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Abstract

We formulate and study the requirement on an allocation rule that no agent should be able to benefit by augmenting his endowment through borrowing resources from the outside world (alternatively, by simply exaggerating it). We show that the Walrasian rule is not "borrowing-proof" even on standard domains. More seriously, no efficient selection from the endowments-lower-bound correspondence, or from the no-envy-in-trades correspondence, or from the egalitarian-equivalent-in-trades correspondence is borrowing-proof. These impossibilities hold even on the domain of economies with homothetic preferences.

Key-works: borrowing-proofness. Endowment lower bound. No-envy-in-trades. Egalitarian-equivalent-in-trades.

JEL classification numbers:

1 Introduction

Allocation rules can often be manipulated by agents misrepresenting their preferences and a large literature has been devoted to understanding this phenomenon. An agent can also benefit by manipulating the resources he controls. It is well-known for instance that the Walrasian rule is not immune to manipulation through "withholding" (Postlewaite, 1979; Thomson, 1987a): by withholding some of his endowment prior to the operation of the rule, and after adding the resources he withheld to the consumption bundle that the rule assigns to him, an agent may end up better off than if he had not withheld. For this rule, an agent may even gain by destroying some of his endowment (Aumann and Peleg, 1974; Hurwicz, 1972, 1978).

We consider here another form of manipulation. Suppose that prior to the operation of the chosen rule, an agent borrows resources to enlarge his endowment. The rule is then applied, the agent receives his assigned consumption bundle, and returns what he had borrowed. The end result may be a bundle that he prefers to the one that he would have been assigned had he not borrowed. Our objective is to study the requirement on a rule that it not be subject to this kind of manipulation. We call it "borrowing-proofness".

The relevance of this property is to situations where certain agents may have access to outside resources. Think of a trader operating on different markets and attempting to exploit arbitrage opportunities. The network literature is concerned with situations of this type, where agents are differently connected, its goal being to understand the advantage (or disadvantage) conferred to an agent by the specific location he occupies in the network (Jackson and Wolinsky, 1996). Here, we do not explicitly model the network but we can think of an agent trading on one market but being linked to another where he can borrow. Alternatively, we can think of an agent simply exaggerating his endowment.

We investigate the implications of borrowing-proofness in the context of exchange economies. In a companion paper (Thomson, 2005), we study it for public good economies.

As is the case for the other forms of manipulation enumerated above, the reason to be concerned with borrowing is that it will deflect the rule from the allocations that it is supposed to achieve, thereby preventing society from achieving its objectives.

Ours is not a model of an inter-temporal monetary economy; there are of course no reason to be concerned with borrowing in situations when borrowing is an instrument to reach inter-temporal efficiency.

Obviously, the rule that assigns to each economy its endowment profile is borrowing-proof. However, it fails efficiency. The rule that assigns to a particular agent, chosen once and for all, the sum of the individual endowments, is borrowing-proof too. It is efficient as well but this rule ignores initial ownership of resources and therefore violates the participation constraints associated with ownership (as commonly interpreted, owning a good comes with the right to use it in any way one wishes). It also violates all of the basic notions of fairness in redistribution that have been proposed in the literature.

The Walrasian rule is efficient and it does recognize individual endowments by always selecting allocations that satisfy most of the requirements of participation and fairness in redistribution to which we just alluded. However, as we show first by means of examples, it is not borrowing-proof. This is hardly surprising given what we know of this rule concerning a range of properties of this kind. It violates virtually all of them, even on relatively narrow domains of economies, including domains on which one would hope that things would work out better.[1]

We then ask whether this negative result is specific to the Walrasian rule. We consider classes of rules satisfying efficiency and one of several requirements of participation and fairness in redistribution. One is the "individual endowments lower bound", which says that each agent should find his net trade at least as desirable as no trade. The other is "no-envy in trades", which says that no agent should prefer the net trade of anyone else to his own (Foley, 1967; Schmeidler and Vind, 1972). The third one is "egalitarian-equivalence in trades" (adapted from a notion proposed by Pazner and Schmeidler, 1978, for the problem of dividing a social endowment on which agents have equal rights), which says that there should be a "reference trade" that each agent finds indifferent to his own trade.

[1]In addition to its unsatisfactory response to the various ways in which endowments can change, documented above, when applied to the problem of fair division, it also does not satisfy elementary fairness conditions that one might have hope for. In particular, it is not *resource monotonic* (Thomson, 1978), that is, when the social endowment increases, some agents may end up worse off. It is not *population monotonic* (Thomson, 1983; Chichilnisky and Thomson, 1987), that is, when population decreases, some of the remaining agents may end up worse off. It does not satisfy *welfare domination under preference replacement* (Thomson, 1996), that is, when the preferences of some agents change, the welfares of the others may be affected in different directions. The study of properties of this type for bargaining solutions is carried out by Chun and Thomson (1988).

Our main results are negative. They state the incompatibility with efficiency and any one of these requirements, of borrowing-proofness.

In practice, there are bounds on how much an agent can borrow: lenders require collateral. If the manipulation is interpreted as exaggeration, there are also bounds on what an agent can pretend to own for his exaggeration to be credible. Thus, one should ask what happens if bounds are placed on the augmentation. The answer is that our results still hold. Indeed, in our proofs, we fix the initial profile of endowments, and for each $\beta > 0$, no matter how small, we show that preferences can be specified so that one of the agents can benefit by borrowing β units of a good. Thus, no matter how small is the amount he is allowed to borrow in relation to his initial holdings, he can still gain by so doing. (Obviously however, the less an agent can borrow, the less he can gain by manipulating).

General results exist stating for private good economies the incompatibility with efficiency and the distributional requirements listed above, of the requirement of non-manipulability through withholding (Postlewaite, 1979; Thomson, 1987a. Results of the same kind are also available for public good economies, Thomson, 1987b, and for economies with indivisible goods, Atlamaz and Klaus, 2007). The results presented here can be seen as complementary. Although one should not be as concerned with manipulation through borrowing as with manipulation through withholding, since the former requires agents to have access to external resources, whereas the former can be carried out on one's own, our results are nevertheless disappointing, all the more so that they hold on classes of otherwise well-behaved economies. Indeed, the proofs, which are by way of counterexamples, involve only two goods and two agents whose preferences are continuous, monotone, convex, and even homothetic. It is natural to expect that sufficiently strong additional restrictions on preferences would lead to more satisfactory conclusions. Identifying such restrictions appears to be an interesting direction for future research.

We raise other questions in the concluding section, where we also discuss variants of our main definitions. Travelling down the path opened by the literature on strategy-proofness, which began with a study of misrepresentation of preferences by a single agent, the first step we take here concerns manipulation by a single agent too. In a second step, it will be important to consider coordinated manipulation by several agents. We identify several forms that such manipulation could take (borrowing by several agents; borrowing from a fellow trader; more generally, transferring resources between traders).

2 Notation and definitions

The model of private good allocation that we study is standard. There are $\ell \in \mathbb{N}$ goods and $n \in \mathbb{N}$ agents. Let $N \equiv \{1, \ldots, n\}$ be the set of agents. Each agent $i \in N$ is equipped with a preference relation on $\mathbb{R}^\ell_+$, denoted by R_i. Let P_i denote the strict preference relation associated with R_i and I_i the corresponding indifference relation. Let $\mathcal{R}$ be a domain of admissible preferences, $R \equiv (R_1, \ldots, R_n) \in \mathcal{R}^N$ being a generic profile of such preferences. Each agent $i \in N$ is endowed with a vector of goods $\omega_i \in \mathbb{R}^\ell_+$, $\omega \equiv (\omega_1, \ldots, \omega_n) \in \mathbb{R}^{\ell N}_+$ being a generic profile of individual endowments. An **economy** is a pair $(R, \omega) \in \mathcal{R}^N \times \mathbb{R}^{\ell N}$. Let $Z(\omega) \equiv \{z \in \mathbb{R}^{\ell N}_+ : \sum z_i = \sum \omega_i\}$ be the **feasible set of (R, ω)**. Let $\mathcal{E}$ be a domain of economies. Let $\mathcal{E}_{cl}$ be the domain of economies in which preferences satisfy the classical assumptions of continuity, monotonicity, and convexity.[2,3] An economy is **quasi-linear** if in addition to the classical assumptions, for each agent, indifference between two bundles is preserved by adding to them the same amount of a particular good (we take it to be good 1). It is **homothetic** if in addition to the classical assumptions, for each agent, indifference between two bundles is preserved by multiplying them by the same positive number. Let $\mathcal{R}_q$ and $\mathcal{R}_h$ be the class of quasi-linear and homothetic preferences respectively, and $\mathcal{E}_q$ and $\mathcal{E}_h$ be the corresponding domains of economies.

Given a domain $\mathcal{E}$ of economies, **a correspondence on $\mathcal{E}$** is a mapping φ associating with each $(R, \omega) \in \mathcal{E}$ a non-empty subset of $Z(\omega)$, denoted $\varphi(R, \omega)$. We use the term **rule** when the mapping is *single-valued*. We apply the phrase **essentially single-valued** to a correspondence φ such that for each $(R, \omega) \in \mathcal{E}$, each pair $\{z, z'\} \subseteq \varphi(R, \omega)$, and each $i \in N$, $z_i \; I_i \; z'_i$.[4] A single-valued rule that always selects among the allocations chosen by a particular correspondence is a **selection from that correspondence.** The **Pareto correspondence, P,** associates with each economy each allocation such that there is no other allocation that each agent finds at least as desirable, and at least one agent prefers: $P(R, \omega) \equiv \{z \in Z(\omega)$: there is no

[2]Vector inequalities: given $a, b \in \mathbb{R}^\ell$, $a \geqq b$ means that for each $k \in \{1, \ldots, \ell\}$, $a_k \geq b_k$; $a \geq b$ means that $a \geqq b$ and $a \neq b$; $a > b$ means that for each k, $a_k > b_k$.

[3]The relation R_i is "monotonic" if for each $\{z_i, z'_i\} \subset \mathbb{R}^\ell_+$ with $z'_i > z_i$, $z'_i \; P_i \; z_i$.

[4]We could also speak of a rule being *single-valued* up to Pareto-indifference.

$z' \in Z(\omega)$ such that for each $i \in N$, $z'_i \; R_i \; z_i$, and for at least one $i \in N$, $z'_i \; P_i \; z_i\}$.

We consider selections from the intersection of the Pareto correspondence with the correspondences that are the most widely discussed in the literature. The **individual-endowments lower bound correspondence**[5]**, $\boldsymbol{B}$**, associates with each economy each allocation that each agent finds at least as desirable as his endowment: $B(R,\omega) \equiv \{z \in Z(\omega)$: for each $i \in N$, $z_i \; R_i \; \omega_i\}$. The **no-envy-in-trades correspondence, $\boldsymbol{F}$,** (Kolm, 1972; Schmeidler and Vind, 1972) associates with each economy each allocation obtained from the endowment profile through a profile of envy-free trades: $F(R,\omega) \equiv \{z \in Z(\omega) \colon z = \omega + t$, where $t \equiv (t_i)_{i \in N} \in \mathbb{R}^{\ell N}$ is such that there is no $\{i,j\} \subseteq N$ for which $(\omega_i + t_j) \; P_i \; z_i\}$. The **egalitarian-equivalence-in-trades correspondence, $\boldsymbol{E}$**, (adapted from a notion proposed by Pazner and Schmeidler, 1978, for the fair division problem) associates with each economy each allocation obtained through a trade profile for which there is a "reference" trade that each agent finds indifferent to his component of the profile: $E(R,\omega) \equiv \{z \in Z(\omega)$: there is $t_0 \in \mathbb{R}^{\ell}$ such that for each $i \in N$, $z_i \; I_i \; (\omega_i + t_0)\}$. The **Walrasian correspondence, $\boldsymbol{W}$,** associates with each economy each allocation that can be "supported" by prices: $W(R,\omega) \equiv \{z \in Z(\omega)$: there is $p \in \Delta^{\ell-1}$ such that for each $i \in N$, $pz_i \leq p\omega_i$, and for each $z'_i \in \mathbb{R}^{\ell}_+$ such that $pz'_i \leq p\omega_i$, we have $z_i \; R_i \; z'_i\}$. Agent i's offer curve from endowment ω_i is the locus of the maximizer of his preferences on his budget set $\{y_i \in \mathbb{R}^{\ell}_+ \colon py_i \leq p\omega_i\}$ as prices p vary. We denote this offer curve by $\boldsymbol{oc(R_i, \omega_i)}$ (in the figures, we use the more compact notation $oc(\omega_i)$).

The notation $\boldsymbol{\sigma(A, B)}$ designates the symmetric image of object A with respect to object B. Object A is a point or a set, and object B is a point or a straight line. The intersection of two solutions φ and φ' is denoted by $\varphi\varphi'$. If φ is a subcorrespondence of φ', we write $\varphi \subseteq \varphi'$. Given a correspondence φ and $i \in N$, φ_i is the projection of φ onto agent i's consumption space: $\varphi_i(R,\omega) \equiv \{a \in \mathbb{R}^{\ell}_+$: there is $z \in \varphi(R,\omega)$ such that $z_i = a\}$. Let $\boldsymbol{\Lambda}$ designate the 45° line and $\boldsymbol{\Delta^{\ell-1}}$ the $(\ell-1)$-dimensional simplex. Given $z_i \in \mathbb{R}^{\ell}_+ \setminus \{0\}$, $\boldsymbol{\rho(z_i)}$ is the ray emanating from the origin and passing through z_i. In an Edgeworth box diagram, there are two origins and to indicate the one from which a ray emanates, we add an extra argument: $\boldsymbol{\rho(0_i, z)}$ is the ray emanating from agent i's origin 0_i and passing through z. In the

[5]It is usually called the individual rationality correspondence.

figures, a small segment centered at a point z_i indicates a line of support to agent i's indifference curve through z_i. The slope of that line is indicated in parentheses next to this segment. The line of slope s passing through the point $a \in \mathbb{R}^2_+$ is denoted $\mathbf{line}(\boldsymbol{s}, \boldsymbol{a})$. Given a list such as $\omega \equiv (\omega_i)_{i \in N}$, we designate by ω_{-i} the sublist obtained by deleting the i-th component.

3 The results

Next is a formal statement of our requirement of immunity to manipulation through borrowing. Although we could write it for correspondences, we will work with allocation rules. (Recall that by this term we mean a mapping that selects a single allocation for each economy.) This is to avoid the difficulty with strategic analysis in situations when to a profile of strategies is associated a set of outcomes, as opposed to a single outcome. Indeed, the issue arises then of specifying how an agent bases his choice of a strategy on comparisons of sets.[6]

Borrowing-proofness: For each $e \equiv (R, \omega) \in \mathcal{E}^N$, each $i \in N$, and each $\omega' \in \mathbb{R}^{\ell N}_+$, if $\omega'_i \geq \omega_i$ and $\omega'_{-i} = \omega_{-i}$, then it is not the case that $(\varphi_i(R, \omega') - (\omega'_i - \omega_i))\ P_i\ \varphi_i(R, \omega)$.

We formulate the property in a negative way because, for an arbitrary vector of borrowed resources, the difference between what the rule assigns to the agent who borrowed and the vector of resources he borrowed may not be a non-negative vector. Of course, an agent would want to borrow only if he can eventually return what he borrowed.

We investigate the existence of well-behaved rules defined over domains of economies with an arbitrary number of commodities and agents, but we establish our general negative results by way of examples with two commodities and two agents. In these examples, preferences are continuous, monotonic, and convex and in fact, homothetic. Operating under this additional requirement makes the construction more difficult in some respects but it helps in other ways because the Pareto efficient set has a simpler structure then. If in

[6] One could allow for *essential single-valuedness*, but Pareto-indifference of the allocations chosen for the reported profile of endowments would not necessarily imply the same property after the amounts borrowed have been returned. Thus, the problem of specifying how an agent compare sets of allocations would still have to be faced. A recent study of manipulation of correspondences is Ching and Zhou (2002).

addition preferences are strictly convex, it is a curve connecting the origins of the Edgeworth box satisfying a certain monotonicity property.[7] Thus, its projection onto the consumption space of either agent is a curve having the origin of that agent's consumption space and the aggregate endowment vector as endpoints, and this curve also satisfies this monotonicity property. In the proof of Theorem 2, we exploit a slightly weaker property of Pareto sets for convex, but not necessarily strictly convex, preferences. This property is stated as Observation 3.

We specify the examples geometrically, introducing them in such a way as to make as intuitive as possible how we arrived at them. We do not give explicit analytical expressions for representations of preferences. Such expressions would not shed any additional light on the nature of the results. Geometric proofs are unavoidable because the distributional criteria we consider cannot be checked on the basis of local information. Assuming differentiability of preferences and examining marginal rates of substitution at various consumption bundles would not help. However, the examples can be easily adapted so as to satisfy this property. Preferences can also be made strictly convex and strictly monotone if desired.

First are examples showing that on two standard domains, the Walrasian rule violates *borrowing-proofness*. These examples are illustrated in Edgeworth boxes. We adopt the following notational and graphical conventions. We start from some endowment profile and imagine agent 1 augmenting his endowment by borrowing. The vector of resources he borrows is of the form $(\beta, 0)$ or $(0, \beta)$ for some $\beta > 0$. To accommodate the resulting increase in the aggregate endowment, we construct a larger Edgeworth box, keeping agent 1's origin fixed and translating agent 2's origin 0_2 by the vector of borrowed resources, to the point marked $0'_2$. Redrawn from this new origin, we denote agent 2's preference map by R'_2.

Example 1 *On the quasi-linear domain, the Walrasian rule is not borrowing-proof.* The example is illustrated in Figure 1a. We choose an arbitrary endowment profile ω, a slope $s < 0$, and an allocation $z \in \text{line}(s, \omega)$ such that $z_{11} < \omega_{11}$. Let $\beta > 0$. Agent 1 borrows the vector $(0, \beta)$. Let $\omega'_1 \equiv \omega_1 + (0, \beta)$, $\omega'_2 \equiv \omega_2$, and $\omega' \equiv (\omega'_1, \omega'_2)$. We choose $s' < s$ and an allocation $z' \in \text{line}(s', \omega')$ such that $z_{12} < z'_{12} < z_{12} + \beta$. Thus, $z'_1 - (0, \beta)$ lies above $\text{line}(s, \omega)$.

[7] See Thomson (2004) for a discussion.

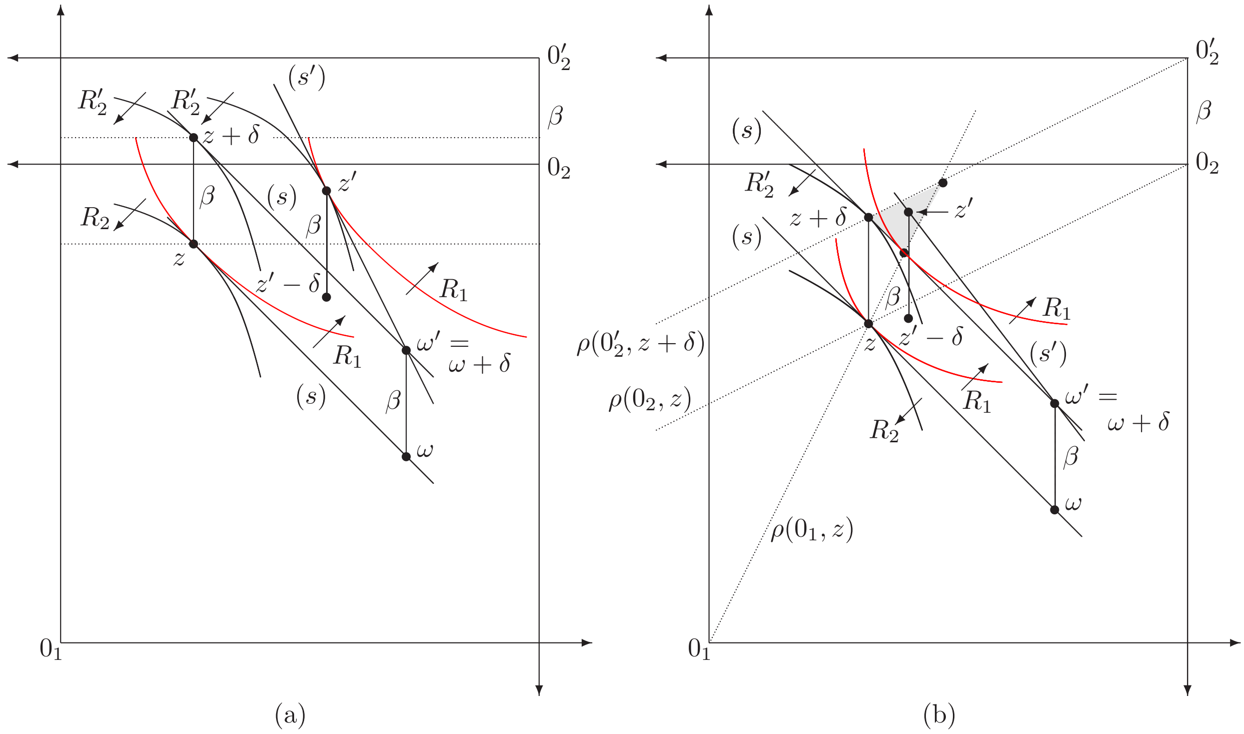

Figure 1: The Walrasian rule is not borrowing-proof. (a) The rule violates the property on the quasi-linear domain (Example 1). (b) It also violates the property on the homothetic domain (Example 2).

Next, we show that preferences $R \in \mathcal{R}_q^N$ can be specified so that $z = W(R, \omega)$ (with a line of support of slope s), $z' = W(R, \omega')$ (with a line of support of slope s'), and $(z_1' - (0, \beta))\ P_1\ z_1$, in violation of *borrowing-proofness*. Since $R \in \mathcal{R}_q^N$, at any point above the horizontal line through z, agent 1's indifference curve should have a line of support whose slope is smaller than s (slopes are measured algebraically, so this means that the line of support is steeper). Moreover, in the augmented Edgeworth box, at any point below the horizontal line through $z + \delta$, where $\delta \equiv ((0, \beta), (0, 0))$ (the bundle z_2 measured from $0_2'$ is the second component of the point marked $z + \delta$), agent 2's indifference curve should have a line of support whose slope is also smaller than s. A relation $R_1 \in \mathcal{R}_q$ exists for which (i) z_1 is a maximizer on line(s, ω), (ii) z_1' is a maximizer of R_1 on line(s', ω'), and (iii) agent 1's indifference curve through z passes below $z' - \delta$. Then indeed, $(z_1' - (0, \beta))\ P_1\ z_1$. Similarly, a relation $R_2 \in \mathcal{R}_q$ exists for which (i) z_2 is a maximizer of R_2 on line(s, ω) in the original Edgeworth box, so that in the augmented Edgeworth box, the second component of $z + \delta$, which is z_2, is a maximizer on line(s, ω'), and (ii) in the augmented Edgeworth box, z_2' is a maximizer on line(s', ω'). Preferences can be specified so as to guarantee uniqueness of the Walrasian allocation.

In Example 1, at the initial Walrasian allocation, agent 1 sells good 1 and buys good 2, and it is by borrowing good 2 that he manages to improve his welfare. This seems to be the intuitively correct way to attempt to manipulate the rule. Indeed, by so doing, he may bring about a decrease in the price of this good, which should be to his advantage. It may be interesting to ask what would happen if he borrowed good 1, the "wrong" good, that is, the good that he sells. As shown next, on the quasi-linear domain, he cannot gain by so doing. This observation, as well as a parallel one below pertaining to the homothetic domain, should help us understand the scope of the problem, and how it interacts with the specification of the domain of preferences.

Observation 1 *On the quasi-linear domain, the Walrasian rule cannot be manipulated by an agent borrowing the good that he sells initially.* Two cases should be distinguished. If the initial Walrasian allocation is interior, it remains so after the change in agent 1's endowment, the same prices remain equilibrium prices, and agent 1's final bundle remains the same. If the initial Walrasian allocation is a boundary allocation, his borrowing good 1 can only make equilibrium prices move against him. In fact, he will be unable to return what he borrowed. For agent 2, his final bundle is invariant under borrowing of good 1, irrespective of whether the initial Walrasian allocation is interior or not.

Next, we turn to the case of homothetic preferences.

Example 2 *On the homothetic domain, the Walrasian rule is not borrowing-proof.* The example is illustrated in Figure 1b. We choose an arbitrary endowment profile ω, a slope $s < 0$, and an allocation $z \in \text{line}(s, \omega)$ such that $z_{11} < \omega_{11}$. Let $\beta > 0$. Agent 1 borrows $(\beta, 0)$. Let $\omega'_1 \equiv \omega_1 + (\beta, 0)$, $\omega'_2 \equiv \omega_2$, and $\omega' \equiv (\omega'_1, \omega'_2)$. We identify the triangle defined by $\rho(0_1, z)$, $\rho(0'_2, z + \delta)$, where $\delta \equiv ((\beta, 0), (0, 0))$, and $\text{line}(s, \omega')$. (It is shaded in the figure.) We choose $s' < s$ so that $\text{line}(s', \omega')$ has a non-degenerate segment of intersection with this triangle. Let z' be a point in the relative interior of this segment.

Next, we show that preferences $R \in \mathcal{R}_h^N$ can be specified so that $z = W(R, \omega)$, $z' = W(R, \omega')$, and $(z'_1 - (\beta, 0))\ P_1\ z_1$, in violation of *borrowing-proofness*. Since $R_1 \in \mathcal{R}_h$, at any point above $\rho(0_1, z)$, agent 1's indifference curve through that point should have a line of support whose slope is smaller than s. Also, in the augmented Edgeworth box, at any point below $\rho(0'_2, z +$

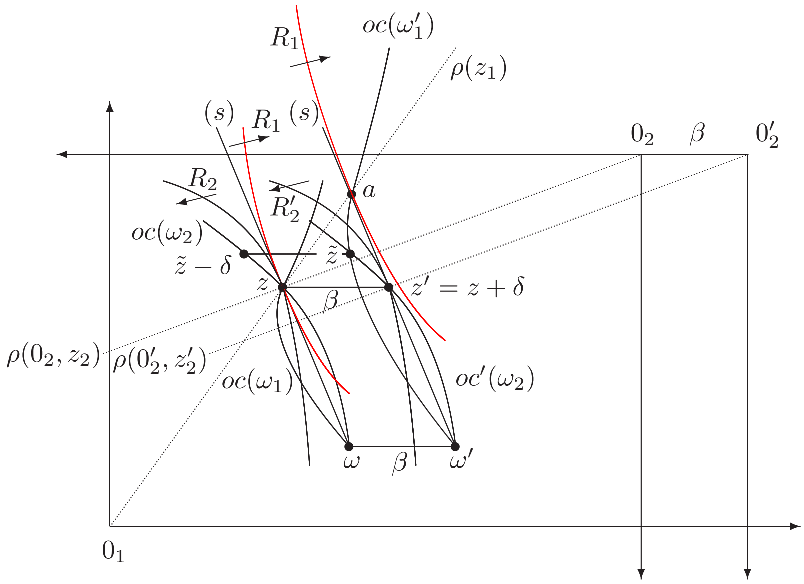

Figure 2: Observation 2. If the endowment profile is ω, at equilibrium, agent 1 sells good 1 in exchange for good 2. The two offer curves from the point ω' obtained when agent 1 borrows β units of good 1, cross below line(s, ω). Thus, agent 1 is not made better off.

δ), agent 2's indifference curve through that point should have a line of support whose slope is also smaller than s. A relation $R_1 \in \mathcal{R}_h$ exists for which (i) z_1 is a maximizer on line(s, ω), (ii) z_1' is a maximizer on line(s', ω'), and (iii) below $\rho(0_1, z)$, agent 1's indifference curve through z is sufficiently close to line(s, ω) that $(z_1' - (0, \beta))\ P_1\ z_1$. Similarly, a relation $R_2 \in \mathcal{R}_h$ exists for which (i) z_2 is a maximizer of R_2 on line(s, ω) in the original Edgeworth box, so that in the augmented Edgeworth box, the second component of $z + \delta$, which is z_2, is a maximizer on line(s, ω'), and (ii) in the augmented Edgeworth box, z_2' is a maximizer on line(s', ω'). Preferences can be specified so as to guarantee uniqueness of the Walrasian allocation.

As on the quasi-linear domain, the Walrasian rule cannot be manipulated by borrowing the "wrong" good.

Observation 2 *On the homothetic domain, the Walrasian rule cannot be manipulated by an agent borrowing the good that he sells initially.* The proof is illustrated in Figure 2. Let $(R, \omega) \in \mathcal{E}^N$, $z \equiv W(R, \omega)$, with a supporting price line of slope s. At z, agent 1 sells good 1 in exchange for good 2. Let then have him borrow the vector $(\beta, 0)$. Let $\omega_1' \equiv \omega_1 + (\beta, 0)$, $\omega_2' \equiv \omega_2$, and $\omega' \equiv (\omega_1', \omega_2')$. Since $R_1 \in \mathcal{R}_h$, agent 1's offer curve from ω_1' passes through the point a on line(s, ω_1') that is proportional to z_1. The bundle z_2 measured from 0_2 is the second component of the allocation marked z'. It is the maximizer of R_2' on line(s, ω'). Thus, in the augmented Edgeworth box, agent 2's offer curve from ω'_2 passes through z'_2. The point a is further from ω'_1 on line(s, ω'_1) than z'_2. Thus, the two agents' offer curves from ω' cross below that line, at the point marked $\tilde{z}$. This implies that $\tilde{z}_1 - (\beta, 0)$ is below line(s, z). Agent 1 is worse off than if he had not borrowed.

We now turn to a general investigation of the implications of *borrowing-proofness*, requiring of rules that they should be efficient and satisfy one or the other of the various distributional requirements defined earlier. We begin with no-envy in trades and the individual-endowments lower bound. The following theorem covers both.

Theorem 1 *On the classical domain, no selection from the no-envy-in-trades and Pareto solution, nor from the individual-endowments lower bound and Pareto solution, is borrowing-proof.*

We will use the following observation.

Observation 3 *In an Edgeworth box economy with homothetic preferences, if an allocation z that differs from either origin is efficient and has no Pareto-indifferent allocation, the entire Pareto set lies in the bow-tie–shaped area defined by the rays through z emanating from the two origins.*

The notation $\mathbf{bt}(z)$ designates the bow-tie–shaped area associated with $z \notin \{0_1, 0_2\}$.

Proof: Let $(R, \omega) \in \mathcal{E}_h^N$ and $z \in P(R, \omega)$, and suppose that $z \notin \{0_1, 0_2\}$. Let s be the slope of a line of support to z. Let $z' \in P(R, \omega) \setminus \text{bt}(z)$. Without loss of generality, suppose that z' lies above both $\rho(0_1, z)$ and $\rho(0_2, z)$. Let s' be the slope of a line of support to z'. Since z' is above $\rho(0_1, z)$, $s' \geq s$, and since z' is above $\rho(0_2, z)$, $s' \leq s$. Thus $s = s'$. This implies that at each point between $\rho(0_1, z)$ and $\rho(0_1, z')$, agent 1's indifference curve through it admits a line of support of slope s, and at each point between $\rho(0_2, z)$ and $\rho(0_2, z')$, agent 2's indifference curve through it admits a line of support of slope s. Let $a \equiv \rho(0_1, z) \cap \text{line}(s, z)$ and $b \equiv \rho(0_2, z) \cap \text{line}(s, z)$. Note that $a \neq z$ and $b \neq z$. Of a and b, let c designate the closest to z. Thus, each allocation in seg$[z, c]$ is in $P(R, \omega)$ with a line of support of slope s. Any such allocation is efficient and Pareto-indifferent to z. □

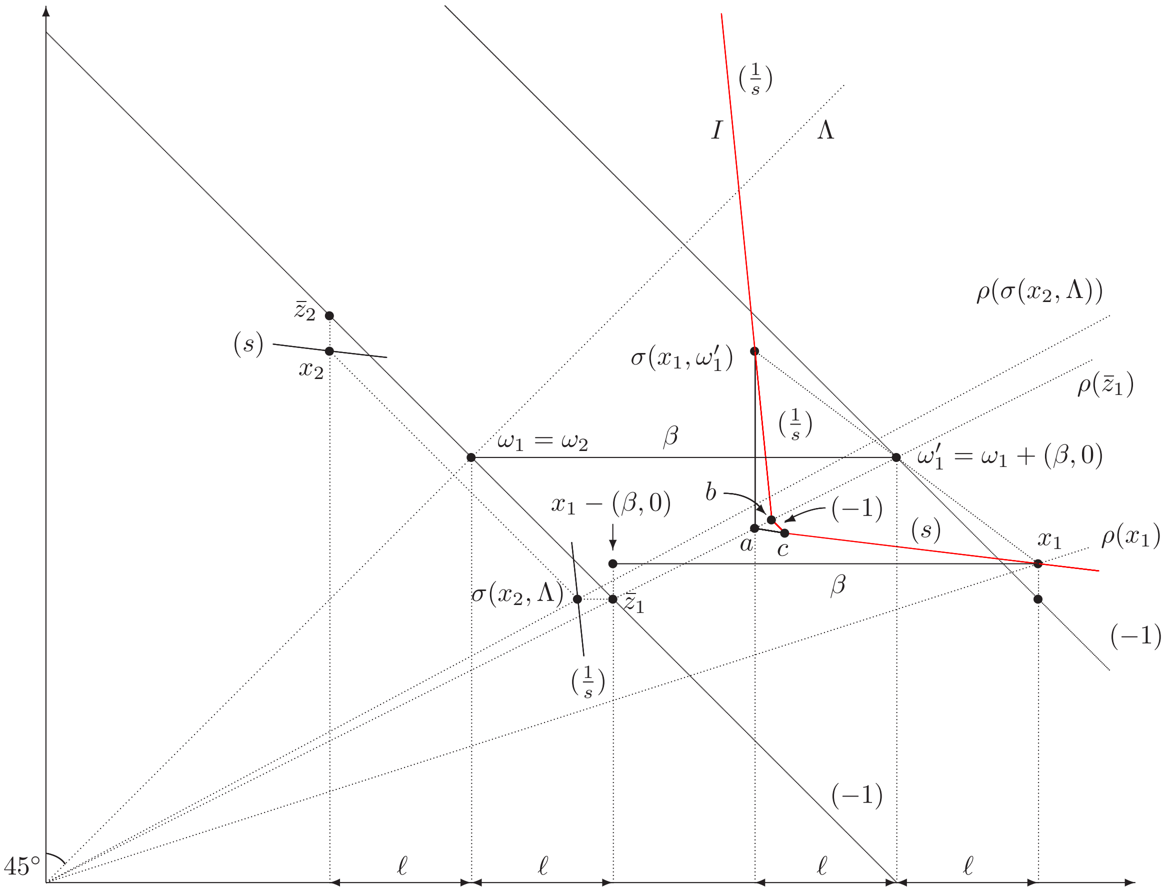

Figure 3: Theorem 1. Describing the economy. The requirements imposed on R_1 are extracted from this figure and reproduced in Figure 4. Figure 5 shows how to construct R_1 that satisfies them.

Proof: (a) The proof of the first statement is by means of an example, illustrated in Figures 3-5, of a two-good, two-agent economy in which the two agents have homothetic preferences that are symmetric of each other with respect to the 45° line (that is, for each pair $\{z_1, z_1'\} \subset \mathbb{R}^2_+$, if $z_1 \; I_1 \; z_1'$, then $\sigma(z_1, \Lambda) \; I_2 \; \sigma(z_1', \Lambda)$). Most of the proof consists in constructing one of agent 1's indifference curves. This curve is denoted I. The specification of R_1 is completed by subjecting I to homothetic transformations. Agent 2's preferences R_2 are obtained from R_1 by symmetry with respect to Λ.

Step 1: Constructing the economy. Let $\omega_1 = \omega_2 \in \Lambda$. Let $\beta > 0$. Agent 1 borrows β units of good 1. Let $\omega_1' \equiv \omega_1 + (\beta, 0)$, $\omega_2' \equiv \omega_2$, and $\omega' \equiv (\omega_1', \omega_2')$.

Let $\bar{z}_1 \equiv \text{line}(-1, \omega_1) \cap \rho(\omega_1')$, $\bar{z}_2 \equiv \omega_1 + \omega_2 - \bar{z}_1$, and $\bar{z} \equiv (\bar{z}_1, \bar{z}_2)$. Let $\ell \equiv \bar{z}_{11} - \omega_{11}$, a the point of intersection of the vertical line of abscissa $\omega_{11}' - \ell$ with $\rho(\omega_1')$ $(= \rho(\bar{z}_1))$ (a is further from the origin along this ray than $\bar{z}_1$), and x_1 be a point of abscissa $\omega_{11}' + \ell$ and ordinate in $]\bar{z}_{12}, a_2[$. The indifference curve I we specify for agent 1 is his indifference curve through x_1 and we reproduce on Figure 4 the requirements we impose on I except for one. Our objective is (i) to make $\bar{z}_1$ the maximizer of R_1 on $\text{line}(-1, \omega_1)$, and (ii) to make x_1 the bundle of $FP(R, \omega')$ that agent 1 likes the least. For the efficiency part of (ii) to hold, I and agent 2's indifference curve through $x_2 \equiv \omega_1' + \omega_2 - x_1$ should have parallel lines of support at x_1 and x_2 respectively. For the fairness part of (ii) to hold (for x to be the worst allocation in $FP(R, \omega')$ for agent 1), I should pass through $\sigma(x_1, \omega_1')$. Let s be the slope of $\text{seg}[a, x_1]$. By the choice of x_1 and a, we have $s > -1$. We give I a line of support of slope s at x_1. Thus, agent 2's indifference curve through x_2 also has a line of support of slope s at x_2. By the symmetry of preferences, agent 1's indifference curve through $\sigma(x_2, \Lambda)$ has a line of support of slope $\frac{1}{s}$ at that point. Since $x_{21} = \bar{z}_{21}$ and $x_{22} < \bar{z}_{22}$, then $\sigma_1(x_2, \Lambda) < \bar{z}_{11}$ and $\sigma_2(x_2, \Lambda) = \bar{z}_{12}$, and therefore $\sigma(x_2, \Lambda)$ is above $\rho(\bar{z}_1)$. Also, since $s > -1$, $\frac{1}{s} < -1$. Since x_1 is below $\rho(\bar{z}_1)$, then $\sigma(x_1, \omega_1')$ is above $\rho(\bar{z}_1)$. Let b be the point of intersection of $\text{line}(\frac{1}{s}, \sigma(x_1, \omega_1'))$ with $\rho(\bar{z}_1)$. Since $a_1 = \sigma_1(x_1, \omega_1')$, then b is further from the origin along $\rho(\bar{z}_1)$ than a. Let c be the point of intersection of $\text{line}(-1, b)$ with the line passing through a and x_1. Observe that $c \in \text{seg}[a, x_1]$.

We now define I satisfying all of the requirements listed so far (Figure 5). We take I to be the union of the following three segments: the extension to the vertical axis of $\text{seg}[b, \sigma(x_1, \omega_1')]$, $\text{seg}[b, c]$, and the extension to the

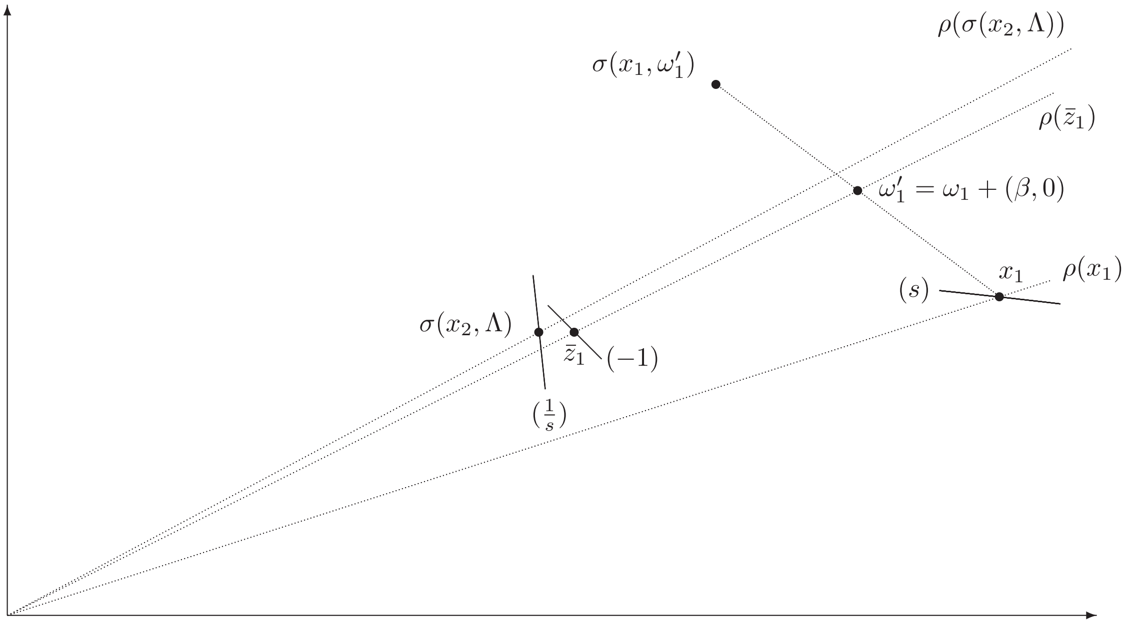

Figure 4: Theorem 1. Listing the requirements imposed on R_1. The map R_1 has to satisfy three slope requirements, along $\rho(\bar{z}_1)$, $\rho(x_1)$, and $\rho(\sigma(x_2,\Lambda))$. Also, the indifference curve through x_1 should pass through $\sigma(x_1,\omega_1')$. A final requirement (not indicated) is that x_1 be the only point at which line(s, x_1) supports the indifference curve passing through x_1, and that $\bar{z}_1$ be the only point at which line$(-1,\bar{z}_1)$ supports the indifference curve passing through $\bar{z}_1$.

horizontal axis of seg$[c, x_1]$. Since $\rho(\sigma(x_2,\Lambda)$ is steeper than $\rho(\bar{z}_1)$, which itself is steeper than $\rho(x_1)$, and the slopes of the lines of support along these three rays, of slopes $\frac{1}{s}$, -1, and s respectively, are ordered in the same manner, the curve I is indeed the indifference curve of a convex relation.

It follows from our construction so far that $\bar{z} \in P(R,\omega)$ and $x \in P(R,\omega')$. The remaining requirement that we impose on agent 1's preferences is there to guarantee that in (R,ω), no allocation is Pareto-indifferent to $\bar{z}$, and that in (R,ω') no allocation is Pareto-indifferent to x. Let $\varepsilon > 0$ and $c^{\varepsilon} \equiv$ line$(-1+\varepsilon, b) \cap$ line$(s-\varepsilon, x_1)$. We replace the curve I of the previous paragraph by the union of the extension to the vertical axis of seg$[b, \sigma(x_1,\omega_1')]$ (as before), seg$[b, c^{\varepsilon}]$, seg$[c^{\varepsilon}, x_1]$, and the extension to the horizontal axis of the segment with left endpoint x_1 that has slope $s+\varepsilon$ (Figure 5). For ε small enough, this (piece-wise linear in four pieces) curve is indeed the indifference curve of a convex relation.

Step 2: Concluding. Let $\varphi \in FP$ and $z \equiv \varphi(R,\omega)$. Since $\bar{z} \in P(R,\omega)$ and $\varphi \in P$, then either $\bar{z}_1 \; R_1 \; z_1$ or $\bar{z}_2 \; R_2 \; z_2$. By the symmetry of the preferences and the equality $\omega_1 = \omega_2$, we can assume, without loss of generality, that the former statement holds. Let $z' \equiv \varphi(R,\omega')$. By (ii), $z_1' \; R_1 \; x_1$. Note that

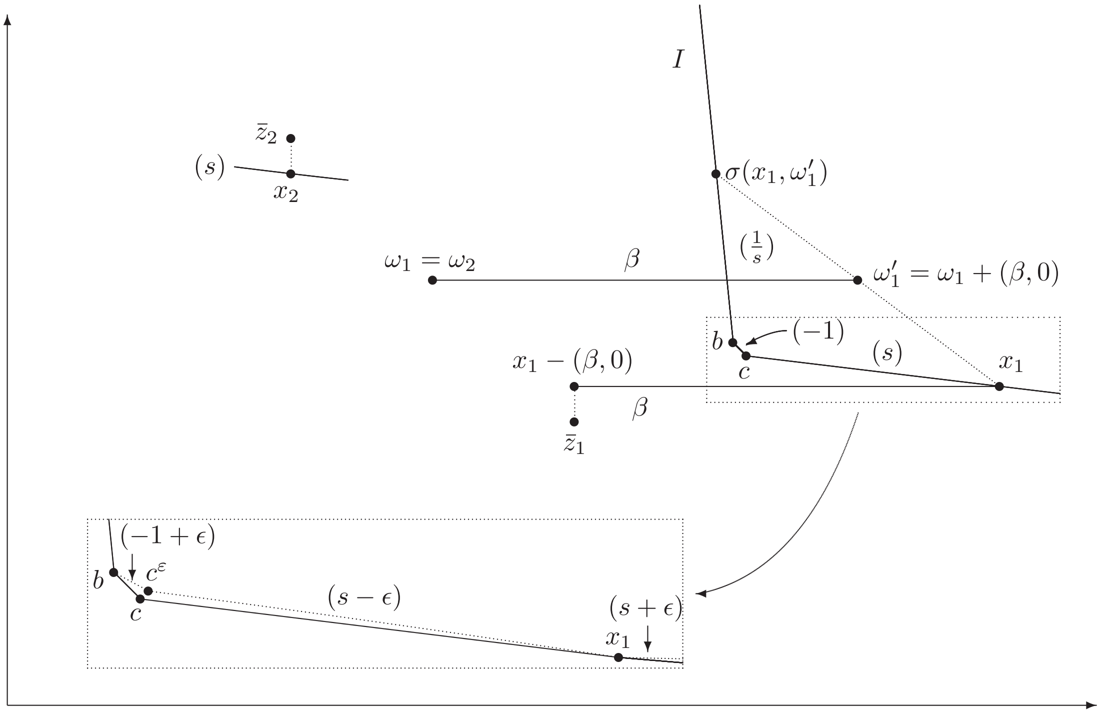

Figure 5: Theorem 1. Constructing a map R_1 satisfying all the requirements. A typical indifference curve is an approximation to the piece-wise linear curve in three pieces denoted I. The approximation is shown in the enlarged box close to the origin as a dotted broken line. The steep segment emanating from b is kept as such. The segment seg$[b, c]$ is rotated slightly counterclockwise around b. The segment seg$[c, x_1]$ is rotated slightly clockwise around x_1. The segment emanating from x_1 to the right of x_1 is rotated slightly counterclockwise around x_1.

line$(-1, \omega_1)$ supports agent 1's indifference curve through $\bar{z}_1$ at that point and $\bar{z}_1$ is the only point of contact of that line with that indifference curve, and line$(-1, \omega_2)$ supports agent 2's upper indifference curve at $\bar{z}_2$. Thus, not only $\bar{z} \in P(R, \omega)$ but in (R, ω), no allocation is Pareto-indifferent to $\bar{z}$. Then, by Observation 3, the projection of $P(R, \omega)$ onto agent 1's consumption space (that is, the consumptions for agent 1 that are the first component of an efficient allocation in (R, ω)) is contained in the projection of $\text{bt}(z)$. Also, line(s, x_1) supports agent 1's indifference curve through x_1 at that point and x_1 is the only point of contact of that line with that indifference curve, and line(s, x_2) supports agent 2's indifference curve through x_2. Thus, not only $x \in P(R, \omega')$ but in (R, ω'), no allocation is Pareto-indifferent to x. Thus, by Observation 3, the projection of $P(R, \omega')$ onto agent 1's consumption space is contained in the projection of $\text{bt}(z')$. Thus, $\bar{z}_1 \geq z_1$ and $z'_1 \geq x_1$. Note that for each $a \geq \bar{z}_1$, $a \; P_1 \; \bar{z}_1$. Altogether then, and since $z'_1 - (\beta, 0) \geq \bar{z}_1$, we obtain $(z'_1 - (\beta, 0)) \; P_1 \; \bar{z}_1 \; R_1 \; z_1$, in violation of *borrowing-proofness*.

(b) In the two-agent case, no-envy-in-trades is a weaker requirement than the individual-endowments lower bound (Kolm, 1972; Thomson, 1987a). Since we proved the negative result pertaining to no-envy by means of a two-agent example, the result pertaining to the individual-endowments lower bound follows as a corollary. □

Next, we consider egalitarian-equivalence and establish a parallel impossibility to that of Theorem 1.

Theorem 2 *On the classical domain, no selection from the egalitarian-equivalence-in-trades and Pareto solution is borrowing-proof.*

We also prove Theorem 2 by means of a two-agent economy. Since in the two-agent case, no-envy-in-trades implies egalitarian-equivalence-in-trades (see Thomson, 1987a, for a proof), Theorem 2 implies Theorem 1. We have chosen to also present the proof for no-envy-in-trades because it is significantly simpler than the one for egalitarian-equivalence-in-trades. We relegate the latter proof to the appendix. Here is an informal explanation of where the complications come from.

In either case, we need to identify for the borrower (agent 1) the lowest indifference curve that contains a bundle that may be his component of an efficient allocation that meets the stated distributional requirement, no-envy-in-trades in one case, egalitarian-equivalence-in-trades in the other.

For no-envy-in-trades, this worst bundle is simply such that agent 1 is indifferent between his trade and agent 2's trade. To verify this property, agent 2's indifference curve through his own bundle need not be constructed. For egalitarian-equivalence-in-trades, this worst bundle—let call it x_1—is such that agent 2's indifference curve through his own bundle—let us call it x_2—when translated by agent 1's trade, lies above agent 1's indifference curve through x_1. To verify this property, agent 2's *entire* indifference curve through x_2 has to be constructed (and translated), but since the two preference relations are symmetric of each other with respect to the 45° line, they cannot be drawn independently of each other: they have to be constructed together.

4 Concluding comments

We conclude with a discussion of the relation between borrowing and withholding, and of variants of our main requirement.

1. Borrowing versus withholding. It is natural to ask whether the vulnerability of an allocation rule to borrowing is related to its vulnerability to withholding. These two behaviors have a strong resemblance, but it is not obvious how to formally relate them and in fact, results obtained on other domains, such as domains of public good economies (Thomson, 2005, 2008) suggest fundamental differences. Intuitively, if an agent is likely to benefit by affecting the scarcity of the goods that he acquires from his fellow traders relative to that of the goods that he provides to them, the circumstances in which he benefits from borrowing a particular good seem to be precisely ones in which he would not benefit from withholding the good.

Both withholding and borrowing have the effect in bringing the trades that an agent will be required to engage in from one region of his consumption space to another region where his preferences "look different". In that way, the two properties essentially amount to a misrepresentation of preferences. The main difference between misrepresenting preferences and misrepresenting endowments is that for the former, a strategic agent has access to any relation satisfying the maintained assumptions on preferences, but for the latter, he only has access to maps that are derived from a single "mother map" (his true map). Furthermore, when distributional requirements are imposed on the possible trades of the other agents, the difference between withholding and borrowing is that by withholding, an agent gains access to submaps that are closer to the origin, whereas in the second case, he gains access to submaps that are further from the origin. Thus, and because the range of submaps obtainable by the former strategy seems to be more limited than the range of submaps obtainable by the latter strategy, one would think that opportunities to gain by borrowing dominate opportunities to gain by withholding. This is not true however when preferences are restricted, by quasi-linearity or homotheticity for example, as we have assumed. Indeed, what preferences look like below the endowment and what they look like above the endowment are related. For instance, homotheticity implies that indifference curves are less curvy above the endowment than below. To compare the possible benefit from withholding to the possible benefit from borrowing, one needs to understand whether it is useful to have access to more curvy preferences or to less curvy ones. Of course, this should in general depend on the rule that is being used. To illustrate, if the rule is the Walrasian rule, then by borrowing sufficiently large amounts, an agent with Cobb-Douglas preferences can make it appear that he has almost linear preferences whose indifference curves have any slope that he chooses. These informal observations may provide some intuition about the differential impact of these two kinds of strategies.

2. Borrowing by groups. One can imagine that agents get together in a group, jointly borrow, and all end up better off. Immunity to such behavior could be called "group borrowing-proofness", a notion that is parallel to the notion of "group strategy-proofness" that has been much studied in the context of preference misrepresentation.

Furthermore, even if a group of agents has no joint borrowing strategy that benefits them all, it may be that after the rule is applied and each of them has returned the resources he borrowed, transfers between them exist that make them better off.

Finally, one need not require that each agent returns exactly the resources he borrowed. As long as the group as a whole returns the sum of the resources its members borrowed, one may argue that it has fulfilled its obligations. A rule could conceivably be *borrowing-proof* but not immune to this type of strategic behavior by groups. The analogy to manipulation through arbitrary transfers of endowments between the members of a group should be noted (see the considerable literature in international trade on the transfer paradox; also Gale, 1974, and Aumann and Peleg, 1974; a recent contribution in the context of economies with indivisible goods is Atlamaz and Klaus, 2007).

We have mainly offered negative results, and matters are of course worse when agents can engage in joint strategic choices of the types just described.

3. Open economy versus closed economy borrowing. In our formulation of *borrowing-proofness*, we have assumed that an agent borrows "from the outside world". Alternatively, we could imagine an agent borrowing from one of his fellow traders. To guarantee the cooperation of an agent from whom he borrows, this lender, after the resources he lent have been returned to him, should be at least as well off as he would have been if he had not lent. For an efficient rule, the option to so manipulate exists only if there are three agents or more. We call a rule that is immune to this kind of manipulation "closed-economy borrowing-proof" using the expression "open-economy borrowing-proofness" for the property to which our theorems pertain. Behavior of this type is akin to the various manipulations by groups enumerated in the previous paragraphs in that it requires cooperation among several agents, whereas in this paper, we have focused on strategic options that individual agents have.

Alternatively, we need not insist that the agent who borrows returns the exact bundle he borrows, but simply that he returns resources that guarantee that the lender ends up at least as well off as he would have been if he had not lent.

Finally, we could also imagine that an agent who borrows does so from more than one agent. Then, after he returns the resources he borrows, each of the lenders should end up at least as well off as he would have been if he had not lent.

4. Other domains. In companion papers devoted to a study of *borrowing-proofness* in economies in which public goods are present (Thomson, 2005; 2007), we have identified a number of situations in which a rule is *borrowing-proof* but not *withholding-proof*, but found no situation where the reverse is true. Atlamaz and Thomson (2005) conduct similar analyses for economies with indivisible goods. Thus a picture of the relative restrictiveness of the two requirements is slowly emerging.

5. What about Walras? We began by criticizing the Walrasian rule for its various failings. However, the general negative results reported here, as well as others, concerning for instance *resource monotonicity* (Moulin and Thomson, 1988), *withholding-proofness* (Postlewaite, 1979; Thomson, 1987a), *population monotonicity* (Kim, 2004), *welfare domination under preference replacement* (Thomson, 1996) can be understood as providing, if not an exon-

eration of the Walrasian rule, at least compelling extenuating circumstances: as soon as efficiency and one or the other of basic distributional requirement are imposed, we face an impossibility. Nevertheless, the main merits of the rule have to do with its informational properties, in terms of dimensionality of message spaces needed for its "realization" (Hurwicz, 1977), as well as its robustness properties with respect to certain changes in population, such as replication invariance and consistency (Thomson, 1988).

6. Manipulation and implementation. Once the problem with manipulation of a rule is recognized, it is natural next to associate with it a manipulation game and to study its equilibria. A study of a Walrasian game when strategies are announcements of endowments is Thomson (1979). The next question is the implementation question. A seminal contribution to implementation theory when endowments and production sets are unknown is Hurwicz, Maskin and Postlewaite (1995).

Appendix

In this appendix, we prove Theorem 2. It is convenient to explain first how to obtain the endpoints of the set of efficient allocations that are egalitarian-equivalent-in-trades. To each pair $(R_i, \omega_i) \in \mathcal{R} \times \mathbb{R}^2_+$ of a preference map over consumption bundles and an individual endowment for agent i can be associated a preference map over trades for him. Geometrically, the latter is obtained from the former by translation by $-\omega_i$. An allocation z is egalitarian-equivalent-in-trades for some economy $(R, \omega) \in \mathcal{E}^N$ if and only if the various agents' indifference curves through their assigned bundles, when translated as just explained, have one point in common. Alternatively, one can use as origin one agent's endowment, say agent 1's, and translate the preference map of each other agent i by the vector $\omega_1 - \omega_i$ (so that agent i's translated endowment coincides with ω_1). Then, z is egalitarian-equivalent-in-trades for $(R, \omega) \in \mathcal{E}^N$ if and only if there is a particular point that, for each $i \in N$ (including agent 1), agent i's indifference curve through his consumption z_i, when translated by the vector $\omega_1 - \omega_i$, contains. We will apply this observation to the two-agent case; then we need to translate the map of only one agent, agent 2. In that case, for an allocation x to be, among the egalitarian-equivalent-in-trades and efficient allocations, one that is least favorable to agent 1, agent 2's indifference curve through x_2, translated by $\omega_1 - \omega_2$, should touch agent 1's indifference curve through x_1 and should lie entirely above it except at the point(s) where they touch.

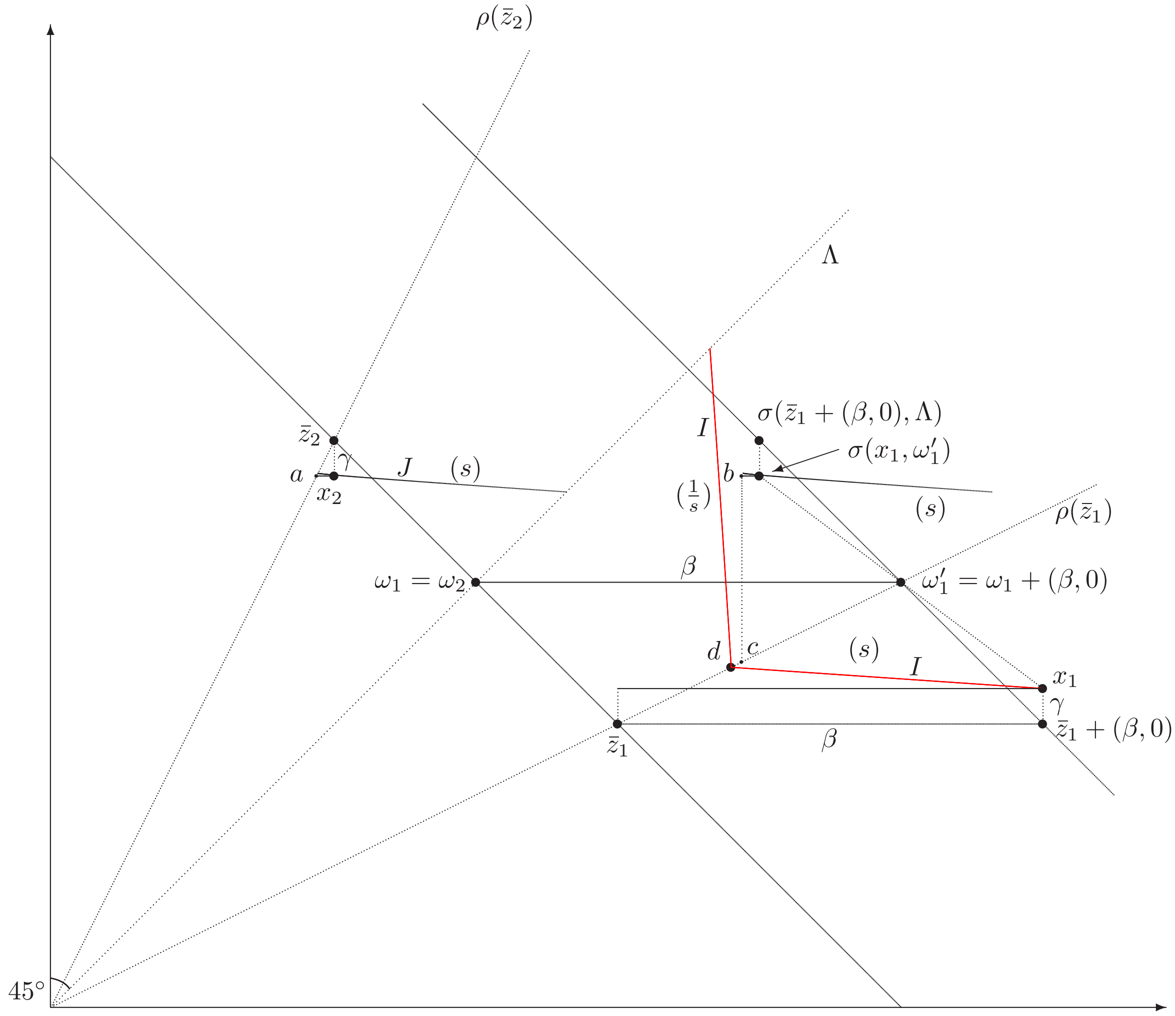

Figure 6: Theorem 2. Identifying critical points.

Proof: (Figure 7) The proof is by means of an example of a two-good, two-agent economy in which agents have homothetic preferences that are symmetric of each other with respect to the 45° line. Most of the proof consists in constructing one of agent 1's indifference curves. The curve we construct is denoted I. The specification of R_1 is completed by subjecting I to homothetic transformations. Agent 2's preferences R_2 are obtained from R_1 by symmetry with respect to Λ.

Step 1: Constructing the economy. Let $\omega_1 = \omega_2 \in \Lambda$. Let $\beta > 0$. Agent 1 borrows β units of good 1. Let $\omega_1' \equiv \omega_1 + (\beta, 0)$, $\omega_2' \equiv \omega_2$, and $\omega' \equiv (\omega_1', \omega_2')$.

The initial steps of the construction are illustrated in Figure 6. Let $\bar{z}_1 \equiv \text{line}(-1, \omega_1) \cap \rho(\omega_1')$, $\bar{z}_2 \equiv \omega_1 + \omega_2 - \bar{z}_1$, and $\bar{z} \equiv (\bar{z}_1, \bar{z}_2)$. Let x_1 be a point of abscissa $\bar{z}_{11} + \beta$ and ordinate $\bar{z}_{12} + \gamma$, where $\gamma > 0$ is chosen as follows. Let $x_2 \equiv \omega_1' + \omega_2 - x_1$. We have $x_2 = \bar{z}_2 - (0, \gamma)$. Let a be the point of $\rho(\bar{z}_2)$ of ordinate x_{22}. (In the figure, the horizontal segment $\text{seg}[a, x_2]$ is hard to distinguish from the longer segment passing through x_2 because this second segment, which will turn out to be part of agent 2's indifference curve through x_2, is almost horizontal. What is important to note is that the abscissa of the leftmost point of that segment, the (unlabelled) point where it meets $\rho(\bar{z}_2)$, is greater than the abscissa of a. Let $b \equiv a + (\beta, 0)$. Let c be the point of $\rho(\bar{z}_1)$ $(= \rho(\omega_1'))$ of abscissa b_1. Let d be a point of $\rho(\bar{z}_1)$ whose abscissa is in $]z_{11}, c_1[$. Now, let γ decrease to 0. The following occurs: $\lim_{\gamma \to 0} x_1 = \bar{z}_1 + (\beta, 0)$, $\lim_{\gamma \to 0} x_2 = \bar{z}_2$, $\lim_{\gamma \to 0} a = \bar{z}_2$, $\lim_{\gamma \to 0} b = \sigma(\bar{z}_1 + (\beta, 0), \Lambda)$, and $\lim_{\gamma \to 0} c_1 = \bar{z}_{21} + \beta$. Since the slope of $\rho(\bar{z}_1)$ is smaller than -1, $\lim_{\gamma \to 0} c_2$ is greater than $\lim_{\gamma \to 0} x_{12}$. Thus, for γ small enough, there is d such that $d_2 > x_{12}$. Let us choose γ and d in this manner.

We now specify agent 1's indifference curve I through x_1 (Figure 7). Our objective is (i) to make $\bar{z}_1$ the maximizer of R_1 on $\text{line}(-1, \omega_1)$, and (ii) to make x_1 the bundle of $EP(R, \omega')$ that agent 1 likes the least. For the efficiency part of (ii) to be met, I and agent 2's indifference curve through x_2, denoted J, should have parallel lines of support at x_1 and x_2 respectively. Let J^t be obtained by translating J by the vector $(\beta, 0) = \omega_1' - \omega_2$. For the fairness part of (ii), (for x to be the point of $EP(R, \omega')$ that agent 1 likes the least), I and J^t should touch, but otherwise, J^t should lie below I (see the comment preceding this proof).

Let s be the slope of $\text{seg}[d, x_1]$. We give I a line of support of slope s at x_1. By the choice of x_1 and d, $s > -1$. Since $x \in P(R, \omega')$, J has a line of support

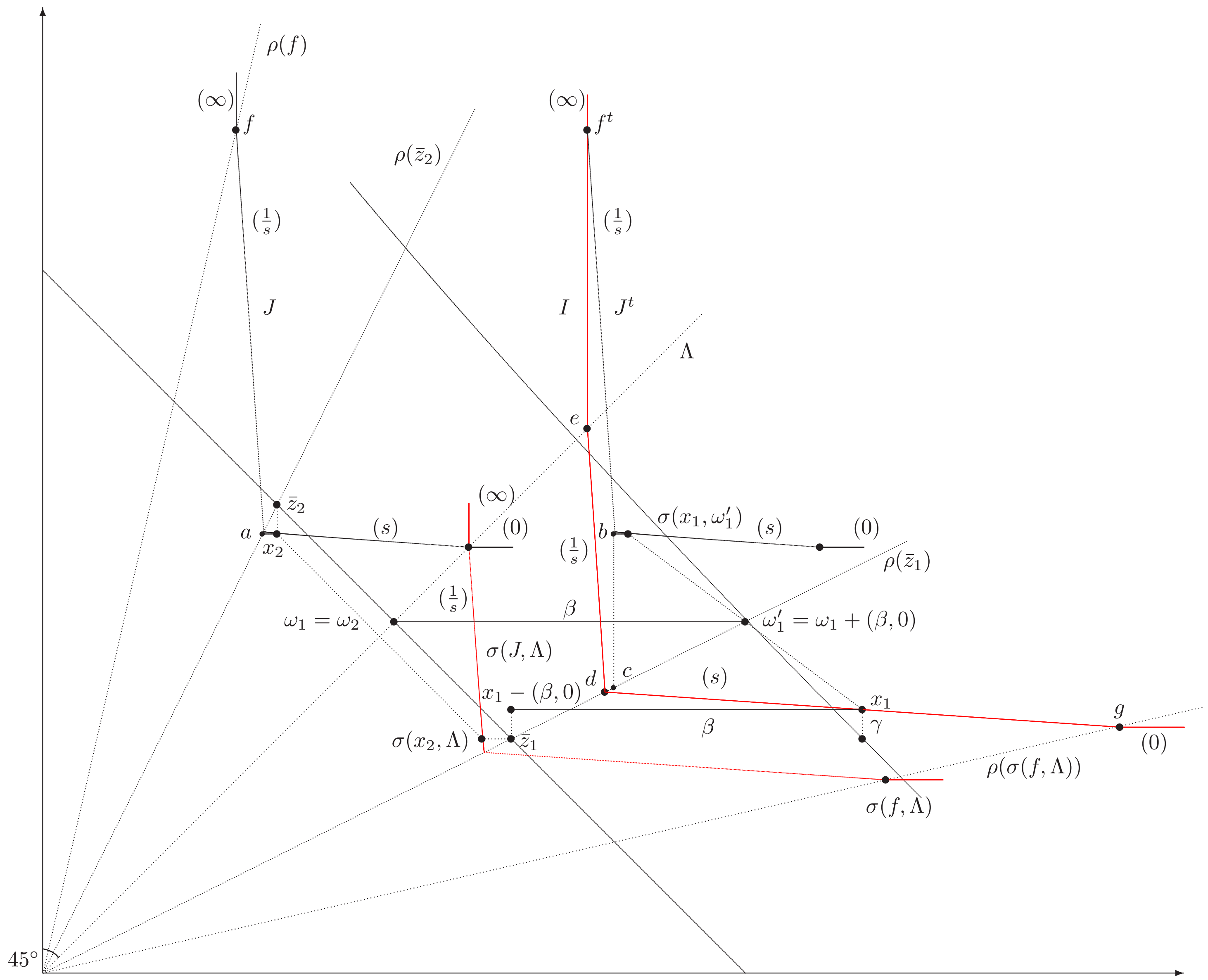

Figure 7: Theorem 2. The preference map for agent 1 has to satisfy three slope requirements, along $\rho(\bar{z}_1)$, $\rho(x_1)$, and $\rho(\sigma(x_2, \Lambda))$. Moreover, his indifference curve through x_1, denoted I, should touch, but otherwise remain underneath, the translate of agent 2's indifference curve through x_2, J, by $(\beta, 0) = \omega_1' - \omega_2$. This translated curve is called J^t and f^t is a point of contact of I and J^t. It is the translate of f by the vector $(\beta, 0)$.

of slope s at x_2. By the symmetry of preferences, agent 1's indifference curve through $\sigma(x_2, \Lambda)$ has a line of support of slope $\frac{1}{s}$ at that point. Since $x_{21} = \bar{z}_{21}$ and $x_{22} < \bar{z}_{22}$, then $\sigma_1(x_2, \Lambda) < \bar{z}_{11}$ and $\sigma_2(x_2, \Lambda) = \bar{z}_{12}$. Thus, $\sigma(x_2, \Lambda)$ is above $\rho(\bar{z}_1)$. Also, since $s > -1$, then $\frac{1}{s} < -1$. Let e be the point of intersection of line$(\frac{1}{s}, d)$ with Λ. We choose (iii) I to contain seg$[d, e]$ and (iv) to continue from e vertically. By the symmetry of preferences and (iii), between Λ and $\rho(\bar{z}_2)$, J is a segment of slope s. By the symmetry of preferences and (iv), below Λ, J is a horizontal half-line. Next, since I contains seg$[d, x_1]$ whose slope is s, J continues upwards from its intersection with $\rho(\bar{z}_2)$ with a segment of slope $\frac{1}{s}$. Let us extend this segment until the point f whose translation by $(\beta, 0)$ belongs to the vertical half-line through e (recall that from (iv), this half-line is part of I). Let f^t be the point of minimal ordinate at which this extension and this half-line meet. Then, $f = f^t - (\beta, 0)$. Thus, J extends linearly from its intersection with $\rho(\bar{z}_2)$ to f and we choose it to continue vertically from f. We now complete the specification of I. By the symmetry of the preferences, we have to extend seg$[d, x_1]$ whose slope is s to the right until it meets $\rho(\sigma(f, \Lambda))$—let us call g this point of intersection—and continue horizontally to the right from g. It remains to show that s can be chosen so that $\rho(\sigma(f, \Lambda))$ passes below x_1. This is so because as $s \to 0$, $f^t_2 \to \infty$. Thus, for s close enough to 0, the ray through f is arbitrarily close to being vertical, and its symmetric image with respect to Λ is arbitrarily close to being horizontal. Thus, for s close enough to 0, this ray does pass below x_1.

We now observe that I and J^t meet at f^t (and at any point on the vertical half-line emanating from f^t). They do not meet anywhere else; indeed J^t lies above the horizontal line of ordinate ω_{12} whereas the part of I that extends to the right of d is entirely below that line.

It follows from our construction so far that $\bar{z} \in P(R, \omega)$ and $x \in P(R, \omega')$. The remaining requirement that we impose on agent 1's preferences is to guarantee that in (R, ω), no allocation is Pareto-indifferent to $\bar{z}$, and that in (R, ω'), no allocation is Pareto-indifferent to x. For that purpose, we modify the curve I defined earlier. Let $\varepsilon > 0$. Instead of choosing I to contain a segment of slope s through x_1, we choose it to contain a segment of slope $s - \varepsilon$ to the left of x_1, its leftmost point d^ε being on $\rho(\bar{z}_1)$, and a segment of slope $s + \varepsilon$ to the right of x_1. (How far this segment extends to the right is determined below.) Above d^ε_1, I continues from d^ε as it continued from d in the previous construction (with a segment of slope $\frac{1}{s}$) until it meets Λ—let e^ε be this meeting point—followed by a vertical segment. By the symmetry of the preferences, J still contains the segment of slope s that extends from Λ to $\rho(\bar{z}_2)$. It has to continue upwards from its intersection with $\rho(\bar{z}_2)$ with a segment of slope $\frac{1}{s-\varepsilon}$ until it meets $\sigma(x_1, \Lambda)$. Let h^ε be the point of intersection. Then, from h^ε, it has to continue upwards with a segment of slope $\frac{1}{s+\varepsilon}$. Thus, J^t continues upwards from the translate of h^ε by $(\beta, 0)$ with a segment of slope $\frac{1}{s-\varepsilon}$ followed by a segment of slope $\frac{1}{s+\varepsilon}$. Let $f^{t\varepsilon}$ be the intersection of the second segment with the vertical half-line emanating from e^ε, which is part of agent 1's indifference curve through x_1. Let $f_2^\varepsilon \equiv f_2^{\varepsilon t} - (\beta, 0)$. The specification of I can now be completed. By the symmetry of the preferences, we have to extend the segment of slope $s + \varepsilon$ emanating from x_1 until it meets $\rho(\sigma(f^\varepsilon, \Lambda))$, and we have to extend I horizontally to the right from $\sigma(f^\varepsilon, \Lambda)$. (Thus, as compared to our previous construction, the ray through x_1 is an additional ray of kinks for the map.)

Step 2: Concluding. Let $\varphi \in EP$ and $z \equiv \varphi(R, \omega)$. Since $\bar{z} \in P(R, \omega)$ and $\varphi \in P$, then either $\bar{z}_1 \; R_1 \; z_1$ or $\bar{z}_2 \; R_2 \; z_2$. By the symmetry of the preferences and because of the equality $\omega_1 = \omega_2$, we can assume, without loss of generality, that the former statement holds. Let $z' \equiv \varphi(R, \omega')$. By (ii), $z'_1 \; R_1 \; x_1$. Note that line$(-1, \bar{z}_1)$ supports agent 1's indifference curve through $\bar{z}_1$ at that point and that $\bar{z}_1$ is the only point of contact of that line with that indifference curve; also, line$(-1, \omega_2)$ supports agent 2's indifference curve at $\bar{z}_2$. Thus, not only $\bar{z} \in P(R, \omega)$ but also in (R, ω), no allocation is Pareto-indifferent to $\bar{z}$. By Observation 3, the projection of $P(R, \omega)$ onto agent 1's consumption space is contained in bt$(\bar{z})$. Also, line$(s, \bar{x}_1)$ supports agent 1's indifference curve through x_1 at that point and x_1 is the only point of contact of that line with that indifference curve; moreover, line(s, x_2) supports agent 2's indifference curve at x_2. Thus, not only $x \in P(R, \omega')$ but also in (R, ω'), no allocation is Pareto-indifferent to x. Thus, by Observation 3, the projection of $P(R, \omega')$ onto agent 1's consumption space is contained in bt(x). Thus, $z'_1 \geq x_1$. Note that for each $a \geq \bar{z}_1$, $a \; P_1 \; \bar{z}_1$. Altogether then, and since $z'_1 - (\omega'_1 - \omega_1) \geq \bar{z}_1$, we obtain $(z'_1 - (\beta, 0)) \; P_1 \; \bar{z}_1 \; R_1 \; z_1$, in violation of *borrowing-proofness*. □

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