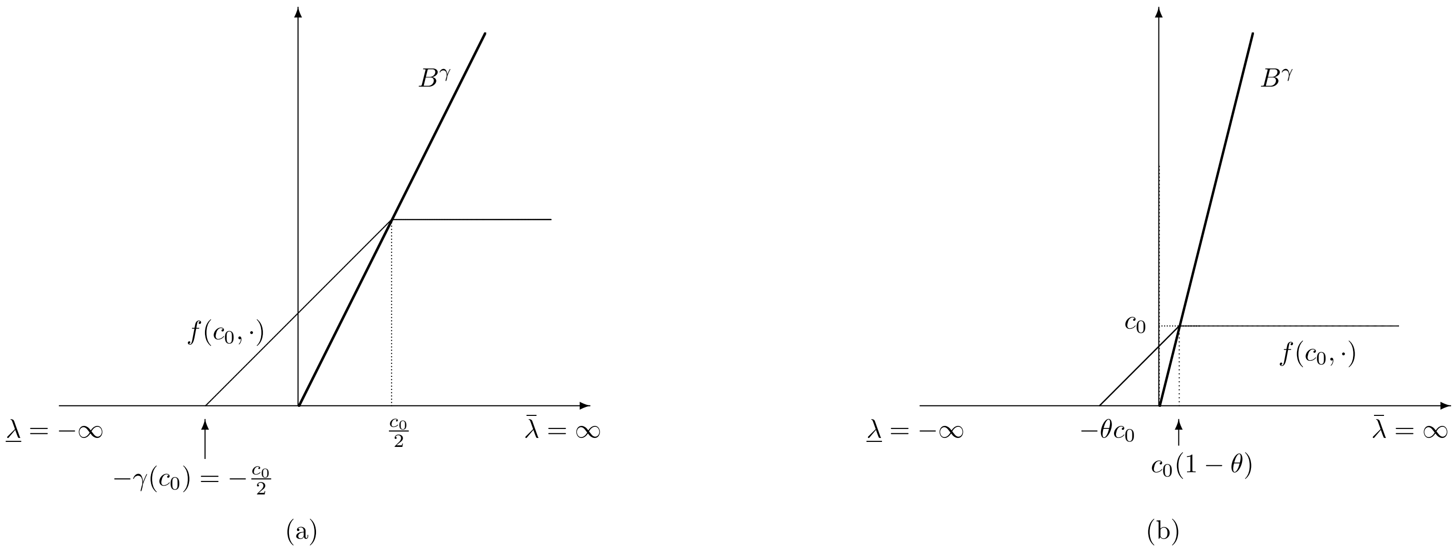


Figure 14: Two other special members of the CIC* family. (a) The reverse Talmud rule (obtained for $\gamma = \frac{e}{2}$). (b) Member of the subfamily $\{U^\theta\}_{\theta \in [0,1]}$ (obtained for $\gamma = \theta e$).