Noisy Stochastic Games

Duggan, John

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John Duggan*†

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Abstract

This paper establishes existence of a stationary Markov perfect equilibrium in general stochastic games with noise — a component of the state that is nonatomically distributed and not directly affected by the previous period's state and actions. Noise may be simply a payoff irrelevant public randomization device, delivering known results on existence of correlated equilibrium as a special case. More generally, noise can take the form of shocks that enter into players' stage payoffs and the transition probability on states. The existence result is applied to a model of industry dynamics and to a model of dynamic partisan electoral competition.

1 Introduction

This paper proves existence of stationary Markov perfect equilibria in a class of stochastic games, a subset of dynamic games in which isomorphic subgames are indexed by a state variable that evolves according to a controlled Markov process. In each period, the current state is publicly observed and determines a stage game in which players simultaneously choose feasible actions, stage payoffs are realized, and a new state is drawn from a distribution depending on the current state and the players' actions. A natural starting point for strategic analysis in this setting is to consider equilibria that reflect the stationary structure of the environment, and so the issue of existence of stationary Markov perfect equilibria in general stochastic games is of central importance. I establish existence of equilibrium in the standard framework by adding noise — a component of the state state variable that is nonatomically distributed and not directly affected by the previous period's state and actions — in each period. I refer to games for which such a decomposition of states is possible as "noisy stochastic games." The presence of such noise is often innocuous from an applied

*Dept. of Political Science and Dept. of Economics, University of Rochester.

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point of view, where shocks to parameters of the game can increase modeling realism and are desirable for purposes of estimation. I give two examples to illustrate the application of the existence theorem: one is a dynamic model of entry, exit, and investment of firms in an industry, where the noise component corresponds to demand or technology shocks, and another is a dynamic model of partisan electoral competition with time-consistent policy choice, where noise is introduced via probabilistic voting, a standard assumption in the literature.

The literature on stochastic games has not yielded general existence results, even under the compactness and continuity conditions familiar from Debreu (1952), Fan (1952), and Glicksberg (1952) for static games. Indeed, the example in Section 2.1 of Harris et al. (1995) shows that compactness and continuity are not sufficient for existence of stationary Markov perfect equilibrium in stochastic games. That example is a relatively simple, two-period game in which the players' action sets are compact and payoffs are continuous in each stage. Though not constructed explicitly as a stochastic game, the example can be formulated as one in which the state in the first period is an exogenous initial state, and the state in the second period is just the profile of actions taken in the first; then the mapping from action profiles in period 1 to the state in period 2 is just the identity mapping. The authors argue that there is no subgame perfect equilibrium in their example, and therefore there is no Markov perfect equilibrium when the game is viewed as a stochastic game. One approach, taken by Harris et al. (1995), is to consider correlated equilibria in history-dependent strategies.

Another approach, followed in the literature on stochastic games, is to strengthen the continuity properties of the transition probability. At issue is the fact that even if stage payoffs are continuous, discontinuities can conceivably be introduced by strategic behavior of the players, for players tomorrow may condition their responses to today's actions in a discontinuous way. The literature has accordingly assumed that next period's state is determined stochastically as a function of the current period's state and actions, and that the distribution of next period's state varies with current actions in a strongly continuous way, e.g., the probability of each measurable set of next period's states is continuous in this period's actions. This precludes deterministic transitions, as in the example of Harris et al. (1995), and it partially addresses the problem of continuity in dynamic games. This stochastic element does not, however, automatically deliver general results on existence of stationary Markov perfect equilibria, for difficult technical problems arise when the set of states is uncountably infinite. Then existence arguments (implicitly or explicitly) involve Bochner integration of Banach-valued correspondences, and in contrast to integrals of correspondences mapping to finite-dimensional Euclidean space, the Bochner integral is not necessarily upper hemicontinuous in its parameters (even if the correspondence being integrated is), and it need not be convex (even if the integrating measure is nonatomic).[1] To obtain these properties, the correspondence being integrated must have convex values: in infinite-dimensional fixed point argu-

[1] See Example 6.1 in Yannelis (1991) for discussion of these points.

ments, the role of convexity is intertwined with upper hemicontinuity.

The papers in the extant literature closest to the current one are Nowak and Raghavan (1992), who prove the existence of correlated stationary Markov perfect equilibria, and Duffie et al. (1994), who additionally deduce ergodicity of equilibrium under stronger conditions. These papers essentially assume that the players observe the outcome of a public randomization device before choosing their actions in each period, convexifying payoffs in every state and delivering the convexity needed to obtain upper hemicontinuity needed for their arguments. The drawback of this approach is that the "sunspot" on which players coordinate is payoff irrelevant and may be unnatural or unmotivated in some applications. The innovation of the current paper is to replace sunspots with shocks that appear explicitly as a component of the state, along with a standard component, and that enter into the stage payoffs of the players and the transition probability on states. I refer to these shocks as a "noise" component of the state, because in contrast to the standard component, the distribution of noise next period is not directly affected by this period's state and actions. (The noise component can be correlated with the standard component, which allows for indirect dependence on the state and actions this period.) The noise component could be simply an iid draw of a payoff-irrelevant, continuously distributed random variable, i.e., a public randomization device, thereby obtaining the existence results of Nowak and Raghavan (1992) and Duffie et al. (1994) as a special case. More generally, the noise component can affect stage payoffs and the state transition, and it arises naturally in many applications, where transitory shocks are desirable on modeling grounds or for purposes of estimation. Even so, it delivers the needed convexity for the existence proof.

The usefulness of the noise component is that it permits the fixed point argument to be framed in the space of "interim" continuation values, which are defined over the standard component alone. The players' interim continuation values can be written as an iterated integral — the outer integral over the noise component of the state, and the inner integral over the players' actions — and integration over the non-atomic noise component functions then serves to convexify the correspondence from interim continuation values to updated interim continuation values. The iterated integral approach is also used by Chakrabarti (1999) in the proof of his Theorem 4 on the existence of stationary equilibria that are semi-Markovian, i.e., players are allowed to condition not just on the current state but also on the previous period's state and actions. There, the fixed point argument takes place in the space of "ex ante" continuation values defined over the previous state and actions, the outside integral is over the current state, and the inside integral is over the players' actions. Assuming the current state is non-atomically distributed, the integration over today's state convexifies the correspondence of updated ex ante continuation values. Thus, although Chakrabarti's assumptions and results are different, there is a connection between the results at a technical level.

Section 2 provides a review of the stochastic games literature. Section 3 presents the noisy stochastic game model and the main existence theorem. In Section 4, I present two applications of the existence theorem, one oriented toward industrial organization and the other toward political economy. Section 5 is devoted to an informal discussion of the proof approach. And Section 6 contains the proof of the main theorem.

2 Literature Review

Existence of stationary Markov perfect equilibrium is a central issue in the literature on stochastic games beginning with Shapley (1953), who proved existence for finite, two-player, zero-sum games. Existence in general finite stochastic games follows from the straightforward application of Kakutani's fixed point theorem in finite dimensions (cf. Fink (1964), Rogers (1969), and Sobel (1971)), while Takahashi (1964) proves existence when the set of states is finite and action sets are compact. Haller and Lagunoff (2000) prove that the set of stationary Markov perfect equilibria in finite games is generically finite; Herings and Peeters (2004) develop an algorithm for computation of equilibrium and use homotopy arguments to show that the number of equilibria is odd; and Doraszelski and Escobar (2010) prove generic strong stability and purifiability of stationary equilibria. Parthasarathy (1973) extends the framework of Shapley to two-player, non-zero sum games with finite action sets and a countable set of states, and Himmelberg et al. (1976) consider two-player, non-zero-sum games with uncountable state and action sets.

General results on existence have been elusive and have relied on the imposition of relatively special structure or departures from the concept of stationary equilibrium,[2] and all of the known results impose some form of strong continuity on transition probabilities. Letting s denote a state and a denote a profile of actions, a transition probability is a measurable mapping $(s, a) \to \mu_t(\cdot|s, a)$ from state-action pairs to a probability measure on the set of states that can conceivably vary with the time period t. Next, in increasing strength, are some assumptions used in the literature.

(A1) μ_t is set-wise continuous in a,[3]

(A2) μ_t is norm-continuous in a,[4]

(A3) μ_t is norm-continuous in a and absolutely continuous with respect to some fixed probability measure ν_t,

[2] Dutta and Sundaram (1998) provide a lucid review of much of the literature on stochastic games and the problem of existence of stationary Markov perfect quilibrium.

[3] For each measurable set Z of states and sequence $a_m \to a$, $\mu(Z|s, a^m) \to \mu(Z|s, a)$.

[4] For each sequence $a^m \to a$, $\mu(Z|s, a^m) \to \mu(Z|s, a)$ uniformly over measurable sets Z of states.

(A4) μ_t is norm-continuous in a and absolutely continuous with respect to a fixed, non-atomic probability measure ν_t,

(A5) μ_t has a density $f(s'|s,a)$ with respect to Lebesgue measure that is continuous with respect to a.

In the analysis of stationary stochastic games, as in the current paper, it is further assumed that the transition probability is fixed across time and the subscript dropped. Note that even the weakest of the above assumptions, (A1), is inconsistent with deterministic transitions (precluding the example of Harris et al. (1995)) when action sets are uncountably infinite. Of course, (A1) and (A2) hold when the players' action sets are finite.

In finite-horizon stochastic games, Rieder (1979) proves existence of Markov perfect equilibrium under (A1). By incorporating time in the state variable of a finite-horizon game, we may in fact view Rieder's equilibrium as stationary. Parthasarathy (1973) proves existence of stationary Markov perfect equilibrium with finite action sets and countable state space, and assuming (A1), Federgruen (1978), Whitt (1980), and Escobar (2006) prove existence for countable state spaces and uncountable action sets. Himmelberg et al. (1976) prove existence of stationary p-equilibria for two-player games with an uncountable state space but assuming finite action sets and strong separability conditions on stage payoffs and the transition probability.[5] Parthasarathy (1982) gives additional conditions under which the result of the latter paper delivers a stationary Markov perfect equilibrium. Existence of equilibrium for multi-player games with uncountable state space is proved in Parthasarathy and Sinha (1989) under the assumptions of finite action sets and state-independent transitions.[6] Under continuity assumptions on the transition probability akin to (A5), Amir (1996, 2002)), Curtat (1996), and Nowak (2007) prove existence of stationary Markov perfect equilibria in games possessing strategic complementarities with uncountable state and action spaces.[7] Assuming that the state transition is a convex combination of a fixed finite set of probability measures, Nowak (2003) gives sufficient conditions related to (A1) for existence of stationary Markov perfect equilibrium.

Otherwise more general results have been obtained by weakening stationarity or considering weaker notions of equilibrium. Most closely related to the current paper, Nowak and Raghavan (1992) prove existence of stationary Markov perfect equilibria with public randomization under (A3), and Duffie et al. (1994) add mutual absolute continuity of transition probabilities and show that the

[5]Here, p is a probability measure on states, and a p-equilibrium is a strategy profile such that players optimize at all but perhaps a set of states with p-measure zero.

[6]The transition to next period's state depends only on current actions, not the current state. Additionally, the authors assume state transitions are non-atomic.

[7]Nowak (2007) also gives conditions based on concavity of the stage game and a decomposition of the transition probability. Other work restricts the way in which players' actions affect each others' payoffs, e.g., Jovanovic and Rosenthal (1988), Bergin and Bernhardt (1992), and Horst (2005).

equilibrium induces an ergodic process. Mertens and Parthasarathy assume (A2) and obtain existence of equilibria that are nearly Markovian. Mertens and Parthasarathy (1991) assume finite action sets and deduce existence of equilibria in which the players' strategies in period t can depend not only on the current state but the previous state as well. Mertens and Parthasarathy (1987, 2003) allow for infinite action sets and deduce equilibria in which players use history-dependent strategies such that each player's mixture over actions is the same following any two histories ending in the same state and generating identical continuation values. Increasing (A2) to (A4), Chakrabarti (1999) proves existence of a stationary equilibrium in semi-Markov perfect strategies, which allow players to condition not only on the current state, but the previous period's state and actions as well.

Building on Rieder's (1979) result for finite-horizon games, Dutta and Sundaram (1998) give a simple proof of the existence of (possibly non-stationary) Markov perfect ϵ-equilibria under (A1). Assuming stage utilities and the state transition are continuous in the state variable, Whitt (1980) gives sufficient conditions related to (A2) for existence of a Markov perfect ϵ-equilibrium in stationary strategies, and Nowak (1985) drops continuity of the state transition in the state variable and increases (A2) to (A4) to obtain a stationary Markov perfect ϵ-equilibrium.

3 Existence Theorem

A *stochastic game* is a list $\Gamma = (N, (S, \mathcal{S}), (X_i, A_i, u_i, \delta_i)_{i\in N}, \mu)$, where N is a finite set of n players, $(S, \mathcal{S})$ is a measurable space of states, X_i is a compact metric space of actions for player i, with $X = \prod_{i\in N} X_i$, $A_i\colon S \rightrightarrows X_i$ is a lower measurable correspondence from S into nonempty, compact feasible sets $A_i(s)$ of actions for player i,[8] $u_i\colon S \times X \to \Re$ is a bounded, measurable stage-payoff function with $u_i(s, a)$ continuous in $a = (a_1, \ldots, a_n) \in X$ for each state s, player i's discount factor is $\delta_i \in [0, 1)$, and $\mu\colon S \times A \times \mathcal{S} \to [0, 1]$ is a transition probability, where $\mu(Z|s, a)$ is the probability that next period's state belongs to Z, given state s and action profile a in the current period. Write $u(s, a) = (u_1(s, a), \ldots, u_n(s, a))$ for the vector of stage payoffs of the players. This is the standard definition of a general stochastic game.

The next step is to introduce a noise structure into the model. To this end, assume that (i) the set of states can be decomposed as $S = Q \times R$ and $\mathcal{S} = \mathcal{Q} \otimes \mathcal{R}$, where Q and R are complete, separable metric spaces and $\mathcal{Q}$ and $\mathcal{R}$ are the respective Borel sigma-algebras. Letting $\mu_q(\cdot|s, a)$ denote the marginal of $\mu(\cdot|s, a)$ on q, assume that (ii) $\mu_q(\cdot|s, a)$ is absolutely continuous with respect to a fixed probability measure κ on $(Q, \mathcal{Q})$. Moreover, assume that (iii) for each s and each sequence $\{a^m\}$ of action profiles converging to some a, the sequence

[8] That is, for every open set $G \subseteq X_i$, the set $\{s \in S \mid A_i(s) \cap G \neq \emptyset\}$ belongs to $\mathcal{S}$.

$\{\mu_q(\cdot|s, a_m)\}$ converges to $\mu_q(\cdot|s, a)$ in the total variation norm. Thus, the first component of the state satisfies the typical norm-continuity condition assumed in the literature.

Furthermore, assume that (iv) conditional on next period's q', the distribution of r' next period is independent of the current state and actions. Specifically, letting $\mu_r(\cdot|s, a, q')$ denote a version of the distribution of r' given state-action pair (s, a) today and conditional on q' next period, assume that $\mu_r(Z|s, a, q') = \mu_r(Z|s', a', q')$ for all $Z \in \mathcal{R}$, all $s, s' \in S$, all $a, a' \in X$, and all $q' \in Q$. Accordingly, I write $\mu_r(\cdot|q)$ for the distribution of r conditional on the realization of q in a given period, which is well-defined up to sets of κ-measure zero. Assume that (v) for κ-almost all q, $\mu_r(\cdot|q)$ has a density $h(r|q)$ with respect to a fixed, atomless probability measure λ on $(R, \mathcal{R})$ and that $h(r|q)$ is jointly measurable in (r, q). For later use, define the product probability measure $\nu = \kappa \otimes \lambda$. A *noisy stochastic game* is a stochastic game satisfying conditions (i)–(v).

A stationary Markov strategy for i is a measurable mapping $\sigma_i : S \to \mathcal{P}(X_i)$, where $\sigma_i(s)$ is a probability measure on X_i that places probability one on $A_i(s)$.[9] Given a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$, let $\sigma(s)$ denote the product probability measure $\sigma_1(s) \otimes \cdots \otimes \sigma_n(s)$ over action vectors induced by the players' strategies, taking Borel measurable sets of action vectors in the argument of $\sigma(\cdot|s)$.[10] Continuation values $v(\cdot; \sigma)$, which are placed in the set $L_1^n(S, \mathcal{S}, \nu)$ of norm-integrable functions from S to $\Re^n$, are uniquely defined by the following recursion:

$$v_i(s; \sigma) \quad = \quad \int_a \left[(1 - \delta_i) u_i(s, a) + \delta_i \int_S v_i(s'; \sigma) \mu(ds'|s, a) \right] \sigma(da|s).$$

A strategy vector σ is a *stationary Markov perfect equilibrium* if each agent i's strategy maximizes i's discounted expected payoff in every state, i.e.,

$$\begin{aligned} & v_i(s; \sigma) \\ & \quad = \quad \max_{a_i \in A_i(s)} \int_{a_{-i}} \left[(1 - \delta_i) u_i(s, a) + \delta_i \int_S v_i(s'; \sigma) \mu(ds'|s, a) \right] \sigma_{-i}(da_{-i}|s), \end{aligned}$$

for all i and all s, where $\sigma_{-i}(s)$ is the product probability measure $\sigma_1(s) \otimes \cdots \otimes \sigma_{i-1}(s) \otimes \sigma_{i+1}(s) \otimes \cdots \otimes \sigma_n(s)$ over vectors a_{-i} of actions of players other than i. By the one-shot deviation principle, every stationary Markov perfect equilibrium is, in particular, subgame perfect.

Theorem: Every noisy stochastic game possesses a stationary Markov perfect equilibrium.

[9] I use the convention that given any separable metric space, $\mathcal{P}(\cdot)$ denotes the set of Borel probability measures endowed with the weak* topology.

[10] By Theorem 4.44 of Aliprantis and Border (2006), the Borel sigma-algebra on action profiles is the product of Borel sigma-algebra's on the players' action sets.

A special case of interest is the situation in which r is identically and independently distributed across periods and u_i is constant in r, so the noise component of the state is payoff-irrelevant. Then r acts as a public randomization device, and every stationary Markov perfect equilibrium of the model can be viewed as a correlated equilibrium in the sense of Nowak and Raghavan (1992) or Duffie et al. (1994), and visa versa. Thus, for any stochastic game satisfying the assumptions of the latter papers, we can extend the game by specifying a noise component r uniformly and independently distributed in each state (and stage payoffs constant in r) to obtain a stationary Markov perfect equilibrium of the extended game, which delivers a correlated equilibrium of the original game. But the formulation of this paper allows for noise that is payoff relevant, which captures many economic and political models of interest.

4 Applications

4.1 Firm Exit, Entry, and Investment

This subsection provides a dynamic model of entry, exit, and investment in an industry. Each period begins with a set of firms active in the market, a vector of capital stocks for each firm, and a vector of demand or technology shocks. Each firm, active or inactive, must decide whether to enter or exit the industry, and conditional on having entered in the previous period, a firm must choose a production plan. The firms' output and investment plans determine profits for the current period and a distribution over capital stocks next period, reflecting uncertain depreciation and returns to investment. The model is comparable to those of Hopenhayn (1992) or Bergin and Bernhardt (2008), where firms make entry and exit decisions over time and are subject to exogenous technology shocks. In contrast, those models assume a continuum of price-taking firms, while here there is a (possibly large) finite number of firms competing oligopolistically; and those models either fix capital or treat it as a variable input, while here firms make investment decisions and accumulate capital over time. In fact, firms' production plans and capital stocks can be multidimensional, and a firm's decisions can affect future production technology through current investment, so that technology evolves endogenously.

Formally, let N be a finite set of n firms (or potential firms) in an industry, and suppose that in each period, firms must decide whether to enter or remain in the market and, conditional on having previously entered the market, must make output and investment decisions. At the beginning of any period, let $z \in \{0,1\}^n$ summarize the firms active in the industry, with $z_i = 1$ indicating that i is active and $z_i = 0$ indicating i is inactive; let $k_i \in \Re^{\ell}_{+}$ denote the capital stock of firm i and $k = (k_1, \ldots, k_n)$ the vector of stocks; and let $r = (r_1, \ldots, r_m)$ be a vector of shocks to demand or production technology belonging to a compact subset $R \subseteq \Re^m$ with positive Lebesgue measure. For tractability, the level of capital stock of each firm is bounded above by $\overline{k}$ in each coordinate. The state of the industry is then summarized by the state variable (z, k, r), where (z, k) is influenced by the actions of the firms, and r is distributed identically and independently across periods. A decision for firm i is a pair (e_i, p_i), where $e_i \in \{0, 1\}$ is firm i's entry/exit decision, with $e_i = 1$ indicating i is will be in the market next period, and $p_i \in \Re^d$ is a multidimensional production plan for firm i. In state (z, k, r), the set of feasible production plans for firm i is a nonempty, compact subset $\phi_i(z, k, r) \subseteq \Re^d$, where $\phi_i \colon \{0, 1\}^n \times [0, \overline{k}]^{n\ell} \times R \rightrightarrows \Re^d$ is lower measurable with compact range $\tilde{X}_i \subseteq \Re^d$. When firm i is inactive, i.e., $z_i = 0$, assume that $\phi_i(z, k, r) = \{0\}$ to indicate that the firm makes no output or investment decision. Then the decisions available to firm i in state (z, k, r) are $(e_i, p_i) \in \{0, 1\} \times \phi_i(z, k, r)$, and the vector of firm decisions is denoted (e, p).

Given current state (z, k, r) and actions (e, p), next period's state, denoted (z', k', r'), is determined as follows. Entry and exit decisions determine each firm's status next period, so $z' = e$, while k' is a random variable assumed to be absolutely continuous with respect to a probability measure $\tilde{\kappa}$, defined as the product measure $\tilde{\kappa} = \tilde{\kappa}_1 \times \cdots \times \tilde{\kappa}_n$, where $\tilde{\kappa}_i$ is the equally weighted average of the uniform distribution on $[0, \overline{k}]^\ell$ and the unit mass on zero. Denote the density of k' with respect to $\tilde{\kappa}$ by $\tilde{g}(k'|(z, k), (e, p))$, which depends on the firms' current capital stocks and decisions. Assume that for all $k, k' \in [0, \overline{k}]$, the density $\tilde{g}(k'|(z, k), (e, p))$ is jointly measurable in its arguments, continuous in (e, p), and independent of the current vector r of shocks. The distribution of next period's capital stock levels reflects the assumption that returns on investment to capital stock are subject to uncertainty, and because $\tilde{\kappa}_i$ places positive probability on zero, the model allows for the possibility that the capital stock of an inactive firm is fixed at zero. Assume $r' \in R$ is identically and independently distributed over time according to a density h with respect to Lebesgue measure.

A firm i remains active in a period if $z_i = e_i = 1$. Let $\pi_i((z, k, r), (e_i, p))$ be the profit of a firm i that remains active given its own entry/exit decision e_i and production plans p in state (z, k, r), and assume $\pi_i((z, k, r), (e_i, p))$ is bounded and jointly measurable in its arguments and continuous in (e_i, p). The payoff of an active firm that decides to leave the market, i.e., $z_i = 1 = e_i + 1$, is $\iota_i((z, k, r), (e_i, p))$, which may reflect the scrap value of a firm leaving the market. Assume $\iota_i((z, k, r), (e_i, p))$ is jointly measurable and continuous in (e_i, p). The payoff of an inactive firm that decides to enter the market, i.e., $z_i = 0 = e_i - 1$, is $\alpha_i(z, k, r)$, a measurable function that may reflect the setup cost of entry into the market. The payoff to an inactive firm that remains inactive, i.e., $z_i = e_i = 0$, is zero. Payoffs are discounted over time by the factor δ_i for each firm.

Though formulated generally, the standard structure can be imposed on these payoffs. In particular, it may be that capital k_i is one-dimensional, that $r = (r_1, \ldots, r_n, r_{n+1}, r_{n+2}) \in \Re^{n+2}$ consists of firm-specific production shocks $(r_1, \ldots, r_n)$, an aggregate output demand shock r_{n+1}, and an aggregate labor supply shock r_{n+2}, and that firms compete in a single output market. A production plan is then a pair $p_i = (\dot{k}_i, \ell_i)$ consisting of levels of capital investment and labor input. Firm i's output is then $y_i = F_i(k_i + \dot{k}_i, \ell_i, r_i)$, total labor demand is $L = \sum_i \ell_i$, and total output is $Y = \sum_i y_i$. The inverse demand for output is given by $P(Y, r_{n+1})$, and inverse supply of labor is $W(L, r_{n+2})$. Then for firms currently active in the market and remaining in the market next period, i.e., $z_i = e_i = 1$, we have

$$\begin{aligned} \pi_i((z,k,r),(1,p)) &= P(Y, r_{n+1})y_i - W(L, r_{n+2})\ell_i \\ Y &= \sum_j F_j(k_j + \dot{k}_j, \ell_j, r_j) \\ L &= \sum_j \ell_j, \end{aligned}$$

with appropriate continuity assumptions on the inverse demand and supply functions, P and W, and bounds on output and investment to obtain compactness. The industry then transitions to (z', k', r'), where z' is determined by entry/exit decisions, k' is drawn from $g(k'|(z,k),(e,p))$ reflecting depreciation on capital and returns to investment, and new shocks r' are drawn independently from h.

At issue is the existence of a stationary Markov perfect equilibrium in this model, which is addressed in the next proposition.

Proposition 1: In the dynamic model of exit, entry, and investment, there exists a stationary Markov perfect equilibrium.

To apply the main theorem of the paper, the model must be recast as a stochastic game and conditions (i)–(v) in the definition of noisy stochastic game verified. The set of players is N, and the set of states is $S = Q \times R$, where $Q = \{0,1\}^n \times [0, \overline{k}]^{n\ell}$ and R is as above, both complete, separable metric spaces endowed with their Borel sigma-algebras. The set of conceivable actions for firm i is $X_i = \{0,1\} \times \tilde{X}_i$, where $\tilde{X}_i$ is a compact subset of $\Re^d$. The correspondence of feasible actions for firm i is defined by $A_i(s) = \{0,1\} \times \phi(z,k,r)$, which is lower measurable with nonempty, compact values contained in X_i, and an action for firm i is $a_i = (e_i, p_i)$. The stage payoff of firm i is then

$$u_i(s,a) = \begin{cases} \pi_i(s,(e_i,p)) & \text{if } z_i = e_i = 1 \\ \iota_i(s,(e_i,p)) & \text{if } z_i = 1, e_i = 0, \\ \alpha_i(s) & \text{if } z_i = 0, e_i = 1, \\ 0 & \text{if } z_i = e_i = 0, \end{cases}$$

which is bounded and measurable and is continuous in a. Discount factors are as given in the original model. To define transition densities fulfilling the definition of noisy stochastic game, let $\hat{\kappa}$ be the uniform distribution on $\{0,1\}^n$, and define $\kappa = \hat{\kappa} \times \tilde{\kappa}$. Let $\hat{g}(\cdot|e)$ be the unit mass on e, i.e., $\hat{g}(z|e) = 1$ if $z = e$ and otherwise, $\hat{g}(z|e) = 0$. Then, given $q = (z,k)$, define the transition density $g(\cdot|s,a)$ by $g(q'|s,a) = \hat{g}(z'|e)\tilde{g}(k'|(z,k),(e,p))$ with respect to κ, which is jointly measurable in (q', s, a) and continuous in a, fulfilling (iii). Letting λ

be the uniform distribution on R, which is nonatomic, and h be as given in the original model, conditions (i)–(v) are satisfied, and the main existence theorem directly yields Proposition 1.

4.2 Partisan Competition and Time-consistent Policy

This subsection provides a dynamic model of elections between two parties in which parties cannot commit to policies prior to an election, and a representative voter sequentially chooses between the two parties.[11] At the beginning of each period, a state of the economy e is given, the voter decides between the two parties, and the winning party chooses a policy p in a policy space $P \subseteq \Re^d$, which then stochastically determines a new state prior to the election next period. To apply the main theorem of the paper, I impose the further structure that the voter's preferences contain an idiosyncratic component that is realized at the beginning of each period prior to the election but after the incumbent's policy choice while in office, so voting is probabilistic; consistent with that assumption, the parties' preferences are also subject to idiosyncratic shocks that are unobserved by the voter at the time of the election. In equilibrium, because they cannot commit, the parties use time-consistent policies, i.e., they choose optimally given the voter's expectations of their choices. In contrast to the standard macroeconomic framework (e.g., Kydland and Prescott (1977)), the parties are in competition with each other, so a party's optimization problem also takes as given the expectations of the voters' future choices between the two parties and the opposing party's future policy choices when elected.

An interesting incentive that arises in equilibrium is that each party will seek to influence future economic states to its advantage. In particular, it may be that one party seeks to "tie the hands" of the other, or to engender economic states in which it is perceived favorably by the voter. The model has antecedents in Alesina (1988), which considers repeated elections with probabilistic voting, but there is no economic state variable in the setting of that paper; there, voting behavior is black-boxed (and does not depend on expectations of the voters of the parties' policy choices); and Alesina considers equilibria in trigger strategies, rather than stationary Markov perfect strategies. Also related is Alesina (1987), who studies stationary equilibria in a model of macroeconomic policy making, where the party in power chooses a level of monetary expansion and rational wage setters anticipate monetary policy. There, however, parties are myopic and voting behavior is exogenous. Dixit et al. (2000) analyze a model in which farsighted parties compete in elections to divide a surplus, and in which a state variable evolves according to an exogenous Markov process.

[11] The assumption of a representative voter is for tractability only. In the equilibrium analysis of voting, it is important that voters eliminate weakly dominated strategies, a refinement that does not generally hold in stationary Markov perfect equilibria. This problem could be finessed in the stochastic game model by having voters vote sequentially, but it is simpler to assume a representative voter.

In contrast, the model of this subsection endogenizes voting behavior, and it allows the possibility that current policy decisions influence future states. As well, voting behavior is exogenous in their model, and those authors focus on efficient equilibria in history-dependent strategies.

Formally, let there be two parties, B and C, and a representative voter, V, and suppose that in each period, the voter must select one party, which then makes a policy decision. Accordingly, each period is divided into two phases: voting and policy making. At the beginning of any period, an economic state e belonging to a subset $E \subseteq \Re^{\ell}$ is given, where E is compact and has positive Lebesgue measure, and also given is a shock $r_V = (\epsilon_B, \epsilon_C)$ belonging to the compact set $R_V = [0,1] \times [0,1]$. In the voting phase, the voter decides between parties B and C by casting a ballot $w \in \{B, C\}$. A new economic state e' is realized according to the density function $\tilde{g}(e'|e,w)$, and preference shocks r_B, r_C belonging to sets $R_B, R_C \subseteq \Re^m$ for the parties are then drawn from the density $\tilde{h}(r_B, r_C|e')$, where R_B and R_C are assumed to be compact and to have positive Lebesgue measure. In the policy making phase, the winning party, w, chooses a policy p belonging to a compact subset $P \subseteq \Re^d$. Finally a new economic state e'' is realized from the density $\tilde{g}(e''|w,e',p)$, which is measurable and assumed continuous in p, new shocks r'_V are drawn according to the density $\tilde{h}(r'_V|e'')$, and another election is held.

In the election phase, the voter receives payoff ϵ_B if $w = B$ and ϵ_C if $w = C$, while the parties receive a zero payoff. In the policy making phase, the voter receives a payoff $u_V(e',p)$ from policy p in state e', and the parties' payoffs are $u_B(e', r_B, p)$ and $u_C(e', r_C, p)$, where all stage utility functions are jointly measurable and continuous in p. Payoffs are discounted after each period (at the end of the policy making phase) by $\tilde{\delta}_V$, $\tilde{\delta}_B$, and $\tilde{\delta}_C$, respectively, for the voter and parties. In the voting phase, the voter must compare the current shocks ϵ_B and ϵ_C from electing either party, together with the parties' policy choices if elected in the current period, plus the discounted future payoffs following the election of either party. Thus, the voter's shocks act to perturb the voter's payoffs from electing either party, and as is common in the literature on probabilistic voting, we interpret these shocks as reflecting non-policy related attributes of the parties (such as the charisma of the parties' candidates) that are unobserved at the time policies are chosen. In the simple formulation above, the economic state e realized in the voting phase does not directly affect payoffs; it is simply a technical device that represents the voter's information about the economic state that will obtain in the subsequent policy making phase.

Proposition 2: In the dynamic model of partisan competition and time-consistent policy, there exists a stationary Markov perfect equilibrium.

To apply the main theorem, this model must be reformulated as a noisy stochastic game. The set of players is $N = \{V, B, C\}$, and the set of states is $S = Q \times R$, where $Q = \{V, B, C\} \times E$ and $R = R_V \cup (R_B \times R_C)$, both complete, separable metric spaces endowed with their Borel sigma-algebras. The sets of conceivable actions are $X_B = X_C = P$ for the parties, both compact subsets of $\Re^d$, and $X_V = \{B, C\}$ for the voter. The correspondence of feasible actions is defined, given $s = (q, r) = ((i, e), r)$, as $A_V(s) = \{B, C\}$ when $i = V$, as $A_B(s) = P$ when $i = B$, and as $A_C(s) = P$ when $i = C$; in states where a player i is inactive, the set of actions of that player can be specified as an arbitrary compact subset of X_i. Given (s, a), where $s = (q, r) = ((i, e), r)$, stage payoffs are defined as

$$u_V(s,a) \quad = \quad \begin{cases} \epsilon_w & \text{if } i = V \text{ and } a_V = w \\ \frac{1}{\delta_V} u_V(e,p) & \text{if } i = B, C \text{ and } a_i = p \end{cases}$$

and

$$u_B(s,a) \quad = \quad \begin{cases} 0 & \text{if } i = V \\ \frac{1}{\delta_B} u_B(e, r_B, p) & \text{if } i = B, C \text{ and } a_i = p, \end{cases}$$

and similarly for party C, and discount factors are the square roots, $\delta_i = \sqrt{\tilde{\delta}_i}$, of the original discount factors. A "period" in the stochastic game formulation corresponds to a "phase" in the original model, so in contrast to the original, discounting now must occur between the voting and policy making phases; this is undone by using square roots of the original discount factors and by inflating stage payoffs in the policy making stage accordingly.

To define transition densities as in the definition of noisy stochastic game, let $\hat{\kappa}$ be uniform on $\{V, B, C\}$ and $\tilde{\kappa}$ be uniform on E, and define $\kappa = \hat{\kappa} \times \tilde{\kappa}$. Given (s, a), where $s = (q, r) = ((i, e), r)$, define the density on $q' = (i', e')$ with respect to κ as follows:

$$g(q'|s,a) \quad = \quad \begin{cases} \tilde{g}(e'|e,w) & \text{if } i = V,\ a_V = w = i' \\ \tilde{g}(e'|w,e,p) & \text{if } i = w,\ a_w = p, \text{ and } i' = V \\ 0 & \text{else.} \end{cases}$$

Thus, the first coordinate of q tracks whether the game is in a voting phase or policy making phase, and the transition on economic states is given by the density $\tilde{g}$. This is jointly measurable and continuous in a. Let $\hat{\lambda}$ and $\tilde{\lambda}$ be uniform on R_V and $R_B \times R_C$, respectively, and let $\lambda = \frac{1}{2}\hat{\lambda} + \frac{1}{2}\tilde{\lambda}$, which is nonatomic. Given $q = (i, e)$, define

$$h(r|q) \quad = \quad \begin{cases} \tilde{h}(r_V|e) & \text{if } i = V, \\ \tilde{h}(r_B, r_C|e) & \text{if } i = B, C. \end{cases}$$

This verifies conditions (i)–(v), and the main existence theorem implies Proposition 2.

5 Discussion of Proof

To describe the method of proof, I begin with the fixed point of argument used by Nowak and Raghavan (1992) to prove existence of stationary Markov

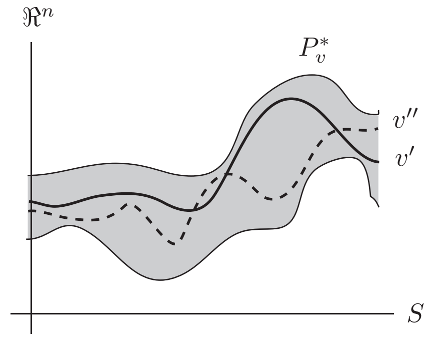

Figure 1: Correlation approach

perfect equilibrium with public randomization. The argument takes place in a compact, convex space V of continuation values $v\colon S \to \Re^n$. Given v, we define the induced game $\Gamma_v(s)$ with actions $A_i(s)$ and payoffs

$$(1-\delta_i)u_i(s,a) + \delta_i \int_{s'} v_i(s')\mu(ds'|s,a).$$

Assuming the transition $\mu(\cdot|s,a)$ is norm-continuous in a, these payoffs are continuous in actions, and the theorem of Debreu (1952)-Fan (1952)-Glicksberg (1952) implies that there is at least one mixed strategy equilibrium of the induced game. Let $P_v(s)$ be the set of mixed strategy equilibrium payoff vectors of the induced game, and let $P_v^*(s)$ be the convex hull of that set. To update continuation values, take all selections v' from the correspondence $s \to P_v^*(s)$. This gives us a nonempty-valued correspondence $v \to E_v$ pictured in Figure 1. Because we are selecting from the convex hull of induced equilibrium payoffs, E_v is clearly convex. Closed graph of $v \to E_v$ follows from both continuity assumptions imposed on the model and convex values of the correspondence; if $v \to E_v$ is not convex-valued, then closed graph does not follow. Thus, the correspondence $v \to E_v$ has a fixed point $v^* \in E(v^*)$, and equilibrium strategies can be backed out from v^*, with care to ensure measurability.

The role of public randomization in convexifying equilibrium payoffs in induced games is critical in the above argument. To eschew correlation, I employ a nonatomically distributed noise component of the state. The argument now takes place in a compact, convex set V of "interim" continuation values, which are conditioned only on the realization of the standard component q, rather than the full state. To convey this notion more precisely, I define the interim continuation $v\colon Q \to \Re^n$ generated by strategy profile σ by the recursion

$$\begin{aligned} & v_i(q;\sigma) \\ &= \int_r \left[\int_a \left[(1-\delta_i)u_i(s,a) + \delta_i \int_{q'} v_i(q';\sigma)\mu_q(dq'|s,a) \right] \sigma(da|s) \right] h(r|q)\lambda(dr), \end{aligned}$$

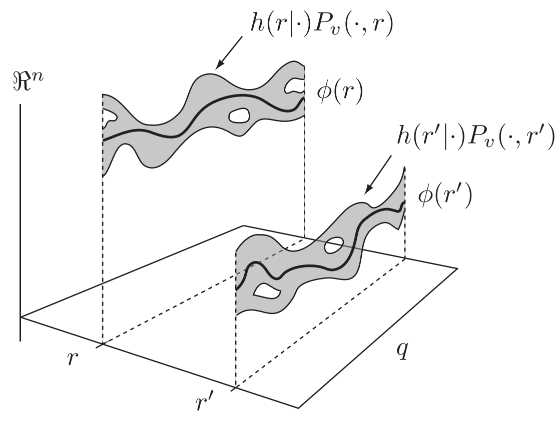

Figure 2: Noise approach

where $s = (q, r)$. Given interim continuation value v, define the induced game $\Gamma_v(s)$ with actions $A_i(s)$ and payoffs

$$(1 - \delta_i)u_i(s, a) + \delta_i \int_{q'} v_i(q')\mu_q(dq'|s, a),$$

which are continuous in actions by norm-continuity of $\mu_q(\cdot|s, a)$ in a. Let $P_v(s)$ be the mixed strategy equilibrium payoffs in $\Gamma_v(s)$, which need not be convex.

To update continuation values, for each $s = (q, r)$, choose an element of $P_v(s)$. Intuitively, we then integrate across the noise component r to get a new interim continuation value v'. Repeating this for all possible ways of selecting from induced equilibrium payoffs, we define a correspondence $v \to E_v$ that maps any v to a set E_v of updated interim continuation values. The crux of the proof is to formalize this idea and establish the usual properties of this correspondence in order to deduce a fixed point. Technically, to define this correspondence, we take a selection $\phi(r) \colon Q \to \Re^n$ for each r of density-weighted equilibrium payoffs, i.e., for a.e. q, $\phi(r)(q) \in h(r|q)P_v(q, r)$. Then, given the measurable mapping $\phi \colon R \to V$, the Bochner integral $v' = \int_r \phi(r)\lambda(dr)$ provides a new interim continuation value, and we repeat this procedure for each function ϕ taking selections of density-weighted equilibrium payoffs. More formally, letting $\Phi_v(r)$ be the set of density-weighted equilibrium payoff selections at r, the set of updated continuation values is $E_v = \int_r \Phi_v(r)\lambda(dr)$, the Bochner integral of the correspondence Φ_v.

The key to the proof is establishing that $v \to E_v$ has convex values and closed graph. Both properties rely on the observation that E_v can be equivalently defined by integrating over selections from $P_v^*(s)$. That is, letting $\Phi_v^*(r)$ be the set of density-weighted mixtures of equilibrium payoff selections as a function of q, we have $E_v = \int \Phi_v^*(r)\lambda(dr)$. The argument for the claim pro-

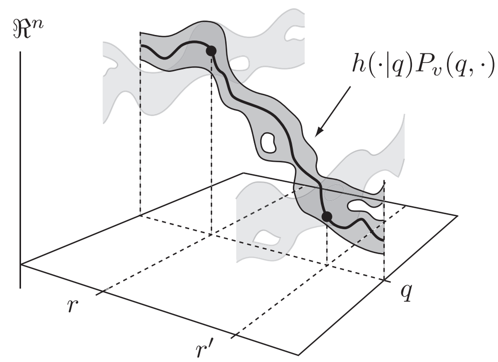

Figure 3: Applying Liapunov

ceeds by arbitrarily choosing $v' \in \int_r \Phi_v^*(r)\lambda(dr)$ and considering each q separately. For almost all q, we have $v'(q) \in \int_r h(r|q)P_v^*(q,r)\lambda(dr)$. Given such q, the correspondence $h(\cdot|q)P_v(q,\cdot)\colon R \to \Re^n$ may have non-convex values, as depicted in Figure 3, but it maps to finite-dimensional Euclidean space. Thus, since λ is nonatomic, a version of Liapunov's theorem yields the equality $\int_r h(r|q)P_v(q,r)\lambda(dr) = \int_r h(r|q)P_v^*(q,r)\lambda(dr)$, and in particular, there is a mapping $\phi(q)\colon R \to \Re^n$ that is measurable on R, integrates to $v'(q)$, and is an almost everywhere selection from $h(\cdot|q)P_v(q,\cdot)$. The selections $\phi(q)$ are chosen independently for each q, and so the mapping $\phi\colon Q \to V$ so-defined need not be measurable, but the theorem of Artstein (1989) allows us to "sew" these selections together in a measurable way, giving us $v' \in \int_r \Phi_v(r)\lambda(dr)$, as required. Then, using the fact that $v \to \Phi_v^*$ has convex values and closed graph, an infinite-dimensional Fatou's lemma due to Yannelis (1990) implies closed graph of the correspondence $v \to E_v$. Therefore, $v \to E_v$ possesses a fixed point $v^* \in E_{v^*}$, and the final step of the proof is to back out equilibrium strategies corresponding to this value.

6 Proof of the Theorem

The formal proof of existence must initially work with functions that are measurable with respect to the completion of λ. Let $\mathcal{R}^*$ denote the sigma-algebra of λ-measurable sets, and let the completion of λ be λ^*, the unique extension of λ to $\mathcal{R}^*$. Since λ is nonatomic, so is λ^*. Endow $L_2^n \equiv L_2^n(Q,\mathcal{Q},\kappa)$ with the weak*, or equivalently the weak, topology $\sigma(L_2^n, L_2^n)$. Let V be the subset of all κ-equivalence classes of functions $v \in L_2^n$ such that $||v(q)|| \leq C$ for κ-almost all q, where C is a fixed constant such that $||u(s,a)|| \leq C$ for all s and a. Obviously, V is nonempty and convex, and it follows from Alaoglu's theorem (see Theorem 6.21 of Aliprantis and Border (2006)) that V is compact. Henceforth, universal quantifiers over continuation value functions are understood to range over V.

For each $v \in V$, let $\Gamma_v(s)$ be the stage game induced by v at s, where player i's action space is $A_i(s)$ and i's payoff from a is

$$U_i(s,a;v) \quad = \quad (1-\delta_i)u_i(s,a) + \delta_i \int_{q'} v_i(q')\mu_q(dq'|s,a),$$

let $U(s,a;v) = (U_1(s,a;v),\ldots,U_n(s,a;v))$ be the vector of payoffs, and note from norm-continuity of $\mu_q(\cdot|s,a)$ in a that $U(s,a;v)$ is continuous in actions. A mixed strategy for player i in $\Gamma_v(s)$ is a Borel probability measure $\alpha_i \in \mathcal{P}(X_i)$ such that $\alpha_i(A_i(s)) = 1$, and mixed strategies for all players determine the product probability measure $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_n \in \bigotimes_i \mathcal{P}(X_i)$. The space $\bigotimes_i \mathcal{P}(X_i)$ of product probability measures is endowed with the relative weak* topology inherited from $\mathcal{P}(\prod_i X_i)$, so convergence of a sequence $\{\alpha^m\}$ to $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_n$ is equivalent to convergence of the marginals $\{\alpha_i^m\}$ to α_i, $i = 1,\ldots,n$. Define the extension $U(\cdot;v)\colon S \times \bigotimes_i \mathcal{P}(X_i) \to \Re^n$ to mixed strategies in the induced game by $U(s,\alpha;v) = \int_a U(s,a;v)\alpha(da)$, with the relative topology on $\bigotimes_i \mathcal{P}(X_i)$ induced by the weak* topology on $\mathcal{P}(X)$. By continuity of $U(s,\alpha;v)$ in α and compactness of each $\mathcal{P}(X_i)$, the Debreu-Fan-Glicksberg theorem implies that the set of mixed strategy Nash equilibria of $\Gamma_v(s)$, denoted $N_v(s)$, is a nonempty, compact subset of $\bigotimes_i \mathcal{P}(X_i)$. Let $P_v(s)$ denote the payoffs generated by equilibria in $N_v(s)$, i.e., $P_v(s) = U(s,N_v(s);v) = \{U(s,\alpha;v) \mid \alpha \in N_v(s)\}$. By continuity of $U(s,\alpha;v)$ in α, $P_v(s)$ is compact. The first lemma, reproduced from Lemma 5 of Nowak and Raghavan (1992), establishes that the correspondence of mixed strategy equilibria is lower measurable in the state, i.e., for every open set $G \subseteq \mathcal{P}(X)$, the set $\{s \in S \mid N_v(s) \cap G \neq \emptyset\}$ belongs to $\mathcal{S}$.

Lemma 1: For each v, the correspondence $s \to N_v(s)$ is lower measurable with respect to $\mathcal{S}$.

For all r, let $\Phi_v(r)$ be the set of $\mathcal{Q}$-measurable, density-weighted, equilibrium payoff selections as a function of q: specifically, $f \in L_1^n \equiv L_1^n(Q,\mathcal{Q},\kappa)$ belongs to $\Phi_v(r)$ if and only if for κ-almost all q, $f(q) \in h(r|q)P_v(q,r)$, i.e., there is an equilibrium payoff vector $y \in P_v(q,r)$ of the stage game induced by v at (q,r) such that $f(q) = h(r|q)y$. Define $E_v = \int_r \Phi_v(r)\lambda^*(dr)$, the Bochner integral of the correspondence $r \to \Phi_v(r)$ with respect to the completion λ^*. The next lemma establishes that the correspondence $v \to E_v$ maps to subsets of V.

Lemma 2: For each v, $E_v \subseteq V$.

Proof: Given $v \in V$, consider any $f \in E_v$. Then there exists an $\mathcal{R}^*$-measurable selection $\phi\colon R \to L_1^n$ such that $\phi(r) \in \Phi_v(r)$ for λ^*-almost all r and such that $f = \int_r \phi(r)\lambda^*(dr)$. Moreover, for all r with $\phi(r) \in \Phi_v(r)$ and for κ-almost all q, $\phi(r)(q) \in h(r|q)P_v(q,r)$. By Aliprantis and Border's (2006) Theorem 11.47, part 1, there is a $\kappa \otimes \lambda^*$-integrable function $F\colon Q \times R \to \Re^n$ such that

(a) for λ^*-almost all $r \in R$, we have $\phi(r) = F(\cdot,r) \in L_1^n$

(b) for κ-almost all $q \in Q$, $F(q,\cdot)$ is λ^*-integrable and

$$\left(\int_r \phi(r)\lambda^*(dr)\right)(q) \quad = \quad \int_r F(q,r)\lambda^*(dr).$$

An implication of (a), with the fact that $\phi(r) \in \Phi_v(r)$ for λ^*-almost all r, is that for κ-almost all q and for λ^*-almost all r, $F(q,r) \in h(r|q)P_v(q,r)$. With (b), this implies that there is a $\mathcal{Q}$-measurable set Q^0 with $\kappa(Q^0) = 0$ such that for all $q \in Q \setminus Q^0$, we have (i) for λ^*-almost all r, $F(q,r) \in h(r|q)P_v(q,r)$, and (ii) $(\int_r \phi(r)\lambda^*(dr))(q) = \int_r F(q,r)\lambda^*(dr)$. Then for all $q \in Q \setminus Q^0$, we have

$$\begin{aligned} ||f(q)|| \quad &= \quad \left|\left|\left(\int_r \phi(r)\lambda^*(dr)\right)(q)\right|\right| \quad = \quad \left|\left|\int_r F(q,r)\lambda^*(dr)\right|\right| \\ &\leq \quad \int_r ||F(q,r)||\lambda^*(dr) \quad \leq \quad \int_r Ch(r|q)\lambda^*(dr) \quad \leq \quad C, \end{aligned}$$

where the first equality follows from $f = \int_r \phi(r)\lambda^*(dr)$, the second equality from (ii), the first inequality from Jensen's inequality (see Theorem 11.24 of Aliprantis and Border (2006)), the second inequality from (i), and the last inequality from the fact that $h(\cdot|q)$ is a density. Of course $\int_q ||f(q)||^2\kappa(dq) < \infty$. This implies $f \in V$, and therefore $E_v \subseteq V$.

It remains to be shown that $v \to E_v$ has nonempty, convex values and closed graph in the weak* topology. The next lemma, a key step in the proof, is Lemma 6 in Nowak and Raghavan (1992). It shows that for every open set $G \subseteq \Re^n$, the set $\{(q,r) \in S \mid P_v(q,r) \cap G \neq \emptyset\}$ belongs to $\mathcal{S}$ and, therefore, also to $\mathcal{Q} \otimes \mathcal{R}$.

Lemma 3: For each v, the correspondence $(q,r) \to P_v(q,r)$ is lower measurable with respect to $\mathcal{S}$, and therefore also with respect to $\mathcal{Q} \otimes \mathcal{R}^*$.

An immediate implication of the preceding lemma, via the Kuratowski-Ryll-Nardzewski selection theorem (see Theorem 18.13 of Aliprantis and Border (2006)), is that for each v, the correspondence $(q,r) \to P_v(q,r)$ admits a $\mathcal{Q}\otimes\mathcal{R}^*$-measurable selection. This implies, as stated in the next lemma, that $v \to E_v$ has nonempty values.

Lemma 4: For each v, $E_v \neq \emptyset$.

Proof: It suffices to deduce a Bochner integrable selection $\phi\colon R \to L_1^n$ such that $\phi(r) \in \Phi_v(r)$ for λ^*-almost all r. By the previous lemma and the Kuratowski-Ryll-Nardzewski selection theorem, there exists a $\mathcal{Q}\otimes\mathcal{R}^*$-measurable mapping $y\colon Q \times R \to \Re$ satisfying $y(q,r) \in P_v(q,r)$ for all q and all r. In particular, $||y(q,r)|| \leq C$ for all q and all r. Obviously, $\int_q \int_r h(r|q)\lambda^*(dr)\kappa(dq) = 1$, and Tonelli's theorem (see Theorem 11.28 of Aliprantis and Border (2006)) implies that $h(r|q)$ is $\kappa \otimes \lambda^*$-integrable and that $\int_r \int_q h(r|q)\kappa(dq)\lambda^*(dr) = 1$. Therefore, it must be that for λ^*-almost all r, we have $\int_q h(r|q)\kappa(dq) < \infty$. Let R^0 be a $\mathcal{R}^*$-measurable set with $\lambda^*(R^0) = 0$ and such that for all $r \in R \setminus R^0$, $\int_q h(r|q)\kappa(dq) < \infty$. Define $\phi\colon R \to L_1^n$ so that $\phi(r) = 0$ for $r \in R^0$ and so that $\phi(r)(q) = h(r|q)y(q,r)$ for $r \in R \setminus R^0$ and all q. Then $\phi(r) \in \Phi_v(r)$ for λ^*-almost all r, and $h(r|q)y(q,r)$ is $\kappa \otimes \lambda^*$-integrable, so Aliprantis and Border's (2006) Theorem 11.47, part 2, implies ϕ is Bochner integrable, as required.

To establish convex values and closed graph of $v \to E_v$, it will be useful to define the following auxiliary correspondence. For each v, let $P_v^*(s)$ denote the convex hull of $P_v(s)$, and let $\Phi_v^*(r)$ be the set of $\mathcal{Q}$-measurable, density-weighted, convex combinations of equilibrium payoff selections as a function of q: specifically, $f \in L_1^n$ belongs to $\Phi_v^*(r)$ if and only if for κ-almost all q, $f(q) \in h(r|q)P_v^*(q,r)$, i.e., there is a convex combination $y \in P_v^*(q,r)$ of equilibrium payoff vectors in the induced game such that $f(q) = h(r|q)y$. In contrast to $\Phi_v(r)$, the set $\Phi_v^*(r)$ must be convex, a property that implies closed graph of $v \to \Phi_v^*(r)$ for all r, as stated in the next lemma, which is adapted from Lemma 7 of Nowak and Raghavan (1992).

Lemma 5: For all r, the correspondence $v \to \Phi_v^*(r)$ has weak* closed graph.

Proof: The proof is exactly that of Nowak and Raghavan's (1992) Lemma 7 after identifying our a with their x, our q with their s, our q' with their t, our $\mu_q(\cdot|s,a)$ with their $q(\cdot|s,x)$, our $U_i(q,r,a;v)$ (with r fixed) with their $u_i(s,x)(v)$, and our correspondence $v \to \Phi_v^*(r)$ with their $v \to M_v$.

The usefulness of the latter lemma lies in the fact that E_v can be written as the Bochner integral of $r \to \Phi_v^*(r)$, i.e., $\int_r \Phi_v(r)\lambda^*(dr) = \int_r \Phi_v^*(r)\lambda^*(dr)$. One direction of this inclusion is obvious. For the less trivial $\supseteq$ inclusion, we would like to apply a version of Liapunov's theorem for correspondences (e.g., Theorem 4, p.64, of Hildenbrand (1974)) with respect to a nonatomic measure: for correspondences mapping to $\Re^n$, Liapunov's theorem implies that the integral of a correspondence with respect to a nonatomic measure is equal to the integral of the convex hull of the correspondence. But Liapunov's theorem does not hold in infinite-dimensional settings, so this direct avenue is not open. Instead, the approach I use implicitly relies on the fact that the correspondence $r \to \Phi_v(r)$ has a product structure, in that $P_v(q,r)$ is defined independently for each q; the selection of equilibrium payoffs in the induced game $\Gamma(q,r)$ does not restrict (beyond considerations of measurability) the selection at $\Gamma(q',r)$. This permits the application of Liapunov's theorem separately for each q.

Thus, the proof "goes down" from the Bochner integral $\int_r \Phi_v^*(r)\lambda(dr)$ to integrals of the correspondence $(q,r) \to P_v^*(q,r)$ defined on (q,r) pairs. I then apply Liapunov's convexity theorem for correspondences mapping to subsets of $\Re^n$, integrating across r one q at a time. Finally the proof "goes up" to the Bochner integral. A technical issue is that in the second step, we have one integral $\int_r P_v(q,r)h(r|q)\lambda^*(dr)$ for each q, and thus one selection from $r \to h(r|q)P_v(q,r)$ for each q. To return to the Bochner integral, we have to "sew up" these selections in a measurable way, a task simplified by a theorem of Artstein (1989).[12]

[12] Alternatively, we can deduce this lemma from part 2 of the more general theorem of

Lemma 6: For each v, $E_v = \int_r \Phi_v^*(r)\lambda^*(dr)$.

Proof: Clearly, $E_v \subseteq \int_r \Phi_v^*(r)\lambda^*(dr)$. Now consider any $f \in \int_r \Phi_v^*(r)\lambda^*(dr)$, so there exists a $\mathcal{R}^*$-measurable mapping $\phi\colon R \to L_1^n$ such that $\phi(r) \in \Phi_v^*(r)$ for λ^*-almost all r and $f = \int_r \phi(r)\lambda^*(dr)$. By Aliprantis and Border's (2006) Theorem 11.47, part 1, there is a $\kappa \otimes \lambda^*$-integrable function $F\colon Q \times R \to \Re^n$ satisfying (a) and (b) in the proof of Lemma 2. An implication of (a), with the fact that $\phi(r) \in \Phi_v^*(r)$ for λ^*-almost all r, is that for κ-almost all q and for λ^*-almost all r, $F(q,r) \in h(r|q)P_v^*(q,r)$. With (b), this implies that there is a $\mathcal{Q}$-measurable set Q^0 with $\kappa(Q^0) = 0$ such that for all $q \in Q \setminus Q^0$, we have (i) for λ^*-almost all r, $F(q,r) \in h(r|q)P_v^*(q,r)$, and (ii) $(\int_r \phi(r)\lambda^*(dr))(q) = \int_r F(q,r)\lambda^*(dr)$. Then for all $q \in Q \setminus Q^0$, we have

$$f(q) = \int_r F(q,r)\lambda^*(dr) \in \int_r P_v^*(q,r)h(r|q)\lambda^*(dr) = \int_r P_v(q,r)h(r|q)\lambda^*(dr),$$

where the first equality follows from $f = \int_r \phi(r)\lambda^*(dr)$ and (ii), the inclusion from (i), and the last equality from Hildenbrand's (1974) Theorem 4 (p.64). It follows that for κ-almost all q, $f(q) \in \int_r P_v(q,r)h(r|q)\lambda^*(dr)$. To apply the theorem of Artstein (1989), note that Q and R are complete separable metric spaces, and the correspondence $(q,r) \to P_v(q,r)$ is lower measurable with respect to $\mathcal{Q} \otimes \mathcal{R}$ and has nonempty, compact values. Artstein's theorem applies to integration with respect to Borel probability measures, but Theorem 10.35 of Aliprantis and Border (2006) implies that every $\mathcal{R}^*$-measurable selection from $r \to P_v(q,r)$ is equivalent to an $\mathcal{R}$-measurable function λ-almost everywhere. Thus, for κ-almost all q, $f(q) \in \int_r P_v(q,r)h(r|q)\lambda(dr)$. Lastly, for all q, $||\int_r h(r|q)P_v(q,r)\lambda(dr)|| \leq C$, so the correspondence $r \to h(r|q)P_v(q,r)$ is λ-integrably bounded. Then Artstein's theorem yields a $\mathcal{S}$-measurable mapping $G\colon Q \times R \to \Re$ such that for κ-almost all q, $f(q) = \int_r G(q,r)\lambda(dr)$ and for λ-almost all r, $G(q,r) \in h(r|q)P_v(q,r)$. In particular, G is $\mathcal{Q}\otimes\mathcal{R}^*$-measurable, and is in fact $\kappa \otimes \lambda^*$-integrable, and $f(q) = \int_r G(q,r)\lambda^*(dr)$ for κ-almost all q. By Aliprantis and Border's (2006) Theorem 11.47, part 2, the mapping $\psi\colon R \to L_1^n$ defined by $\psi(r)(q) = G(q,r)$ is Bochner integrable with respect to λ^*, and for κ-almost all q,

$$f(q) = \int_r G(q,r)\lambda^*(dr) = \left(\int_r \psi(r)\lambda^*(dr)\right)(q),$$

so $f = \int_r \psi(r)\lambda^*(dr)$. Furthermore, $\psi(r) \in \Phi_v(r)$ for λ^*-almost all r, and we conclude that $f \in \int_r \Phi_v(r)\lambda^*(dr) = E_v$, as required.

The preceding lemma immediately implies that the correspondence $v \to E_v$ is convex-valued.

Lemma 7: For each v, E_v is convex.

We can now establish that the correspondence $v \to E_v$ has closed graph by

Mertens (2003). The proof of Lemma 6 here is straightforward relative to Mertens' proof.

an application of a version of Fatou's lemma on upper hemicontinuity of the integral of Banach-valued correspondences, due to Yannelis (1990).

Lemma 8: The correspondence $v \to E_v$ has closed graph.

Proof: Note that $(R, \mathcal{R}^*, \lambda^*)$ is a complete, finite measure space, and L_2^n is a Banach space with the usual norm. Furthermore, since Q is a separable metric space, it follows that L_2^n is separable in the norm topology. (See Theorem 8.3.27 of Corbae et al. (2009).) A further implication, by Aliprantis and Border's (2006) Theorem 6.30, is that V is metrizable in the weak* topology. Furthermore, the correspondence $r \to \Phi_v^*(r)$ has nonempty, convex values, and $\Phi_v^*(r)$ is closed in L_2^n with the weak* topology and, therefore, in the norm topology. Thus, the background conditions of Yannelis's (1990) Theorem 3.2 are satisfied. For fixed r, we have seen that the correspondence $v \to \Phi_v^*(r)$ has weak* closed graph, fulfilling condition (i) of the latter theorem; and the correspondence has weak* compact range and is integrably bounded by the function taking the constant value C, fulfilling condition (ii). Thus, Theorem 3.2 of Yannelis (1990) implies that $v \to \int_r \Phi_v^*(r)\lambda^*(dr) = E_v$ has weak* closed graph, as required.

By the Debreu-Fan-Glicksberg theorem, there exists a fixed point $v \in E_v$. The final step of the proof is to construct a stationary Markov perfect equilibrium. Let $\phi\colon R \to L_1^n$ be an $\mathcal{R}^*$-measurable mapping such that $\phi(r) \in \Phi_v(r)$ for λ^*-almost all r and such that $v = \int_r \phi(r)\lambda^*(dr)$. Let $F\colon Q \times R \to \Re^n$ be a $\mathcal{Q} \otimes \mathcal{R}^*$-measurable function satisfying (a) and (b) in the proof of Lemma 2. In particular, for λ^*-almost all r, $F(\cdot, r) \in \Phi_v(r)$, and for κ-almost all q, $v(q) = \int_r F(q,r)\lambda^*(dr)$. The former condition implies there is a $\mathcal{Q} \otimes \mathcal{R}^*$-measurable set S^0 with $(\kappa \otimes \lambda^*)(S^0) = 0$ and such that for all $s = (q,r) \in S \setminus S^0$, we have $F(q,r) \in h(r|q)P_v(q,r)$. Recalling that $\nu = \kappa \otimes \lambda$, Aliprantis and Border's (2006) Theorem 10.47 implies that the product sigma-algebra $\mathcal{Q} \otimes \mathcal{R}^*$ is contained in the sigma-algebra of ν-measurable sets, denoted $\mathcal{S}^*$, and therefore F is measurable with respect to $\mathcal{S}^*$. By Theorem 10.35 of Aliprantis and Border (2006), there is a $\mathcal{S}$-measurable mapping $G\colon Q \times R \to \Re^n$ such that $G(q,r) = F(q,r)$ for ν-almost all (q,r). In particular, for κ-almost all q, we have $v(q) = \int_r G(q,r)\lambda(dr)$, and there is a $\mathcal{S}$-measurable set S^1 with $S^0 \subseteq S^1$ and $\nu(S^1) = 0$ such that for all $(q,r) \in S \setminus S^1$, we have $G(q,r) \in h(r|q)P_v(q,r)$. Using the fact that $(q,r) \to P_v(q,r)$ is lower measurable with respect to $\mathcal{S}$, and therefore admits a $\mathcal{S}$-measurable selection, we can specify that $G(q,r) \in h(r|q)P_v(q,r)$ for all $(q,r) \in S^1$ as well.

Recall that $U(\cdot; v)\colon S \times \bigotimes_i \mathcal{P}(X_i) \to \Re^n$ is a Caratheodory function, i.e., $U(s, \alpha; v)$ is jointly measurable in (s, α) and is continuous in α, and that $s \to N_v(s)$ is lower measurable with respect to $\mathcal{S}$. Moreover, for all $s = (q,r)$, there exists $\alpha \in N_v(s)$ such that $G(q,r) = h(r|q)U(q,r,\alpha;v)$. Then Filippov's implicit function theorem (see Aliprantis and Border's (2006) Theorem 18.17) yields a $\mathcal{S}$-measurable function $\xi\colon S \to \bigotimes_i \mathcal{P}(X_i)$ such that for all s, we have $\xi(s) \in N_v(s)$ and $G(s) = h(r|q)U(s, \xi(s); v)$. Define the strategy $\sigma_i\colon S \to \mathcal{P}(X_i)$ for each player i so that for all s, $\sigma_i(s)$ is the marginal of $\xi(s)$ on X_i, and write $\sigma(s)$ for

$\xi(s)$, the product of the players' mixed strategies. By Aliprantis and Border's (2006) Theorem 19.7, $\sigma_i \colon S \to \mathcal{P}(X_i)$ is indeed measurable, so these strategies are well-defined.

We have left to confirm that σ is an equilibrium. For κ-almost all q and all i, we have

$$\begin{aligned} v_i(q) &= \int_r G_i(s)\lambda(dr) = \int_r U_i(s,\sigma(s);v)h(r|q)\lambda(dr) \\ &= \int_r \left[\int_a \left[(1-\delta_i)u_i(s,a) + \delta_i \int_{q'} v_i(q')\mu_q(dq'|s,a)\right]\sigma(da|s)\right]\mu_r(dr|q), \end{aligned}$$

where $s=(q,r)$. Now define $w \in L_1^n(S,\mathcal{S},\nu)$ so that for ν-almost all q and all i,

$$w_i(s) = \int_a \left[(1-\delta_i)u_i(s,a) + \delta_i \int_{q'} v_i(q')\mu_q(dq'|s,a)\right]\sigma(da|s). \tag{1}$$

Note that for all s, all a, and all i,

$$\int_{q'} v_i(q')\mu_q(dq'|s,a) = \int_{s'} w_i(s')\mu(ds'|s,a), \tag{2}$$

and then for ν-almost all s and all i, we obtain

$$w_i(s) = \int_a \left[(1-\delta_i)u_i(s,a) + \delta_i \int_{s'} w_i(s')\mu(ds'|s,a)\right]\sigma(da|s),$$

so w satisfies the recursion that uniquely defines $v(\cdot;\sigma)$. Therefore, $w = v(\cdot;\sigma)$. Furthermore, using (1), continuation values can be written in terms of payoffs in the game induced by v at s as

$$v_i(s;\sigma) = \int_a \left[(1-\delta_i)u_i(s,a) + \delta_i \int_{q'} v_i(q')\mu_q(dq'|s,a)\right]\sigma(da|s).$$

Then, with (2), the fact that $\sigma_i(s)$ is a best response to $\sigma_{-i}(s)$ in $\Gamma_v(s)$ implies

$$\begin{aligned} &v_i(s;\sigma) \\ &= \max_{a_i \in A_i(s)} \int_{a_{-i}} \left[(1-\delta_i)u_i(s,a) + \delta_i \int_{s'} v_i(s';\sigma)\mu(ds'|s,a)\right]\sigma_{-i}(da_{-i}|s), \end{aligned}$$

for all i and all s. Therefore, σ is a stationary Markov perfect equilibrium.

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