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Bertrand competition in markets with fixed costs*

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Abstract

This paper provides necessary and sufficient conditions for the existence of a pure strategy Bertrand equilibrium in a model of price competition with fixed costs. It unveils an interesting and unexplored relationship between Bertrand competition and natural monopoly. That relationship points out that the non-subadditivity of the cost function at the output level corresponding to the oligopoly break-even price, denoted by $D(p_L(n))$, is sufficient to guarantee that the market supports a (not necessarily symmetric) Bertrand equilibrium in pure strategies with two or more firms supplying at least $D(p_L(n))$. Conversely, the existence of a pure strategy equilibrium ensures that the cost function is not subadditive at every output greater than or equal to $D(p_L(n))$. JEL Classification: D43, L13.

Key words: Bertrand competition; cost subadditivity; fixed costs; natural monopoly.

1 Introduction

In Industrial Organization, the simplest model of price competition, called 'Bertrand competition' in honor of its initiator Joseph Bertrand, studies the

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market of a homogenous good in which a small number of firms simultaneously post a price and commit to sale the quantity of the firm's product that consumers demand given those posted prices.

The classical result in the literature on Bertrand competition is the well known Bertrand's paradox, according to which, if firms are identical, the average cost is constant, and total revenues are bounded, all Nash equilibria in the mixed extension of the pricing game are characterized by two or more firms charging the marginal cost (Harrington, 1989).

With unbounded revenues, there are also mixed strategy equilibria in the game where prices always exceed the marginal cost (Baye and Morgan, 1999; Kaplan and Wettstein, 2000). However, such equilibria are ruled out by the usual assumptions on the demand function, namely, continuity and a finite choke-off price. Thus, under reasonable market conditions, the message coming out from Bertrand competition is that the perfectly competitive outcome, with price equal to marginal cost and zero equilibrium profits, is achieved independently of the number of firms in the market.[1]

In recent years, there has been a renewed interest for reexamining the Bertrand's paradox under different cost conditions. A remarkable work in this literature is offered by Dastidar (1995), who showed that with an increasing average cost, consistent with decreasing returns production technology, the paradox does not hold. To be precise, Dastidar proved that, if firms have identical, continuous, and convex cost functions, price competition à la Bertrand typically leads to multiple pure strategy Nash equilibria.[2] If the costs are sufficiently convex, even the joint profit-maximizing price can be in the range of equilibrium prices; and, it may be actually easier to achieve that outcome when there are more firms in the market (Dastidar, 2001).

The driving force behind the existence of multiple pure strategy equilibria in Bertrand competition with convex costs is easy to picture. Since the average cost increases too fast, each firm has an incentive to avoid being the only firm charging the lowest price and supplying the whole market. This explains why undercutting the other firms is not profitable in equilibrium even if they price above the marginal cost.

[1] If firms cannot observe their rivals' costs, Spulber (1995) showed that all firms pricing above the marginal cost and getting positive expected profits is an equilibrium. However, as the number of firms increases, equilibrium prices converge to the average cost.

[2] If firms have asymmetric costs, a pure strategy equilibrium always exists; it could be unique or not; and in any such equilibrium firms with positive sales charge the same price.

The same force also operates in the mixed extension. Therefore, under decreasing returns to scale, there is not only multiple pure strategy equilibria, but also a continuum of mixed strategy equilibria with continuous support (Hoernig, 2002). Further, any finite set of equilibrium prices that lead to positive equilibrium profits can be supported in a mixed strategy equilibrium. Unbounded returns are not necessary to derive this result.

Interestingly, the set of Nash equilibria becomes suddenly smaller if firms can cooperate in the pricing game. To elaborate, if firms possess an identical and increasing average cost, Bertrand competition admits a unique and symmetric coalition-proof Nash equilibrium (Chowdhury and Sengupta, 2004). The equilibrium price is decreasing in the number of firms; and, in the limit, it converges to the competitive price. On the other hand, if firms have asymmetric costs and they share the market according to the competitive supply of each firm, a coalition-proof Nash equilibrium always exists. The minimum price charged in any equilibrium where firms do not use weakly dominated strategies is above the competitive price; but, it converges to the marginal cost as the number of firms increases.[3]

Regarding Bertrand competition with decreasing average costs, the literature argues that the existence of a Nash equilibrium is problematic. Indeed, if the cost function is concave, Dastidar (2006) has recently shown that under the standard 'equal sharing' tie-breaking rule, which roughly means that consumers split equally among the firms that charge the lowest price, Bertrand competition does not possess a Nash equilibrium in pure strategies.[4] The existence of mixed strategy equilibria remains an open question.[5] Since oligopoly theory is most relevant in markets with significant scale economies, Shapiro (1989, pgs. 344-345) reckoned that the nonexistence of equilibria is a serious drawback of the model.

By contrast, the model admits a Nash equilibrium if prices vary over a grid. For instance, in a symmetric duopoly with linear demand, a constant

[3]The limiting properties of the set of coalition-proof Nash equilibria contrast with those of the set of pure strategy Nash equilibria. In effect, if the average cost is increasing, Novshek and Chowdhury (2003) have shown that the limit set of pure equilibrium prices includes the competitive price, but it is not a singleton.

[4]Under the less common 'winner-take-all' tie-breaking rule, a zero profit Nash equilibrium exists if and only if the monopoly profit function has an initial break-even price. In addition, if the function is left lower semi-continuous and bounded from above, the zero profit's outcome is unique (Baye and Morgan, 1999).

[5]Hoernig (2007, pg. 582) offers an example with increasing returns to scale and the equal sharing rule where neither pure nor mixed strategy equilibria exist.

marginal cost, and an avoidable fixed cost, Chaudhuri (1996) showed that in the limit, as the size of the grid becomes very small, there is a unique equilibrium with a single firm pricing at the average cost, supplying the whole market, and earning zero profits. This result has been extended later on by Chowdhury (2002) to the case with asymmetric firms, finding that as the size of the grid approaches zero, the equilibrium prices converge to the limit-pricing outcome where the price charged by the most efficient firm is just low enough to prevent entry.

Surprisingly, the analysis of Bertrand competition under the more familiar case of U-shaped average costs has not received much attention in the literature. To the best of our knowledge, there are only two papers that deal with this matter. The first article, due to Novshek and Chowdhury (2003), finds that as the market becomes large the equilibrium set is empty for some parameter values, and it comprises a whole interval of prices for others. The lower bound of this interval is bounded away from the minimum average cost. No conditions are provided to guarantee equilibrium existence.

The second article, due to Yano (2006a), studies a pricing game with a more complex set of strategies. Specifically, the strategy of each firm is a paring of a unit price and the set of quantities that the firm is indifferent to sell at that unit price.[6] When the total amount that buyers wish to acquire at a given price is different from the amount that firms offer to sell at that price, each agent on the long side gets to trade proportionately to the amount that he desires in such a way that the equilibrium between demand and supply is reestablished. Yano argues that, by incorporating this rationing process, the resulting pricing game may be thought of as belonging to the family of Bertrand-Edgeworth price games.[7] Several equilibria arise in this framework, including the standard equilibrium in Bertrand competition and the contestable outcome (see also Yano, 2006b).

Taking Dastidar (1995) as a benchmark for our analysis, in this paper we reexamine price competition in a homogenous good market with U-shaped average costs. Like Dastidar, we suppose that the total cost function $C(\cdot)$ exhibits an increasing marginal cost. However, following Grossman (1981), we assume that the total cost is given by the sum of a continuous and con-

[6]The firm is indifferent between any two quantities at a given price if they give rise to the same profit.

[7]For the difference between Bertrand and Bertrand-Edgeworth competition, see Vives (1999, Chap. 5).

vex variable cost function, $VC(\cdot)$, and a fixed cost, $F \geq 0$. Telser (1991) calls this type of markets, with a U-shaped average cost curve, 'Viner industries'. Since we do not restrict a priori the nature of the fixed cost, the paper accommodates cases where the fixed cost is (i) completely avoidable, (ii) partially avoidable, and (iii) unavoidable, in which case our model coincides with Dastidar's (1995). In contrast with what happens in the latter case, the cost functions corresponding to the first two situations suffer from discontinuities and non-convexities around the origin, making the analysis of equilibrium existence a nontrivial matter.

Within the framework briefly depicted above, this paper investigates necessary and sufficient conditions for the existence of a pure strategy Nash equilibrium. When the fixed cost is fully avoidable, we find an interesting relationship between Bertrand competition and cost subadditivity.[8] That relationship says that the non-subadditivity of the cost function at the output level corresponding to the oligopoly break-even price, denoted by $D(p_L(n))$, is sufficient to guarantee that the market supports a (not necessarily symmetric) equilibrium in pure strategies with two or more firms supplying at least $D(p_L(n))$. Conversely, the existence of a pure strategy equilibrium ensures that the cost function is not subadditive at every output greater than or equal to $D(p_L(n))$. As a by-product, the latter implies that the average cost is not decreasing over the mentioned range of outputs.

In addition, this work also reconsiders under the cost conditions specified above the relationship between Bertrand equilibrium and price-taking or competitive equilibrium. We find that the latter is sufficient but not necessary for Bertrand competition to possess an equilibrium in pure strategies. Thus, given that in our framework a price-taking equilibrium always exists when the fixed cost is fully unavoidable, we derive as a corollary from the previous statement an existence result for the case with decreasing returns to scale much simpler than Dastidar (1995).

The rest of the paper is organized as follows. Section 2 describes the model and the equilibrium concept, referred to as Bertrand equilibrium. Section 3 deals with the relationship between price-taking equilibria and Bertrand equilibria. Section 4 contains the main results of this article, asso-

[8] A cost function $C(\cdot)$ is subadditive at $q \in \mathbb{R}$ if the cost of producing q with a single firm is smaller than the sum of the costs of producing it separately with a group of two or more identical firms. As Baumol (1977) pointed out, subadditivity of the cost function is a necessary and sufficient condition for natural monopoly.

ciating the existence of Bertrand equilibria with the non-subadditivity of the cost function. For expositional convenience, some of the proofs of Section 4 as well as an example of the model are displayed in the Appendix, which appears at the end of the paper. Final remarks are done in Section 5.

2 The model

Consider the market of a homogenous good, with a unit price P and an aggregate demand $D(P)$. Let $N = \{1, 2, \ldots, n\}$, $n \geq 2$, be the set of firms operating in the market. Suppose each firm $i \in N$ competes for the market demand $D(\cdot)$ by simultaneously and independently proposing to the costumers a price p_i from the interval $[0, \infty)$. Let $q_i = q_i(p_i, \mathbf{p}_{-i})$ denote firm i's output supply as a function of $(p_i, \mathbf{p}_{-i})$, where $\mathbf{p}_{-i} = (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n)$ is the list of prices chosen by the other firms.

The following assumptions complete the description of the model.

Assumption 1 The aggregate demand $D(\cdot)$ is bounded on $\mathbb{R}_+$; that is, there exist $K > 0$ and $\overline{P} > 0$ such that $D(0) = K$ and $D(P) = 0$ for all $P \geq \overline{P}$. In addition, $D(\cdot)$ is twice continuously differentiable and decreasing on $(0, \overline{P})$; i.e., $\forall P \in (0, \overline{P})$, $D'(P) < 0$.

Assumption 2 For each firm $i \in N$, the production cost associated with any output level $q_i \in \mathbb{R}_+$ is given by

$$C(q_i) = \begin{cases} VC(q_i) + F & \text{if } q_i > 0, \\ C(0) & \text{if } q_i = 0, \end{cases}$$

where $F \geq 0$ represents a fixed cost, $C(0) \in [0, F]$, and $VC(\cdot)$ is a variable cost function, which is twice continuously differentiable, increasing and convex on $\mathbb{R}_+$, with $VC(0) = 0$ and $0 \leq VC'(0) < \overline{P}$.

Although Assumption 2 does not specify the *nature* of the fixed cost, which is determined by the value of $C(0)$, in the rest of the paper we consider two possibilities. The first case takes place when $C(0) = F$, which means that the fixed cost F is unavoidable. The second possibility occurs when F is positive and $C(0) \in [0, F)$, implying that the fixed cost can be completely or partially avoided by producing no output. In contrast with the first case, which has been studied in Dastidar (1995), in the second the cost function

$C(\cdot)$ is not only discontinuous but also non-convex around the origin. As we explain in Section 3, these two scenarios result in quite different predictions regarding the existence of Nash equilibria in Bertrand competition.

Our next assumption determines the individual demand faced by each firm for every possible profile of prices. To do that, we adopt the standard market sharing rule used in the literature on price competition, according to which the market demand is equally split between the firms that charge the lowest price, and the remaining firms sell nothing.[9]

Assumption 3 For each firm $i \in N$ and every $(p_i, \mathbf{p}_{-i}) \in [0, \infty)^n$, the demand of i at $(p_i, \mathbf{p}_{-i})$, denoted by $d_i(p_i, \mathbf{p}_{-i})$, is defined as follows:

$$d_i(p_i, \mathbf{p}_{-i}) = \begin{cases} D(p_i) & \text{if } p_i < p_j \ \forall j \in N \backslash \{i\}, \\ \frac{D(p_i)}{m} & \text{if } p_i \leq p_j \ \forall j \ \& \ p_i = p_{k_t} \ \forall t = 1, \ldots, m-1, \\ 0 & \text{if } p_i > p_j \text{ for some } j \in N \backslash \{i\}. \end{cases} \quad (1)$$

As is usually the case in Bertrand competition, we assume that each firm always meets all its demand at the price it has announced. Formally,

Assumption 4 For all $i \in N$, and all $(p_i, \mathbf{p}_{-i}) \in [0, \infty)^n$, $q_i(p_i, \mathbf{p}_{-i}) = d_i(p_i, \mathbf{p}_{-i})$.

Let $H : [0, \overline{P}] \times N \to \mathbb{R}$ be such that, for all $p \in [0, \overline{P}]$ and all $m \in N$,

$$H(p, m) = p \, \frac{D(p)}{m} - C\left(\frac{D(p)}{m}\right).$$

Assumption 5 For each $m \in N$, $H(\cdot, m)$ is strictly quasi-concave on $(0, \overline{P})$, with $p^h(m) = \arg\max_{p \in (0, \overline{P})} H(p, m)$; and, for all $m \neq 1$, $0 < H(p^h(m), m) < H(p^M, 1)$, where $p^M = p^h(1)$.

Assumption 5 guarantees that, for every $m \in N$, $H(\cdot, m)$ has an interior maximum. This is because $H(0, m) = -VC(K/m) - F < 0$ and $H(\overline{P}, m) = -C(0) \leq 0$. In addition, it also ensures that the monopoly obtains the greatest maximal benefits.

The model of price competition described above follows Dastidar (1995).[10] The only difference is that in our framework F is a fixed cost which

[9] For price competition under alternative sharing rules, see among others Baye and Morgan (2002), Dastidar (2006), and a recent article by Hoernig (2007).

[10] See Appendix A.1 for an example of the model with two firms and a linear demand.

may or may not be avoided by producing zero output. On the contrary, in Dastidar (1995) only unavoidable fixed costs are considered, although it is not explicitly stated in that way. Apart from this, the two models are similar.

Let $\pi_i(p_i, \mathbf{p}_{-i}) = p_i\, d_i(p_i, \mathbf{p}_{-i}) - C(d_i(p_i, \mathbf{p}_{-i}))$ be firm i's profit function. We denote by $G_n = \langle [0,\infty),\, \pi_i \rangle_{i \in N}$ the price competition game defined by Assumptions 1 – 4. A ***pure strategy Bertrand equilibrium*** (PSBE) for G_n is a profile of prices $(p_i, \mathbf{p}_{-i}) \in [0,\infty)^n$ such that, for each $i \in N$ and all $\hat{p}_i \in [0,\infty)$, $\pi_i(p_i, \mathbf{p}_{-i}) \geq \pi_i(\hat{p}_i, \mathbf{p}_{-i})$. We denote by $\mathcal{B}(G_n)$ the set of all such equilibria, and by $\mathcal{S}(G_n) \subseteq \mathcal{B}(G_n)$ the subset of *symmetric* pure strategy equilibria, where for all $(p_1, \ldots, p_n) \in \mathcal{S}(G_n)$ and all $i, j \in N$, $p_i = p_j$.

3 Price-taking and Bertrand equilibria

We begin this section by showing that, independently of the nature of the fixed cost, the existence of a price-taking equilibrium (yet to be defined) in the homogenous good market described in Section 2 is a sufficient condition for a pure strategy Bertrand equilibrium to exist.

To do that, let $E_n = \langle N, D(\cdot), C(\cdot) \rangle$ represent the homogenous good market where every firm $i \in N$ maximizes the function $\Pi_i(P, Q_i) = P\, Q_i - C(Q_i)$ with respect to $Q_i \in \mathbb{R}_+$ taking the price $P > 0$ as given. Suppose as before that $D(\cdot)$ and $C(\cdot)$ satisfy Assumptions 1 and 2, respectively. A ***price-taking equilibrium*** (PTE) for E_n is a price $P^C \in (0, \overline{P})$ and a profile of outputs $(Q_1^C, \ldots, Q_n^C) \in \mathbb{R}_+^n$ with the property that, for each firm $i \in N$,

$$Q_i^C \in \arg\max_{Q_i \in \mathbb{R}_+} \; \Pi_i(P^C, Q_i), \tag{2}$$

and

$$\sum_{i=1}^{n} Q_i^C = D(P^C). \tag{3}$$

Notice that, by Assumption 2, for all $i \in N$ and any $P^C \in (0, \overline{P})$, $\Pi_i(P^C, Q_i) = P^C\, Q_i - C(Q_i)$ is concave on $Q_i \in \mathbb{R}_{++}$, and $\Pi_i(P^C, 0) = P^C\, 0 - C(0) = -C(0)$. Hence, a unique output $Q_i^C \in \mathbb{R}_+$ satisfying (2) always exists. Moreover, since firms are identical, $Q_1^C = \ldots = Q_n^C$. Denote this common value by Q^C. By equation (3), $Q^C = D(P^C)/n$. Hence, abusing the notation, in what follows we denote a PTE by the pair (P^C, Q^C).

Proposition 1 *Let Assumptions 1–4 hold. If* (P^C, Q^C) *is a price-taking equilibrium for* $E_n = \langle N, D(\cdot), C(\cdot)\rangle$*, then* $(p_1, \ldots, p_n) = (P^C, \ldots, P^C)$ *is a pure strategy Bertrand equilibrium for* $G_n = \langle [0, \infty), \pi_i \rangle_{i \in N}$.

Proof Let (P^C, Q^C) be a PTE for $E_n = \langle N, D(\cdot), C(\cdot)\rangle$, where $P^C \in (0, \overline{P})$ and $Q^C = D(P^C)/n$. By Assumption 1, $Q^C > 0$. Therefore, (2) implies that $P^C Q^C - C(Q^C) \geq P^C 0 - C(0) = -C(0)$. Consider the game $G_n = \langle [0, \infty), \pi_i \rangle_{i \in N}$ and the strategy profile $\mathbf{p}^C = (P^C, \ldots, P^C)$. Notice that, for all $i \in N$, $\pi_i(P^C, \ldots, P^C) = P^C \frac{D(P^C)}{n} - C\left(\frac{D(P^C)}{n}\right)$. Hence, for all $i \in N$, $\pi_i(P^C, \ldots, P^C) \geq -C(0)$. Suppose, by contradiction, $\mathbf{p}^C \notin \mathcal{B}(G_n)$. Then, there must exist a firm $i \in N$ and a price $\hat{p}_i \in [0, \infty)$ such that $\pi_i(\hat{p}_i, \mathbf{p}^C_{-i}) > \pi_i(P^C, \mathbf{p}^C_{-i})$. If $\hat{p}_i > P^C$, then $d_i(\hat{p}_i, \mathbf{p}^C_{-i}) = 0$, meaning that $\pi_i(\hat{p}_i, \mathbf{p}^C_{-i}) = \hat{p}_i 0 - C(0) = -C(0)$, which stands in contradiction with the fact that $\pi_i(\hat{p}_i, \mathbf{p}^C_{-i}) > \pi_i(P^C, \mathbf{p}^C_{-i})$. Therefore, $\hat{p}_i < P^C$; and, by (1), $d_i(\hat{p}_i, \mathbf{p}^C_{-i}) = D(\hat{p}_i) > 0$. If $\hat{p}_i = 0$, then $\hat{q}_i = K$ and $\pi_i(\hat{p}_i, \mathbf{p}^C_{-i}) = -C(K) < -C(0)$, a contradiction. Thus, $\hat{p}_i > 0$. Let $\hat{Q}_i = \arg\max_{Q \in \mathbb{R}_+} \Pi_i(\hat{p}_i, Q)$. Note that, since $\hat{p}_i < P^C$, $\Pi_i(\hat{p}_i, \hat{Q}_i) = \max_{Q \in \mathbb{R}_+}\{\hat{p}_i \cdot Q - C(Q)\} \leq \max_{Q \in \mathbb{R}_+}\{P^C \cdot Q - C(Q)\} = \Pi_i(P^C, Q^C)$. However, $\Pi_i(P^C, Q^C) = \pi_i(P^C, \mathbf{p}^C_{-i}) < \pi_i(\hat{p}_i, \mathbf{p}^C_{-i})$. Therefore, $\max_{Q \in \mathbb{R}_+}\{\hat{p}_i \cdot Q - C(Q)\} < \hat{p}_i \cdot D(\hat{p}_i) - C(D(\hat{p}_i))$, a contradiction. Thus, $(P^C, \ldots, P^C) \in \mathcal{B}(G_n)$. ∎

The previous proposition shows that the existence of a price-taking equilibrium in a homogenous good market, with a finite number of identical firms and demand and cost functions that satisfy our assumptions, is sufficient to guarantee the existence of a pure strategy equilibrium when firms compete in prices à la Bertrand, instead of taking the market price as given.

A similar result has been previously stated in Vives (1999, pg. 120) for the case where firms have identical, increasing, smooth, and convex costs. The contribution of Proposition 1 is to show that the claim is also valid in markets with fixed costs, regardless of whether the fixed cost is unavoidable or (totally or partially) avoidable.[11] A direct implication is therefore that in our framework the continuity and convexity around the origin of the cost function $C(\cdot)$ are not required to derive the result in question.

[11] This paper allows F to be equal to 0. Thus, Proposition 1 and Corollary 1 below also hold in the more familiar case with decreasing returns to scale and no fixed cost.

A second and perhaps more interesting implication obtained from Proposition 1 is summarized in Corollary 1 below. This corollary offers an existence result for the case analyzed in Dastidar (1995) that is simpler and more intuitive than the existing one. In short, it shows that when the fixed cost is unavoidable a price-taking equilibrium always exists and, consequently, that the set of symmetric pure strategy Bertrand equilibria is nonempty.[12]

Corollary 1 *Let Assumptions 1–4 hold. If the fixed cost is unavoidable, then the set of symmetric pure strategy equilibria* $\mathcal{S}(G_n)$ *is nonempty.*

Proof Consider the homogenous good market $E_n = \langle N, D(\cdot), C(\cdot)\rangle$ introduced above, where each firm $i \in N$ maximizes $\Pi_i(P, Q_i)$ with respect to $Q_i \in \mathbb{R}_+$ taking the price $P > 0$ as given. Suppose that $C(0) = F$. We wish to prove that E_n has a PTE.

Fix any price $P \in (0, \overline{P})$ and any firm $i \in N$, and let $Q_i^*(P) = \arg\max_{Q_i > 0} \Pi_i(P, Q_i)$. By Assumption 2, $Q_i^*(P)$ exists and is unique (recall that $\Pi_i(P, \cdot)$ is concave on $\mathbb{R}_{++}$). Moreover, using the first order condition, $Q_i^*(P) = MC^{-1}(P)$, where $MC^{-1}(\cdot)$ denotes the inverse of $VC'(\cdot)$, which exists because $VC'(\cdot)$ is increasing on $\mathbb{R}_+$. Notice that,

$$\Pi_i(P, Q_i^*(P)) = Q_i^*(P)\left[P - \frac{VC(Q_i^*(P))}{Q_i^*(P)}\right] - F > -F = \Pi_i(P, 0),$$

because by the first order condition, $P = VC'(Q_i^*(P))$ and, by Assumption 2, $VC'(Q_i^*(P)) > \frac{VC(Q_i^*(P))}{Q_i^*(P)}$. Therefore, for every price $P \in (0, \overline{P})$ and every firm $i \in N$, the optimal output supply of i at P is given by $Q_i^*(P) = MC^{-1}(P)$; and, since firms are identical, the total market supply is $S(P) = \sum_{i \in N} Q_i^*(P) = n\,[MC^{-1}(P)]$.

We can now calculate the equilibrium price by solving the equation $D(P) = n\,[MC^{-1}(P)]$, which has a solution on $(0, \overline{P})$ due to our assumptions on the demand and cost functions. Denote this value by P^*. Clearly, $(P^*, Q_i^*(P^*))$ is a PTE for E_n. Hence, by Proposition 1, the profile $(p_1, \ldots, p_n) = (P^*, \ldots, P^*) \in \mathcal{S}(G_n)$. ∎

The reader may now wonder whether or not the existence result stated in Corollary 1 can be extended to the case where the fixed is avoidable.

[12] Actually, the set of symmetric PSBE is typically a whole interval of prices. For experimental evidence regarding this result, see Klaus and Brandts (2008).

Unfortunately, the answer to that query is that it is relatively simple to construct examples where neither a price-taking equilibrium nor a pure strategy Bertrand equilibrium exist. In Appendix A.1 we analyze in detail one of these examples. We show that in a duopoly with linear demand and quadratic variable costs, if the fixed cost can be completely avoided by producing no output, then depending upon its value the set of pure strategy Bertrand equilibria may be empty.[13]

The example discussed in the appendix is also useful to illustrate that, in our framework with a U-shaped average cost curve, Baye and Morgan's (1999) condition, namely, the existence of an initial break-even price in the monopoly profit function $H(\cdot, 1)$, is not enough to rule out the nonexistence of equilibria. Thus, a sensible question to ask is what conditions (if any) prevent this from happening. Finding these conditions will occupy the remainder of the paper.

4 Cost subadditivity and Bertrand equilibrium

To begin the analysis of the problem pointed out above requires us to define a key property of the cost function, namely, subadditivity. Following Panzar (1989, pg. 23), we say that a cost function $C(\cdot)$ is ***subadditive*** at $q \in \mathbb{R}_+$ if for *every* list of outputs $q_1, \ldots, q_n$, with $q_i \in \mathbb{R}_+$ and $q_i \neq q$ for all $i = 1, \ldots, n$, it is the case that $C(q) < \sum_{i=1}^{n} C(q_i)$ whenever $\sum_{i=1}^{n} q_i = q$.

In words, $C(\cdot)$ is subadditive at q if the cost of producing q with a single firm is smaller than the sum of the costs of producing it separately with a group of two or more identical firms. As Baumol (1977) noted, this property is a necessary and sufficient condition for a natural monopoly to exist. However, subadditivity is a local property, in the sense that it refers to a particular point on the cost curve. Thus, it is possible for a market to be a natural monopoly for a certain output level but not for others.

When the cost function $C(\cdot)$ is twice differentiable and the marginal cost is increasing, there is a simple necessary condition for subadditivity. In effect, under these conditions, any output q is divided in positive portions

[13] When the marginal cost is constant, this problem has been noted by Shapiro (1989), Vives (1999) and Baye and Kovenock (2008). In fact, Baye and Kovenock (2008) showed that, with a constant marginal cost, a fully avoidable fixed cost may preclude the existence of mixed strategy equilibria too. The difference with our example is that we consider a U-shaped average cost, whereas in the previous references the average cost is decreasing.

most cheaply among n identical firms if each firm produces the same amount $q_i = q/n$. Hence, since the *minimized* cost corresponding to output q for the n-firm market is $\sum_{i=1}^{n} C(q_i) = n\,C(q/n)$, it follows that $C(\cdot)$ is subadditive at $q \in \mathbb{R}_+$ only if $C(q) < n\,C(q/n)$. If there are only two firms, this condition is also sufficient, since the requirement embedded in the definition of subadditivity that $q_i \neq q\,\forall i \in N$ implies, when $n = 2$, that $q_i \neq 0\,\forall i = 1, 2$.

We claim below in Proposition 2 that if the fixed cost is avoidable, then a necessary condition for the existence of a symmetric pure strategy Bertrand equilibrium is for the cost function *not* to be subadditive at the total demand corresponding to the n-firm oligopoly break-even price (yet to be defined).

As we will immediately see, the proof of Proposition 2 relies on three preliminary results, which are summarized for convenience in Lemmas 1, 2, and 3, respectively. The first of these lemmas shows that regardless of the number of firms in the market, there is always a price with positive demand such that, if the firms share the market equally at that price, then each of them is indifferent between staying in operation and exit the market.

Lemma 1 *Let Assumptions 1, 2 and 5 hold. For every $m \in N$, there is a price $\hat{p}(m) \in (0, p^h(m))$ such that $H(\hat{p}(m), m) = -C(0)$.*

Proof Fix any $m \in N$. By Assumptions 1 and 2, $H(0, m) = -VC(K/m) - F$. Thus, $H(0, m) < -F$; and, since $C(0) \leq F$, we have that $H(0, m) < -C(0)$. On the other hand, by Assumption 5, $H(p^h(m), m) > 0 \geq -C(0)$. Hence, by the intermediate value theorem, there is a price $\hat{p}(m) \in (0, p^h(m))$ such that $H(\hat{p}(m), m) = -C(0)$. ∎

Fix now any number of firms $m \in N$, and let $p_L(m) = \min\{\hat{p}(m) \in (0, \overline{P}) : H(\hat{p}(m), m) = -C(0)\}$. By Lemma 1, $p_L(m)$ is well defined. By Assumption 1, $D(p_L(m)) > 0$. Our next result says that for any possible number of firms $m \in N$ the corresponding break-even price $p_L(m)$ is lower than or equal to the monopoly profit-maximizing price p^M.

Lemma 2 *Let Assumptions 1, 2 and 5 hold. For each $m \in N$, $p_L(m) \leq p^M$.*

Proof See Appendix A.2 ∎

Finally, our last preliminary result demonstrates that whenever the set of symmetric pure strategy Bertrand equilibria is nonempty, the oligopoly break-even price $p_L(n)$ belongs to that set.

Lemma 3 *Let Assumptions 1–5 hold. If $\mathcal{S}(G_n) \neq \emptyset$, then the strategy profile $(p_1, \ldots, p_n) = (p_L(n), \ldots, p_L(n)) \in \mathcal{S}(G_n)$.*

Proof See Appendix A.3 ∎

We are now ready to state Proposition 2.

Proposition 2 *Let Assumptions 1–5 hold and suppose that $C(0) = 0$. If the set of symmetric pure strategy equilibrium $\mathcal{S}(G_n)$ is nonempty, then the cost function $C(\cdot)$ is not subadditive at $D(p_L(n))$.*

Proof Assume, by contradiction, that $C(\cdot)$ is subadditive at $D(p_L(n))$. (Recall that $D(p_L(n)) > 0$.) Then, it must be that producing $D(p_L(n))$ with a single firm is cheaper than producing it with n identical firms. That is,

$$C(D(p_L(n))) < n\, C\left(\frac{D(p_L(n))}{n}\right). \tag{4}$$

Adding the term $-p_L(n)\, D(p_L(n))$ to both sides of (4), we have that

$$-p_L(n)\, D(p_L(n)) + C(D(p_L(n))) < -p_L(n)\, D(p_L(n)) + n\, C\left(\frac{D(p_L(n))}{n}\right),$$

which can be rewritten as

$$p_L(n) D(p_L(n)) - C(D(p_L(n))) > n\left[p_L(n)\frac{D(p_L(n))}{n} - C\left(\frac{D(p_L(n))}{n}\right)\right]. \tag{5}$$

By the definition of $p_L(n)$, the right hand side of (5) is equal to $-n\, C(0)$. Hence, since $C(0) = 0$, (5) implies that $p_L(n)\, D(p_L(n)) - VC(D(p_L(n))) - F > 0$. By the continuity of $p\, D(p) - VC(D(p)) - F$ in p at $p_L(n)$, there exists a price $p' < p_L(n)$ such that $p'\, D(p') - VC(D(p')) - F > 0$. Fix any firm $i \in N$, and consider firm i's strategy $p'_i = p'$. By Assumption 3, $\pi_i(p'_i, (p_L(n))_{-i}) = p'_i\, D(p'_i) - VC(D(p'_i)) - F$. Hence, $\pi_i(p'_i, (p_L(n))_{-i}) > 0$. On the other hand, $\pi_i(p_L(n), \ldots, p_L(n)) = H(p_L(n), n) = 0$. Thus, firm i can profitably deviate from $p_L(n)$ to p'_i at $(p_L(n), (p_L(n))_{-i})$, contradicting that, by Lemma 3, the profile $(p_L(n), (p_L(n))_{-i}) \in \mathcal{S}(G_n)$. Therefore, $C(\cdot)$ is not subadditive at $D(p_L(n))$. ∎

Proposition 2 formalizes the intuitive idea that, if the fixed cost is avoidable (or, there is no fixed cost at all), then a necessary condition for the existence of a symmetric pure strategy Bertrand equilibrium in a homogenous good market is for the market *not* to be a natural monopoly at the output level corresponding to the oligopoly break-even price.[14]

What about the converse of Proposition 2? Can we say that the nonexistence of a natural monopoly at the oligopoly break-even price is also *sufficient* for $\mathcal{S}(G_n) \neq \emptyset$? As we argue in Proposition 3, the answer to this question is affirmative if there is a price-taking equilibrium in the market, because in that case Proposition 1 ensures that the set of symmetric pure strategy equilibria is always nonempty.

More interestingly, the converse of Proposition 2 also holds in a duopoly, independently of the nature of the fixed cost (i.e., regardless of the value of $C(0)$). The reason behind this result is found in the next lemma.

Lemma 4 *Let Assumptions 1–5 hold. If* $C(D(p_L(n))) \geq n\, C\left(\frac{D(p_L(n))}{n}\right)$*, then the profile* $(p_1, \ldots, p_n) = (p_L(n), \ldots, p_L(n))$ *constitutes a pure strategy Bertrand equilibrium for* G_n*.*

Proof Suppose, by contradiction, $(p_L(n), \ldots, p_L(n)) \notin \mathcal{S}(G_n)$. Then, there must be a price $\tilde{p} \in (0, p_L(n))$ such that $H(\tilde{p}, 1) > -C(0) = H(p_L(n), n)$. By the definition of $p_L(n)$, $D(p_L(n)) > 0$. Thus, the hypothesis in Lemma 4, i.e., $C(D(p_L(n))) \geq n\, C\left(\frac{D(p_L(n))}{n}\right)$, can be rewritten as

$$VC(D(p_L(n))) \geq n\, VC\left(\frac{D(p_L(n))}{n}\right) + (n-1)F. \tag{6}$$

Using the definition of $p_L(n)$, it is easy to see that the right hand side of (6) is equal to

$$p_L(n)\, D(p_L(n)) - F + n\, C(0). \tag{7}$$

Hence, substituting (7) into (6), it follows that $H(p_L(n), 1) \leq -C(0)$. However, this contradicts that, by Assumption 5, $H(\cdot, 1)$ is quasi-concave on $(0, \overline{P})$, because $p_L(n) \in (\tilde{p}, p^M)$ and $H(p_L(n), 1) \leq -C(0) < \min\{H(\tilde{p}, 1),\, H(p^M, 1)\}$. ∎

Overall Lemma 4 and Proposition 1 allow us to state the following result:

[14] This result does not hold if $C(0) \neq 0$. In effect, suppose that $n = 3$ and that $D(P) = 10 - P$, and let $C(q_i) = \frac{3}{22} q_i^2 + \frac{15}{2}$ if $q_i > 0$, and $C(0) = \frac{15}{2}$. Routine calculations show that $p_L(3) = \frac{10}{23}$ and that $H(10/23, 1) \approx -15.82$. Hence, $(p_1, p_2, p_3) = \left(\frac{10}{23}, \frac{10}{23}, \frac{10}{23}\right)$ is a PSBE. However, $C(\cdot)$ is subadditive at $D(10/23)$, because $C(D(10/23)) \approx 19.98$, $2\, C(D(10/23)/2) \approx 21.24$, and $3\, C(D(10/23)/3) \approx 26.66$.

Proposition 3 *Let Assumptions 1–5 hold and suppose that either* (P^C, Q^C) *is a price-taking equilibrium for* $E_n = \langle N, D(\cdot), C(\cdot)\rangle$*, or that there are only two firms in the market. Then, if* $C(\cdot)$ *is not subadditive at* $D(p_L(n))$*, the set of symmetric pure strategy equilibria* $\mathcal{S}(G_n)$ *is nonempty.*

Proof If (P^C, Q^C) is a PTE for $E_n = \langle N, D(\cdot), C(\cdot)\rangle$, then the desired result follows from Proposition 1. On the other hand, if $n = 2$, then $C(\cdot)$ is not subadditive at $D(p_L(2))$ if and only if $C(D(p_L(2))) \geq 2\, C\left(\frac{D(p_L(2))}{2}\right)$. Hence, by Lemma 4, $(p_L(2), p_L(2)) \in \mathcal{S}(G_2)$. ∎

In words, Proposition 3 indicates that if the homogenous good market is not a natural monopoly and either (i) there is a price-taking equilibrium, or (ii) there are exactly two firms, then the market supports a symmetric pure strategy equilibrium where firms compete in prices à la Bertrand and supply a total output that leaves each of them indifferent between staying in operation and exit the market. In particular, this result holds when the market has an unavoidable fixed cost, since in that case a PTE always exists.

By contrast, if there are more than two firms and the fixed cost is completely or partially avoidable, then the result in Proposition 3 is no longer valid. To illustrate, consider the demand and the cost function corresponding to the example analyzed in Appendix A.1. That is, let $D(P) = 10 - P$ and suppose that $C(q_i) = 1/2q_i^2 + F$ if $q_i > 0$, and that $C(0) = 0$. Assume $n = 5$ and $F = 4.3$. Then, $p_L(5) \approx 4.3983$ and $H(p_L(5), 1) \approx 4.65 > 0 = H(p_L(5), 5)$. Therefore, $(p_1, \ldots, p_5) = (p_L(5), \ldots, p_L(5))$ is not a PSBE for G_5; and, by Lemma 3, we can conclude that $S(G_5) = \emptyset$. However, it is easy to verify in this numerical example that $C(\cdot)$ is not subadditive at $D(p_L(5)) \approx 5.6017$. Indeed, producing $D(p_L(5))$ with a single firm generates a cost equal to $C(D(p_L(5))) \approx 19{,}9895$, whereas producing it with two identical firms costs $2 \cdot C(D(p_L(5))/2) \approx 16.4447$.

So, is there anything to say about Bertrand competition when the n-firm market is not a natural monopoly and there is no price-taking equilibrium? Indeed, we show next that, if the fixed cost is avoidable, then the non-subadditivity of the cost function $C(\cdot)$ at $D(p_L(n))$ is sufficient to guarantee that the market supports a (not necessarily symmetric) pure strategy Bertrand equilibrium where two or more identical firms jointly supply at least $D(p_L(n))$. And, conversely, the existence of a pure strategy Bertrand equilibrium ensures that the market is not a natural monopoly at every output greater than or equal to $D(p_L(n))$. In particular, the latter implies that the average cost is not decreasing on $[D(p_L(n)), K)$, (see Corollary 2 below and the discussion following that result).

Theorem 1 *Let Assumptions 1–5 hold and suppose* $C(0) = 0$. *If the cost function* $C(\cdot)$ *is not subadditive at* $D(p_L(n))$, *then there exist a pure strategy Bertrand equilibrium* $(p_1, \ldots, p_n) \in \mathcal{B}(G_n)$ *where* $\sum_{i\in N} q_i(p_1, \ldots, p_n) \geq D(p_L(n))$. *Conversely, if a pure strategy Bertrand equilibrium exists, then the cost function* $C(\cdot)$ *is not subadditive over* $[D(p_L(n)), K)$.

Proof See Appendix A.4. ▮

Given any output $\overline{q} > 0$, the average cost at $\overline{q}$ is defined as $AC(\overline{q}) = C(\overline{q})/\overline{q}$. The average cost function $AC(\cdot)$ is decreasing at $\overline{q}$ if there exists a $\delta > 0$ such that for all $q', q'' \in (\overline{q} - \delta, \overline{q} + \delta)$, with $q' < q''$, $AC(q'') < AC(q')$. Additionally, $AC(\cdot)$ is said to decrease *through* $\overline{q}$ if for all $q', q'' \in (0, \overline{q}]$, with $q' < q''$, $AC(q'') < AC(q')$, (Panzar, 1989, pg. 24). If, like in our case, $C(\cdot)$ is twice continuously differentiable on $\mathbb{R}_{++}$, then (i) $AC(\cdot)$ is decreasing at $\overline{q}$ if $\frac{\partial AC(\overline{q})}{\partial q} < 0$; and (ii) $AC(\cdot)$ is decreasing *through* $\overline{q}$ if for all $q' \in (0, \overline{q}]$, $\frac{\partial AC(q')}{\partial q} < 0$, (i.e., if $AC(\cdot)$ is decreasing on $(0, \overline{q}]$).

Lemma 5 *If the average cost* $AC(\cdot)$ *is decreasing through* q, *then the cost function* $C(\cdot)$ *is subadditive at* q, *but not conversely.*

Proof See Appendix A.5. ▮

Corollary 2 *Let Assumptions 1–5 hold and suppose* $C(0) = 0$. *The set of pure strategy Bertrand equilibria* $\mathcal{B}(G_n)$ *is nonempty only if the average cost* $AC(\cdot)$ *is not decreasing on* $[D(p_L(n)), K)$.

Proof Immediate from Theorem 1 and Lemma 5. ▮

The second part of Theorem 1 and its implication in Corollary 2 are closely related with Dastidar's (2006) Proposition 3, which says that the set of Bertrand equilibria $\mathcal{B}(G_n)$ is nonempty only if $C(\cdot)$ is not concave. Hence, before closing this section, it may be worthy to underline some differences between these results.

First of all, let's emphasize that the necessary condition for equilibrium existence stated in the second part of Theorem 1 considerably sharpens

Dastidar's (2006) condition, because concavity implies subadditivity, but not conversely. Thus, we could have a cost function which is non-concave and subadditive at the same time. A function like that would violate our necessary condition for existence, whereas it wouldn't do so with Dastidar's. Secondly, in Theorem 1 we allow for avoidable fixed costs and, therefore, for discontinuities in the cost function around the origin. On the contrary, in Dastidar (2006) the cost function is continuous and $F = 0$. Finally, Theorem 1 provides not only a necessary condition for $\mathcal{B}(G_n) \neq \emptyset$, but also a sufficient condition. Instead, Dastidar (2006) only offers a necessary condition.

5 Final remarks

The main contributions of this article can be summarized by restating Propositions 1, 2 and 3, and Theorem 1. By looking at Proposition 1 (and the example in Section A.1) we see that in a market with convex variable costs and fixed costs, the existence of a price-taking equilibrium is sufficient but not necessary for a pure strategy Bertrand equilibrium to exist.[15]

This work relates the existence of Bertrand equilibria with the nonexistence of natural monopoly. In Proposition 2, we show that if the fixed cost is avoidable and there is a symmetric PSBE, then the cost function is not subadditive at the total demand corresponding to the oligopoly break-even price $p_L(n)$. This is equivalent to say that, under the previous conditions, the market cannot be a natural monopoly at $D(p_L(n))$. As Proposition 3 posits, the reverse of that statement is also true if there are exactly two firms or if there is a PTE. Numerical examples confirm, however, that the converse of Proposition 2 does not hold in other cases.

The last and the most important result of this paper is found Theorem 1, which generalizes Propositions 2 and 3 to markets with more than two firms and without a price-taking equilibrium. Theorem 1 shows that, when the fixed cost is avoidable and the cost function $C(\cdot)$ is not subadditive at $D(p_L(n))$, there always exists a Bertrand equilibrium in pure strategies, though it need not be a symmetric one. Conversely, if a pure strategy equilibrium exists, then $C(\cdot)$ is not subadditive for all of the output levels greater than or equal to $D(p_L(n))$.

[15] Grossman (1981) found a related result, but for a different equilibrium concept. Using a model similar to ours, he showed that if a PTE exists, then it is a *supply function equilibrium*. However, the latter may exist even if there is no PTE in the market.

As a final remark, notice that the results of this article could be used to determine the maximum number of firms that a homogenous good market can support under price competition. That is, they could be helpful to the theory of endogenous industry structure. This is because the break-even price $p_L(\cdot)$ is typically increasing in the number of firms that operate in the market. Thereby, a cost function could be subadditive at $D(p_L(n+1))$, but not at $D(p_L(n))$. A more comprehensive analysis of this matter as well as the study of more general forms of non-convexities and discontinuities in the cost function are left for a future research.

A Appendix

A.1 Example

Let $D(P) = 10 - P$ and $N = \{1, 2\}$. Suppose $C(q_i) = 1/2q_i^2 + F$ if $q_i > 0$, and let $C(0) = 0$. If a PTE exists, then (2) and (3) imply that $P^C = 10/3$ and $Q^C = 10/3$. However, $\Pi_i(P^C, Q^C) \geq 0$ $(= \Pi_i(P^C, 0))$ only if $F \leq 50/9\,(\approx 5.55^{'})$. Thus, if $F > 50/9$, the market does not possess a PTE.

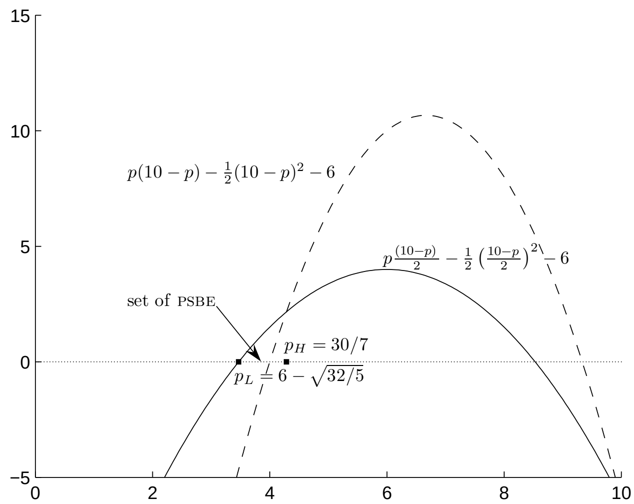

Figure 1: Existence of Bertrand equilibria ($F = 6$)

Regarding Bertrand equilibria, a pair of prices $(p_1^*, p_2^*) \in [0, \infty)^2$ constitutes a symmetric pure strategy equilibrium if and only if $p_1^* = p_2^*$ and, for all

$i \in N$, (a) $H(p_i^*, 2) \geq 0$, and (b) for all $\hat{p}_i < p_i^*$, $H(p_i^*, 2) \geq H(\hat{p}_i, 1)$. Notice that the latter condition requires that $1/2 p_i^*(10 - p_i^*) - 3/8(10 - p_i^*)^2 \leq 0$, which is satisfied whenever $p_i^* \leq 30/7\,(\approx 4.2858)$. On the other hand, from (a), it follows that $p_i^* \geq 6 - \sqrt{16 - 8/5\,F}$. Hence, a price p_i^* simultaneously satisfying both conditions exists if and only if $F \leq 400/49\,(\approx 8.1633)$.

For instance, when $F = 6$ the price $p_i^* = 4$ is a solution for (a) and (b). Therefore, the profile $(p_1^*, p_2^*) = (4, 4) \in \mathcal{S}(G_2)$. Actually, if the fixed cost $F \leq 50/9$, then the set of symmetric Bertrand equilibria includes the price-taking equilibrium; that is, $(10/3, 10/3) \in \mathcal{S}(G_2)$. However, when $F \in (50/9, 400/49)$, a PTE does not exist, but the game possesses multiple PSBE. Indeed, as Figure 1 illustrates, when $F = 6$ any price between the lower bound $p_L = 6 - \sqrt{32/5}$ (≈ 3.4702) and the upper bound $p_H = 30/7$ satisfies conditions (a) and (b) and, therefore, is part of a symmetric PSBE.

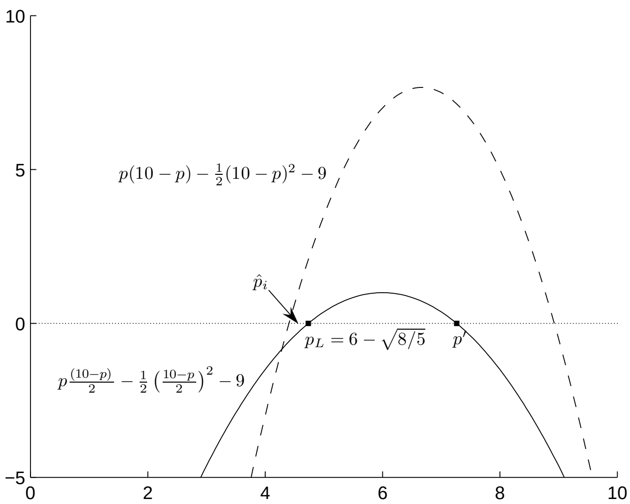

Figure 2: Nonexistence of Bertrand equilibria ($F = 9$)

It also comes to light in this example that if $F \in \left(\frac{400}{49}, 10\right]$, then a symmetric equilibrium in pure strategies does not exist. For $F = 9$, this is illustrated in Figure 2, where it can be easily seen that, for any price p for which the solid curve $H(p, 2)$ is over the horizontal axis, the dashed curve $H(p, 1)$ lies above. This implies that, whenever both firms choose any price $p \in [0, \overline{P}]$ satisfying the condition $H(p, 2) \geq 0$, there is a deviation $\hat{p}_i < p$

for one firm, say for firm i, such that $H(p, 2) < H(\hat{p}_i, 1)$.

Finally, observe that when $F = 9$ our example not only fails to possess a symmetric pure strategy equilibrium, but also a PSBE with $p_1 \neq p_2$. To see this, assume, by contradiction, such equilibrium exists. Without loss of generality, suppose that $p_1 < p_2$. Notice that $p_1 \leq p^M = 20/3$. Otherwise, firm 1 can profitably deviate to p^M. Then, by Assumption 3, $\pi_1(p_1, p_2) = H(p_1, 1)$ and $\pi_2(p_1, p_2) = 0$. Suppose first $H(p_1, 1) > 0$. Then, $q_1(p_1, p_2) > 0$; and, by the continuity of $H(p, 1) = p\,(10-p) - \frac{1}{2}\,(10-p)^2 - 9$ in p at p_1, there is a price $p_2' < p_1$ such that $H(p_2', 1) > 0 = \pi_2(p_1, p_2)$, contradicting that p_2 is firm 2's best response to p_1.

Next, observe that if $(p_1, p_2) \in \mathcal{B}(G_2)$, then $H(p_1, 1)$ cannot be negative. This is because $\pi_1(10, \hat{p}_2) = 0$ for all $\hat{p}_2 \in [0, \infty)$. Therefore, $H(p_1, 1) = 0$; and, given the shape of $H(\cdot, 1)$ displayed in Figure 2, it has to be that $p_1 = (20-\sqrt{46})/3$ (≈ 4.4059). If $H(p_2, 1) > 0$, then using the continuity of $H(p, 1)$ in p at p_2, there must be a price $p_1' < p_2$ such that $H(p_1', 1) > 0 = \pi_1(p_1, p_2)$, which would contradict that firm 1 is playing his best response against p_2. Thus, $H(p_2, 1) \leq 0$. But, since $p_2 > p_1$ and $H(p^M, 1) \approx 7.66^{'} > 0$, this implies $p_2 > p^M$. Hence, firm 1 can profitably deviate to p^M, meaning that (p_1, p_2) is not a PSBE for G_2.

A.2 Proof of Lemma 2

Suppose, by way of contradiction, that $p_L(m) > p^M$ for some $m \in N$. Then, $m \neq 1$. By Assumption 5, $H(\cdot\,, 1)$ is non-increasing at $p_L(m)$; i.e., $\frac{\partial H(p_L(m),1)}{\partial p} \leq 0$. Hence,

$$p_L(m) \geq VC'(D(p_L(m))) - \frac{D(p_L(m))}{D'(p_L(m))}. \tag{8}$$

Similarly, by Assumption 5 and the fact that, by definition, $p_L(m) < p^h(m)$, $H(\cdot\,, m)$ is non-decreasing at $p_L(m)$; i.e., $\frac{\partial H(p_L(m),m)}{\partial p} \geq 0$. Therefore,

$$p_L(m) \leq VC'\left(\frac{D(p_L(m))}{m}\right) - \frac{D(p_L(m))}{D'(p_L(m))}. \tag{9}$$

Finally, since the marginal cost $VC'(\cdot)$ is increasing and $-\frac{D(p_L(m))}{D'(p_L(m))}$ is positive, $VC'\left(\frac{D(p_L(m))}{m}\right) - \frac{D(p_L(m))}{D'(p_L(m))} < VC'(D(p_L(m))) - \frac{D(p_L(m))}{D'(p_L(m))}$; and, by (8) and (9), we get the desired contradiction. ∎

A.3 Proof of Lemma 3

To prove Lemma 3, we first derive the following auxiliary result.

Lemma 6 *Let Assumptions 1 and 2 hold. For all* $p < p^M$, $H(p,1) - H(p,n) = 0$ *implies that* $\frac{\partial[H(p,1)-H(p,n)]}{\partial p} > 0$.

Proof For every price $p < p^M$, we have that

$$H(p,1) - H(p,n) = \frac{n-1}{n}\, p\, D(p) - VC(D(p)) + VC\left(\frac{D(p)}{n}\right). \tag{10}$$

Taking the derivative of (10) with respect to p,

$$\begin{aligned} \frac{\partial[H(p,1) - H(p,n)]}{\partial p} = \frac{n-1}{n}\, D(p) + [p - VC'(D(p))]\, D'(p) - \\ - \left[p - VC'\left(\frac{D(p)}{n}\right)\right] \frac{D'(p)}{n}. \end{aligned} \tag{11}$$

Consider any $p \in [0, p^M)$ with the property that $H(p,1) - H(p,n) = 0$. Then,

$$\frac{n-1}{n}\, p\, D(p) = VC(D(p)) - VC\left(\frac{D(p)}{n}\right). \tag{12}$$

By the convexity of $VC(\cdot)$,

$$VC(D(p)) - VC\left(\frac{D(p)}{n}\right) < \frac{n-1}{n}\, D(p)\, VC'(D(p)), \tag{13}$$

and

$$VC(D(p)) - VC\left(\frac{D(p)}{n}\right) > \frac{n-1}{n}\, D(p)\, VC'\left(\frac{D(p)}{n}\right). \tag{14}$$

Thus, combining (12) and (13), we have that $p < VC'(D(p))$; and, from the expressions in (12) and (14), it also follows that $p > VC'\left(\frac{D(p)}{n}\right)$. Therefore, since $(n-1)/n\, D(p) > 0$ and $D'(p) < 0$, the right hand side of (11) is greater than zero; i.e., $\frac{\partial[H(p,1)-H(p,n)]}{\partial p} > 0$. ∎

Proof of Lemma 3 Suppose, by way of contradiction, that the strategy profile $(p_1, \ldots, p_n) = (p_L(n), \ldots, p_L(n)) \notin \mathcal{S}(G_n)$. Then, there must be a firm $i \in N$ and a price $\tilde{p}_i < p_L(n)$ such that $\pi_i(\tilde{p}_i, (p_L(n))_{-i}) > \pi_i(p_L(n), (p_L(n))_{-i})$, where $(p_L(n))_{-i}$ denotes the sub-profile of prices in which everybody except firm i chooses $p_L(n)$. Notice that $\pi_i(\tilde{p}_i, (p_L(n))_{-i}) = H(\tilde{p}_i, 1)$ and $\pi_i(p_L(n), (p_L(n))_{-i}) = H(p_L(n), n)$. Thus, $H(\tilde{p}_i, 1) -$

$H(p_L(n), n) > 0$. Moreover, since $H(p_L(n), n) = -C(0)$ and $H(\tilde{p}_i, n) < -C(0)$,[16] it also follows that $H(\tilde{p}_i, 1) - H(\tilde{p}_i, n) > 0$. Therefore, given that $H(0, 1) - H(0, n) = -VC(K) + VC(K/n) < 0$ and $H(\cdot, 1) - H(\cdot, n)$ is continuous on $[0, \tilde{p}_i]$, there must be a price $p' \in (0, \tilde{p}_i)$ such that $H(p', 1) - H(p', n) = 0$.

Next, recall that, by hypothesis, $\mathcal{S}(G_n) \neq \emptyset$. That is, there is a price $p^* \in (p_L(n), p^M)$ such that $H(p^*, n) \geq -C(0)$ and, for all $p < p^*$, $H(p, 1) - H(p^*, n) \leq 0$. Since p can be chosen arbitrarily close to p^*, by continuity, it must be that $H(p^*, 1) - H(p^*, n) \leq 0$. On the other hand, by Assumption 5, $H(p^M, 1) - H(p^M, n) > 0$. So, there has to be a price $p'' \in (p_L(n), p^M)$ such that $H(p'', 1) - H(p'', n) = 0$.

In summary, if $\mathcal{S}(G_n) \neq \emptyset$ and $(p_L(n), \ldots, p_L(n)) \notin \mathcal{S}(G_n)$, the previous two paragraphs indicate that the curves $H(\cdot, 1)$ and $H(\cdot, n)$ must intersect each other at least twice on $(0, p^M)$. Therefore, in order to show that $(p_L(n), \ldots, p_L(n))$ is indeed a symmetric pure strategy Bertrand equilibrium for G_n, it is enough to prove that there is only one such intersection; i.e., it is sufficient to show that there is a unique price $p \in (0, p^M)$ for which $H(p, 1) - H(p, n) = 0$.

Without of generality, assume that there is a pair of prices $p^\alpha, p^\beta \in (0, p^M)$, with $p^\alpha < p^\beta$, such that $H(p^\alpha, 1) - H(p^\alpha, n) = 0$ and $H(p^\beta, 1) - H(p^\beta, n) = 0$. Notice that, by Lemma 6, for $\epsilon^1 > 0$ small enough, $H(p^\alpha, 1) - H(p^\alpha, n) = 0$ implies that $H(p^\alpha + \epsilon^1, 1) - H(p^\alpha + \epsilon^1, n) > 0$. In the same way, by Lemma 6, for $\delta > 0$ small enough, $H(p^\beta, 1) - H(p^\beta, n) = 0$ implies that $H(p^\beta - \delta, 1) - H(p^\beta - \delta, n) < 0$. Hence, since $H(\cdot, 1) - H(\cdot, n)$ is continuous on $(0, p^M)$, there must be a price $p^{\alpha+1} \in (p^\alpha, p^\beta)$ such that $H(p^{\alpha+1}, 1) - H(p^{\alpha+1}, n) = 0$.

Repeating the argument of the previous paragraph, for $\epsilon^2 > 0$ small enough, $H(p^{\alpha+1}, 1) - H(p^{\alpha+1}, n) = 0$ implies that $H(p^{\alpha+1} + \epsilon^2, 1) - H(p^{\alpha+1} + \epsilon^2, n) > 0$. Hence, there must be a price $p^{\alpha+2} \in (p^{\alpha+1}, p^\beta)$ such that $H(p^{\alpha+2}, 1) - H(p^{\alpha+2}, n) = 0$. And using the same reasoning over and over again, we can construct a sequence of prices $\{p^{\alpha+s}\}_{s=1}^{\infty} \subset (p^\alpha, p^\beta)$ with the property that $H(p^{\alpha+s}, 1) - H(p^{\alpha+s}, n) = 0$ for all $s = 1, \ldots, \infty$. Observe that, by construction, each term $p^{\alpha+s}$ of the sequence is closer to p^β than

[16] Note that, $H(\tilde{p}_i, n) \neq -C(0)$ because $\tilde{p}_i < p_L(n)$ and, by definition, $p_L(n)$ is the smallest price for which $H(\cdot, n)$ equals $-C(0)$. On the other hand, since $H(0, n) < -C(0)$, $H(\tilde{p}_i, n)$ cannot be greater than $-C(0)$. Otherwise, there would be a price $p \in (0, \tilde{p}_i)$ such that $H(p, n) = -C(0)$, contradicting the definition of $p_L(n)$. Thus, $H(\tilde{p}_i, n) < -C(0)$.

what it was $p^{\alpha+s-1}$. Therefore, by Lemma 6, for some $s \geq 1$ sufficiently high, there must exist $\epsilon \in (0, \delta)$ and a price $\bar{\bar{p}} \in (p^{\alpha+s} + \epsilon, p^{\beta} - \epsilon)$ such that $H(\bar{\bar{p}}, 1) - H(\bar{\bar{p}}, n) > 0$ and $H(\bar{\bar{p}}, 1) - H(\bar{\bar{p}}, n) < 0$, which provides the desired contradiction. ∎

A.4 Proof of Theorem 1

To prove Theorem 1, the following auxiliary result will be useful.

Lemma 7 *Let Assumptions 1–2 hold. For all* $p < \overline{P}$, $\frac{\partial[H(p,1) - m\,H(p,m)]}{\partial p} > 0$.

Proof For every $p < \overline{P}$, we have that $H(p, 1) - m\,H(p, m) = -VC(D(p)) - F + m\,VC\left(\frac{D(p)}{m}\right) + m\,F$. Taking the derivative with respect to p,

$$\frac{\partial[H(p, 1) - m\,H(p, m)]}{\partial p} = D'(p)\left[VC'\left(\frac{D(p)}{m}\right) - VC'(D(p))\right], \quad (15)$$

which is positive because $D'(p) < 0$ and $VC'(\cdot)$ is increasing. ∎

Proof of Theorem 1 Assume the cost function $C(\cdot)$ is not subadditive at $D(p_L(n))$. Then, there must exist $m \in \{2, \ldots, n\}$ such that

$$C(D(p_L(n))) \geq m\,C\left(\frac{D(p_L(n))}{m}\right). \quad (16)$$

If $m = n$, we are done. By Lemma 4, $(p_1, \ldots, p_n) = (p_L(n), \ldots, p_L(n)) \in \mathcal{S}(G_n) \subseteq \mathcal{B}(G_n)$. Moreover, $\sum_{i \in N} q_i(p_1, \ldots, p_n) = n\,\frac{D(p_L(n))}{n} = D(p_L(n))$. So, suppose that

$$C(D(p_L(n))) < n\,C\left(\frac{D(p_L(n))}{n}\right). \quad (17)$$

Adding the term $-p_L(n)\,D(p_L(n))$ to both sides of (16), it follows that $-p_L(n)\,D(p_L(n)) + C(D(p_L(n))) \geq -p_L(n)\,D(p_L(n)) + m\,C\left(\frac{D(p_L(n))}{m}\right)$, which implies that

$$H(p_L(n), 1) \leq m\,H(p_L(n), m). \quad (18)$$

By following the same argument as above, it is easy to see from (17) that

$$H(p_L(n), 1) > n\,H(p_L(n), n). \quad (19)$$

Therefore, since $C(0) = 0$, combining (18) and (19), we get that both $H(p_L(n), 1) > 0$ and $m\,H(p_L(n), m) > 0$. Further, by Lemma 1, there is a price $p(m) \in (0, p^h(m))$ such that $H(p(m), m) = 0$. Hence, since $p_L(m)$ is the smallest of all such prices, and $H(p_L(n), m) > 0$ and $H(0, m) < 0$, it follows that $p_L(m) < p_L(n)$.

By Lemma 2, $p_L(m)$ is smaller than p^M too. Suppose, by contradiction, there is a price $\hat{p} < p_L(m)$ such that

$$H(\hat{p}, 1) > H(p_L(m), m). \tag{20}$$

Since $H(\cdot\,, 1)$ is quasi-concave on $(0, \overline{P})$, $p_L(m) \in (\hat{p}, p^M)$ implies that $H(p_L(m), 1) \geq \min\{H(\hat{p}, 1), H(p^M, 1)\} = H(\hat{p}, 1)$. Hence, using (20), $H(p_L(m), 1) > H(p_L(m), m)$. Moreover, since $H(p_L(m), m) = 0$, $m\,H(p_L(m), m) = H(p_L(m), m)$. Therefore,

$$H(p_L(m), 1) > m\,H(p_L(m), m). \tag{21}$$

Given that $H(\cdot\,, 1) - m\,H(\cdot\,, m)$ is continuous on $[0, \overline{P})$, the expressions in (18) and (21) imply that there is a price $p^\alpha \in (p_L(m), p_L(n)]$ such that

$$H(p^\alpha, 1) - m\,H(p^\alpha, m) = 0. \tag{22}$$

By Lemma 7, there exists $\epsilon > 0$ small enough with the property that $H(p^\alpha - \epsilon, 1) - m\,H(p^\alpha - \epsilon, m) < 0$. But then, using (21) once again, it follows that there is a price $p^{\alpha+1} \in (p_L(m), p^\alpha - \epsilon)$ such that $H(p^{\alpha+1}, 1) - m\,H(p^{\alpha+1}, m) = 0$. And repeating the argument over and over again, we get a sequence of prices $\{p^{\alpha+s}\}_{s=1}^{\infty} \subset (p_L(m), p^\alpha)$ with the property that for all $s = 1, \ldots, \infty$,

$$H(p^{\alpha+s}, 1) - m\,H(p^{\alpha+s}, m) = 0. \tag{23}$$

Notice that each term $p^{\alpha+s}$ of the sequence is closer to $p_L(m)$ than what it was $p^{\alpha+s-1}$. Therefore, invoking Lemma 7 together with the expressions in (21) and (23), we conclude that for some $s \geq 1$ sufficiently high, there must exist $\epsilon > 0$ and a price $\bar{\bar{p}} \in (p_L(m), p^{\alpha+s} - \epsilon)$ for which $H(\bar{\bar{p}}, 1) - m\,H(\bar{\bar{p}}, m)$ is simultaneously positive and negative, a contradiction. This contradiction was obtained by assuming the existence of a price $\hat{p} < p_L(m)$ that verifies (20). Hence, for all $\hat{p} < p_L(m)$, $H(\hat{p}, 1) \leq 0 = H(p_L(m), m)$.

Let $p_L(m^*) \equiv \min\{p_L(s), s \in \{2, \ldots, n\}\}$. Clearly, $p_L(m^*) \leq p_L(m)$. Thus, since by definition $H(p_L(m^*), m^*) = 0$, for all $\hat{p} \leq p_L(m^*)$, $H(\hat{p}, 1) \leq 0 = H(p_L(m^*), m^*)$. Next, suppose, by contradiction, there exists $s \in \{m^* + 1, \ldots, n\}$ such that $H(p_L(m^*), s) > 0 = H(p_L(m^*), m^*)$. Since $H(0, s) < 0$ and $H(\cdot\,, s)$ is continuous on $[0, \overline{P})$, there must exist $p' \in (0, p_L(m^*))$ such that $H(p', s) = 0$, contradicting the definition of $p_L(m^*)$. Therefore, for all $s \in \{m^* + 1, \ldots, n\}$, $H(p_L(m^*), s) \leq 0$.

Finally, we claim that the strategy profile $\mathbf{p} = (p_1, \ldots, p_n) \in [0, \overline{P})^n$, with the property that (i) for all $i = 1, \ldots, m^*$, $p_i = p_L(m^*)$, and (ii) for all $j = m^* + 1, \ldots, n$, $p_j > p_L(m^*)$, constitutes a PSBE for G_n. Indeed, if $i \in \{1, \ldots, m^*\}$, then $\pi_i(p_i, \mathbf{p}_{-i}) = H(p_L(m^*), m^*) = 0$. Consider a deviation $\hat{p}_i \neq p_i$ for firm i. If $\hat{p}_i > p_i$, then $\pi_i(\hat{p}_i, \mathbf{p}_{-i}) = 0$. Instead, if $\hat{p}_i < p_i$, then $\pi_i(\hat{p}_i, \mathbf{p}_{-i}) = H(\hat{p}_i, 1) \leq 0$, where the last inequality follows from the fact that, according with the analysis in the previous paragraph, for all $\hat{p} \leq p_L(m^*)$, $H(\hat{p}, 1) \leq 0$.

On the other hand, if $i \in \{m^* + 1, \ldots, n\}$, then $\pi_i(p_i, \mathbf{p}_{-i}) = 0$. Again, consider a deviation $\hat{p}_i \neq p_i$ for firm i. If $\hat{p}_i > p_L(m^*)$, then $\pi_i(\hat{p}_i, \mathbf{p}_{-i}) = 0$. If $\hat{p}_i < p_L(m^*)$, then $\pi_i(\hat{p}_i, \mathbf{p}_{-i}) = H(\hat{p}_i, 1) \leq 0$. Lastly, if $\hat{p}_i = p_L(m^*)$, then $\pi_i(\hat{p}_i, \mathbf{p}_{-i}) = H(p_L(m^*), m^* + 1)$, which we have already shown is smaller than or equal to 0. Therefore, $\mathbf{p} = (p_1, \ldots, p_n) \in \mathcal{B}(G_n)$. And, since $p_L(m^*) \leq p_L(n)$, $\sum_{i \in N} q_i(p_1, \ldots, p_n) = m^* \frac{D(p_L(m^*))}{m^*} \geq D(p_L(n))$.

Now, let's prove the second part of Theorem 1. That is, let's show that if $\mathcal{B}(G_n) \neq \emptyset$, then the assertion "the cost function $C(\cdot)$ is subadditive at every output $q \in [D(p_L(n)), K)$" is false. Clearly, if $\mathcal{S}(G_n) \neq \emptyset$, the result follows from Proposition 2. Hence, assume $\mathcal{S}(G_n) = \emptyset$.

Fix any equilibrium profile $\mathbf{p} = (p_1, \ldots, p_n) \in \mathcal{B}(G_n)$ and suppose, by contradiction, there is a firm $k \in N$ whose reported price $p_k < p_j$ for all $j \in N \backslash \{k\}$. Without loss of generality, denote by p_h, $h \neq k$, the second smallest price; i.e., let $p_h = \min\{p_j;\, j \neq k\}$. Then,

(a) If $p_k > p^M$, firm k can profitably deviate to the monopoly profit-maximizing price p^M;

(b) If $p_k = p^M$, Assumption 5 and the continuity of $H(\cdot\,, 1)$ at p^M imply that there exists $\epsilon > 0$ such that $H(p_k - \epsilon, 1) > 0$. Thereby, since $\pi_h(p_1, \ldots, p_n) = 0$, firm h can do better by proposing $p_k - \epsilon$ instead of p_h, a contradiction;

(c) Finally, if $p_k < p^M$, depending upon the location of p_h the following happens. If $p_h > p^M$, firm k can profitably deviate to p^M as before. Otherwise, if $p_h \leq p^M$, then by Assumption 5 there is $\epsilon > 0$ such that $H(p_h - \epsilon, 1) > H(p_k, 1) = \pi_k(p_k, \mathbf{p}_{-k})$, which contradicts that $(p_1, \ldots, p_n) \in \mathcal{B}(G_n)$.

Hence, using (a)–(c), we conclude that if $(p_1, \ldots, p_n) \in \mathcal{B}(G_n)$ and $\mathcal{S}(G_n) = \emptyset$, there must be at least two firms which tie at the lowest price, say p^*, and another firm proposing a price above p^*.[17] That is, there must exist $m \in \{2, \ldots, n-1\}$, $n > 2$, and $p^* \in [0, \infty)$ such that (i) for all $i \in \{i_1, \ldots, i_m\} \subset N$, $p_i = p^*$; and (ii) for all $j \notin \{i_1, \ldots, i_m\}$, $p_j > p^*$. By following the same reasoning as in (a) and (b), it is easy to see that $p^* < p^M$.

Notice that, since $\mathcal{S}(G_n) = \emptyset$, $H(p_L(n), n) < H(p_L(n), 1)$, which implies that $H(p_L(n), 1) > 0$. Thus, given that $H(0, 1) = -C(K) < 0$, it follows that $p_L(1) < p_L(n)$. We show next, using the argument behind the proof of Lemma 3, that $p_L(m) \leq p_L(1)$. In effect, assume, by contradiction, $p_L(m) > p_L(1)$. (Recall that by Lemma 2 $p_L(m) \leq p^M$.) By Assumption 5, $H(p_L(m), 1) > H(p_L(1), 1) = 0$. Thus,

$$H(p_L(m), 1) - H(p_L(m), m) > 0. \tag{24}$$

On the other hand,

$$H(0, 1) - H(0, m) = -C(K) + C(K/m) < 0. \tag{25}$$

Therefore, from (24) and (25) and the continuity of $H(\cdot, 1) - H(\cdot, m)$ on $[0, p^M]$, there has to be a price $p^\alpha \in (0, p_L(m))$ such that

$$H(p^\alpha, 1) - H(p^\alpha, m) = 0. \tag{26}$$

Recall that, at the equilibrium $(p_1, \ldots, p_n) \in \mathcal{B}(G_n)$, m firms tie at the lowest price p^*; hence, $H(p^*, m) \geq 0$. Otherwise, any of these firms can profitably deviate to $\overline{P}$. Moreover, $p^* \geq p_L(m)$ because $H(\cdot, m)$ is negative below $p_L(m)$. In addition, $(p_1, \ldots, p_n) \in \mathcal{B}(G_n)$ implies $H(p^*, m) \geq H(p^*, 1)$, which is equivalent to $H(p^*, 1) - H(p^*, m) \leq 0$. Thus, using (24), we conclude that $p^* \neq p_L(m)$. Finally, by Assumption 5, $H(p^M, 1) - H(p^M, m) > 0$.

[17] Recall that, since by supposition $\mathcal{S}(G_n) = \emptyset$, at most $n-1$ firms can tie at p^*.

Therefore, there exists a price $p^\beta \in [p^*, p^M)$ such that

$$H(p^\beta, 1) - H(p^\beta, m) = 0. \tag{27}$$

Summarizing, by assuming that $p_L(m) > p_L(1)$, (26) and (27) indicate that the curves $H(\cdot\,, 1)$ and $H(\cdot\,, m)$ must intersect each other at least twice on $(0, p^M)$. An argument analogous to that used in the proof of Lemma 3 shows that this assertion is false. (The proof is available upon request.) Thus, $p_L(m) \leq p_L(1)$. Furthermore, $p_L(m) < p_L(n)$ because we already showed that $p_L(1) < p_L(n)$. Consequently, $D(p_L(m)) > D(p_L(n))$.

So, it remains to be proved that $C(\cdot)$ is not subadditive at $D(p_L(m))$. To do that, note that $H(\cdot\,, 1)$ is negative below $p_L(m)$, because $p_L(1) \geq p_L(m)$. That means $0 = m\,H(p_L(m), m) > H(p_L(m), 1)$, which renders the desired result; i.e., $m\,C\left(\frac{D(p_L(m))}{m}\right) < C(D(p_L(m)))$. Therefore, $C(\cdot)$ is not subadditive on $[D(p_L(n)), K)$. ∎

A.5 Proof of Lemma 5

The proof is based on Panzar (1989, pg. 25). Fix any $q > 0$ and assume $AC(\cdot)$ is decreasing through q. Consider any division $q_1, \ldots, q_n$ of q with the property that: (i) $\forall i \in N$, $0 \leq q_i < q$, and (ii) $\sum_{i \in N} q_i = q$. Let $N^+ = \{i \in N : q_i > 0\}$. Then, for all $i \in N^+$, $AC(q) < AC(q_i)$, which is equivalent to $C(q_i) > (q_i/q) \cdot C(q)$. Summing over N^+, we have $\sum_{i \in N^+} C(q_i) > C(q)$. Therefore, since $C(0) \geq 0$, it follows that $\sum_{i \in N} C(q_i) > C(q)$. Finally, since $q_1, \ldots, q_n$ was arbitrarily chosen, this implies that $C(\cdot)$ is subadditive at q.

To show that subadditivity does not imply decreasing average costs, consider the cost function $C(q) = 1/2 \cdot q^2 + 100$ for all $q \geq 0$. It is easy to see that $AC(\cdot)$ is not decreasing at $q = 15$. However, if $n = 2$, then $C(\cdot)$ is subadditive at 15. ∎

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