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Abstract

We analyze Bertrand's price competition in a homogenous good market with a fixed cost and an increasing marginal cost (i.e., with variable returns to scale). If the fixed cost is avoidable, we show that the non-subadditivity of the cost function at the output corresponding to the oligopoly break-even price, denoted by $D(p_L(n))$, is sufficient to guarantee that the market supports an equilibrium in pure strategies with two or more active firms supplying at least $D(p_L(n))$. Conversely, the existence of a pure strategy equilibrium ensures that the cost function is not subadditive at every output greater than or equal to $D(p_L(n))$. As a by-product, the latter implies that the average cost cannot be decreasing over the range of outputs mentioned before. In addition, we also prove that the existence of a price-taking equilibrium is sufficient, but not necessary, for Bertrand's price competition to possess an equilibrium in pure strategies. This provides a simple existence result for the case where the fixed cost is fully unavoidable. JEL Classification: D43, L13.

1 Introduction

In Industrial Organization, the simplest model of price competition, called 'Bertrand's price competition' in honor of its initiator the French mathematician Joseph Bertrand, studies the market of a homogenous good in which a small number of firms simultaneously post a price and commit to sale the quantity of the firm's product that consumers demand given those posted prices. The classical result in the literature on Bertrand's competition is the well known Bertrand's paradox, which says that, if firms are identical, the average cost is constant, and total revenues are bounded, all Nash equilibria in the mixed extension of the pricing game are characterized by two or more firms charging the marginal cost (Harrington, 1989).[1]

With unbounded revenues, there are also mixed strategy equilibria where prices always excess the marginal cost (Baye and Morgan, 1999; Kaplan and Wettstein, 2000). However, such equilibria are ruled out by the usual assumptions on the demand function, namely, continuity and a finite choke-off price. Thus, under reasonable market conditions, the message coming

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[1]If one firm has an absolute cost advantage over its rivals, it prices at the marginal cost of the next to lowest cost firm and captures the entire market. All other firms earn zero profit.

out from Bertrand's competition is that the perfectly competitive outcome, with price equal to marginal cost and zero equilibrium profits, is achieved independently of the number of firms in the market.[2]

In recent years, there has been a renewed interest for examining the Bertrand's paradox under different cost conditions. A remarkable work within this literature is Dastidar (1995), who has shown that with decreasing returns to scale, i.e., with an increasing average cost, the result does not hold. Specifically, Dastidar has proved that, for firms with identical, continuous, and convex cost functions, price competition à la Bertrand typically leads to multiple pure strategy Nash equilibria.[3] Furthermore, if the cost function is sufficiently convex, even the joint profit-maximizing price can be in the range of equilibrium prices; and, it may actually be easier to arrive to that outcome when there are more firms in the market (Dastidar, 2001).

The reason behind the existence of multiple pure strategy equilibria is simple. With decreasing returns to scale, being the only firm charging the lowest price and supplying the whole market leads to lower profits because the average cost increases too fast. Therefore, there is an incentive to join the group. This explains why undercutting the rest is not profitable in equilibrium even if the other firms price above the marginal cost. Hoernig (2002) has also found that the same logic applies in the mixed extension. Consequently, there is a continuum of mixed strategy Nash equilibria with continuous support. Moreover, any finite set of pure equilibrium prices that lead to positive equilibrium profits can be supported in a mixed strategy equilibrium. Unbounded returns are not necessary for this result.

Interestingly, the set of equilibria gets drastically smaller when there is the possibility of limited cooperation among the firms. Indeed, if firms possess an identical and increasing average cost, Bertrand's price competition admits a unique and symmetric coalition-proof Nash equilibrium (Chowdhury and Sengupta, 2004). The equilibrium price is decreasing in the number of firms; and, in the limit, it converges to the competitive price. If firms have asymmetric costs and they share the market according to capacity, i.e., according to the competitive supply of each firm, a coalition-proof Nash equilibrium always exists. Moreover, if firms do not use weakly dominated strategies, the minimum price charged in any of such equilibria is always above the competitive price, but it converges to the marginal cost as the number of firms increases.[4]

Regarding Bertrand's competition with increasing returns to scale, the literature indicates that the existence of a Nash equilibrium, either in pure or mixed strategies, is problematic. On one hand, if the marginal cost is decreasing, Dastidar (2006) has recently shown that, under the usual 'equal sharing' tie-breaking rule, which roughly means that consumers split equally among the firms that charge the lowest price, Bertrand competition does not possess a Nash

[2]If firms cannot observe their rivals' costs, the precision to price slightly below the rivals disappears. Thus, Spulber (1995) showed that all firms pricing above the marginal cost and getting positive expected profits is an equilibrium. However, as the number of firms increases equilibrium prices converge to the average cost.

[3]For firms with asymmetric costs, pure strategy equilibria always exists; it could be unique or non-unique; and in any equilibrium all firms with positive sales charge the same price.

[4]The limiting properties of the set of coalition-proof Nash equilibria are interesting because Novshek and Chowdhury (2003) have shown that the multiplicity of pure strategy Nash equilibria holds even when the market is large. In particular, if the average cost is increasing, they have proved that the limit equilibrium set includes the perfectly competitive price, but it is not a singleton.

equilibrium in pure strategies. The existence of mixed equilibria remains an open question.[5]

On the other hand, when the marginal cost is constant, but there exists an avoidable fixed cost, existing works, including Vives (1999, pg. 118) and Baye and Kovenock (2008), point out that pure strategy Nash equilibria may or may not exist. The reason is firms as usual have the incentive to undercut each other in order to increase their sales; but they may prefer to exit rather than to pay the fixed cost and produce a positive amount of output. Since oligopoly theory is most relevant in markets with significant scale economies, Shapiro (1989, pgs. 344-345) reckoned that the nonexistence of an equilibrium is a serious drawback of the model.

Nonetheless, Bertrand's price competition with a constant marginal cost and an avoidable fixed cost does indeed possess a pure strategy Nash equilibrium when prices vary over a grid. Moreover, for a symmetric duopoly with linear demand, Chaudhuri (1996) has shown that, in the limit, as the size of the grid becomes very small, there is a unique equilibrium that converges to the contestable outcome; that is, in the limit, there is average cost pricing with a single firm supplying the whole market and earning zero profits. This result has been extended later on by Chowdhury (2002) to the case with asymmetric firms, finding among other things that as the size of the grid approaches zero, the equilibrium prices converge to the limit-pricing outcome where the price charged by the most efficient firm is just low enough to prevent entry.

With a sunk entry cost, instead of an avoidable fixed cost, a two stage, simultaneous move game of entry and pricing also shows nonexistence of pure strategy Nash equilibria (Sharkey and Sibley, 1993). By contrast, mixed strategy equilibria always exist. If firms are symmetric, as more firms become potential competitors, the equilibrium price distribution places greater weight on high prices, contradicting the usual intuition from perfect competition. Instead, if entrants face different sunk costs, the equilibrium of the game lead to blockaded entry for higher cost firms (Marquez, 1997).

Surprisingly, the analysis of Bertrand's competition under the more familiar case of variable returns to scale has not received enough attention in the literature. To the best of our knowledge, there are only two papers that deal with this matter. The first article, due to Novshek and Chowdhury (2003), finds that, with a continuous and U-shaped average cost, as the market becomes large, (i.e., as the number of firms increases or, alternatively, as the market size is taken to infinity), the equilibrium set is empty for some parameter values, and it comprises a whole interval of prices for others. The lower bound of this interval is bounded away from the minimum average cost. No conditions are provided to ensure equilibrium existence.

The second article, due to Yano (2006a), studies a pricing game with a more complex set of strategies. Specifically, the strategy of each firm is a paring of a unit price and the set of quantities that the firm is indifferent to sell at that unit price.[6] The unsatisfied demand is then proportional rationed; that means, if the total amount that buyers wish to acquire at a given price is different from that which firms offer to sell at that price, each agent on the long side

[5]Under the less common 'winner-take-all' tie-breaking rule, a zero profit Nash equilibrium exists if and only if the monopoly profit function has an initial break-even price. In addition, if the function is left lower semi-continuous and bounded from above, the zero profit's outcome is unique (Baye and Morgan, 1999).

[6]The firm is indifferent between any two quantities at a given price if they give rise to the same profit.

gets to trade proportionately to the amount that he desires in such a way that the equilibrium between demand and supply is reestablished. Yano argues that, by incorporating this rationing process, the resulting pricing game may be thought of as belonging to the family of Bertrand-Edgeworth price games.[7] Several equilibria arise in this framework, including the equilibrium of the standard Bertrand's price competition and the contestable outcome (see also Yano, 2006b).

Taking Dastidar's (1995) model as the benchmark, in this paper we reexamine Bertrand's price competition under variable returns to scale. Like in Dastidar, we suppose that the total cost function $C(\cdot)$ exhibits an increasing marginal cost. However, following Grossman (1981), we assume that the total cost is the sum of a continuous and convex variable cost, $VC(\cdot)$, and a fixed cost, $F \geq 0$. Telser (1991) calls this type of markets, with U-shaped average cost, 'Viner industries'. Since we do not restrict a priori the nature of the fixed cost, the paper accommodates cases where the fixed cost is (i) completely avoidable, i.e., $C(0) = 0$; (ii) partly avoidable, i.e., $C(0) \in (0, F)$; and (iii) unavoidable, i.e., $C(0) = F$, in which case we are back to Dastidar's (1995) scenery. In contrast with the latter case, the first two situations give rise to discontinuities and non-convexities in the firms' payoff functions, making the analysis of equilibrium existence a nontrivial exercise.

Within the framework briefly depicted above, this paper investigates necessary and sufficient conditions to guarantee the existence of pure strategy Nash equilibria. Remarkably, when the fixed cost is fully avoidable, we find an interesting and unexplored relationship between Bertrand's competition and cost subadditivity.[8] That relationship indicates that the non-subadditivity of the cost function at the output corresponding to the oligopoly break-even price, denoted by $D(p_L(n))$, is sufficient to guarantee that the market supports a (not necessarily symmetric) equilibrium in pure strategies with two or more firms supplying at least $D(p_L(n))$. Conversely, the existence of a pure strategy equilibrium ensures that the cost function is not subadditive at every output greater than or equal to $D(p_L(n))$. As a by-product, the latter implies that the average cost cannot be decreasing over the mentioned range of outputs.

In addition to the previous analysis, under the cost conditions specified before, this work also reexamines the relationship between the existence of a pure strategy equilibrium in Bertrand's competition and of a price-taking or competitive equilibrium in the market. We find that the latter is sufficient but not necessary for the Bertrand price game to possess an equilibrium in pure strategies. In particular, since in our framework the former always exists when the fixed cost is unavoidable, this provides an existence result much simpler than Dastidar (1995).

The rest of the paper is organized as follows. Section 2 describes the model and the equilibrium concept, referred to as Bertrand equilibrium. Section 3 deals with the relationship between price-taking equilibria and Bertrand equilibria. Section 4 contains the main results of the article, linking symmetric and nonsymmetric Bertrand equilibria with cost subadditivity. For expositional convenience, some of the proofs of this section are relegated to the Appendix, which is displayed as usual at the end of the paper. Final remarks are done in Section 5.

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[7] For the difference between Bertrand and Bertrand-Edgeworth competition, see Vives (1999, Chap. 5).

[8] A cost function $C(\cdot)$ is subadditive at $q \in \mathbb{R}$ if the cost of producing q with a single firm is smaller than the sum of the costs of producing it separately with a group of two or more identical firms.

2 The model

Consider the market of a homogenous good, with a unit price P and an aggregate demand $D(P)$. Let $N = \{1, 2, \dots, n\}$, $n \geq 2$, be the set of firms operating in the market. Suppose each firm $i \in N$ competes for the market demand $D(\cdot)$ by simultaneously and independently proposing to the costumers a price p_i from the interval $[0, \infty)$. Let $q_i = q_i(p_i, \mathbf{p}_{-i})$ denote firm i's output supply as a function of $(p_i, \mathbf{p}_{-i})$, where $\mathbf{p}_{-i} = (p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$ is the list of prices chosen by the other firms.

The following assumptions complete the description of the model.

Assumption 1 The aggregate demand $D(\cdot)$ is bounded on $\mathbb{R}_+$; that is, there exist $K > 0$ and $\overline{P} > 0$ such that $D(0) = K$ and $D(P) = 0$ for all $P \geq \overline{P}$. In addition, $D(\cdot)$ is twice continuously differentiable and decreasing on $(0, \overline{P})$; i.e., $\forall P \in (0, \overline{P})$, $D'(P) < 0$.

Assumption 2 For each firm $i \in N$, the production cost associated with any $q_i \in \mathbb{R}_+$ is

$$C(q_i) = \begin{cases} VC(q_i) + F & \text{if } q_i > 0, \\ C(0) & \text{if } q_i = 0, \end{cases}$$

where $F \geq 0$ represents a fixed cost, $C(0) \in [0, F]$, and $VC(\cdot)$ is a variable cost function, which is twice continuously differentiable, increasing and convex on $\mathbb{R}_+$, with $VC(0) = 0$ and $0 \leq VC'(0) < \overline{P}$.

Even though Assumption 2 does not specify the *nature* of the fixed cost and, consequently, the exact value of $C(0)$, in the rest of the paper we consider two possibilities. The first case takes place when F is unavoidable, meaning that $C(0) = F$. In this case, the cost function $C(\cdot)$ is continuous and convex on $\mathbb{R}_+$. The second possibility occurs when F is positive and can be completely or partially avoided by producing no output, so that $C(0) < F$. In contrast with the first case, in the second the cost function $C(\cdot)$ is not only discontinuous at 0, but also non-convex around the origin. As we show in Section 3 these two scenarios result in quite different predictions regarding the existence of equilibria.

Our next assumption determines the individual demand faced by each firm for every possible profile of prices. To do that, we adopt the standard market sharing rule used in the literature on price competition, according to which the market demand is equally split between the firms that charge the lowest price, and the remaining firms sell nothing.[9]

Assumption 3 For each firm $i \in N$ and every $(p_i, \mathbf{p}_{-i}) \in [0, \infty)^n$, the individual demand of i at $(p_i, \mathbf{p}_{-i})$, denoted by $d_i(p_i, \mathbf{p}_{-i})$, is defined as follows:

$$d_i(p_i, \mathbf{p}_{-i}) = \begin{cases} D(p_i) & \text{if } p_i < p_j \ \forall j \in N \backslash \{i\}, \\ \frac{1}{m} D(p_i) & \text{if } p_i \leq p_j \ \forall j \in N \backslash \{i\} \ \& \ p_i = p_{k_t} \ \forall t = 1, \dots, m-1, \\ 0 & \text{if } p_i > p_j \text{ for some } j \in N \backslash \{i\}. \end{cases} \tag{1}$$

[9] For price competition under alternative sharing rules, see among others Baye and Morgan (2002), Dastidar (2006), and a recent article by Hoernig (2007).

As usual in Bertrand's competition, we assume that each firm always meets all the demand at the price it announces. More formally,

Assumption 4 For all $i \in N$, and all $(p_i, \mathbf{p}_{-i}) \in [0, \infty)^n$, $q_i(p_i, \mathbf{p}_{-i}) = d_i(p_i, \mathbf{p}_{-i})$.

Let $H : [0, \overline{P}] \times N \to \mathbb{R}$ be such that, for all $p \in [0, \overline{P}]$ and all $m \in N$,

$$H(p, m) = p \, \frac{D(p)}{m} - C\left(\frac{D(p)}{m}\right).$$

Assumption 5 For each $m \in N$, $H(\cdot\,, m)$ is strictly quasi-concave on $(0, \overline{P})$, with $p^h(m) = \arg \max_{p \in (0, \overline{P})} H(p, m)$; and, for all $m \neq 1$, $0 < H(p^h(m), m) < H(p^M, 1)$, where $p^M = p^h(1)$.

Assumption 5 guarantees that, for every $m \in N$, $H(\,\cdot\,, m)$ has an interior maximum. This is because $H(0, m) = -VC(K/m) - F < 0$ and $H(\overline{P}, m) = -C(0) \leq 0$. In addition, it also ensures that the monopoly receives the greatest maximal benefits.

The model of price competition described above follows Dastidar (1995). The only difference is that in our framework F is a fixed cost which may or may not be avoided by producing zero output. On the contrary, in Dastidar (1995) only unavoidable fixed costs are considered, although it is not explicitly stated in that way. Apart from this, the two models are similar.

Let $\pi_i(p_i, \mathbf{p}_{-i}) = p_i \, d_i(p_i, \mathbf{p}_{-i}) - C(d_i(p_i, \mathbf{p}_{-i}))$ be firm i's profit function. We denote by $G_n = \langle [0, \infty), \pi_i \rangle_{i \in N}$ the price competition game defined by Assumptions 1 – 5. A ***pure strategy Bertrand equilibrium*** (PSBE) for G_n is a profile of prices $(p_i, \mathbf{p}_{-i}) \in [0, \infty)^n$ such that, for each $i \in N$ and all $\hat{p}_i \in [0, \infty)$, $\pi_i(p_i, \mathbf{p}_{-i}) \geq \pi_i(\hat{p}_i, \mathbf{p}_{-i})$. We denote by $\mathcal{B}(G_n)$ the set of all such equilibria, and by $\mathcal{S}(G_n) \subseteq \mathcal{B}(G_n)$ the subset of *symmetric* pure strategy equilibria, where for all $(p_1, \ldots, p_n) \in \mathcal{S}(G_n)$ and all $i, j \in N$, $i \neq j$, $p_i = p_j$.

3 Price-taking equilibrium and Bertrand equilibrium

We begin this section by showing that, independently of the nature of the fixed cost, the existence of a price-taking equilibrium (yet to be defined) in the homogenous good market described in Section 2 is a sufficient condition for a pure strategy Bertrand equilibrium to exist; i.e., it is sufficient for $\mathcal{B}(G_n) \neq \emptyset$.

Let $E_n = \langle N, D(\cdot), C(\cdot) \rangle$ represent the homogenous good market where every firm $i \in N$ maximizes the function $\Pi_i(P, Q_i) = P \, Q_i - C(Q_i)$ with respect to $Q_i \in \mathbb{R}_+$ taking the price $P > 0$ as given. Suppose as before $D(\cdot)$ and $C(\cdot)$ satisfy Assumptions 1 and 2, respectively. Then, a ***price-taking equilibrium*** (PTE) for E_n is a price $P^C \in (0, \overline{P})$ and a profile of outputs $(Q_1^C, \ldots, Q_n^C) \in \mathbb{R}_+^n$ with the property that, for each firm $i \in N$,

$$Q_i^C \in \arg \max_{Q_i \in \mathbb{R}_+} \; \Pi_i(P^C, Q_i), \tag{2}$$

and

$$\sum_{i=1}^{n} Q_i^C = D(P^C). \tag{3}$$

Notice that, by Assumption 2, for all $i \in N$ and any $P^C \in (0, \overline{P})$, $\Pi_i(P^C, Q_i) = P^C\, Q_i - C(Q_i)$ is concave on $\mathbb{R}_{++}$, and $\Pi_i(P^C, 0) = P^C\, 0 - C(0) = -C(0)$. Hence, a unique output $Q_i^C \in \mathbb{R}_+$ satisfying (2) always exists. Moreover, since firms are identical, $Q_1^C = \ldots = Q_n^C$. Denote this common value by Q^C. By equation (3), $Q^C = D(P^C)/n$. Hence, abusing the notation, in what follows we denote a PTE by the pair (P^C, Q^C).

Proposition 1 *If (P^C, Q^C) is a price-taking equilibrium for $E_n = \langle N, D(\cdot), C(\cdot)\rangle$, then $(p_1, \ldots, p_n) = (P^C, \ldots, P^C)$ is a pure strategy Bertrand equilibrium for $G_n = \langle [0, \infty), \pi_i\rangle_{i \in N}$.*

Proof. Let (P^C, Q^C) be a PTE for $E_n = \langle N, D(\cdot), C(\cdot)\rangle$, where $P^C \in (0, \overline{P})$ and $Q^C = D(P^C)/n$. By Assumption 1, $Q^C > 0$. Therefore, (2) implies that $P^C\, Q^C - C(Q^C) \geq P^C\, 0 - C(0) = -C(0)$. Consider the game $G_n = \langle [0, \infty), \pi_i\rangle_{i \in N}$ and the strategy profile $\mathbf{p}^C = (P^C, \ldots, P^C)$. Notice that, for all $i \in N$, $\pi_i(P^C, \ldots, P^C) = P^C\, \frac{D(P^C)}{n} - C\left(\frac{D(P^C)}{n}\right)$. Hence, for all $i \in N$, $\pi_i(P^C, \ldots, P^C) \geq -C(0)$. Suppose, by contradiction, $\mathbf{p}^C \notin \mathcal{B}(G_n)$. Then, there must exist a firm $i \in N$ and a price $\hat{p}_i \in [0, \infty)$ such that $\pi_i(\hat{p}_i, \mathbf{p}_{-i}^C) > \pi_i(P^C, \mathbf{p}_{-i}^C)$. If $\hat{p}_i > P^C$, then $d_i(\hat{p}_i, \mathbf{p}_{-i}^C) = 0$, meaning that $\pi_i(\hat{p}_i, \mathbf{p}_{-i}^C) = \hat{p}_i\, 0 - C(0) = -C(0)$, which stands in contradiction with the fact that $\pi_i(\hat{p}_i, \mathbf{p}_{-i}^C) > \pi_i(P^C, \mathbf{p}_{-i}^C)$. Therefore, $\hat{p}_i < P^C$ and, by (1), $d_i(\hat{p}_i, \mathbf{p}_{-i}^C) = D(\hat{p}_i) > 0$. If $\hat{p}_i = 0$, then $\hat{q}_i = K$ and $\pi_i(\hat{p}_i, \mathbf{p}_{-i}^C) = -C(K) < -C(0)$, a contradiction. Thus, $\hat{p}_i > 0$. Let $\hat{Q}_i = \arg\max_{Q \in \mathbb{R}_+} \Pi_i(\hat{p}_i, Q)$. Note that, since $\hat{p}_i < P^C$, $\Pi_i(\hat{p}_i, \hat{Q}_i) = \max_{Q \in \mathbb{R}_+}\{\hat{p}_i \cdot Q - C(Q)\} \leq \max_{Q \in \mathbb{R}_+}\{P^C \cdot Q - C(Q)\} = \Pi_i(P^C, Q^C)$. However, $\Pi_i(P^C, Q^C) = \pi_i(P^C, \mathbf{p}_{-i}^C) < \pi_i(\hat{p}_i, \mathbf{p}_{-i}^C)$. Therefore, $\max_{Q \in \mathbb{R}_+}\{\hat{p}_i \cdot Q - C(Q)\} < \hat{p}_i \cdot D(\hat{p}_i) - C(D(\hat{p}_i))$, a contradiction. Thus, $(P^C, \ldots, P^C) \in \mathcal{B}(G_n)$. ■

The previous proposition shows that the existence of a price-taking equilibrium in a homogenous good market, with a finite number of identical firms and demand and cost functions that satisfy our assumptions, is a sufficient condition to guarantee the existence of a pure strategy equilibrium in the market when firms compete in prices à la Bertrand, instead of taking the market price as given. A similar result has been previously stated by Vives (1999, pg. 120) for the case where all firms have identical, increasing, smooth and convex cost functions. The contribution of Proposition 1 is to show that that assertion also holds in markets with fixed costs, regardless of whether the fixed cost is unavoidable or (totally or partially) avoidable.[10] Thus, a natural implication is that, in our framework, the continuity and convexity of $C(\cdot)$ at the origin are not required to ensure the validity of the assertion.

Another consequence of Proposition 1 is that, when F is an unavoidable fixed cost, the set of (symmetric) pure strategy Bertrand equilibria is always nonempty.[11]

Corollary 1 *If $C(0) = F$, the set of symmetric pure strategy equilibria $\mathcal{S}(G_n)$ is nonempty.*

Proof. Let $C(0) = F$. Consider the homogenous good market $E_n = \langle N, D(\cdot), C(\cdot)\rangle$ introduced above, where each firm $i \in N$ maximizes $\Pi_i(P, Q_i)$ with respect to $Q_i \in \mathbb{R}_+$ taking the price

[10] This paper allows F to be equal to 0. Thus, Proposition 1 and Corollary 1 below also hold in the more familiar case where there are decreasing returns to scale and no fixed cost.

[11] Actually, as Dastidar (1995) have shown, a whole interval of prices can be typically supported as pure strategy equilibrium. See also Klaus and Brandts (2008) for experimental results.

$P > 0$ as given. Suppose $D(\cdot)$ and $C(\cdot)$ satisfy Assumptions 1 and 2, respectively. We wish to prove that E_n has a PTE.

Fix any price $P \in (0, \overline{P})$ and any firm $i \in N$, and let $Q_i^*(P) = \arg\max_{Q_i > 0} \Pi_i(P, Q_i)$. By Assumption 2, $Q_i^*(P)$ exists and is unique (recall that $\Pi_i(P, \cdot)$ is concave on $\mathbb{R}_{++}$). Moreover, by the first order condition, $Q_i^*(P) = MC^{-1}(P)$, where $MC^{-1}(\cdot)$ denotes the inverse of $VC'(\cdot)$, which exists because $VC'(\cdot)$ is increasing on $\mathbb{R}_+$. Notice that,

$$\Pi_i(P, Q_i^*(P)) = Q_i^*(P)\left[P - \frac{VC(Q_i^*(P))}{Q_i^*(P)}\right] - F > -F = \Pi_i(P, 0),$$

because, by the first order condition, $P = VC'(Q_i^*(P))$ and, by Assumption 2, $VC'(Q_i^*(P)) > \frac{VC(Q_i^*(P))}{Q_i^*(P)}$. Therefore, for every price $P \in (0, \overline{P})$ and every firm $i \in N$, the optimal output supply of i at P is given by $Q_i^*(P) = MC^{-1}(P)$. Since firms are identical, the market supply is $S(P) = \sum_{i \in N} Q_i^*(P) = n\,[MC^{-1}(P)]$. Thus, the equilibrium price is obtained by solving the equation $D(P) = n\,[MC^{-1}(P)]$, which has a solution on $(0, \overline{P})$ due to our assumptions on the demand and cost functions. Denote this value by P^*. It is immediate to see that $(P^*, Q_i^*(P^*))$ constitutes a PTE for E_n. Hence, by Proposition 1, the profile $(p_1, \ldots, p_n) = (P^*, \ldots, P^*) \in \mathcal{S}(G_n)$. ■

Now, the reader may wonder what happens when F is an avoidable fixed cost. Indeed, it is relatively simple to construct examples where neither a price-taking equilibrium nor a pure strategy Bertrand equilibrium exist. Here is one. Let $D(P) = 10 - P$ and $N = \{1, 2\}$. Suppose $C(q_i) = 1/2q_i^2 + F$ if $q_i > 0$, and let $C(0) = 0$ otherwise. If a PTE exists, then (2) and (3) imply that $P^C = 10/3$ and $Q^C = 10/3$. However, $\Pi_i(P^C, Q^C) \geq 0$ $(= \Pi_i(P^C, 0))$ only if $F \leq 50/9\,(\approx 5.55^{'})$. Thus, if $F > 50/9$, the market does not possess a PTE.

Regarding Bertrand equilibria, a pair of prices $(p_1^*, p_2^*) \in [0, \infty)^2$ constitutes a symmetric pure strategy equilibrium if and only if $p_1^* = p_2^*$ and, for all $i \in N$, (a) $H(p_i^*, 2) \geq 0$, and (b) for all $\hat{p}_i < p_i^*$, $H(p_i^*, 2) \geq H(\hat{p}_i, 1)$. Notice that the latter condition requires that $1/2p_i^*(10 - p_i^*) - 3/8(10 - p_i^*)^2 \leq 0$, which is satisfied whenever $p_i^* \leq 30/7\,(\approx 4.2858)$. (See Figure 1 for the case where $F = 6$.) On the other hand, from (a), it follows that $p_i^* \geq 6 - \sqrt{16 - 8/5\,F}$. Hence, a price p_i^* simultaneously satisfying both conditions exists if and only if $F \leq 400/49\,(\approx 8.1633)$.

For instance, when $F = 6$, the price $p_i^* = 4$ is a solution of (a) and (b). Therefore, the profile $(p_1^*, p_2^*) = (4, 4) \in \mathcal{S}(G_2)$. Actually, if the fixed cost $F \leq 50/9$, then the set of symmetric Bertrand equilibria includes the price-taking equilibrium; that is, $(10/3, 10/3) \in \mathcal{S}(G_2)$. However, when $F \in (50/9, 400/49)$, a PTE does not exist, but the game possesses multiple PSBE.[12] Indeed, as Figure 1 illustrates, when $F = 6$ any price between the lower bound $p_L = 6 - \sqrt{32/5}$ (≈ 3.4702) and the upper bound $p_H = 30/7$ satisfies conditions (a) and (b) and, therefore, constitutes a symmetric PSBE.

[12] A similar result appears in Grossman (1981), but associated with a different equilibrium concept. Using a model similar to ours, Grossman has shown that a PTE, if it exists, is a *supply function equilibrium.* However, the latter may exist even if there is no PTE in the market.

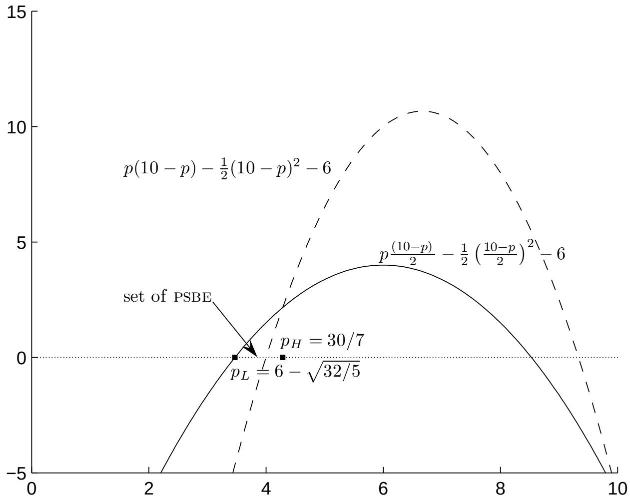

Figure 1: Existence of Bertrand equilibria ($F = 6$)

In addition, it also comes to light from the previous example that, if $F \in \left(\frac{400}{49}, 10\right]$, then a symmetric equilibrium in pure strategies does not exist. For $F = 9$, this is illustrated in Figure 2, where it can be easily seen that, for any price p for which the solid curve $H(p, 2)$ is over the horizontal axis, the dashed curve $H(p, 1)$ lies above. This implies that, whenever both firms choose any price $p \in [0, \overline{P}]$ satisfying the condition $H(p, 2) \geq 0$, there is a deviation $\hat{p}_i < p$ for one firm, say for firm i, such that $H(p, 2) < H(\hat{p}_i, 1)$. In effect, for the case in which $p_{-i} = p_L$, the diagram shows that firm i's strategy $\hat{p}_i < p_L$ dominates p_L, in the sense that

$$\pi_i(\hat{p}_i, p_L) = \hat{p}_i\,(10 - \hat{p}_i) - \frac{1}{2}\,(10 - \hat{p}_i)^2 - 9 >$$

$$> p_L\,\frac{(10 - p_L)}{2} - \frac{1}{2}\left(\frac{10 - p_L}{2}\right)^2 - 9 = \pi_i(p_L, p_L).$$

Therefore, $(p_L, p_L) \notin \mathcal{S}(G_2)$. And, since the same reasoning applies for every price $p \in (p_L, p']$, it follows that $\mathcal{S}(G_2) = \emptyset$. (Notice that $(\overline{P}, \overline{P}) = (10, 10)$ is not an equilibrium either, because any of the two firms can profitably deviate to the monopoly price $p^M = 20/3$, which renders a payoff of $H(p^M, 1) \approx 7.66' > 0 = \pi_i(10, 10)$.)

Finally, observe that when $F = 9$ our example not only fails to possess a symmetric pure strategy equilibrium, but also a PSBE with $p_1 \neq p_2$. To see this, assume, by contradiction, such equilibrium exists. Without loss of generality, suppose that $p_1 < p_2$. Note that $p_1 \leq p^M = 20/3$. Otherwise, firm 1 can profitably deviate to p^M. Then, by Assumption 3, $\pi_1(p_1, p_2) = H(p_1, 1)$ and $\pi_2(p_1, p_2) = 0$. Suppose first $H(p_1, 1) > 0$. Then, $q_1(p_1, p_2) > 0$; and, by continuity of $H(p, 1) = p\,(10 - p) - \frac{1}{2}\,(10 - p)^2 - 9$ in p at p_1, there is a price $p_2' < p_1$ such that $H(p_2', 1) > 0 = \pi_2(p_1, p_2)$, contradicting that p_2 is firm 2's best response to p_1.

Next, observe that, if $(p_1, p_2) \in \mathcal{B}(G_2)$, then $H(p_1, 1)$ cannot be negative. This is because

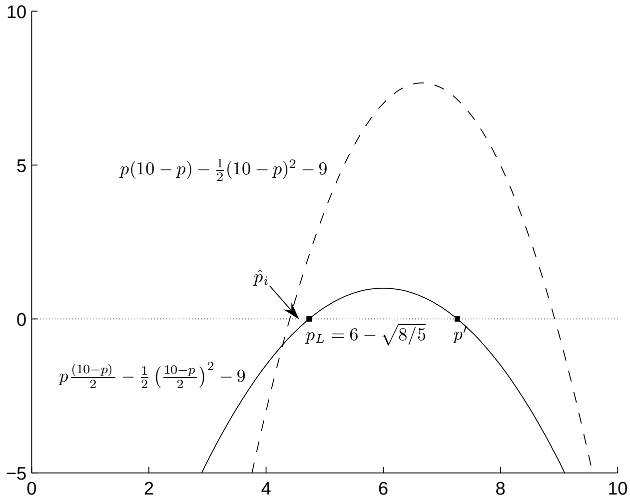

Figure 2: Nonexistence of Bertrand equilibria ($F = 9$)

$\pi_1(10, \hat{p}_2) = 0$ for all $\hat{p}_2 \in [0, \infty)$. Therefore, $H(p_1, 1) = 0$; and, given the shape of $H(\cdot, 1)$ displayed in Figure 2, it has to be that $p_1 = (20 - \sqrt{46})/3$ (≈ 4.4059). If $H(p_2, 1) > 0$, then using the continuity of $H(p, 1)$ in p at p_2, there must be a price $p_1' < p_2$ such that $H(p_1', 1) > 0 = \pi_1(p_1, p_2)$, which would contradict that firm 1 is playing his best response against p_2. Thus, $H(p_2, 1) \leq 0$. But, since $p_2 > p_1$ and $H(p^M, 1) > 0$, this implies $p_2 > p^M$. Hence, firm 1 can profitably deviate to p^M, meaning that (p_1, p_2) is not a PSBE for G_2.[13]

4 Cost subadditivity and Bertrand equilibrium

The example discussed above shows that, if the fixed cost can be completely avoided by producing no output, then depending upon its value the set of pure strategy Bertrand equilibria may be empty. Under a constant marginal cost, this problem has been previously noted by Shapiro (1989), Vives (1999) and Baye and Kovenock (2008), among others.[14] A sensible question to ask is therefore what conditions (if any) prevent this from happening. Finding these conditions will occupy the remainder of the paper.

To begin to analyze this matter requires us to define a key property of the cost function, namely, subadditivity. Following Panzar (1989, pg. 23), we say that a cost function $C(\cdot)$ is ***subadditive*** at $q \in \mathbb{R}_+$ if for *every* list of outputs $q_1, \ldots, q_n$, with $q_i \in \mathbb{R}_+$ and $q_i \neq q$ for all $i = 1, \ldots, n$, it is the case that $C(q) < \sum_{i=1}^n C(q_i)$ whenever $\sum_{i=1}^n q_i = q$.

In words, $C(\cdot)$ is subadditive at q if the cost of producing q with a single firm is smaller than the sum of the costs of producing it separately with a group of two or more identical

[13] Incidentally, note that the example shows that, in our framework, Baye and Morgan's (1999) condition, i.e., the existence of an initial break-even price in the profit function $H(\cdot, 1)$, is not sufficient to guarantee a PSBE.

[14] In fact, Baye and Kovenock (2008) have also shown that, with a constant marginal cost, an avoidable fixed cost may preclude the existence of mixed strategy equilibria as well.

firms. As Baumol (1977) pointed out, subadditivity of the cost function is a necessary and sufficient condition for a natural monopoly to exist. Notice, however, that subadditivity is a local property in that it refers to a particular point on the cost curve. Thus, it is possible for a market to be a natural monopoly for a certain output, but not for others.

When the cost function $C(\cdot)$ is twice continuously differentiable and the marginal cost is increasing, there is a simple necessary condition for subadditivity. In effect, under the cost conditions stated before, any output q is divided in positive portions most cheaply among n identical firms if each firm produces the same amount $q_i = q/n$. Hence, since the *minimized* cost corresponding to output q for a n-firm market is $\sum_{i=1}^{n} C(q_i) = n\,C(q/n)$, it follows that $C(\cdot)$ is subadditive at $q \in \mathbb{R}_+$ only if $C(q) < n\,C(q/n)$. If there are only two firms, then this condition is also sufficient. This is because the requirement embedded in the definition of subadditivity that $q_i \neq q$ for all $i \in N$ implies, when $n = 2$, that $q_i \neq 0$ for all $i = 1, 2$.

We claim now that, if the fixed cost is fully avoidable, then a necessary condition for the existence of a symmetric pure strategy Bertrand equilibrium in the price competition game defined in Section 2 is for the cost function *not* to be subadditive at the output corresponding to the oligopoly break-even price. To formally prove this assertion, the following preliminary results will be useful.

Lemma 1 *For every $m \in N$, there is a price $\hat{p}(m) \in (0, \overline{P})$ such that $H(\hat{p}(m),\, m) = -C(0)$.*

Proof. Fix any $m \in N$. By Assumptions 1 and 2, $H(0, m) = -VC(K/m) - F$. Thus, $H(0, m) < -F$; and, since $C(0) \leq F$, we have that $H(0, m) < -C(0)$. On the other hand, by Assumption 5, $H(p^h(m), m) > 0 \geq -C(0)$. Hence, by the intermediate value theorem, there is a price $\hat{p}(m) \in (0, p^h(m))$ such that $H(\hat{p}(m),\, m) = -C(0)$. ■

Fix $m \in N$ and let $p_L(m) = \min\{\hat{p}(m) \in (0, \overline{P}) : H(\hat{p}(m),\, m) = -C(0)\}$. By Lemma 1, $p_L(m)$ is well defined. By Assumption 1, $D(p_L(m)) > 0$. Suppose now $p_L(m) > p^M$. Then, $m \neq 1$; and, by Assumption 5, $H(\cdot\,, 1)$ is non-increasing at $p_L(m)$; i.e., $\frac{\partial H(p_L(m),1)}{\partial p} \leq 0$. Hence,

$$p_L(m) \geq VC'(D(p_L(m))) - \frac{D(p_L(m))}{D'(p_L(m))}. \tag{4}$$

Similarly, by Assumption 5 and the fact that, by definition, $p_L(m) < p^h(m)$, $H(\cdot\,, m)$ is non-decreasing at $p_L(m)$; i.e., $\frac{\partial H(p_L(m),m)}{\partial p} \geq 0$. Therefore,

$$p_L(m) \leq VC'\left(\frac{D(p_L(m))}{m}\right) - \frac{D(p_L(m))}{D'(p_L(m))}. \tag{5}$$

Finally, since $VC'(\cdot)$ is increasing and $-\frac{D(p_L(m))}{D'(p_L(m))}$ is positive,

$$VC'\left(\frac{D(p_L(m))}{m}\right) - \frac{D(p_L(m))}{D'(p_L(m))} < VC'(D(p_L(m))) - \frac{D(p_L(m))}{D'(p_L(m))};$$

and, by (4) and (5), we get that $p_L(m) < p_L(m)$, a contradiction. Thus, for all $m \in N$, $p_L(m) \leq p^M$.

Lemma 2 *For all $p < p^M$, $H(p,1) - H(p,n) = 0$ implies that* $\frac{\partial[H(p,1)-H(p,n)]}{\partial p} > 0$.

Proof. For every price $p < p^M$, we have that

$$H(p,1) - H(p,n) = \frac{n-1}{n}\, p\, D(p) - VC(D(p)) + VC\left(\frac{D(p)}{n}\right). \tag{6}$$

Taking the derivative of (6) with respect to p,

$$\begin{aligned}\frac{\partial[H(p,1)-H(p,n)]}{\partial p} = \frac{n-1}{n}\, D(p) + \\ + \left[p - VC'(D(p))\right] D'(p) - \left[p - VC'\left(\frac{D(p)}{n}\right)\right] \frac{D'(p)}{n}.\end{aligned} \tag{7}$$

Consider any price $p \in [0, p^M)$ with the property that $H(p,1) - H(p,n) = 0$. Then,

$$\frac{n-1}{n}\, p\, D(p) = VC(D(p)) - VC\left(\frac{D(p)}{n}\right). \tag{8}$$

By convexity of $VC(\cdot)$,

$$VC(D(p)) - VC\left(\frac{D(p)}{n}\right) < \frac{n-1}{n}\, D(p)\, VC'(D(p)), \tag{9}$$

and

$$VC(D(p)) - VC\left(\frac{D(p)}{n}\right) > \frac{n-1}{n}\, D(p)\, VC'\left(\frac{D(p)}{n}\right). \tag{10}$$

Thus, combining (8) and (9), we have that $p < VC'(D(p))$; and, from the expressions in (8) and (10), it also follows that $p > VC'\left(\frac{D(p)}{n}\right)$. Therefore, since $(n-1)/n\, D(p) > 0$ and $D'(p) < 0$, the right hand side of (7) is greater than zero; i.e., $\frac{\partial[H(p,1)-H(p,n)]}{\partial p} > 0$. ■

Lemma 3 *If $\mathcal{S}(G_n) \neq \emptyset$, then the strategy profile $(p_1, \ldots, p_n) = (p_L(n), \ldots, p_L(n)) \in \mathcal{S}(G_n)$.*

Proof. Suppose, by contradiction, the strategy profile $(p_1, \ldots, p_n) = (p_L(n), \ldots, p_L(n)) \notin \mathcal{S}(G_n)$. Then, there must be a firm $i \in N$ and a price $\tilde{p}_i < p_L(n)$ such that $\pi_i(\tilde{p}_i, (p_L(n))_{-i}) > \pi_i(p_L(n), (p_L(n))_{-i})$, where $(p_L(n))_{-i}$ denotes the sub-profile of prices in which everybody except firm i chooses $p_L(n)$. Notice that $\pi_i(\tilde{p}_i, (p_L(n))_{-i}) = H(\tilde{p}_i, 1)$ and $\pi_i(p_L(n), (p_L(n))_{-i}) = H(p_L(n), n)$. Thus, $H(\tilde{p}_i, 1) - H(p_L(n), n) > 0$. Moreover, since $H(p_L(n), n) = -C(0)$ and $H(\tilde{p}_i, n) < -C(0)$,[15] it also follows that $H(\tilde{p}_i, 1) - H(\tilde{p}_i, n) > 0$. Therefore, given that $H(0,1) - H(0,n) = -VC(K) + VC(K/n) < 0$ and $H(\cdot, 1) - H(\cdot, n)$ is continuous on $[0, \tilde{p}_i]$, there must be a price $p' \in (0, \tilde{p}_i)$ such that $H(p', 1) - H(p', n) = 0$.

Next, recall that, by hypothesis, $\mathcal{S}(G_n) \neq \emptyset$. That is, there is a price $p^* \in (p_L(n), p^M)$ such that $H(p^*, n) \geq -C(0)$ and, for all $p < p^*$, $H(p,1) - H(p^*, n) \leq 0$. Since p can be chosen

[15] Note that $H(\tilde{p}_i, n) \neq -C(0)$, because $\tilde{p}_i < p_L(n)$ and, by definition, $p_L(n)$ is the smallest price for which $H(\cdot, n)$ equals $-C(0)$. On the other hand, since $H(0,n) < -C(0)$, $H(\tilde{p}_i, n)$ cannot be greater than $-C(0)$. Otherwise, there would be a price $p \in (0, \tilde{p}_i)$ with the property that $H(p,n) = -C(0)$, which again contradicts the definition of $p_L(n)$. Thus, $H(\tilde{p}_i, n) < -C(0)$.

arbitrarily close to p^*, by continuity, it must be that $H(p^*,1) - H(p^*,n) \leq 0$. On the other hand, by Assumption 5, $H(p^M,1) - H(p^M,n) > 0$. So, there must be a price $p'' \in (p_L(n), p^M)$ such that $H(p'',1) - H(p'',n) = 0$.

In summary, if $\mathcal{S}(G_n) \neq \emptyset$ and $(p_L(n),\ldots,p_L(n)) \notin \mathcal{S}(G_n)$, the previous two paragraphs indicate that the curves $H(\cdot\,,1)$ and $H(\cdot\,,n)$ must intersect each other at least twice on $(0,p^M)$. Therefore, in order to show that $(p_L(n),\ldots,p_L(n))$ is indeed a symmetric pure strategy Bertrand equilibrium for G_n, it is enough to prove that there is only one such intersection; i.e., it is sufficient to show that there is a unique price $p \in (0,p^M)$ for which $H(p,1) - H(p,n) = 0$.

Without of generality, assume that there is a pair of prices $p^\alpha, p^\beta \in (0,p^M)$, $p^\alpha < p^\beta$, such that $H(p^\alpha,1) - H(p^\alpha,n) = 0$ and $H(p^\beta,1) - H(p^\beta,n) = 0$. Notice that, by Lemma 2, for $\epsilon^1 > 0$ small enough, $H(p^\alpha,1) - H(p^\alpha,n) = 0$ implies that $H(p^\alpha + \epsilon^1,1) - H(p^\alpha + \epsilon^1,n) > 0$. In the same way, by Lemma 2, for $\delta > 0$ small enough, $H(p^\beta,1) - H(p^\beta,n) = 0$ implies that $H(p^\beta - \delta,1) - H(p^\beta - \delta,n) < 0$. Hence, since $H(\cdot\,,1) - H(\cdot\,,n)$ is continuous on $(0,p^M)$, there must be a price $p^{\alpha+1} \in (p^\alpha,p^\beta)$ such that $H(p^{\alpha+1},1) - H(p^{\alpha+1},n) = 0$. Repeating the previous argument, for $\epsilon^2 > 0$ small enough, $H(p^{\alpha+1},1) - H(p^{\alpha+1},n) = 0$ implies that $H(p^{\alpha+1} + \epsilon^2,1) - H(p^{\alpha+1} + \epsilon^2,n) > 0$. Hence, there must be a price $p^{\alpha+2} \in (p^{\alpha+1},p^\beta)$ such that $H(p^{\alpha+2},1) - H(p^{\alpha+2},n) = 0$.

Repeating these steps over and over again, we get a sequence of prices $\{p^{\alpha+s}\}_{s=1}^{\infty} \subset (p^\alpha,p^\beta)$ with the property that $H(p^{\alpha+s},1) - H(p^{\alpha+s},n) = 0$ for all $s = 1,\ldots,\infty$. Observe that, by construction, each term $p^{\alpha+s}$ of the sequence is closer to p^β than what it was $p^{\alpha+s-1}$. Therefore, by Lemma 2, for some $s \geq 1$ sufficiently high, there must exist $\epsilon \in (0,\delta)$ and a price $\bar{\bar{p}} \in (p^{\alpha+s} + \epsilon, p^\beta - \epsilon)$ such that $H(\bar{\bar{p}},1) - H(\bar{\bar{p}},n) > 0$ and $H(\bar{\bar{p}},1) - H(\bar{\bar{p}},n) < 0$, which provides the desired contradiction. ∎

Now, we are ready to state and prove Proposition 2.

Proposition 2 *Suppose $C(0) = 0$. If the set of symmetric pure strategy equilibrium $\mathcal{S}(G_n)$ is nonempty, then the cost function $C(\cdot)$ is not subadditive at $D(p_L(n))$.*

Proof. Suppose, by contradiction, that $C(\cdot)$ is subadditive at $D(p_L(n))$. (Recall that, by definition of $p_L(n)$, $D(p_L(n)) > 0$.) Then, it must be that producing $D(p_L(n))$ with a single firm is cheaper than producing it with n identical firms; that is,

$$C(D(p_L(n))) < n\, C\left(\frac{D(p_L(n))}{n}\right). \tag{11}$$

Adding the term $-p_L(n)\, D(p_L(n))$ to both sides of (11), it follows that

$$-p_L(n)\, D(p_L(n)) + C(D(p_L(n))) < -p_L(n)\, D(p_L(n)) + n\, C\left(\frac{D(p_L(n))}{n}\right),$$

which can be rewritten as

$$p_L(n)\, D(p_L(n)) - C(D(p_L(n))) > n \left[p_L(n)\, \frac{D(p_L(n))}{n} - C\left(\frac{D(p_L(n))}{n}\right)\right]. \tag{12}$$

By definition of $p_L(n)$, the right hand side of (12) is equal to $-n\,C(0)$. Hence, if $C(0) = 0$, then (12) implies that $p_L(n)\,D(p_L(n)) - VC(D(p_L(n))) - F > 0$. By continuity of $p\,D(p) - VC(D(p)) - F$ in p at $p_L(n)$, there is a price $p' < p_L(n)$ such that $p'\,D(p') - VC(D(p')) - F > 0$. Fix any firm $i \in N$, and consider firm i's strategy $p'_i = p'$. By Assumption 3, $\pi_i(p'_i, (p_L(n))_{-i}) = p'_i\,D(p'_i) - VC(D(p'_i)) - F$. Hence, $\pi_i(p'_i, (p_L(n))_{-i}) > 0$. On the other hand, $\pi_i(p_L(n), \ldots, p_L(n)) = H(p_L(n), n) = 0$. Thus, firm i can profitably deviate at $(p_L(n), (p_L(n))_{-i})$, from $p_L(n)$ to p'_i, contradicting that, by Lemma 3, the profile $(p_L(n), (p_L(n))_{-i}) \in \mathcal{S}(G_n)$. Therefore, $C(\cdot)$ is not subadditive at $D(p_L(n))$. ■

Proposition 2 formalizes the intuitive idea that, if the fixed cost is avoidable (or, there is no fixed cost at all), then a necessary condition for the existence of a symmetric pure strategy Bertrand equilibrium in a homogenous good market is for the market *not* to be a natural monopoly at the output corresponding to the oligopoly break-even price. Unfortunately, this result does not hold if $C(0) \neq 0$. To see this, suppose that $n = 3$ and $D(P) = 10 - P$, and let $C(q_i) = \frac{3}{22}q_i^2 + \frac{15}{2}$, if $q_i > 0$, and $C(0) = \frac{15}{2}$ otherwise. Routine calculations show that $p_L(3) = \frac{10}{23}$ and $H(10/23, 1) \approx -15.82$. Hence, $(p_1, p_2, p_3) = \left(\frac{10}{23}, \frac{10}{23}, \frac{10}{23}\right)$ is a symmetric PSBE. However, $C(\cdot)$ is subadditive at $D(10/23)$, because $C(D(10/23)) \approx 19.98$, $2\,C(D(10/23)/2) \approx 21.24$, and $3\,C(D(10/23)/3) \approx 26.66$.

Inspired by the example examined at the end of Section 3, where the lack of a symmetric pure strategy equilibrium and the subadditivity of the cost function at $D(p_L)$ occur for the same range of values of F, (namely, for all $F \in \left(\frac{400}{49}, 10\right]$),[16] a natural and interesting question to ask is whether or not the converse of Proposition 2 holds. As we state in Proposition 3, if there is a price-taking equilibrium in the market, then the answer to this question is obviously affirmative, simply because Proposition 1 ensures that the set of symmetric pure strategy equilibria is always nonempty. More interestingly, it also holds in a duopoly, independently of the nature of the fixed cost (i.e., regardless of the value of $C(0)$). The reason behind this last result can be found in the following lemma.

Lemma 4 *If* $C(D(p_L(n))) \geq n\,C\left(\frac{D(p_L(n))}{n}\right)$, *then the strategy profile* $(p_1, \ldots, p_n) = (p_L(n), \ldots, p_L(n))$ *constitutes a pure strategy Bertrand equilibrium for* G_n.

Proof. Suppose, by contradiction, $(p_L(n), \ldots, p_L(n)) \notin \mathcal{S}(G_n)$. Then, there must be a price $\tilde{p} \in (0, p_L(n))$ such that $H(\tilde{p}, 1) > -C(0) = H(p_L(n), n)$. By definition of $p_L(n)$, $D(p_L(n)) > 0$. Thus, the hypothesis in Lemma 4, i.e., $C(D(p_L(n))) \geq n\,C\left(\frac{D(p_L(n))}{n}\right)$, can be rewritten as

$$VC(D(p_L(n))) \geq n\,VC\left(\frac{D(p_L(n))}{n}\right) + (n-1)F. \tag{13}$$

Using the definition of $p_L(n)$, it is easy to see that the right hand side of (13) is equal to

$$p_L(n)\,D(p_L(n)) - F + n\,C(0). \tag{14}$$

[16]In the example in question, the cost function is subadditive at $D(p_L) = 10 - (6 - \sqrt{16 - 8/5F})$ if and only if $VC(D(p_L)) < 2VC(D(p_L)/2) - F$. Routine calculations show that this inequality is satisfied if and only if $F \in \left(\frac{400}{49}, 10\right]$.

Hence, substituting (14) into (13), it follows that $H(p_L(n), 1) \leq -C(0)$. However, this contradicts that, by Assumption 5, $H(\cdot, 1)$ is quasi-concave on $(0, \overline{P})$, because $p_L(n) \in (\tilde{p}, p^M)$ and $H(p_L(n), 1) \leq -C(0) < \min\{H(\tilde{p}, 1),\, H(p^M, 1)\}$. ■

Proposition 3 *Suppose that either (P^C, Q^C) is a price-taking equilibrium for $E_n = \langle N, D(\cdot), C(\cdot)\rangle$, or that there are only two firms in the market. Then, if $C(\cdot)$ is not subadditive at $D(p_L(n))$, the set of symmetric pure strategy equilibria $\mathcal{S}(G_n)$ is nonempty.*

Proof. If (P^C, Q^C) is a PTE for $E_n = \langle N, D(\cdot), C(\cdot)\rangle$, then the desired result follows from Proposition 1. On the other hand, if $n = 2$, then $C(\cdot)$ is not subadditive at $D(p_L(2))$ if and only if $C(D(p_L(2))) \geq 2\,C\left(\frac{D(p_L(2))}{2}\right)$. Hence, by Lemma 4, $(p_L(2), p_L(2)) \in \mathcal{S}(G_2)$. ■

In short, Proposition 3 tells us that, if a homogenous good market is not a natural monopoly and either, there is a price-taking equilibrium, or there are only two firms, then the market supports a symmetric pure strategy equilibrium where firms compete in prices à la Bertrand and supply a total output which leaves each of them indifferent between staying in operation and exit the market. In particular, this holds when the market has an unavoidable fixed cost, because in that case a price-taking equilibrium always exists.

By contrast, if there are more than two firms and the fixed cost is completely or partially avoidable, then the converse of Proposition 2 is not true. To illustrate this, consider again the demand and cost function corresponding to the example analyzed in Section 3. Assume that $n = 5$, and let $F = 4.3$. Then, $p_L(5) \approx 4.3983$ and $H(p_L(5), 1) \approx 4.65 > 0 = H(p_L(5), 5)$. Therefore, $(p_1, \ldots, p_5) = (p_L(5), \ldots, p_L(5))$ is not a (symmetric) PSBE for G_5; and, by Lemma 3, we can conclude that $S(G_5) = \emptyset$. However, it is easy to verify in this numerical example that $C(\cdot)$ is not subadditive at $D(p_L(5)) \approx 5.6017$. Indeed, producing $D(p_L(5))$ with a single firm generates a cost equal to $C(D(p_L(5))) \approx 19,9895$, whereas producing it with two identical firms costs $2 \cdot C(D(p_L(5))/2) \approx 16.4447$.

So, is there something to say about Bertrand's price competition when the n-firm market is not a natural monopoly and there is no price-taking equilibrium? Indeed, we show next that, if the fixed cost is avoidable, then the non-subadditivity of the cost function $C(\cdot)$ at $D(p_L(n))$ is sufficient to guarantee that the market supports a (not necessarily symmetric) pure strategy Bertrand equilibrium where two or more identical firms jointly supply at least $D(p_L(n))$. And, conversely, the existence of a pure strategy Bertrand equilibrium ensures that the market is not a natural monopoly at every output greater than or equal to $D(p_L(n))$. In particular, the latter implies that the average cost cannot be decreasing on $[D(p_L(n)), K)$, (see Corollary 2 below and the discussion following this result).[17]

Theorem 1 *Suppose $C(0) = 0$. If the cost function $C(\cdot)$ is not subadditive at $D(p_L(n))$, then there exist a pure strategy Bertrand equilibrium $(p_1, \ldots, p_n) \in \mathcal{B}(G_n)$ where $\sum_{i \in N} q_i(p_1, \ldots, p_n) \geq D(p_L(n))$. Conversely, if a pure strategy Bertrand equilibrium exists, then the cost function $C(\cdot)$ is not subadditive on the interval $[D(p_L(n)), K)$.*

[17] An example where under increasing returns to scale and equal sharing rule neither pure nor mixed strategy equilibria exist is exhibited in Hoernig (2007, pg. 582).

Proof. See the Appendix. ■

Given any output $\overline{q} > 0$, the average cost at $\overline{q}$ is defined as $AC(\overline{q}) = C(\overline{q})/\overline{q}$. The average cost function $AC(\cdot)$ is decreasing at $\overline{q}$ if there exists a $\delta > 0$ such that for all $q', q'' \in (\overline{q} - \delta, \overline{q} + \delta)$, with $q' < q''$, $AC(q'') < AC(q')$. Additionally, $AC(\cdot)$ is said to decrease *through* $\overline{q}$ if for all $q', q'' \in (0, \overline{q}]$, with $q' < q''$, $AC(q'') < AC(q')$, (Panzar, 1989, pg. 24). If, like in our case, $C(\cdot)$ is twice continuously differentiable on $\mathbb{R}_{++}$, then $AC(\cdot)$ is decreasing at $\overline{q}$ if $\frac{\partial AC(\overline{q})}{\partial q} < 0$; and $AC(\cdot)$ is decreasing *through* $\overline{q}$ if for all $q' \in (0, \overline{q}]$, $\frac{\partial AC(q')}{\partial q} < 0$, (i.e., if $AC(\cdot)$ is decreasing on $(0, \overline{q}]$).

Lemma 5 *If the average cost $AC(\cdot)$ is decreasing through q, then the cost function $C(\cdot)$ is subadditive at q, but not conversely.*

Proof. The proof is based on Panzar (1989, pg. 25). Fix any $q > 0$ and assume $AC(\cdot)$ is decreasing through q. Consider any division $q_1, \ldots, q_n$ of q, with the property that (i) $\forall i \in N$, $0 \leq q_i < q$, and (ii) $\sum_{i \in N} q_i = q$. Let $N^+ = \{i \in N : q_i > 0\}$. Then, for all $i \in N^+$, $AC(q) < AC(q_i)$, which is equivalent to $C(q_i) > (q_i/q) \cdot C(q)$. Summing over N^+, we have $\sum_{i \in N^+} C(q_i) > C(q)$. Therefore, since $C(0) \geq 0$, it follows that $\sum_{i \in N} C(q_i) > C(q)$. Finally, since $q_1, \ldots, q_n$ was arbitrarily chosen, this implies that $C(\cdot)$ is subadditive at q.

To show that subadditivity does not imply decreasing average costs, consider the cost function $C(q) = 1/2 \cdot q^2 + 100$ for all $q \geq 0$. It is easy to see that $AC(\cdot)$ is not decreasing at $q = 15$. However, if $n = 2$, then $C(\cdot)$ is subadditive at 15. ■

Corollary 2 *Suppose $C(0) = 0$. The set of pure strategy Bertrand equilibrium $\mathcal{B}(G_n)$ is nonempty only if the average cost $AC(\cdot)$ is not decreasing on $[D(p_L(n)), K)$.*

Proof. Immediate from Theorem 1 and Lemma 5. ■

The second part of Theorem 1 and its implication in Corollary 2 are closely related with Dastidar's (2006) Proposition 3, which says that the set of Bertrand equilibria $\mathcal{B}(G_n)$ is nonempty only if $C(\cdot)$ is not concave. Hence, before closing this section, it may be worthy to underline some differences between these results.

First of all, let's emphasize that the necessary condition for equilibrium existence stated in the second part of Theorem 1 considerably sharpens Dastidar's (2006) condition, because concavity implies subadditivity, but not conversely. Thus, we could have a cost function which is non-concave and subadditive at the same time. A function like that would violate our necessary condition for existence, whereas it wouldn't do so with Dastidar's. Secondly, in Theorem 1 we allow for avoidable fixed costs and, therefore, for discontinuities in the cost function around the origin. On the contrary, in Dastidar (2006) the cost function is continuous and $F = 0$. Finally, Theorem 1 provides not only a necessary condition for $\mathcal{B}(G_n) \neq \emptyset$, but also a sufficient condition. Instead, Dastidar (2006) only gives a necessary condition.

5 Concluding remarks

The main conclusions of this paper can be summarized by restating Propositions 1, 2 and 3, and Theorem 1. By looking at Proposition 1 we see that, in a market with convex variable costs and fixed costs, the existence of a price-taking equilibrium is a sufficient but not a necessary condition for a pure strategy Bertrand equilibrium to exist. That means it may be perfectly the case that a PTE does not exist, while the set of symmetric PSBE is nonempty, (see, for instance, the example at the end of Section 3).

The nonexistence of a price-taking equilibrium in markets with convex variable costs and fixed costs has been examined by the literature on 'empty-core markets', (Telser, 1991). It has also appeared in Grossman (1981), who studied supply function equilibria for oligopolies with avoidable fixed costs and convex variable costs. Our results in Section 3 are to some extent similar to Grossman's, because he has shown that in a setting similar to ours supply function equilibria may exist even when a PTE does not. However, neither Telser nor Grossman have addressed the existence of Bertrand equilibria in those markets.

As we show in Section 4, the existence of a symmetric PSBE when variable costs are convex and the fixed cost is avoidable is related with the subadditivity of the cost function at the oligopoly break-even price $p_L(n)$. In Proposition 2, we demonstrate that, if $C(0) = 0$ and a symmetric PSBE exists, then firms' cost functions cannot be subadditive at $D(p_L(n))$. This is equivalent to say that, under the previous conditions, the market cannot be a natural monopoly when firms break-even and demand is in equilibrium. As Proposition 3 points out, the reverse of that statement is also true if there are only two firms in the market, or if there is a PTE. Unfortunately, numerical examples show that it does not hold in other situations.

The last result of this work, and perhaps the most important, is Theorem 1, which generalize Propositions 2 and 3 to cases where equilibria are not symmetric, there are no PTE and $n > 2$. In short, Theorem 1 says that, when the fixed cost is fully avoidable and the cost function $C(\cdot)$ is not subadditive at $D(p_L(n))$, there always exists a Bertrand equilibrium in pure strategies, though it need not be a symmetric one. Conversely, if a PSBE exists, then $C(\cdot)$ cannot be subadditive for all quantities greater than or equal to $D(p_L(n))$.

The results of this article relate the existence of Bertrand equilibrium with the nonexistence of natural monopoly. By doing so, the paper explores a relationship which has not been previously considered in the literature. Our findings can also be taken as a contribution to the theory of endogenous industry structure. This is because $p_L(n)$ is typically increasing in the number of firms that operate in the market. Hence, under variable returns to scale, it is possible that a cost function be subadditive at $D(p_L(n+1))$ and not at $D(p_L(n))$. In that case, a market with n firms or less might have a symmetric PSBE, but a market with $n+1$ firms or more might not. That could be useful to figure out the maximum number of active firms that a homogenous good market can support under price competition. The analysis of this conjecture and the study of more general forms of non-convexities and discontinuities of the cost function and their impact on Bertrand's competition are left for a future research.

6 Appendix: Proof of Theorem 1

In order to prove Theorem 1, first we show the following auxiliary result:

Lemma 6 *For all* $p < \overline{P}$, $\frac{\partial[H(p,1) - m\,H(p,m)]}{\partial p} > 0$.

Proof. For every price $p < \overline{P}$, we have that $H(p,1) - m\,H(p,m) = -VC(D(p)) - F + m\,VC\left(\frac{D(p)}{m}\right) + m\,F$. Taking the derivative with respect to p,

$$\frac{\partial[H(p,1) - m\,H(p,m)]}{\partial p} = D'(p)\left[VC'\left(\frac{D(p)}{m}\right) - VC'(D(p))\right], \tag{15}$$

which is positive because $D'(p) < 0$ and $VC'(\cdot)$ is increasing. ■

Proof of Theorem 1. Assume the cost function $C(\cdot)$ is not subadditive at $D(p_L(n))$. Then, there must be a $m \in \{2, \ldots, n\}$ such that

$$C(D(p_L(n))) \geq m\,C\left(\frac{D(p_L(n))}{m}\right). \tag{16}$$

If $m = n$, we are done. By Lemma 4, $(p_1, \ldots, p_n) = (p_L(n), \ldots, p_L(n)) \in \mathcal{S}(G_n) \subseteq \mathcal{B}(G_n)$. Moreover, $\sum_{i \in N} q_i(p_1, \ldots, p_n) = n\,\frac{D(p_L(n))}{n} = D(p_L(n))$. So, suppose that

$$C(D(p_L(n))) < n\,C\left(\frac{D(p_L(n))}{n}\right). \tag{17}$$

Adding the term $-p_L(n)\,D(p_L(n))$ to both sides of (16), it follows that $-p_L(n)\,D(p_L(n)) + C(D(p_L(n))) \geq -p_L(n)\,D(p_L(n)) + m\,C\left(\frac{D(p_L(n))}{m}\right)$, which implies that

$$H(p_L(n),1) \leq m\,H(p_L(n),m). \tag{18}$$

Following the same steps, it is easy to see from (17) that

$$H(p_L(n),1) > n\,H(p_L(n),n). \tag{19}$$

Therefore, since $C(0) = 0$, combining (18) and (19), we get that both $H(p_L(n),1) > 0$ and $m\,H(p_L(n),m) > 0$.

Recall that, by Lemma 1, there is a price $p(m) \in (0, p^h(m))$ such that $H(p(m),m) = 0$. Since $p_L(m)$ is the smallest of such prices, and $H(p_L(n),m) > 0$ and $H(0,m) < 0$, it follows that $p_L(m) < p_L(n)$. Indeed, $p_L(m)$ is also smaller than p^M, because Assumption 5 implies that $p_L(n) \leq p^M$. Suppose, by contradiction, there is a price $\hat{p} < p_L(m)$ such that

$$H(\hat{p},1) > H(p_L(m),m). \tag{20}$$

Since $H(\cdot\,,1)$ is quasi-concave on $(0,\overline{P})$, $p_L(m) \in (\hat{p}, p^M)$ implies that $H(p_L(m),1) \geq \min\{H(\hat{p},1), H(p^M,1)\} = H(\hat{p},1)$. Hence, using (20), $H(p_L(m),1) > H(p_L(m),m)$. Moreover, since $H(p_L(m), m) = 0$, $m\, H(p_L(m), m) = H(p_L(m), m)$. Therefore,

$$H(p_L(m), 1) > m\, H(p_L(m), m). \tag{21}$$

Given that $H(\cdot\,, 1) - m\, H(\cdot\,, m)$ is continuous on $[0, \overline{P})$, the expressions in (18) and (21) imply that there is a price $p^\alpha \in (p_L(m), p_L(n)]$ such that

$$H(p^\alpha, 1) - m\, H(p^\alpha, m) = 0. \tag{22}$$

Thus, by Lemma 6 and (22), there must exist $\epsilon > 0$ small enough with the property that $H(p^\alpha - \epsilon, 1) - m\, H(p^\alpha - \epsilon, m) < 0$. But then, using (21) once again, it follows that there is a price $p^{\alpha+1} \in (p_L(m), p^\alpha - \epsilon)$ such that $H(p^{\alpha+1}, 1) - m\, H(p^{\alpha+1}, m) = 0$. And repeating the argument over and over again, we get a sequence of prices $\{p^{\alpha+s}\}_{s=1}^{\infty} \subset (p_L(m), p^\alpha)$ with the property that for all $s = 1, \ldots, \infty$,

$$H(p^{\alpha+s}, 1) - m\, H(p^{\alpha+s}, m) = 0. \tag{23}$$

Notice that each term $p^{\alpha+s}$ of the sequence is closer to $p_L(m)$ than what it was $p^{\alpha+s-1}$. Therefore, invoking Lemma 6 together with the expressions in (21) and (23), we conclude that for some $s \geq 1$ sufficiently high, there must exist $\epsilon > 0$ and a price $\bar{\bar{p}} \in (p_L(m), p^{\alpha+s} - \epsilon)$ for which $H(\bar{\bar{p}}, 1) - m\, H(\bar{\bar{p}}, m)$ must simultaneously be positive and negative, a contradiction. This contradiction was obtained by assuming the existence of a price $\hat{p} < p_L(m)$ that verifies (20). Hence, for all $\hat{p} < p_L(m)$, $H(\hat{p}, 1) \leq 0 = H(p_L(m), m)$.

Let $p_L(m^*) \equiv \min\{p_L(s), s \in \{2, \ldots, n\}\}$. Clearly, $p_L(m^*) \leq p_L(m)$. Thus, since by definition $H(p_L(m^*), m^*) = 0$, for all $\hat{p} \leq p_L(m^*)$, $H(\hat{p}, 1) \leq 0 = H(p_L(m^*), m^*)$. Next, suppose, by contradiction, there is a $s \in \{m^* + 1, \ldots, n\}$ such that $H(p_L(m^*), s) > 0 = H(p_L(m^*), m^*)$. Since $H(0, s) < 0$ and $H(\cdot\,, s)$ is continuous on $[0, \overline{P})$, there must exist $p' \in (0, p_L(m^*))$ such that $H(p', s) = 0$, contradicting the definition of $p_L(m^*)$. Therefore, for all $s \in \{m^* + 1, \ldots, n\}$, $H(p_L(m^*), s) \leq 0$.

Finally, we claim that the strategy profile $\mathbf{p} = (p_1, \ldots, p_n) \in [0, \overline{P})^n$, with the property that (i) for all $i = 1, \ldots, m^*$, $p_i = p_L(m^*)$, and (ii) for all $j = m^* + 1, \ldots, n$, $p_j > p_L(m^*)$, constitutes a PSBE for G_n. Indeed, if $i \in \{1, \ldots, m^*\}$, then $\pi_i(p_i, \mathbf{p}_{-i}) = H(p_L(m^*), m^*) = 0$. Consider a deviation $\hat{p}_i \neq p_i$ for firm i. If $\hat{p}_i > p_i$, then $\pi_i(\hat{p}_i, \mathbf{p}_{-i}) = 0$. Instead, if $\hat{p}_i < p_i$, then $\pi_i(\hat{p}_i, \mathbf{p}_{-i}) = H(\hat{p}_i, 1) \leq 0$, where the last inequality follows from the fact that, according with the analysis in the previous paragraph, for all $\hat{p} \leq p_L(m^*)$, $H(\hat{p}, 1) \leq 0$.

On the other hand, if $i \in \{m^* + 1, \ldots, n\}$, then $\pi_i(p_i, \mathbf{p}_{-i}) = 0$. Once again, consider a deviation $\hat{p}_i \neq p_i$ for firm i. If $\hat{p}_i > p_L(m^*)$, then $\pi_i(\hat{p}_i, \mathbf{p}_{-i}) = 0$. If $\hat{p}_i < p_L(m^*)$, then $\pi_i(\hat{p}_i, \mathbf{p}_{-i}) = H(\hat{p}_i, 1) \leq 0$. Lastly, if $\hat{p}_i = p_L(m^*)$, then $\pi_i(\hat{p}_i, \mathbf{p}_{-i}) = H(p_L(m^*), m^* + 1)$, which we have already shown is smaller than or equal to 0. Therefore, $\mathbf{p} = (p_1, \ldots, p_n) \in \mathcal{B}(G_n)$. And, since $p_L(m^*) \leq p_L(n)$, $\sum_{i \in N} q_i(p_1, \ldots, p_n) = m^* \frac{D(p_L(m^*))}{m^*} \geq D(p_L(n))$.

Now, let's prove the second part of Theorem 1. That is, let's show that if $\mathcal{B}(G_n) \neq \emptyset$, then the assertion "the cost function $C(\cdot)$ is subadditive at every output $q \in [D(p_L(n)), K)$" is false.

Clearly, if $\mathcal{S}(G_n) \neq \emptyset$, the result follows from Proposition 2. Hence, assume $\mathcal{S}(G_n) = \emptyset$.

Fix any equilibrium profile $\mathbf{p} = (p_1, \ldots, p_n) \in \mathcal{B}(G_n)$ and suppose, by contradiction, there is a firm $k \in N$ whose reported price $p_k < p_j$ for all $j \in N \backslash \{k\}$. Without loss of generality, denote by p_h, $h \neq k$, the second smallest price among the announced prices $(p_1, \ldots, p_n)$; i.e., let $p_h = \min\{p_j;\, j \neq k\}$. Then,

(a) If $p_k > p^M$, firm k can profitably deviate to the monopoly profit-maximizing price p^M;

(b) If $p_k = p^M$, Assumption 5 and continuity of $H(\cdot\,, 1)$ at p^M imply that there exists $\epsilon > 0$ such that $H(p_k - \epsilon, 1) > 0$. But, since $\pi_h(p_1, \ldots, p_n) = 0$, this means that firm h can do better by proposing $p_k - \epsilon$ instead of p_h, a contradiction;

(c) Finally, if $p_k < p^M$, depending upon the location of p_h the following happens. If $p_h > p^M$, firm k can profitably deviate to p^M as before. Otherwise, if $p_h \leq p^M$, then by Assumption 5 there is $\epsilon > 0$ such that $H(p_h - \epsilon, 1) > H(p_k, 1) = \pi_k(p_k, \mathbf{p}_{-k})$, which contradicts that $(p_1, \ldots, p_n) \in \mathcal{B}(G_n)$.

Hence, using (a)-(c), we conclude that, if $(p_1, \ldots, p_n) \in \mathcal{B}(G_n)$ and $\mathcal{S}(G_n) = \emptyset$, there must be at least two firms which tie at the lowest price, say p^*, and another firm proposing a price above p^*.[18] That is, there must exist $m \in \{2, \ldots, n-1\}$, $n > 2$, and $p^* \in [0, \infty)$ such that (i) For all $i \in \{i_1, \ldots, i_m\} \subset N$, $p_i = p^*$; and (ii) For all $j \notin \{i_1, \ldots, i_m\}$, $p_j > p^*$. Following a similar reasoning than in (a) and (b), it is easy to see that $p^* < p^M$.

Notice that, since $\mathcal{S}(G_n) = \emptyset$, $H(p_L(n), n) < H(p_L(n), 1)$, which implies that $H(p_L(n), 1) > 0$. Thus, given that $H(0, 1) = -C(K) < 0$, it follows that $p_L(1) < p_L(n)$. Next, using the argument behind the proof of Lemma 3, we show that $p_L(m) \leq p_L(1)$.

In effect, assume, by contradiction, $p_L(m) > p_L(1)$. (Recall that before Lemma 2 we proved $p_L(m) \leq p^M$.) By Assumption 5, $H(p_L(m), 1) > H(p_L(1), 1) = 0$. Thus,

$$H(p_L(m), 1) - H(p_L(m), m) > 0. \tag{24}$$

On the other hand,

$$H(0, 1) - H(0, m) = -C(K) + C(K/m) < 0. \tag{25}$$

Therefore, from (24) and (25) and continuity of $H(\cdot\,, 1) - H(\cdot\,, m)$ on $[0, p^M]$, there exists a price $p^\alpha \in (0, p_L(m))$ such that

$$H(p^\alpha, 1) - H(p^\alpha, m) = 0. \tag{26}$$

Recall that, at the equilibrium $(p_1, \ldots, p_n) \in \mathcal{B}(G_n)$, m firms tie at the lowest price p^*; hence, $H(p^*, m) \geq 0$. Otherwise, any of these firms can profitably deviate to $\overline{P}$. Moreover, $p^* \geq p_L(m)$, because $H(\cdot\,, m)$ is negative below $p_L(m)$. In addition, $(p_1, \ldots, p_n) \in \mathcal{B}(G_n)$

[18] Recall that, since by supposition $\mathcal{S}(G_n) = \emptyset$, at most $n-1$ firms can tie at p^*.

implies $H(p^*, m) \geq H(p^*, 1)$, which is equivalent to $H(p^*, 1) - H(p^*, m) \leq 0$. Thus, using (24), we conclude that $p^* \neq p_L(m)$. Finally, by Assumption 5, $H(p^M, 1) - H(p^M, m) > 0$. Therefore, there exists a price $p^\beta \in [p^*, p^M)$ such that

$$H(p^\beta, 1) - H(p^\beta, m) = 0. \tag{27}$$

Summarizing, by assuming that $p_L(m) > p_L(1)$, (26) and (27) indicate that the curves $H(\cdot\,, 1)$ and $H(\cdot\,, m)$ must intersect each other at least twice on $(0, p^M)$. An argument analogous to the one used in the proof of Lemma 3 shows that this assertion is false.[19] Thus, $p_L(m) \leq p_L(1)$. Furthermore, since we have already shown that $p_L(1) < p_L(n)$, we get $p_L(m) < p_L(n)$; and, more importantly, $D(p_L(m)) > D(p_L(n))$.

So, it remains to show that $C(\cdot)$ is not subadditive at $D(p_L(m))$. To do that, note that $H(\cdot\,, 1)$ is negative below $p_L(m)$, because $p_L(1) \geq p_L(m)$. That means $0 = m\, H(p_L(m), m) > H(p_L(m), 1)$, which renders the desired result; i.e., $m\, C\left(\frac{D(p_L(m))}{m}\right) < C(D(p_L(m)))$. Therefore, $C(\cdot)$ is not subadditive on $[D(p_L(n)), K)$. ■

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[19] The argument exploits a claim similar to Lemma 2, obtained by replacing n with m. The complete proof is available upon request.

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