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# Extremal Choice Equilibrium: Existence and Purification with Infinite-Dimensional Externalities

Barelli, Paulo and John Duggan

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# EXTREMAL CHOICE EQUILIBRIUM: EXISTENCE AND PURIFICATION WITH INFINITE-DIMENSIONAL EXTERNALITIES

PAULO BARELLI AND JOHN DUGGAN

Abstract. We prove existence and purification results for equilibria in which players choose extreme points of their feasible actions in a class of strategic environments exhibiting a product structure. We assume finite-dimensional action sets and allow for infinite-dimensional externalities. Applied to large games, we obtain existence of Nash equilibrium in pure strategies while allowing a continuum of groups and general dependence of payoffs on average actions across groups, without resorting to saturated measure spaces. Applied to games of incomplete information, we obtain a new purification result for Bayes-Nash equilibria that permits substantial correlation across types, without assuming conditional independence given the realization of a finite environmental state. We highlight our results in examples of industrial organization, auctions, and voting.

## 1. Introduction

We study a general class of strategic environments exhibiting a product structure and prove existence of equilibrium in an abstract setting; adding an assumption of nonatomicity, we further show that every equilibrium can be purified in the sense that there exists an equivalent equilibrium in which players choose extreme points of their feasible actions. When the product structure is imposed on the set of players in a large game, we identify a player with the group she belongs to and a nonatomically distributed personal characteristic, and we assume payoffs depend on own actions and the profile of average actions of the groups. The space of groups is general, we allow for infinite-dimensional externalities across groups, and we obtain existence of Nash equilibria in which players choose from the extreme sets of their actions. Ours is the first such result that does not make use of saturated measure spaces; the cost is that action sets are finite-dimensional. When the product structure is imposed on type spaces in a Bayesian game, we view a player's type as consisting of a general component together with an atomless, conditionally independent, private

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values component. The general component is allowed to be correlated among players, and we obtain a purification result for games with finitely many actions, the first such result that allows for substantially correlated information, without assuming conditional independence given the realization of a finite environmental state.

**Examples:** Consider a market composed of a large number of firms, where each firm is characterized by its location $t$ and technological characteristic $u$. Assume that there are infinitely many locations and infinitely many technologies. Each firm $(t, u)$ produces a vector $q(t, u) \in \Re^d$ of commodities belonging to a production set $Y(t, u)$. Let $\alpha(t) = \int_u q(t, u)du$ denote the aggregate production vector at location $t$, averaging over firm technologies $u$. We assume implicitly that prices are determined by product and factor market clearing in each location, where consumers and workers may (at some cost) travel to transact in markets at different locations. Thus, prices and firm profits depend on the aggregate production function $\alpha$. For simplicity, we write the profit of firm $(t, u)$ from production vector $q(t, u)$ given aggregate production $\alpha$ as $\pi(t, u, q(t, u); \alpha)$. Assume types and locations lie in complete, separable metric spaces; production sets $Y(t, u)$ are nonempty, compact, and lower measurable; profits are jointly measurable and continuous in $(q(t, u), \alpha)$; and that firm types $u$ are nonatomically distributed. Our result for large games (Proposition 1) ensures that a Nash equilibrium exists. The innovation with respect to the literature (see, for instance, Yu and Zhu (2005)) is in allowing for the information of infinitely many locations to affect the price received by a given firm.

Next, consider a multi-unit auction for bonds, drilling rights, etc. Assume that there are $n$ bidders, indexed by $j$. Each bidder $j$ performs a private investigation to determine the value of different portfolios, summarized by a multidimensional signal $t_j$, and submits a menu of bids as a function of holdings, with prices determined by any order statistic over bids. Although privately observed by the bidder, her signal potentially contains information relevant to the other bidders, as bidders may have different signal technologies, and there may be inherent randomness in testing; so the expected value of the objects for bidder $j$ depends on the entire profile $(t_1, \ldots, t_n)$. Moreover, assume that bidder $j$ has an additional private characteristic $u_j$ that affects the value of the objects for bidder $j$ only (due, e.g., to aspects of the bidder's production technology or product market), and that $u_j$ is conditionally independent given $t_j$. Assume types belong to complete, separable metric spaces; bids are in discrete (and bounded) monetary amounts, so there are only finitely many feasible bids, that the distribution of signals $t_j$ satisfies the standard diffuseness condition; and the private value types $u_j$ are nonatomically distributed. Our result for Bayesian games (Proposition 2) ensures existence of Bayes-Nash equilibria, and moreover that each equilibrium in mixed strategies can be purified. While existence of mixed strategy equilibria in this model follows from known results (see Milgrom and Weber (1985)), the innovation with respect to the literature on purification is in allowing for very general correlation among the signals $(t_1, \ldots, t_n)$. The implied existence of pure strategy equilibria extends Athey's (2001) Theorem 1 for finite games and McAdams' (2006) Theorem 1 for multi-unit auctions to settings with a private-value component: the existence of (possibly non-monotonic) pure strategy Bayes-Nash equilibria in these settings does not require single-crossing conditions.

Finally, consider a voting game among $n$ voters who must choose between two alternatives, $A$ and $Q$, using majority rule or other quota rule. A state variable $s$ is selected by nature, and conditional on $s$, each voter $j$ receives a signal $t_j$ drawn (independently conditional on $s$) from a countable signal space. In addition, each voter is characterized by a preference parameter $u_j$ that is independent of the other voters' types. Then the payoff of voter $j$ from outcome $x = A, Q$, given preference parameter $u_j$ and state $s$, is written $U_j(x, s, u_j)$. Assuming that states and preference parameters belong to complete, separable metric spaces and that preference parameters are nonatomically distributed, our result for Bayesian games yields existence and purification of Bayes-Nash equilibria. When the state $s$ is discrete, the purification theorem of Milgrom and Weber (1985) applies, but we allow for a continuously distributed state variable. As such, we generalize the existence result Proposition 1 of Feddersen and Pesendorfer (1997), who assume a one-dimensional state and finite signal space, and we do so without imposing the monotonicity conditions used to obtain equilibria in cutoff strategies. In fact, our result does not use the assumption of two alternatives, and so it extends to any number of alternatives and voting mechanism in which voters choose messages from a finite set.

**Analytical framework:** Our general framework is formulated abstractly, without an immediate interpretation in terms of a game; there are, for example, no players and no payoff functions. It can be viewed, rather, as a fixed point theorem that exploits a special kind of product structure on its domain. This product structure allows us to apply the iterated integral approach used by Duggan (2011b) to prove existence of stationary Markov perfect equilibria in noisy stochastic games. To convey the idea, we define a choice function $\gamma$ as assigning to each pair $(t, u)$ a choice in $\mathfrak{R}^d$. We then calculate the corresponding average choice function, $\alpha$, by taking the marginal, $\alpha(t) = \int_u \gamma(t, u) du$, of $\gamma$ pointwise for each $t$. We then assign a choice set $M(t, u; \alpha)$ to each pair $(t, u)$, where by construction these sets depend only on average choices, and we define a choice equilibrium as a mapping $\gamma$ such that for almost all $(t, u)$, $\gamma(t, u)$ belongs to the choice set $M(t, u; \alpha)$ determined by the corresponding average choices. Our existence result for choice equilibria is non-nested with Theorem 2.2.1 of Balder (2002), who establishes existence of equilibria in pseudogames that are more general than our model in that his action sets may be infinite-dimensional, but less general in that he assumes externalities are finite-dimensional. Beyond existence, assuming $u$ is nonatomically distributed, we provide a purification result: for every choice equilibrium, there is an extremal choice equilibrium $\hat{\gamma}$ that chooses from the (closure of) extreme points of choice sets $M(t,u;\hat{\alpha})$; moreover, $\hat{\gamma}$ is equivalent to $\gamma$ in the sense that it determines the same average choices and, therefore, the same choice sets for all $(t,u)$. In the general framework, we impose further product structure on the general component $t$ and choice sets to obtain Bayesian environments as special cases.

The existence argument takes place in the space of average choice functions. We define $S(\alpha)$ as the set of selections of the correspondence $t \mapsto \int_u M(t,u;\alpha)du$, and we prove existence of a fixed point $\alpha^* \in S(\alpha^*)$ that is generated by an equilibrium choice function $\gamma^*$. The fixed point argument surmounts a number of technical challenges. To ensure sequential upper hemicontinuity of $S$, we apply results of Yannelis (1990, 1991) on properties of selections of correspondences, and as the space of average choice functions is not necessarily (weakly) compact or metrizable, we apply a recent result of Agarwal and O'Regan (2002) to obtain a fixed point, $\alpha^*$. Finally, we employ the theorem of Artstein (1989) to back out an equilibrium choice function $\gamma^*$ consistent with $\alpha^*$. Our purification argument relies on an application of a version of Lyapunov's theorem (see Hildenbrand (1974)) pointwise for each $t$, using nonatomicity of $u$; we then apply Arstein's theorem again to back out an extremal choice function. Of note, the latter step relies on a result (Lemma 10, in the appendix) establishing lower measurability of the extreme points of a lower measurable correspondence with nonempty, compact values in $\mathfrak{R}^d$.

**Related literature:** Our existence results for Nash equilibria in large games is non-nested with respect to the results in Martins-da-Rocha and Topuzu (2008) and Balder (2002), as we allow for infinite-dimensional externalities at the cost of finite-dimensional action sets. With respect to Khan, Rath, and Sun (1997), we provide a modeling approach that allows us to handle infinite-dimensional externalities without relying on an infinite-dimensional version of Lyapunov's theorem. In particular, letting $\sigma$ denote a strategy profile and $\sigma(t,u)$ denote the action of player $(t,u)$, the standard approach would be to condense externalities to the finite-dimensional statistic $\beta = \int_{(t,u)} \sigma(t,u)d(t,u)$, which means that two strategy profiles $\sigma$ and $\hat{\sigma}$ with $\beta = \hat{\beta}$ are considered equivalent by all players. In contrast, in our model, it is the infinite-dimensional statistic $\alpha(\cdot) = \int_u \sigma(\cdot,u)du$ on which players condition their choices; it is obviously possible to have $\alpha \neq \hat{\alpha}$ while $\beta = \hat{\beta}$, so players react to a richer set of "societal statistics" in our formulation. This modeling strategy at the same time circumvents the failure of Lyapunov's theorem in infinite dimensions, without resorting to saturated (or super-nonatomic) measure spaces, as in Podczeck (2008).

Our existence result for pure-strategy Bayesian equilibrium is non-nested with the application of Balder's (2002) results to pure-strategy Bayesian equilibrium. His Theorem 3.2.1 gives conditions for existence of a pure-strategy Bayesian equilibrium; in comparison to our result, he allows for a measure space of players and infinite-dimensional action sets (we assume a finite set of players and finite-dimensional action spaces), but he assumes a countable set of states of the world, convex action sets, and concave payoff functions (we allow for a general type component and assume finite action sets). In comparison to the purification results of Migrom and Weber (1985), Khan, Rath, and Sun (2006), and Balder (2008), we generalize conditional independence of types and drop finiteness of their "environmental variable," $t_0$. Instead, we assume a product structure on player types by decomposing player types into a general component (which are possibly correlated and distributed very generally) and an atomless, private value component (which is independently drawn conditional on the profile of general types). Viewing one of the players as Nature, with trivial action space, we can easily incorporate an environmental variable $t_0$ that lives in a general complete, separable metric space. Compared to the recent purification results based on saturated measure spaces (Loeb and Sun (2006), Podczeck (2009), and Wang and Zhang (2010)), we obtain similar improvements: only the private component is assumed conditionally independent, and the environmental variable need not be finite or countable. There is no need to resort to saturated measure spaces, as we apply the classical Lyapunov result; but the cost, as before, is the finite-dimensionality of the action space.

**Organization:** In Section 2, we present the abstract framework, and Section 3 contains the statement and proofs of our main existence and purification results. Section 4 provides an application of our general results to large games, and Section 5 takes up the case of Bayesian games. The appendix contains Lemma 10, on the lower measurability of the extreme points of a correspondence.

## 2. Abstract Framework

Let $(N, \mathcal{N}, \mu)$ be a measure space, where $N$ is a complete, separable metric space, $\mathcal{N}$ the Borel sigma-algebra, and $\mu = (\mu_j)_{j=1}^n$ a $\mathfrak{R}^n$-valued, Borel vector probability measure on $N$. Assume:

(A1) $N = T_1 \times \cdots \times T_n \times U$, where $T_j$, $j = 1, \ldots, n$, and $U$ are complete, separable metric spaces.

Let $\mathcal{T}_j$, $j = 1, \ldots, n$, and $\mathcal{U}$ be the respective Borel sigma-algebras. Let $T = \times_{j=1}^{n} T_j$ and $\mathcal{T} = \otimes_{j=1}^{n} \mathcal{T}_j$, with generic element $t = (t_1, \ldots, t_n) \in T$, and note that $\mathcal{N} = \mathcal{T} \otimes \mathcal{U}$ (see Theorem 4.44 of Aliprantis and Border (2006), henceforth AB). Let $\kappa_j$ denote the marginal of $\mu_j$ on $T_j$.

Assume: for each $j = 1, \ldots, n$,

(A2) there is a Borel transition probability $\nu_j(\cdot|\cdot) : T_j \times \mathcal{U} \to [0,1]$ such that for all $Q = R \times S \in \mathcal{T} \otimes \mathcal{U}$,

$$\mu_j(Q) \quad = \quad \int_{T_j} \nu_j(S|t_j) \kappa_j(dt_j),$$

so that $\nu_j(\cdot|t_j)$ is a Borel probability measure on $U$ for $\kappa_j$-almost all $t_j \in T_j$ and $t_j \mapsto \mu_j(E|t_j)$ is a $\mathcal{T}_j$-measurable function for all $E \in \mathcal{U}$. In particular, the mapping $t_j \mapsto \nu_j(\cdot|t_j)$ is Borel measurable with the weak* topology on the space of Borel probability measures on $U$ (see Theorem 19.7 of AB). In terms of standard notation, $\mu_j$ is the extension of $\nu_j(\cdot|\cdot) \otimes \kappa_j$ to $N$. An implication is that the distribution of $u$ according to $\mu_j$ is independent of $t_{-j}$. For each $t \in T$, we therefore write $\mu(\cdot|t) = (\nu_j(\cdot|t_j))_{j=1}^{n}$ as the vector of conditional probabilities.

For each $j = 1, \ldots, n$ and $(t_j, u) \in T_j \times U$, let $A_j(t_j, u) \subseteq \mathfrak{R}^d$ denote a set of *feasible alternatives.* A *choice function for* $j$ is a $\mathcal{T}_j \times \mathcal{U}$-measurable mapping $\gamma_j : T_j \times U \to \mathfrak{R}^d$ such that for $\nu_j(\cdot|\cdot) \otimes \kappa_j$-almost all $(t_j, u)$, we have $\gamma_j(t_j, u) \in A_j(t_j, u)$. For each $j = 1, \ldots, n$, let $\overline{U}_j \in \mathcal{U}$ contain the atoms of $\{\nu_j(\cdot|t_j) : t_j \in T_j\}$, and assume:

(A3) for all $(t_j, u) \in T_j \times U$, $A_j(t_j, u)$ is nonempty and compact; and for each $(t_j, u) \in T_j \times \overline{U}_j$, the set $A_j(t_j, u)$ is convex,

(A4) the correspondence $A_j : T_j \times U \rightrightarrows \mathfrak{R}^d$ is lower measurable, i.e., for all open $G \subseteq \mathfrak{R}^d$, the set $\{(t_j, u) \in T_j \times U : A_j(t_j, u) \cap G \neq \varnothing\}$ is $\mathcal{T}_j \otimes \mathcal{U}$-measurable.

We make use of the following:

**Lemma 1:** *For each* $j = 1, \ldots, n$, *the mapping* $(t, u) \to \sup \|A_j(t, u)\|$ *is* $\mathcal{N}$-*measurable.*

**Proof:** Note three observations: with continuity of the Euclidean norm $\|\cdot\|$, (A6) implies that the correspondence $(t, u) \mapsto \|A_j(t, u)\|$ is lower measurable with nonempty, closed values in $\mathfrak{R}$; as a consequence, there is a sequence $\{f_n\}$ of $\mathcal{N}$-measurable functions $f_n : N \to \mathfrak{R}$ such that for all $(t, u)$, $\|A_j(t, u)\| = \mathrm{cl}\{f_n(t, u)\}$ (see Corollary 18.14 in AB); and the pointwise limit of a sequence of measurable functions into a complete, separable metric space is itself measurable (see Lemma 4.29 of AB). Therefore, $(t,u) \mapsto \sup \|A_j(t,u)\| = \sup\{f_n(t,u)\}$ is $\mathcal{N}$-measurable. This completes the proof of the lemma.

Let $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ be fixed for the remainder of the paper. Assume: for each $j = 1, \ldots, n$,

(A5) for $\kappa_j$-almost all $t_j$, the mapping $u \mapsto \sup \|A_j(t_j,u)\|$ is $p$-integrably bounded, i.e.,

$$\int_u \sup \|A_j(t_j,u)\|^p \nu_j(du|t_j) \quad < \quad \infty.$$

For later use, we record the following strengthening of assumption (A5):

(A5′) the mapping $(t_j,u) \mapsto \sup \|A_j(t_j,u)\|$ is $p$-integrably bounded, i.e.,

$$\int_{(t_j,u)} \sup \|A_j(t_j,u)\|^p (\nu_j(\cdot|\cdot) \otimes \kappa_j)(d(t_j,u)) \quad < \quad \infty.$$

When $p = 1$, assumption (A5′) (and therefore (A5)) is automatically satisfied if the feasible sets are bounded by a fixed, compact subset of $\mathfrak{R}^d$, but we allow in principle for arbitrarily large action sets $A(i)$. In particular, (A5) does not preclude the possibility that action sets grow large "quickly" as we vary $t_j$.

A *choice function* is an ordered $n$-tuple $\gamma = (\gamma_j)_{j=1}^n$ of choice functions for $j = 1, \ldots, n$. A choice function $\gamma_j$ for $j$ determines an *average choice function for* $j$, denoted $\alpha_j : T_j \to \mathfrak{R}^d$, as follows: for each $t_j \in T_j$, we define

$$\alpha_j(t_j) \quad = \quad \int_u \gamma_j(t_j,u)\nu_j(du|t_j)(du),$$

which is Borel measurable by (A2). More precisely, given $\gamma_j$, define the $\mathcal{T}_j$-measurable function $\alpha_j(\cdot|\gamma_j) : T_j \to \mathfrak{R}^d$ by $\alpha_j(t_j|\gamma_j) = \int_u \gamma_j(t_j,u)\nu_j(du|t_j)$. Then the set of average choice functions for $j$ consists of any mapping that is equivalent to some $\alpha_j(\cdot|\gamma_j)$ up to a set of $\kappa_j$-measure zero:

$$\mathfrak{A}_j = \left\{ \alpha_j : T_j \to \mathfrak{R}^d : \begin{array}{c} \alpha_j(t_j) = \alpha_j(t_j|\gamma_j) \text{ for } \kappa_j\text{-almost all } t_j \in T_j \\ \text{and for some choice function } \gamma_j \text{ for } j \end{array} \right\}.$$

We will sometimes suppress dependence of $\alpha_j(\cdot|\gamma_j)$ on $\gamma_j$ without confusion in the sequel. An *average choice function* is an ordered $n$-tuple $\alpha = (\alpha_j)_{j=1}^n$ of average choice functions for $j = 1, \ldots, n$. Note that because (A5) is stated pointwise for each $t_j$, it does not imply compactness of the space of average choice functions for $j$ in the weak topology.

Given any $\alpha \in \mathfrak{A}$ and $j = 1, \ldots, n$, let $M_j(\cdot;\alpha) : T_j \times U \rightrightarrows \mathfrak{R}^d$ be a *choice correspondence.* Assume: for each $\alpha \in \mathfrak{A}$ and $j = 1, \ldots, n$, (A6) for all $(t_j, u) \in T_j \times U$, $M_j(t_j, u; \alpha) \subseteq A_j(t_j, u)$;

(A7) for all $(t_j, u) \in T_j \times U$, the set $M_j(t_j, u; \alpha)$ is nonempty and compact; and for all $(t_j, u) \in T_j \times \overline{U}_j$, the set $M_j(t_j, u; \alpha)$ is convex;

(A8) the correspondence $(t_j, u) \mapsto M_j(t_j, u; \alpha)$ is lower measurable, i.e., for all open $G \subseteq \mathfrak{R}^d$, the set $\{(t_j, u) \in T_j \times U : M_j(t_j, u; \alpha) \cap G \neq \varnothing\}$ is $\mathcal{T}_j \otimes \mathcal{U}$-measurable;

(A9) the correspondence $(t_j, u) \mapsto M_j(t_j, u; \alpha)$ is uniformly bounded by a $p$-integrable correspondence, i.e., there exists a lower measurable correspondence $\Upsilon_j : T_j \times U \rightrightarrows \mathfrak{R}^d$ with compact and convex values such that for all $\alpha \in \mathfrak{A}$ and all $(t_j, u) \in T_j \times U$, we have $M_j(t_j, u; \alpha) \subseteq \Upsilon_j(t_j, u)$, and

$$\int_{(t_j,u)} \sup \|\Upsilon_j(t_j, u)\|^p (\nu_j(\cdot|\cdot) \otimes \kappa_j)(d(t_j, u)) \quad < \quad \infty.$$

Note, in particular, that we impose convexity only on the atoms of $U$; if the probability measures $\{\nu_j(\cdot|t_j) : t_j \in T_j\}$ are nonatomic, then (A7) demands only that choice sets be nonempty and compact. Also note that if we strengthen (A5) to (A5′), assumption (A9) is implied by our other assumptions by taking $\Upsilon_j(t_j, u) = A_j(t_j, u)$ for all $(t_j, u) \in T_j \times U$.

We have not as yet shown the existence of average choice functions that are $p$-integrable; the existence of such functions does not follow from (A5), because that assumption does not restrict feasible action sets across $t_j$, but it does follow from (A6)–(A9). This is established in the next lemma. Henceforth, let $\mathfrak{A}_j^p = \{\alpha_j \in \mathfrak{A}_j : \|\alpha_j\|_p < \infty\}$ denote the subset of $p$-integrable average choice functions for $j$.

**Lemma 2:** *For each $j = 1, \ldots, n$, $\mathfrak{A}_j^p$ is nonempty.*

**Proof:** Since $A_j$ is lower measurable with closed values, it admits a measurable selection $\gamma_j$ (see Theorem 18.13 in AB), and then $\alpha_j = \alpha_j(\cdot|\gamma_j)$ is an average choice function for $j$, and $\alpha = (\alpha_j)_{j=1}^n \in \mathfrak{A}$. Then (A7) and (A8) imply that $M_j(\cdot; \alpha)$ admits a measurable selection $\tilde{\gamma}_j$; (A6) implies that $\tilde{\alpha}_j = \alpha_j(\cdot|\tilde{\gamma}_j)$ is an average choice function for $j$; and (A9) implies $\|\tilde{\alpha}_j\|_p < \infty$. Therefore, $\tilde{\alpha}_j \in \mathfrak{A}_j^p$. This completes the proof of the lemma.

We endow $\mathfrak{A}_j^p$ with the topology inherited from the weak topology $\sigma(L_j^p, L_j^q)$ on $L_j^p \equiv L^p(T_j, \mathcal{T}_j, \kappa_j)$, where $L^p(T_j, \mathcal{T}_j, \kappa_j)$ is the set of $\kappa_j$-equivalence classes of $\mathcal{T}_j$-measurable mappings $\alpha_j : T_j \to \mathfrak{R}^d$ such that $\|\alpha_j\|_p \equiv \int_{t_j} \|\alpha_j(t_j)\|^p \kappa_j(dt_j) < \infty$. Convergence in $L_j^p$ is defined with reference to the dual space, $L_j^q \equiv L^q(T_j, \mathcal{T}_j, \kappa_j)$, so that given any net $\{\alpha_j^\nu\}$, we have $\alpha_j^\nu \to \alpha_j$ if and only if for all $\varphi_j \in L_j^q$, $\int_{t_j} \varphi_j(t_j) \cdot [\alpha_j^\nu(t_j) - \alpha_j(t_j)]\kappa_j(dt_j) \to 0$. Let $\mathfrak{A}^p = \times_{j=1}^n \mathfrak{A}_j^p$, and endow this space with the product topology.

Finally, we impose the natural assumption that choice sets are (sequentially) upper hemicontinuous on the space of $p$-integrable average choice functions.[1] Assume: for each $j = 1, \ldots, n$,

(A10) for all $(t_j, u) \in T_j \times U$, the correspondence $\alpha \mapsto M_j(t_j, u; \alpha)$ is sequentially upper hemicontinuous on $\mathfrak{A}^p$.

Thus, the degree $p$ of integrability controls the tradeoff between our boundedness assumptions on $A_j(t_j, \cdot)$ and $M_j(\cdot; \alpha)$ (in (A5) and (A9)) and our continuity assumption on $M_j(t_j, u; \cdot)$ (in (A10)); of course, higher $p$ strengthens boundedness and weakens continuity.

## 3. Main Result

A choice function $\gamma^*$ is a *choice equilibrium* if $\gamma_j^*(t_j, u) \in M_j(t_j, u; \alpha^*)$ for $\nu_j(\cdot|\cdot) \otimes \kappa_j$-almost all $(t_j, u)$ and $\alpha_j^*(t_j) = \int_u \gamma_j^*(t_j, u)\nu_j(du|t_j)$ for $\kappa_j$-almost all $t_j$, and all $j = 1, \ldots, n$. An *extremal choice equilibrium* is a choice equilibrium $\gamma^*$ such that for each $j = 1, \ldots, n$ and $\nu_j(\cdot|\cdot) \otimes \kappa_j$-almost all $(t_j, u)$, we have $\gamma_j^*(t_j, u) \in \text{ext} M_j(t_j, u; \alpha^*)$, where $\text{ext} M_j(t_j, u; \alpha^*)$ is the set of extreme points of $M_j(t_j, u; \alpha^*)$. We denote by $\overline{\text{ext}} M_j(t_j, u; \alpha^*)$ the closure of the set of extreme points of the choice correspondence. Our main theorem asserts existence of a choice equilibrium and a partial purification result: given any choice equilibrium, there is a choice equilibrium such that choices are made from the closure of extreme points of choice sets for almost all $(t_j, u)$ with $u$ in the nonatomic part of $\overline{U}_j$ and that is equivalent the choice equilibrium, in the sense that the equilibria determine the same average actions (and therefore same choice sets) up to a set of measure zero.

**Theorem:** *Assume (A1)–(A10). (a) A choice equilibrium exists; (b) for every choice equilibrium $\gamma^*$, there exists a choice equilibrium $\hat{\gamma}$ such that (i) $\gamma^*$ and $\hat{\gamma}$ determine equivalent average actions, i.e., for each $j = 1, \ldots, n$ and $\kappa_j$-almost all $t_j$, $\alpha_j^*(t_j) = \hat{\alpha}_j(t_j)$; and (ii) $\hat{\gamma}$ chooses from the closure of extreme points of choice sets for the nonatomic part of $U$, i.e., for each $j = 1, \ldots, n$ and $\nu_j(\cdot|\cdot) \otimes \kappa_j$-almost all $(t_j, u) \in T_j \times (U \smallsetminus \overline{U}_j)$, we have $\hat{\gamma}_j(t_j, u) \in \overline{ext} M_j(t_j, u; \hat{\alpha})$.*

Obviously, if the probabilities $\{\nu_j(\cdot|t_j) : t_j \in T_j\}$ are nonatomic and the sets of extreme points are almost always closed, then extremal choice equilibria exist,

[1] Given Banach space $X$ and set $Y \subseteq X$, a correspondence $\psi : Y \rightrightarrows Y$ is *sequentially upper hemicontinuous* if for all weakly closed sets $F \subseteq X$, the lower inverse $\psi^\ell(F) = \{x \in Y : \psi(x) \cap F \neq \varnothing\}$ is sequentially closed in the weak topology relative to $Y$, i.e., every sequence in $\psi^\ell(F)$ that converges in the relative topology on $Y$ has limit in $\psi^\ell(F)$.

and we can strengthen part (b) of the theorem to obtain a full purification result. Closedness of the set of extreme points does not hold generally (see Figure 7.4 of AB), but it does hold widely.

**Corollary:** *Assume that (A1)–(A10) hold; that for each $j = 1,\ldots,n$, we have $\overline{U}_j = \varnothing$; and for each $\alpha \in \mathfrak{A}$ and for $\nu_j(\cdot|\cdot)\otimes\kappa_j$-almost all $(t_j,u)$, ext$M_j(t_j,u;\alpha)$ is closed. (a) An extremal choice equilibrium exists; (b) for every choice equilibrium, there exists an extremal choice equilibrium $\hat{\gamma}$ that determines equivalent average actions, i.e., for each $j = 1,\ldots,n$ and $\kappa_j$-almost all $t_j$, $\alpha_j^*(t_j) = \hat{\alpha}_j(t_j)$.*

The remainder of this section consists of the proof of the theorem, and we assume throughout that conditions (A1)–(A10) hold. To begin, we define two useful correspondences. For $j = 1,\ldots,n$, let $A_j^*: T_j \rightrightarrows \mathfrak{R}^d$ be defined by

$$A_j^*(t_j) \quad = \quad \int_u A_j(t_j,u)\nu_j(du|t_j),$$

and for each $\alpha \in \mathfrak{A}$, define $M_j^*(\cdot;\alpha): T_j \rightrightarrows \mathfrak{R}^d$ by

$$M_j^*(t_j;\alpha) \quad = \quad \int_u M_j(t_j,u)\nu_j(du|t_j).$$

These are, respectively, the Aumann integrals of the feasible action correspondence $A_j(t_j,\cdot)$, and of the choice correspondence $M_j(t_j,\cdot;\alpha)$, with respect to $u$. Note that (A6) implies that $M_j^*(t_j;\alpha) \subseteq A_j^*(t_j)$. Also, because we are only interested in almost everywhere properties, it is without loss to assume that $(T_j,\mathcal{T}_j,\kappa_j)$, $j = 1,\ldots,n$, is a complete measure space. More precisely, (A5) and (A10) ensure that we will work with integrably bounded measurable functions, so the integrals does not change when we consider the completion of $(T_j,\mathcal{T}_j,\kappa_j)$, $j = 1,\ldots,n$.

The next lemma characterizes the average choice functions for $j$ in terms of the correspondence $A_j^*$, and it characterizes the almost everywhere selections from $M_j(\cdot;\beta)$ (for any given average choice function $\beta$) in terms of $M_j^*(\cdot;\beta)$.

**Lemma 3:** *For $j = 1,\ldots,n$ and each $\mathcal{T}_j$-measurable $\alpha_j: T_j \to \mathfrak{R}^d$, (a) $\alpha_j$ is a $\kappa_j$-almost everywhere selection from $A_j^*$ if and only if $\alpha_j \in \mathfrak{A}_j$; (b) for each $\beta \in \mathfrak{A}$, $\alpha_j$ is a $\kappa_j$-almost everywhere selection from $M_j^*(\cdot;\beta)$ if and only if there exists a choice function $\gamma_j$ for $j$ with $\gamma_j(t_j,u) \in M_j(t_j,u;\beta)$ for $\nu_j(\cdot|\cdot)\otimes\kappa_j$-almost all $(t_j,u)$ such that for $\kappa_j$-almost all $t_j$, $\alpha_j(t_j) = \int_u \gamma_j(t_j,u)\nu_j(du|t_j)$.*

**Proof:** To prove (a), note that the "if" direction is immediate from the definition of average choice function for $j$. Indeed, letting $\alpha_j \in \mathfrak{A}_j$ be determined as $\alpha_j = \alpha_j(\cdot|\gamma_j)$ for the choice function $\gamma_j$ for $j$, it follows that for all $t_j \in T_j$, $\gamma_j(t_j,\cdot): U \to \mathfrak{R}^d$ is a selection from $A_j(t_j,\cdot): U \rightrightarrows \mathfrak{R}^d$, and therefore $t_j \mapsto \int_u \gamma_j(t_j,u)\nu_j(du|t_j)$

is a $\mathcal{T}_j$-measurable selection from $A_j^*$. Since $\alpha_j(t_j) = \int_u \gamma_j(t_j,u)\nu_j(du|t_j)$ for $\kappa_j$-almost all $t_j$, this direction is proved. For the "only if" direction, let $\alpha_j$ be a $\kappa_j$-almost everywhere selection from $A_j^*$. Then the theorem of Artstein (1989) yields a $\mathcal{T}_j \otimes \mathcal{U}$-measurable mapping $\gamma_j: T_j \times U \to \mathfrak{R}^d$ such that for $\kappa_j$-almost all $t_j$, we have: $\alpha_j(t_j) = \int_u \gamma_j(t_j,u)\nu_j(du|t_j)$, and for $\nu_j(\cdot|t_j)$-almost all $u$, $\gamma_j(t_j,u) \in A_j(t_j,u)$. In particular, his assumptions (i)–(vi) are fulfilled, respectively, by (A1) (twice), the assumption that $\nu_j(\cdot|\cdot): \mathcal{U} \times T_j \to [0,1]$ is a transition probability, (A3), (A4), and (A5). Thus, $\alpha_j$ is determined as $\alpha_j = \alpha_j(\cdot|\gamma_j)$ for the choice function $\gamma_j$ for $j$. The proof of (b) is parallel, using $M_j(\cdot;\beta)$ and $M_j^*(\cdot;\beta)$ instead of $A_j$ and $A_j^*$. This completes the proof of the lemma.

**Lemma 4:** *For $j = 1,\ldots,n$, (a) the correspondence $A_j^*$ has nonempty, compact, and convex values; (b) for each $\alpha \in \mathfrak{A}$, the correspondence $M_j^*(\cdot;\alpha)$ has nonempty, compact, and convex values.*

**Proof:** Nonemptiness in (a) follows from (A3) and (A4), which imply that $A_j$ is lower measurable with nonempty, closed values, and so it admits a measurable selection (see Theorem 18.13 of AB); nonemptiness in (b) follows similarly from (A7) and (A8). For compactness in (a) and (b), note that (A5) implies that for $\kappa_j$-almost all $t_j$, the correspondence $u \mapsto A_j(t_j,u)$ is $p$-integrably bounded with respect to $\nu_j(\cdot|t_j)$. By a version of Fatou's lemma (see Proposition 7 (p.73) of Hildenbrand (1974)), the integral $A_j^*(t_j) = \int_u A_j(t_j,u)\nu_j(du|t_j)$ of this correspondence is compact. Similarly, the integral $M_j^*(t_j) = \int_u M_j(t_j,u;\alpha)\nu_j(du|t_j)$ is compact. Now note that for each $t_j \in T_j$,

$$A_j^*(t_j) \quad = \quad \int_{u \notin \bar{U}_j} A_j(t_j,u)\nu_j(du|t_j) + \int_{u \in \bar{U}_j} A_j(t_j,u)\nu_j(du|t_j),$$

so convexity in (a) follows because the first term on the right-hand side is convex by a version of Lyapunov's theorem (see Theorem 3 (p.62) of Hildenbrand (1974)), and the second term is convex by (A3). The argument for convexity in (b) is parallel, using (A7) instead of (A3). This completes the proof of the lemma.

The next lemma elaborates on convexity if $M_j^*(t_j;\alpha)$. Note that because the latter set is nested between the integral of extreme points of $M_j(t_j,u;\alpha)$ and the integral of the convex hull, the lemma implies equality of all three sets. A further implication, since $M_j(t_j,u;\alpha)$ is convex for all $u \in \overline{U}_j$ by (A7), is that for each $j = 1,\ldots,n$, each $\alpha \in \mathfrak{A}$, and each $t_j \in T_j$, we have $M_j^*(t_j;\alpha) = \int_u \mathrm{co} M_j(t_j,u;\alpha)\nu_j(du|t_j)$.

**Lemma 5:** *For each $j = 1,\ldots,n$, each $\alpha \in \mathfrak{A}$, and each $t_j \in T_j$, we have*

$$\int_{U \setminus \overline{U}_j} ext M_j(t_j,u;\alpha)\nu_j(du|t_j) \quad = \quad \int_{U \setminus \overline{U}_j} co M_j(t_j,u;\alpha)\nu_j(du|t_j).$$

**Proof:** Fix $\alpha \in \mathfrak{A}$ and $t_j \in T_j$. It is clear that the integral of extreme points of $M_j(t_j, u; \alpha)$ is contained in the integral of the convex hull. For the opposite inclusion, note that $u \mapsto M_j(t_j, u; \alpha)$ is $p$-integrably bounded by (A5) and (A6). Then

$$\begin{aligned}\int_{U \setminus \overline{U}_j} \mathrm{ext} M_j(t_j, u; \alpha)\nu_j(du|t_j) &= \int_{U \setminus \overline{U}_j} \mathrm{coext} M_j(t_j, u; \alpha)\nu_j(du|t_j) \\ &= \int_{U \setminus \overline{U}_j} \mathrm{co} M_j(t_j, u; \alpha)\nu_j(du|t_j),\end{aligned}$$

where the first equality follows from nonatomicity of $\nu_j(\cdot|t_j)$ on $U \setminus \overline{U}_j$ and a version of Lyapunov's theorem (see Theorem 4 (p.64) of Hildenbrand (1974)),[2] and the second follows from (A7) and the observation that the convex hull of a compact set $C$ is equal to the convex hull of the extreme points of co$C$ (co$C$ = coextco$C$ by the Krein-Milman theorem, Theorem 7.68 of AB), and the fact that $C$ and its convex hull possess the same extreme points (extco$C$ = ext$C$). This completes the proof of the lemma.

**Lemma 6:** *For each $j = 1, \ldots, n$, the set $\mathfrak{A}^p_j$ is convex and norm-closed.*

**Proof:** Convexity follows from Lemmas 3 and 4, which establish that $\mathfrak{A}$ consists of all measurable selections from the convex-valued correspondence $A^*_j$. To prove norm-closedness, assume the sequence $\{\alpha^m_j\}$ in $\mathfrak{A}^p$ converges to $\alpha_j$ in $L^p_j$. Then $\alpha^m_j \to \alpha_j$ in measure (AB, Theorem 13.39), and therefore there is a subsequence (still indexed by $m$) and a $\kappa_j$-measure zero set $R_j \in \mathcal{T}_j$ such that for all $t_j \notin R_j$, $\alpha^m_j(t_j) \to \alpha_j(t_j)$ (AB, Theorem 13.38). Given any $t_j \notin R_j$, since $\alpha^m_j(t_j) \in A^*_j(t_j)$ for all $m$, and since $A^*_j(t_j)$ is compact by Lemma 4, it follows that $\alpha_j(t_j) \in A^*_j(t_j)$. Then Lemma 3 yields $\alpha_j \in \mathfrak{A}^p_j$. This completes the proof of the lemma.

**Lemma 7:** *For each $t_j \in T_j$, the correspondence $\alpha \mapsto M^*_j(t_j; \alpha)$ is sequentially upper hemicontinuous on $\mathfrak{A}^p$.*

**Proof:** Fix $t_j \in T_j$. Note that for each $\alpha \in \mathfrak{A}$, we have $M^*_j(t_j; \alpha) \subseteq A^*_j(t_j)$, the latter compact by Lemma 4. Thus, the correspondence $\alpha \mapsto M^*_j(t_j; \alpha)$ has closed values (by Lemma 4) and compact range, and it suffices to prove sequentially closed graph on $\mathfrak{A}^p$. Furthermore, by Lemma 5, it suffices to show $\alpha \mapsto \int_u \mathrm{co} M_j(t_j, u; \alpha)$ has sequentially closed graph. By Lemma 4, it has nonempty, convex, and closed values. Furthermore, (A10) implies that for each $u \in U$, the correspondence $\alpha \mapsto \mathrm{co} M_j(t_j, u; \alpha)$ is sequentially upper hemicontinuous on $\mathfrak{A}^p$ (see Theorem 17.35 of AB). Now, let $\{\alpha^m\}$ be a sequence in converging to $\alpha$ in $\mathfrak{A}^p$, and let $y^m \in M^*_j(t_j, \alpha^m)$

[2]Note that Hildenbrand assumes the correspondence is bounded below; see, however, Aumann's (1965) discussion following his Theorem 3 to the effect that integrable boundedness is sufficient.

for each $m$ and $y^m \to y$. We apply Theorem 6.5 of Yannelis (1991) to conclude that $y \in M_j^*(t_j,\alpha)$. In particular, to fulfill the assumptions of that theorem, we identify our $(U,\mathcal{U},\nu_j(\cdot|t_j))$ with Yannelis' finite measure space $(T,\tau,\mu)$; $\Re^d$ with his $X$; our correspondence $u \mapsto \Upsilon_j(t_j,u)$ is his correspondence $t \mapsto K(t)$; and our correspondence $M_j(t_j,\cdot;\alpha_m)$ is his $\phi_n$. Then $\limsup M_j^*(t_j,\alpha_m)$ is contained in the closure of $\int_u \limsup M_j(t_j,u;\alpha_m)\nu_j(du|t_j)$. By (A10), we have $\limsup M_j(t_j,u;\alpha_m) \subseteq M_j(t_j,u;\alpha)$ for all $u \in U$. Since $M_j^*(t_j;\alpha)$ is closed, by Lemma 4, we conclude that $y \in \limsup M_j^*(t_j,\alpha_m) \subseteq \mathrm{cl}\int_u \limsup M_j(t_j,u;\alpha_m)\nu_j(du|t_j) \subseteq M_j^*(t_j;\alpha)$. This completes the proof of the lemma.

Define $S_j\colon \mathfrak{A}^p \rightrightarrows L_j^p$ so that $S_j(\alpha)$ consists of all $\mathcal{T}_j$-measurable, $\kappa_j$-almost everywhere selections from $M_j^*(\cdot;\alpha)$. The product correspondence $S = \times_{j=1}^n S_j$ will be the subject of our fixed point argument.

**Lemma 8:** *For each $j = 1,\ldots,n$, the range of $S_j$, $S_j(\mathfrak{A}^p)$, is a relatively compact subset of $\mathfrak{A}_j^p$.*

**Proof:** Let $S_j^*$ consist of all $\mathcal{T}_j \otimes \mathcal{U}$-measurable, $\nu(\cdot|\cdot) \otimes \kappa_j$-almost everywhere selections from $\Upsilon_j$. By Theorem 3.1 and Remark 3.1 of Yannelis (1991) with $\Re^d$ for his $X$, $S_j^*$ is compact in $L_j^p(T_j \times U, \mathcal{T}_j \times \mathcal{U}, \nu_j(\cdot|\cdot) \otimes \kappa_j)$ endowed with the weak topology induced by $L_j^q(T_j \times U, \mathcal{T}_j \times \mathcal{U}, \nu_j(\cdot|\cdot) \otimes \kappa_j)$. Now define the mapping $\phi\colon S_j^* \to L_j^p$ by $\phi(\beta)(t_j) = \int_u \beta(t_j,u)\nu_j(du|t_j)$ for all $\beta \in S^*$. Indeed, the range is well-specified because

$$\int_{t_j} \|\phi(\beta)(t_j)\|^p \kappa_j(dt_j) \le \int_{(t_j,u)} \|\beta(t_j,u)\|^p (\nu_j(\cdot|\cdot)\otimes\kappa_j)(d(t,u)) < \infty,$$

where the first inequality follows from Jensen's inequality and the second from (A9). We claim that $\phi$ is continuous. Consider a net $\{\beta^\nu\}$ in $S_j^*$ such that $\beta^\nu \to \beta \in S_j^*$ in the weak topology, and consider any $\varphi \in L_j^q$. Then

$$\begin{aligned}
&\int_{t_j} [\varphi(t_j)\cdot(\phi(\beta^\nu)(t_j) - \phi(\beta)(t_j))]\kappa_j(dt_j) \\
&\quad = \int_{(t_j,u)} [\varphi(t_j)\cdot(\beta^\nu(t_j,u) - \beta(t_j,u))](\nu_j(\cdot|\cdot)\otimes\kappa_j)(d(t_j,u)) \\
&\quad \to 0,
\end{aligned}$$

as claimed. Therefore, $\phi(S_j^*)$ is a compact subset of $L_j^p$, and because $S_j(\alpha) \subseteq \phi(S_j^*)$ for all $\alpha \in \mathfrak{A}^p$, it follows that the range $S_j(\mathfrak{A}^p)$ is relatively compact. That is is a subset of $\mathfrak{A}^p$ follows from Lemma 3 and the observation that $M_j^*(t_j;\alpha) \subseteq A_j^*(t_j)$ for all $t_j$. This completes the proof of the lemma.

**Lemma 9:** *For each $j = 1,\ldots,n$, $S_j$ has nonempty, closed, convex values and is sequentially upper hemicontinuous.*

**Proof:** For nonemptiness, consider any $\alpha \in \mathfrak{A}^p$, and note that (A7) and (A10) imply that $M_j(\cdot;\alpha)$ admits a measurable selection $\gamma_j$ (see Theorem 18.13 of AB). Defining the average choice function $\beta_j$ for $j$ by $\beta_j(t_j) = \int_u \gamma_j(t_j,u)\nu_j(du|t_j)$, Lemma 3 implies $\beta_j \in S_j(\alpha)$. To prove that $S_j(\alpha)$ is weakly closed in $L^p_j$, note that $M^*_j(\cdot;\alpha)$ has nonempty, compact, and convex values by Lemma 4, and it is $p$-integrably bounded by (A9). Then the result follows from Theorem 3.1 and Remark 3.1 of Yannelis (1991). Convexity follows from Lemma 4. To show sequential upper hemicontinuity, note that $S_j$ has compact range by Lemma 8, so we must show sequentially closed graph. Let $\{\alpha^m\}$ be a sequence converging to $\alpha$ in $\mathfrak{A}^p$, and let $\beta^m \in S_j(\alpha^m)$ for each $m$ and $\beta^m \to \beta$. We apply Theorem 5.5 of Yannelis (1991) to conclude that $\beta \in S_j(\alpha)$. In particular, to fulfill the assumptions of that theorem, we equip the set $P = \{\alpha\} \cup \{\alpha^m\}$ with the metric $\rho$ defined as follows: for each $m$, let $\rho(\alpha,\alpha^m) = \frac{1}{m}$, and for $\ell \neq m$, let $\rho(\alpha^\ell,\alpha^m) = |\frac{1}{\ell} - \frac{1}{m}|$; then the sequence $\{\alpha^m\}$ trivially converges to $\alpha$ in the metric topology on $P$; our $(T_j,\mathcal{T}_j,\kappa_j)$ is Yannelis' complete, finite separable measure space $(T,\tau,\mu)$; $\mathfrak{R}^d$ is his $X$; our correspondence $t_j \mapsto \int_u \Upsilon_j(t_j,u)\nu_j(du|t_j)$ is his correspondence $t \mapsto K(t)$; and our correspondence $(t_j,\alpha) \mapsto M^*_j(t_j;\alpha)$ is his $(t,p) \mapsto \psi(t,p)$. Note that the latter correspondence has nonempty, closed, convex values by Lemma 4, and it is sequentially upper hemicontinuous in $\alpha$ from Lemma 7. (Inspection of the proof of Yannelis' Theorem 5.5 reveals that sequential upper hemicontinuity is all that is required.) Because $\alpha$ and the sequence $\{\alpha^m\}$ are arbitrary, sequential upper hemicontinuity follows. This completes the proof of the lemma.

We can now complete the proof of the theorem. For (a), endow $L^p \equiv \times_{j=1}^n L^p_j$ with the product norm, making it a Banach space. We observe that the correspondence $S{:}\,\mathfrak{A}^p \rightrightarrows L^p$ satisfies the conditions of Theorem 2.3 of Agarwal and O'Regan (2002). In particular, Lemma 6 implies that $\mathfrak{A}^p$ is a nonempty, convex, norm-closed subset of $L^p$; Lemma 9 implies that $S(\alpha)$ is nonempty, weakly closed, and convex for each $\alpha \in \mathfrak{A}^p$, and that the correspondence is sequentially upper hemicontinuous; furthermore, Lemma 8 implies that the range of $S$ is a relatively compact subset of $\mathfrak{A}^p$ with the weak topology. Then there exists $\alpha^* \in \mathfrak{A}^p$ satisfying $\alpha^* \in S(\alpha^*)$. Since $\alpha^*_j$ is a selection from $M^*_j(\cdot;\alpha^*)$ for each $j = 1,\ldots,n$, Lemma 3 yields choice functions $\gamma^*_j$ for each $j$ such that $\alpha^*$ is determined by $\gamma^* = (\gamma^*_j)_{j=1}^n$, and therefore $\gamma^*$ is a choice equilibrium.

For part (b), let $\gamma^*$ be a choice equilibrium with corresponding average choice function $\alpha^*$. For each $j = 1,\ldots,n$, define the correspondence $\hat{M}_j{:}\,T_j \times U \rightrightarrows \mathfrak{R}^d$ by

$$\hat{M}_j(t_j,u) \quad = \quad \begin{cases} \overline{\text{ext}} M_j(t_j,u;\alpha^*) & \text{if } u \notin \overline{U}_j \\ M_j(t_j,u;\alpha^*) & \text{else.} \end{cases}$$

By (A7) and Lemma 10 (in the appendix), it follows that $\overline{\text{ext}} M_j(\cdot;\alpha^*)$ is lower measurable with nonempty, compact values. As the measurable splicing of two such correspondences, $\hat{M}_j$ is lower measurable with nonempty, compact values. Since (A7) implies $\text{ext} M_j(t_j,u;\alpha^*) \subseteq \overline{\text{ext}} M_j(t_j,u;\alpha^*) \subseteq \text{co} M_j(t_j,u;\alpha^*)$, Lemma 5 implies that

$$\begin{aligned} &\int_u \hat{M}_j(t_j,u)\nu_j(du|t_j) \\ &= \int_{U\setminus\overline{U}_j} \overline{\text{ext}} M_j(t_j,u;\alpha^*)\nu_j(du|t_j) + \int_{\overline{U}_j} M_j(t_j,u;\alpha^*)\nu_j(du|t_j) \\ &= M_j^*(t_j;\alpha^*), \end{aligned}$$

and therefore for $\kappa_j$-almost all $t_j$, we have $\alpha_j^*(t_j) \in \int_u \hat{M}_j(t_j,u)\nu_j(du|t_j)$. The correspondence $\hat{M}_j$ satisfies the conditions of Artstein's (1989) theorem, and therefore there exists a $\mathcal{T}_j \otimes \mathcal{U}$-measurable mapping $\hat{\gamma}_j : T_j \times U \to \mathfrak{R}^d$ such that for $\kappa_j$-almost all $t_j$, we have: $\alpha_j^*(t_j) = \int_u \hat{\gamma}_j(t_j,u)\nu_j(du|t_j) = \alpha_j(\cdot|\hat{\gamma}_j)$, and for $\nu_j(\cdot|t_j)$-almost all $u$, $\hat{\gamma}_j(t_j,u) \in \hat{M}_j(t_j,u)$. Then $\hat{\gamma} = (\hat{\gamma}_j)_{j=1}^n$ is an extremal choice equilibrium as called for by the theorem.

## 4. Large Games

The goal of this section is to formulate a class of large games as a special case of the abstract framework. We endow the set $N$ of players with a product structure, so that a player is described by a general component $t$ and a private characteristic $u$, where the latter are distributed independently conditional on $t$ in the space of players. The abstract framework from Section 2 specialized to $n = 1$ is readily seen as a large game framework. Given a measure space $(T,\mathcal{T},\kappa)$, let $\mathfrak{M}(T,\mathcal{T},\kappa,\mathfrak{R}^d)$ denote the set of $\kappa$-equivalence classes of measurable functions mapping $T$ to $\mathfrak{R}^d$. A *product large game* is described by a tuple $(T,U,A,P,\kappa,\nu)$ such that

- $N = T \times U$, with sigma-algebra $\mathcal{T} \otimes \mathcal{U}$ is the player space,
- $A: N \rightrightarrows \mathfrak{R}^d$ is the feasible action correspondence,
- $P: N \times \mathfrak{R}^d \times \mathfrak{M}(T,\mathcal{T},\kappa,\mathfrak{R}^d) \rightrightarrows \mathfrak{R}^d$ is the preference correspondence,
- $\kappa$ is a Borel probability measure on $T$,
- $\nu: T \times \mathcal{U} \to [0,1]$ is a Borel transition probability.

Here, $\kappa$ is the distribution of general components $t$ in the player space; the transition probability $\nu(\cdot|t)$ gives the distribution of private characteristic $u$ conditional on $t$; and $\mu = \nu(\cdot|\cdot) \otimes \kappa$ gives the overall distribution of players in the game. Denote a generic player by $i = (t,u) \in N$.

Given the above, the player space is succinctly described by $(N, \mathcal{N}, \mu)$ with $N = T \times U$ and $\mathcal{N} = \mathcal{T} \otimes \mathcal{U}$, as in the abstract framework. Assumption (A2), with $n = 1$, is already embedded in the definition of a product large game. Letting $\overline{U}$ contain the atoms of $\{\nu(\cdot|t) : t \in T\}$, we make the following basic assumption to connect the large game model to Section 2:

(L1) assumptions (A1), (A3), (A4) and (A5′) from Section 2 hold with $n = 1$,

A *strategy profile* is an $\mathcal{N}$-measurable mapping $\sigma : N \to \Re^d$ such that $\sigma(i) \in A(i)$ for $\mu$-almost all players $i$. Given a strategy profile $\sigma$, the implied average action is a function $\alpha : T \to \Re^d$ satisfying $\alpha(t) = \int_u \sigma(t,u)\nu(du|t)$ for $\kappa$-almost all $t$ (where we identify functions equivalent up to sets of measure zero). Let $\mathfrak{A}$ denote the space of average actions. A possible interpretation is that players are characterized by their private characteristics, $u$, and by the groups to which they belong, $t$, and the externality (or "societal responses") are captured by the average actions $\alpha$ across groups; thus, the influence of "society" on a player's outcome, given a strategy $\sigma$, is captured by the infinite dimensional object $\alpha$. Returning to the example of the intro, it may be that players are firms, that $t$ is the market in which a firm competes, and $u$ is a technological characteristic of the firm.

We interpret the set $P(i, a; \alpha)$ as the set of actions that are strictly preferred to $a$ by player $i$ given externality $\alpha$.[3] Let $R : N \times \Re^d \times \mathfrak{M}(T, \mathcal{T}, \kappa, \Re^d) \rightrightarrows \Re^d$ denote the weak preference correspondence corresponding to $P$, i.e., $R(i, a; \alpha) = \{a' \in \Re^d : a \notin P(i, a'; \alpha)\}$, and let $R^{-1}$ denote its inverse. Fix player $i$ and externality $\alpha$. We say $P(i, \cdot; \alpha)$ is *irreflexive* if for all $a \in \Re^d$, we have $a \notin P(i, a; \alpha)$. Following Duggan (2011a), we say that a set $Y \subset A(i)$ is *finitely dominant* if it is finite and for all $x \in A(i)$ there exists $y \in Y$ with $y \in P(i, x; \alpha)$. Given $v \in A(i)$, we say that $P(i, \cdot; \alpha)$ is *finitely subordinated* to $v$ if there is a finitely dominant set $Y$ with $v \in Y$ and such that there exists $z \in Y$ with $v \in P(i, z; \alpha)$ and $Y \setminus \{v\} \subseteq R^{-1}(t, u, z; \alpha)$. We say that $P(i, \cdot; \alpha)$ satisfies the *finite-subordination property* if there is no $v \in A(i)$ such that $P(i, \cdot; \alpha)$ is finitely subordinated to $v$. We make the following further assumptions:

(L2) for all $i \in N$ and all $\alpha \in \mathfrak{A}$, $P(i, \cdot; \alpha)$ is irreflexive and satisfies the finite-subordination property,

(L3) for all $i \in T \times \overline{U}$, all $a \in A(i)$, and all $\alpha \times \mathfrak{A}$, $R(i, a; \alpha) \cap A(i)$ is convex,

[3]We remark that the formulation of the preference correspondence $P$ is not subject to the critique in Balder (2000) (see Martins-da-Rocha and Topuzu (2008)). We note also that Martins-da-Rocha and Topuzu (2008) provide general sufficient conditions on $P$ that yield well-behaved choice correspondences $M$. We simply offer a set of such sufficient conditions, without attempting to generalize other approaches in the literature.

(L4) for all $\alpha \in \mathfrak{A}$, the correspondence $i \mapsto \{a \in A(i): P(i,a;\alpha) = \varnothing\}$ is lower measurable,

(L5) for all $i \in N$, the set $\{(a,\alpha) \in A(i) \times \mathfrak{A} : P(i,a;\alpha) \neq \varnothing\}$ is open in the product topology, where $\mathfrak{A}$ is endowed with the weak topology (consistent with $p$ from (L1)).

For all $i \in N$ and all $\alpha \in \mathfrak{A}$, let $M(i;\alpha) = \{a \in A(t,u) : P(t,u,a;\alpha) \cap A(i) = \varnothing\}$ denote the maximal feasible actions for player $i$ given externality $\alpha$. A strategy profile $\sigma^*$ is a *Nash equilibrium* if $\sigma^*(i) \in M(i;\alpha^*)$ for $\mu$-almost all $i$ and $\alpha^*(t) = \int_u \sigma^*(t,u)\nu(du|t)$ for $\kappa$-almost all $t$. This is readily seen as the specialization of a choice equilibrium from Section 2 for the case $n = 1$. Assume that (L1)–(L5) hold. Then we have:

**Proposition 1:** *Assume (L1)–(L5). (a) A Nash equilibrium exists; (b) for every Nash equilibrium $\sigma^*$, there exists a Nash equilibrium $\hat{\sigma}$ such that (i) $\sigma^*$ and $\hat{\sigma}$ determine equivalent externalities, i.e., for $\kappa$-almost all $t$, $\alpha^*(t) = \hat{\alpha}(t)$; and (ii) $\hat{\sigma}$ chooses from the closure of extreme points of choice sets for the nonatomic part of $U$, i.e., for $\mu$-almost all $i \in T \times (U \setminus \overline{U})$, we have $\hat{\sigma}(i) \in \overline{ext} M(i;\hat{\alpha})$.*

The result follows from the correspondence between product large games and the abstract framework, with $n = 1$, upon verifying (A1)–(A10). We have noted (A2), and remaining assumptions (A1)–(A5′) follow directly from (L1), with (A5′) giving us (A9) as well. The definition of $M(i;\alpha)$ immediately yields (A6). Nonemptiness of $M(i;\alpha)$ follows from compactness of $A(i)$ (from (L1)), (L2), (L5), and Theorem 1 of Duggan (2011a); and compactness of $M(i;\alpha)$ follows immediately from the open graph assumption (L5). Given $i \in T \times \overline{U}$, irreflexivity (from (L2)) and convexity (from (L3)) of $P(i,\cdot;\alpha)$ yield convexity of $M(i;\alpha)$. Indeed, suppose suppose $M(i;\alpha)$ is not convex, so there exist distinct $x, y \in M(i;\alpha)$ and $\lambda \in (0,1)$ such that $z = \lambda x + (1-\lambda)y \notin M(i;\alpha)$. Then there exists $w \in A(i)$ such that $w \in P(i,z;\alpha)$. Since $x \in M(i;\alpha)$, we have $x \in R(i,w;\alpha)$, and similarly $y \in R(i,w;\alpha)$. But then convexity of $R(i,w;\alpha)$ implies $z \in R(i,w;\alpha)$, a contradiction. We conclude that (A7) holds. Then (L4) implies (A8), and (L5) implies (A10) by standard continuity arguments.

Proposition 1 generalizes Theorem 2 and Remark 2 in Schmeidler (1973) from the model with a finite number of groups to the general model with a continuum of groups. More importantly, the assumption that best responses depend on the distribution across groups of average actions within groups, rather than the overall average action, puts Proposition 1 in an intermediate position compared to other results in the literature. The externality $\alpha$ is an infinite-dimensional object, as opposed to the finite-dimensional externalities found in the literature, either the overall average action as in Rath (1992), or the finite-dimensional image of a function of the overall average action, as in Balder (2002), Martins-da-Rocha and Topuzu (2008), and Yu and Zhu (2005).[4] So we allow for players to respond to a much richer set of "societal" variables, weakening considerably the implied notion of anonymity. Because we restrict the analysis to finite-dimensional action sets, whereas the literature allows for arbitrary compact action sets, our result is intermediate.

Proposition 1 occupies a non-existent position in Table 1 of Khan, Rath, and Sun (1997): the rightmost column of that table indicates that in games with uncountable action spaces and infinite-dimensional externalities, there is no pure-strategy Nash equilibrium. Here, we do have an uncountable action space and infinite-dimensional average actions $\alpha$, the product structure of the player-type space, together with the result of Artstein (1989), allow us to work around the failure of Lyapunov's theorem in infinite dimensions, without having to move into Loeb or super nonatomic measure spaces (see, among others, Podczeck (2008)).

For completeness, we consider the case that preferences are represented by a numerical payoff function $\pi\colon N \times \mathfrak{R}^d \times \mathfrak{M}(T, \mathcal{T}, \kappa, \mathfrak{R}^d) \to \mathfrak{R}$. As before, we simply offer a (standard) set of sufficient conditions, without attempting to generalize other approaches in the literature (see Balder (2002) and Martins-da-Rocha and Topuzu (2008)). Maintaining (L1), assume that for all $i \in N$, $\pi(i, \cdot, \cdot)$ is jointly continuous on $A(i) \times \mathfrak{M}(T, \mathfrak{R}^d)$; and that for all $i \in T \times \overline{U}$, $A(i)$ is convex and, moreover, for all $\alpha \in \mathfrak{M}(T, \mathcal{T}, \kappa, \mathfrak{R}^d)$, $\pi(i, \cdot, \alpha)$ is quasiconcave on $A(i)$. Also assume that for each $(a, \alpha) \in \mathfrak{R}^d \times \mathfrak{M}(T, \mathfrak{R}^d)$, the mapping $i \mapsto \pi(i, a, \alpha)$ is $\mathcal{N}$-measurable. We imbed this in the product large game framework in the obvious way, defining $P(i, a; \alpha) = \{a' \in A(i) : \pi(i, a'; \alpha) > \pi(i, a; \alpha)\}$, so that best responses are the payoff-maximizing feasible actions:

$$M(i; \alpha) = \{a \in A(i) : P(i, a; \alpha) = \varnothing\} = \arg\max_{a' \in A(i)} \pi(i, a'; \alpha).$$

Then conditions (L2)–(L5) obtain. Indeed, irreflexivity of $P(i, \cdot; \alpha)$ follows by construction, and the finite subordination property is implied by the fact that $P(i, \cdot; \alpha)$ is a weak order of the set of feasible actions; thus, (L2) follows. Quasiconcavity of $\pi$ on $A(i)$ for all $i \in T \times \overline{U}$ implies (L3). From the assumption that the payoff function $\pi(\cdot, \cdot; \alpha)$ is Carathéodory for each $\alpha$, and from the assumption that $i \mapsto A(i)$ is lower measurable, a measurable version of the maximum theorem (AB Theorem 18.19) implies that the correspondence $M(\cdot; \alpha)\colon N \rightrightarrows \mathfrak{R}^d$ is measurable. Thus, it is

[4]More precisely, the former two papers allow for infinite-dimensional externalities on the purely atomic part of the player space, while the externality on the non atomic part must be finite-dimensional. With nonatomicity, externalities do not have any infinite-dimensional component.

lower measurable (AB, Lemma 18.2), and (L4) follows. Finally, (L5) follows from continuity of $\pi(i,\cdot;\cdot)$ for each $i \in N$.

## 5. Bayesian Games

We now present a general class of Bayesian games that can be obtained as a special case of the abstract framework of Section 2. The type of a player $j$ has two components, $t_j$ and $u_j$, the first a general component that may be payoff relevant for all players, the second a private value component that, conditional on $t_j$, is independent of the other players' types. We allow the action sets to be type dependent. Formally, the class of *product Bayesian games* is described by a tuple $(T_j, U_j, A_j, \pi_j, \kappa, \nu_j)_{j=1}^n$ indexed by the set $\{1,\dots,n\}$ of players and such that for each player $j = 1,\dots,n$,

- $T_j \times U_j$, with sigma-algebra $\mathcal{T}_j \otimes \mathcal{U}_j$, is $j$'s type space,
- $A_j: T_j \times U_j \rightrightarrows \Re^d$ is $j$'s feasible action correspondence,
- $\pi_j: \Re^{nd} \times T \times U_j \to \Re$ is $j$'s payoff function,
- $\kappa$ is a Borel probability measure on $T = \times_{j=1}^n T_j$,
- $\nu_j : T_j \times \mathcal{U}_j \to [0,1]$ is a Borel transition probability,

where $T = \times_{j=1}^n T_j$ is the set of profiles of general types, denoted $t = (t_1,\dots,t_n)$, and $\kappa$ is $j$'s prior probability measure on $T = \times_{j=1}^n T_j$. As usual, $\mathcal{T} = \otimes_{j=1}^n \mathcal{T}_j$ and $\mathcal{U} = \otimes_{j=1}^n \mathcal{U}_j$ are the product sigma-algebras. We define the transition probability $\nu: T \otimes \mathcal{U} \to [0,1]$ as follows: for each $t \in T$ and each $S = S_1 \times \cdots \times S_n \in \mathcal{U}$, $\nu(S|t) = \otimes_{j=1}^n \nu_j(S_j|t_j)$. Let $\kappa_j$ denote the marginal of $\kappa$ on $T_j$.

We let $\mu = \nu(\cdot|\cdot) \otimes \kappa$ represent the common prior of the players and $\mu_j$ the marginal of $\mu$ on $T_j \times U_j$. Note that, conditional on $t_j$, the random variable $u_j$ is independent of $(t_{-j}, u_{-j})$. Assume: for each $j = 1,\dots,n$,

(B1) $T_j$ and $U_j$, $j = 1,\dots,n$, are complete, separable metric spaces with their Borel sigma-algebras,
(B2) for all $(t_j,u_j) \in T_j \times U_j$, the feasible set $A_j(t_j,u_j)$ is a face of the unit simplex in $\Re^d$,
(B3) the correspondence $A_j: T_j \times U_j \rightrightarrows \Re^d$ is lower measurable,
(B4) $\pi_j(a,t,u_j)$ is Borel measurable in $(a,t,u_j)$ and multilinear in $a = (a_j)_{j=1}^n$,
(B5) the mapping $t \mapsto \sup_{(a,u_j) \in \Delta^n \times U_j} |\pi_j(a,t,u_j)|$ is integrable, i.e.,

$$\int_t \sup_{(a,u_j) \in \Delta^n \times U_j} |\pi_j(a,t,u_j)| \kappa(dt) < \infty,$$

where $\Delta$ is the unit simplex in $\mathfrak{R}^d$. An interpretation, suggested by (B2) and (B4), is that an action $a_j$ is a probability distribution over a finite set of pure strategies corresponding to the vertices of the unit simplex in $\mathfrak{R}^d$. Here, we allow for $A_j(t_j, u_j)$ to be a lower-dimensional face of the unit simplex (we assume only that the number of feasible actions is bounded above by $d$ across players and types), in which case pure strategies corresponding to vertices outside $A_j(t_j, u_j)$ are necessarily given probability zero.

Finally, we impose the assumption of absolutely continuous information, introduced by Milgrom and Weber (1985), on the general type component.

(B6) $\kappa$ is absolutely continuous with respect to $\bigotimes_{j=1}^n \kappa_j$.

For each $j = 1, \ldots, n$, a *strategy* for player $j$ is a $\mathcal{T}_j \otimes \mathcal{U}_j$-measurable function $\sigma_j : T_j \times U_j \to \mathfrak{R}^d$ such that $\sigma_j(t_j, u_j) \in A_j(t_j, u_j)$ for $\kappa_j$-almost all $(t_j, u_j)$. We view a *strategy profile* $(\sigma_1, \ldots, \sigma_n)$ as a mapping $\sigma : T \times U \to \mathfrak{R}^{nd}$ defined by $\sigma(t, u) = (\sigma_j(t_j, u_j))_{j=1}^n$. Let $\Sigma_j$ denote the set of strategies for $j$ and $\Sigma = \bigtimes_{j=1}^n \Sigma_j$ the set of strategy profiles. Player $j$'s ex ante expected payoff from a profile $\sigma$ of strategies is

$$\Pi_j(\sigma) = \int_{(t,u)} \pi_j(\sigma(t,u), t, u_j)\mu(d(t,u)).$$

A *Bayes-Nash equilibrium* is a strategy profile $\sigma^*$ such that for each $j = 1, \ldots, n$, $\Pi_j(\sigma^*) = \sup_{\sigma_j \in \Sigma_j} \Pi_j(\sigma_j, \sigma^*_{-j})$. We say $\sigma^*$ is a *pure strategy Bayes-Nash equilibrium* if it is a Bayes-Nash equilibrium such that for each $j = 1, \ldots, n$ and for $\mu_j$-almost all $(t_j, u_j)$, we have $\sigma^*_j(t_j, u_j) \in \operatorname{ext} A_j(t_j, u_j)$.

Our main contribution in this section is an existence and purification result for pure strategy equilibria. To define our notion of equivalence between strategy profiles, note that for each $j = 1, \ldots, n$, the expected action of player $j$ determined by strategy $\sigma_j$, conditional on general component $t_j$, is

$$\alpha_j(t_j|\sigma_j) \equiv \int_{u_j} \sigma_j(t_j, u_j)\nu_j(du_j|t_j),$$

where $\alpha_j(\cdot|\sigma_j) \in L^1_j$ by (B2). Let $\mathfrak{A}_j = \{\alpha_j(\cdot|\sigma_j) : \sigma_j \in \Sigma_j\}$ denote the space of expected actions for player $j$, and endow it with the weak topology $\sigma(L^1_j, L^\infty_j)$. By Fubini's theorem and multilinearity, from (B4), we have

$$\begin{aligned}
\Pi_j(\sigma) &= \int_t \int_u \pi_j(\sigma(t,u), t, u_j) \bigotimes_{k=1}^n \nu_j(du|t)\kappa(dt) \\
&= \int_t \int_{u_j} \pi_j(\sigma_j(t_j, u_j), \left( \int_{u_k} \sigma_k(t_k, u_k)\nu_k(du_k|t_k) \right)_{k \neq j}, t, u_j)\nu_j(du_j|t_j)\kappa(dt) \\
&= \int_t \int_{u_j} \pi_j(\sigma_j(t_j, u_j), \alpha_{-j}(t_{-j}|\sigma_{-j}), t, u_j)\nu_j(du_j|t_j)\kappa(dt),
\end{aligned}$$

where $\alpha_{-j}(t_{-j}|\sigma_{-j}) = (\alpha_k(t_k|\sigma_k))_{k\neq j}$. Because the expected payoff depends on $\sigma_{-j}$ only through expected actions, we can redefine the ex ante payoff function for player $j$ as the mapping $\Pi_j\colon \Sigma_j \times \mathfrak{A}_{-j} \to \mathfrak{R}$ given by

$$\Pi_j(\sigma_j, \alpha_{-j}) \quad = \quad \int_t \int_{u_j} \pi_j(\sigma_j(t_j, u_j), \alpha_{-j}(t_{-j}), t, u_j)\nu_j(du_j|t_j)\kappa(dt),$$

where $\mathfrak{A}_{-j} = \times_{k\neq j} \mathfrak{A}_k$ is endowed with the product topology. Accordingly, we replace the optimization of $\Pi_j(\sigma_j, \sigma^*_{-j})$ with $\Pi_j(\sigma_j, \alpha_{-j}(\cdot|\sigma^*_{-j}))$ in the definition of Bayes-Nash equilibrium. We then say two strategy profiles $\sigma, \sigma'$ are *equivalent* if they determine the same expected actions, i.e., for each $j = 1, \ldots, n$ and for $\kappa_j$-almost all $t_j$, we have $\alpha_j(t_j|\sigma) = \alpha_j(t_j|\sigma')$.

**Proposition 2:** *Assume (B1)–(B6). (a) A Bayes-Nash equilibrium exists; (b) if the probability measures $\{\nu_j(\cdot|t_j) : t_j \in T_j\}$ are nonatomic for each $j = 1, \ldots, n$, then for every Bayes-Nash equilibrium $\sigma^*$, there exists an equivalent pure strategy Bayes-Nash equilibrium.*

Before proceeding to the proof, several remarks are in order.

**Remark 1:** *We can add a complete, separable metric space $T_0$ of environmental states at no cost.* For this, we simply add an artificial player 0, whose general type component $t_0$ corresponds to the environmental state, with trivial action set $A(t_0, u_0) \equiv \{0\}$ for all $(t_0, u_0)$.

**Remark 2:** *The purification result in Proposition 2(b) is, to the best of our knowledge, the first that allows for a general space of environmental states.* From Remark 1, we generalize the purification result of Milgrom and Weber (1985) by allowing for an uncountable set of states. Typical results in the literature (e.g., Milgrom and Weber (1985), Khan, Rath, and Sun (2006), and Balder (2008)) assume the existence of an environmental variable $t_0$, taking at most countably many values and such that player types are independent conditional on $t_0$. Such a result is easily obtained as a corollary of Proposition 3(b) by trivializing $T_j$, using the private type component $u_j$ to represent the information of player $j$, and letting $T_0$ be countable.

**Remark 3:** *We also extend the previous literature on purification by allowing a non-private values type component that is correlated across players.* There is no need to assume the general type spaces $T_j$ of the players $j \neq 0$ are singletons, as in the last remark; rather, we allow $T_j$ to be a complete, separable metric space, and we permit arbitrary correlation subject only to the diffuseness condition (B6).

**Remark 4:** *Proposition 2 provides an existence result for mixed strategy Bayes-Nash equilibria in the framework of Milgrom and Weber (1985).* As mentioned, we can add a Polish space $T_0$ of states to our framework at no cost. In contrast to Milgrom and Weber (1985), we assume types have a conditionally independent, private-value component $u_j$, in addition to the standard component; action sets are finite-dimensional, rather than complete, separable metric spaces; and we do not assume equicontinuous payoffs. As a special case, however, the private components $u_j$ may be payoff-irrelevant and uniformly and independently distributed over $[0,1]$, so they serve only as randomization devices; then any pure strategy equilibrium of our model (in which player $j$ conditions her action on $u_j$) corresponds to a mixed strategy equilibrium in the game with no private components. This improves Milgrom and Weber (1985) by dropping equicontinuity but assuming finite-dimensional actions; this result, however, specializes Section 3.4 of Balder (2002).[5]

The remainder of this section is devoted to the proof of Proposition 2, which consists of mapping the product Bayesian game model into the abstract framework, verifying (A1)–A(10), and applying Corollary 1. We first define $\tilde{U} = \dot{\bigcup}_{j=1}^{n} U_j$ as the disjoint union of $U_j$, $j = 1,\ldots,n$, and for notational simplicity, we henceforth treat the $U_j$ as disjoint subsets of $\tilde{U}$. We metrize $\tilde{U}$ so that elements of $U_j$ and $U_k$, $j \neq k$, are at distance one, and we let $\tilde{\mathcal{U}}$ be the corresponding Borel sigma-algebra, which is just finite unions of Borel sets in $\mathcal{U}_j$, $j = 1,\ldots,n$. Defining $N = T \times \tilde{U}$, (A1) is satisfied. To fulfill (A2), we convert the transition probability $\nu_j\colon T_j \times \mathcal{U}_j \to [0,1]$ to $\tilde{\nu}_j\colon T_j \times \tilde{\mathcal{U}} \to [0,1]$ in the obvious way: given any $S \in \tilde{\mathcal{U}}$, $\tilde{\nu}_j(S|t_j) = \nu_j(S \cap U_j|t_j)$. Thus, when integrating with respect to $\tilde{\nu}_j(\cdot|t_j)$, realizations $u_k$, $k \neq j$, of the private value component for other players receive no weight. We then define $\tilde{\mu} = (\tilde{\mu}_j)_{j=1}^{n}$ so that $\tilde{\mu}_j$ is the extension of $\tilde{\nu}_j(\cdot|\cdot) \otimes \kappa_j$ to $N$.

We convert feasible action correspondences $A_j\colon T_j \times U_j \rightrightarrows \mathfrak{R}^d$ to $\tilde{A}_j\colon T_j \times \tilde{U} \rightrightarrows \mathfrak{R}^d$ by specifying $\tilde{A}_j(t_j,u) = A_j(t_j,u)$ if $u \in U_j$, otherwise set $\tilde{A}_j(t_j,u) = \{e^1\}$, where $e^1$ is the first unit coordinate vector, if $u \notin U_j$. Then (B2) and (B3) immediately imply (A3)–(A5). A choice function is then $\gamma = (\gamma_j)_{j=1}^{n}$, where $\gamma_j\colon T_j \times U \to \mathfrak{R}^d$ satisfies $\gamma_j(t_j,u) \in \tilde{A}_j(t_j,u)$ for $\tilde{\nu}_j(\cdot|\cdot) \otimes \kappa_j$-almost all $(t_j,u)$. We define expected actions as above, now integrating over $u \in \tilde{U}$ with respect to $\tilde{\nu}_j(\cdot|t_j)$, e.g., $\tilde{\alpha}_j(t_j|\gamma_j) \equiv \int_u \gamma_j(t_j,u)\tilde{\nu}_j(du|t_j)$, and the average choice functions for $j$ determined by $\gamma_j$, denoted $\tilde{\mathfrak{A}}_j$, are the mappings that equal $\tilde{\alpha}_j(\cdot|\gamma_j)$ up to a $\kappa_j$-measure zero set. We imbed $\tilde{\mathfrak{A}}_j$ in $L_j^1$ with the weak topology, and of course, $\tilde{\mathfrak{A}} = \times_{j=1}^{n} \tilde{\mathfrak{A}}_j$ is endowed with the product weak topology.

[5] We allow for action sets to be type-dependent, whereas Balder's (2002) Theorem 3.4.1 fixes action sets independently of type; but that extra generality can be achieved using his methods. Note that his Theorem 3.2.1 generalizes a result of Kim and Yannelis (1997), and his Theorem 3.4.1 generalizes results of Balder (1988) and Balder and Rustichini (1994).

Given $\tilde{\alpha} \in \tilde{\mathfrak{A}}$, we define the choice correspondence $M_j(\cdot;\tilde{\alpha}): T_j \times U \rightrightarrows \mathfrak{R}^d$ as the actions maximizing the player's interim expected payoff conditional on $(t_j, u)$. Convert the payoff function $\pi_j: \mathfrak{R}^{nd} \times T \times U_j \to \mathfrak{R}$ to $\tilde{\pi}_j: \mathfrak{R}^{nd} \times T \times \tilde{U} \to \mathfrak{R}$ by specifying $\tilde{\pi}_j(a,t,u) = \pi_j(a,t,u)$ if $u \in U_j$, and otherwise set $\tilde{\pi}_j(a,t,u) = 0$ if $u \notin U_j$. Since $T$ is a complete, separable metric space, Theorem 5.3.7 of Rao (1993) allows us to choose a regular conditional probability $\kappa: T_j \times \mathcal{T}_{-j} \to [0,1]$, i.e., a transition probability such that for all $t_j$ outside a $\kappa_j$-measure zero set $T'_j$, $\kappa(\cdot|t_j)$ is a conditional probability on $T_{-j}$. Note that (B6) implies that for $t_j$ outside a $\kappa_j$-measure zero set $T''_j$, $\kappa(\cdot|t_j)$ is absolutely continuous with respect to $\bigotimes_{k\neq j} \kappa_k$. Note further that (B5) implies

$$\begin{aligned}\int_t \sup_{(a,u_j)\in\Delta^n\times U_j} |\tilde{\pi}_j(a,t,u_j)|\kappa(dt) &= \int_{t_j}\int_{t_{-j}} \sup_{(a,u)\in\Delta^n\times\tilde{U}} |\tilde{\pi}_j(a,t,u)|\kappa(t_{-j}|t_j)\kappa_j(dt_j) \\ &< \infty,\end{aligned}$$

where we use the generalization of Fubini's theorem in Proposition 2.3.2 of Rao (1993). Thus, $\int_{t_{-j}} \sup_{a\in\Delta^n} |\tilde{\pi}_j(a,t,u)|\kappa(t_{-j}|t_j) < \infty$ for all $t_j$ outside a $\kappa_j$-measure zero set $T'_j \cup T'''_j$ and all $u \in \tilde{U}$. For future use, define $\tilde{T}_j = T'_j \cup T''_j \cup T'''_j$.

Our arguments will apply to $t_j \in T_j \setminus \tilde{T}_j$, fixing choice sets arbitrarily for $t_j \in \tilde{T}_j$. Thus, define interim expected payoffs as $\tilde{\Pi}_j(\cdot;\tilde{\alpha}): \mathfrak{R}^d \times T_j \times \tilde{U} \to \mathfrak{R}$ by

$$\tilde{\Pi}_j(a_j,t_j,u;\tilde{\alpha}) = \int_{t_{-j}} \tilde{\pi}_j(a_j,\tilde{\alpha}_{-j}(t_{-j}),t,u)\kappa(t_{-j}|t_j),$$

and define the corresponding choice correspondence $M_j(\cdot;\tilde{\alpha}): T_j \times \tilde{U} \rightrightarrows \mathfrak{R}^d$ as

$$M_j(t_j,u;\tilde{\alpha}) = \arg\max\{\tilde{\Pi}_j(a_j,t_j,u;\tilde{\alpha}) : a_j \in \tilde{A}_j(t_j,u)\}$$

when $t_j \in T_j \setminus \tilde{T}_j$, and otherwise define $M_j(t_j,u;\tilde{\alpha})$ to consist of the first unit coordinate vector when $t_j \in \tilde{T}_j$. Then (A6) and (A9) are trivially satisfied, and (B4) immediately implies (A7).

Before establishing (A8), we claim that $\tilde{\Pi}_j(\cdot;\tilde{\alpha})$ is Borel measurable. Indeed, define $\tilde{\kappa}(\cdot|t_j,u)$ by extending $\kappa(\cdot|t_j)$ to $\mathcal{T}\otimes\tilde{\mathcal{U}}$ with unit mass on $(t_j,u)$, i.e., given $Q = R\times S = (\times_{j=1}^n R_j)\times S \in \mathcal{T}\otimes\tilde{\mathcal{U}}$, we specify $\tilde{\kappa}(Q|t_j,u) = \kappa(R_{-j}|t_j)$ if $(t_j,u) \in R_j \times S$, and otherwise $\tilde{\kappa}(R\times S|t_j,u) = 0$ if $(t_j,u) \notin R_j \times S$. Then $\tilde{\kappa}(\cdot|\cdot): (T_j \times U) \times (\mathcal{T}\otimes\tilde{\mathcal{U}}) \to [0,1]$ is a transition probability. By Theorem 19.12 of AB, the mapping $(t_j,u) \mapsto \tilde{\kappa}(\cdot|t_j,u)$ is Borel measurable with the weak* topology on the space of Borel probability measures on $T \times U$, and Theorem 19.7 of AB implies that for every bounded, Borel measurable $f: T\times U \to \mathfrak{R}$, the mapping $(t_j,u) \mapsto \int_{(t,u)} f(t,u)\tilde{\kappa}(d(t,u)|t_j,u) = \int_{t_{-j}} f(t,u)\kappa(dt_{-j}|t_j)$ is $\mathcal{T}_j\otimes\tilde{\mathcal{U}}$-measurable. By (B5), $\tilde{\pi}_j(a_j,\tilde{\alpha}_{-j}(t_{-j}),t,u)$ is $\tilde{\nu}_j(\cdot|\cdot)\otimes\kappa$-integrable, and therefore it is approximated pointwise by the sequence $\{f^m\}$ of truncations defined by $f^m(t,u) = \min\{m, \max\{-m, \tilde{\pi}_j(a_j,\tilde{\alpha}_{-j}(t_{-j}),t,u)\}\}$. For each $m$, the mapping $(t_j,u) \mapsto \int_{(t,u)} f^m(t,u)\kappa(dt_{-j}|t_j)$ is $\mathcal{T}_j\otimes\tilde{\mathcal{U}}$-measurable, and therefore, so is the pointwise limit (see Theorem 4.27 of AB), which is $\tilde{\Pi}_j(a_j,\cdot;\tilde{\alpha})$ by Lebesgue's dominated convergence theorem. This argument holds when $a_j$ is any unit coordinate vector, $e^\ell$, $\ell = 1,\ldots,d$, and in general (B4) yields

$$\tilde{\Pi}_j(a_j,t_j,u;\tilde{\alpha}) \quad = \quad \sum_{\ell=1}^{d} p_\ell \tilde{\Pi}_j(e^\ell,t_j,u;\tilde{\alpha}),$$

where $a_j = (p_\ell)_{\ell=1}^d$. As the composition of a linear function with a finite number of measurable functions, $\tilde{\Pi}_j(a_j,t_j,u;\tilde{\alpha})$ is Borel measurable, as claimed. Clearly, the interim expected payoff of player $j$ is in fact continuous in $a_j$, so $\tilde{\Pi}_j(\cdot;\tilde{\alpha})$ is a Carathéodory function. Finally, we can now apply a measurable version of the maximum theorem (see Theorem 18.19 of AB) to conclude that $M_j(\cdot;\tilde{\alpha})$ is lower measurable, fulfilling (A8).

The last assumption to verify is (A10). Fix $(t_j,u)\in T_j\times\tilde{U}$. Obviously, $M_j(t_j,u;\cdot)$ is sequentially upper hemicontinuous when $t_j\in\tilde{T}_j$. So suppose $t_j\in T_j\setminus\tilde{T}_j$. Letting $Z=\{1,\ldots,d\}$ and $z_j\in Z$, we claim that for each unit coordinate vector $e^{z_j}$, $\tilde{\Pi}_j(e^{z_j},t_j,u;\tilde{\alpha})$ is continuous in $\tilde{\alpha}$. Indeed, consider any net $\{\tilde{\alpha}^\nu\}$ converging weakly to $\tilde{\alpha}$ in $\tilde{\mathfrak{A}}$. We apply results of Balder (1988) for transition probabilities by viewing the unit simplex in $\mathfrak{R}^d$, $\Delta$, as the set of probability distributions over $Z$ and viewing expected actions $\tilde{\beta}_k\colon T_k\times 2^Z\to[0,1]$, $k=1,\ldots,n$, as transition probabilities. Let $R_k\in\mathcal{T}_k$ and (continuous) $g\colon Z\to\mathfrak{R}$ be given. Using the equivalent vector representations $g=(g_\ell)_{\ell=1}^d\in\mathfrak{R}^d$, $\tilde{\alpha}_k^\nu(t_k)=(\tilde{\alpha}_k^\nu(t_k)_\ell)_{\ell=1}^d$, and $\tilde{\alpha}_k(t_k)=(\tilde{\alpha}_k(t_k)_\ell)_{\ell=1}^d$, note that

$$\begin{aligned}\int_{t_k\in R_k}\int_{z_k} g(z_k)\tilde{\alpha}_k^\nu(t_k,dz_k)\kappa_k(dt_k) &= \int_{t_k\in R_k} g\cdot\tilde{\alpha}_k^\nu(t_k)\kappa_k(dt_k)\\ \to\quad \int_{t_k\in R_k} g\cdot\tilde{\alpha}_k(t_k)\kappa_k(dt_k) &= \int_{t_k\in R_k}\int_{z_k} g(z_k)\tilde{\alpha}_k(t_k,dz_k)\kappa_k(dt_k).\end{aligned}$$

By part (c) of Balder's (2002) Theorem 2.2, because $R_k$ and $g$ were arbitrary, it follows that convergence of expected actions $\{\tilde{\alpha}_k^\nu\}$ to $\tilde{\alpha}_k$ in the weak topology on $\tilde{\mathfrak{A}}_k$ implies convergence in the narrow topology (also called the "weak topology") on transition probabilities.

Given any expected actions $(\tilde{\beta}_k)_{k\neq j}$ for players other than $j$, we can define the product transition $\bigotimes_{k\neq j}\tilde{\beta}_k\colon T\times 2^{Z^n}\to[0,1]$ as follows: for all $t\in T$ and all $Y=\times_{k\neq j}Y_k\subseteq Z^n$, $(\bigotimes_{k\neq j}\tilde{\beta}_k)(t,Y)=\prod_{k\neq j}\tilde{\beta}_k(t_k,Y_k)$. Balder's (1988) Theorem 2.5 implies that the product mapping $(\tilde{\beta}_k)_{k\neq j}\mapsto\bigotimes_{k\neq j}\tilde{\beta}_k$ is continuous in the narrow topology, where the range is the set of transition probabilities from $T_{-j}$, endowed with the product measure $\bigotimes_{k\neq j}\kappa_k$, into the probability distributions on $Z^n$. Returning to the continuity argument, the net $\{\bigotimes_{k\neq j}\tilde{\alpha}_k^\nu\}$ of product transitions therefore converges in the narrow sense to $\bigotimes_{k\neq j}\tilde{\alpha}_k$. Using (B6) and the assumption that $t_j\notin\tilde{T}_j$, $\kappa(\cdot|t_j)$

has density $h(\cdot|t_j): T_{-j} \to \mathfrak{R}$ with respect to $\bigotimes_{k \neq j} \kappa_k$, so we have

$$\tilde{\Pi}_j(e^{z_j}, t_j, u; \tilde{\alpha}^\nu) = \int_{t_{-j}} \tilde{\pi}_j(e^{z_j}, \tilde{\alpha}^\nu_{-j}(t_{-j}), t, u) h(t_{-j}|t_j) \bigotimes_{k \neq j} \kappa_k(dt_{-j}).$$

We represent a profile of pure strategies for players other than $j$ by an $(n-1)$-tuple $z_{-j} = (z_k)_{k \neq j} \in Z^{n-1}$, and we write $\phi_j(z_{-j}, t, u) = \tilde{\pi}_j(e^{z_j}, (e^{z_k})_{k \neq j}, t, u)$ for the vector of player $j$'s payoff when, given $(t, u)$, $j$ chooses $z_j$ and the other players choose $z_{-j}$. By multilinearity, from (B4), and Fubini's theorem, we then have

$$\tilde{\Pi}_j(e^{z_j}, t_j, u; \tilde{\alpha}^\nu) = \int_{t_{-j}} \int_{z_{-j}} \phi_j(z_{-j}, t, u) h(t_{-j}|t_j) \bigotimes_{k \neq j} \tilde{\alpha}^\nu_k(t_{-j}, dz_{-j}) \bigotimes_{k \neq j} \kappa_k(dt_{-j}).$$

Note that the integrand $\phi_j(z_{-j}, t, u) h(t_{-j}|t_j)$ above is (trivially) continuous in $z_{-j}$ and jointly measurable in $(t_{-j}, z_{-j})$. Furthermore, we have $|\phi_j(z_{-j}, t, u) h(t_{-j}|t_j)| \leq \sup_{(a,u) \in \Delta^n \times \tilde{U}} |\tilde{\pi}_j(a, t, u) h(t_{-j}|t_j)|$ for all $(t_{-j}, z_{-j})$ and by $t_j \notin \tilde{T}_j$, we have

$$\begin{aligned} \int_{t_{-j}} \sup_{(a,u) \in \Delta^n \times \tilde{U}} |\tilde{\pi}_j(a, t, u) h(t_{-j}|t_j)| \bigotimes_{k \neq j} \kappa_k(dt_{-j}) &= \int_{t_{-j}} \sup_{(a,u) \in \Delta^n \times \tilde{U}} |\tilde{\pi}_j(a, t, u)| \kappa(dt_{-j}|t_j) \\ &< \infty. \end{aligned}$$

Therefore, it is a Carathéodory integrand (Balder (1988)), and by definition of the narrow topology, we have

$$\begin{aligned} \tilde{\Pi}_j(e^{z_j}, t_j, u; \tilde{\alpha}^\nu) &\to \int_{t_{-j}} \int_{z_{-j}} \phi_j(z_{-j}, t, u) h(t_{-j}|t_j) \bigotimes_{k \neq j} \tilde{\alpha}_k(t_{-j}, dz_{-j}) \bigotimes_{k \neq j} \kappa_k(dt_{-j}) \\ &= \tilde{\Pi}_j(e^{z_j}, t_j, u; \tilde{\alpha}), \end{aligned}$$

establishing continuity of $\tilde{\Pi}_j(e^{z_j}, t_j, u; \tilde{\alpha})$ in $\tilde{\alpha}$, as claimed. For joint continuity, recall that in general,

$$\tilde{\Pi}_j(a_j, t_j, u; \tilde{\alpha}) = \sum_{z_j \in Z} p_\ell \tilde{\Pi}_j(e^{z_j}, t_j, u; \tilde{\alpha}),$$

where $a_j = (p_\ell)_{\ell=1}^d$, which is continuous in $(a_j, \tilde{\alpha})$.

To finish the argument for (A10), consider any sequence $\{\tilde{\alpha}^m\}$ converging weakly to $\tilde{\alpha}$ in $\tilde{\mathfrak{A}}$, and any sequence $\{a_j^m\}$ converging to $a_j$ in $\tilde{A}_j(t_j, u)$. Since $M_j(t_j, u; \cdot)$ has compact range, by (B2), it suffices to show that $a_j \in M_j(t_j, u; \tilde{\alpha})$. Since $\tilde{\Pi}_j(a_j, t_j, u; \tilde{\alpha})$ is jointly continuous in $(a_j, \tilde{\alpha})$, this follows immediately from theorem of the maximum (AB, Theorem 17.31).

To prove (a), we apply Theorem 1 to select a choice equilibrium $\gamma^* = (\gamma_j^*)_{j=1}^n$ in the abstract framework. From $\gamma_j^*: T_j \times \tilde{U} \to \mathfrak{R}^d$, $j = 1, \ldots, n$, we define the strategy $\sigma_j^*: T_j \times U_j \to \mathfrak{R}^d$ in the obvious way by $\sigma_j^*(t_j, u_j) = \gamma_j^*(t_j, u_j)$ for all $(t_j, u_j) \in T_j \times U_j$, i.e., $\sigma_j^*$ is simply the restriction of $\gamma_j^*$ to $T_j \times U_j$. We claim that the corresponding strategy profile $\sigma^* = (\sigma_j^*)_{j=1}^n$ is a Bayes-Nash equilibrium. If not, then there is some player $j$ with a strategy $\sigma_j$ such that $\Pi_j(\sigma_j, \sigma^*_{-j}) > \Pi_j(\sigma^*)$. Letting $\alpha^* = \alpha(\cdot|\sigma^*)$ and $\tilde{\alpha}^* = \tilde{\alpha}(\cdot|\gamma^*)$ be the corresponding expected actions, and letting $\gamma_j$ be any extension of $\sigma_j$ to $T_j \times \tilde{U}$, we have

$$\begin{aligned}\Pi_j(\sigma_j, \alpha^*_{-j}) &= \int_t \int_u \tilde{\pi}_j(\gamma_j(t_j, u), \tilde{\alpha}^*_{-j}(t_{-j}), t, u)\tilde{\nu}_j(du|t_j)\kappa(dt) \\ &> \int_t \int_u \tilde{\pi}_j(\gamma^*_j(t_j, u), \tilde{\alpha}^*_{-j}(t_{-j}), t, u)\tilde{\nu}_j(du|t_j)\kappa(dt) \\ &= \Pi_j(\sigma^*_j, \alpha^*_{-j}).\end{aligned}$$

Moreover, using Proposition 2.3.2 of Rao (1993), we can decompose $\Pi_j(\sigma_j, \alpha^*_{-j})$ as

$$\begin{aligned}&\int_t \int_u \tilde{\pi}_j(\gamma_j(t_j, u), \tilde{\alpha}^*_{-j}(t_{-j}), t, u)\tilde{\nu}_j(du|t_j)\kappa(dt) \\ &= \int_{(t_j,u)} \int_{t_{-j}} \tilde{\pi}_j(\gamma_j(t_j, u), \tilde{\alpha}^*_{-j}(t_{-j}), t, u) h(t_{-j}|t_j)\kappa(dt_{-j}|t_j)(\tilde{\nu}_j(\cdot|\cdot) \otimes \kappa_j)(d(t_j, u)) \\ &= \int_{(t_j,u)} \tilde{\Pi}_j(\gamma_j(t_j, u), t_j, u; \tilde{\alpha}^*)(\tilde{\nu}_j(\cdot|\cdot) \otimes \kappa_j)(d(t_j, u)),\end{aligned}$$

with a similar decomposition for $\Pi_j(\sigma^*_j, \alpha^*_{-j})$. But then there is a set of $Q \in \mathcal{T}_j \otimes \tilde{\mathcal{U}}$ with positive $\tilde{\nu}_j(\cdot|\cdot) \otimes \kappa_j$-measure such that for all $(t_j, u) \in Q$, we have $t_j \notin \tilde{T}_j$ and

$$\tilde{\Pi}_j(\gamma_j(t_j, u), t_j, u; \tilde{\alpha}^*) \quad > \quad \tilde{\Pi}_j(\gamma^*_j(t_j, u), t_j, u; \tilde{\alpha}^*).$$

Thus, for all $(t_j, u) \in Q$, we have $\gamma^*_j(t_j, u) \notin M_j(t_j, u; \tilde{\alpha}^*)$, contradicting the fact that $\gamma^*$ is a choice equilibrium. This establishes (a).

For (b), assume that $\{\nu_j(\cdot|t_j) : t_j \in T_j\}$ are nonatomic, which implies $\{\tilde{\nu}_j(\cdot|t_j) : t_j \in T_j\}$ are as well. Let $\sigma^*$ be any Bayes-Nash equilibrium, and define the choice function $\gamma^* = (\gamma^*_j)^n_{j=1}$ by extending $\sigma^*_j$ to $T_j \times \tilde{U}$, $j = 1, \ldots, n$, arbitrarily. By arguments similar to the above, $\gamma^*$ is a choice equilibrium. Then Corollary 1 yields an extremal choice equilibrium $\hat{\gamma}$ that is equivalent to $\gamma^*$. Finally, defining the strategy profile $\hat{\sigma} = (\hat{\sigma}_j)^n_{j=1}$ by restricting each $\hat{\gamma}_j$ to $T \times U_j$, the resulting strategy profile is a pure strategy Bayes-Nash that is equivalent to $\sigma^*$, as required.

## Appendix A. Lower Measurability of Extreme Points

This appendix contains a lemma establishing, among other things, lower measurability of the closure of extreme points of a correspondence. Note that by Lemma 18.3 of AB, this result extends to the correspondence $\text{ext}\varphi(\cdot)$ of extreme points as well (although this correspondence may not have closed values).

**Lemma 10:** *Let $(S, \Sigma)$ denote a measurable space, and assume $\varphi\colon S \rightrightarrows \Re^d$ is lower measurable with nonempty and compact values. Then the correspondence $s \mapsto \overline{\text{ext}}\varphi(s)$ is lower measurable with nonempty and compact values.*

**Proof:** Nonempty and compact values follow from compactness of $\varphi(s)$. To prove lower measurability, let $\psi(s) = \text{co}\varphi(s)$, and note that these sets possess the same extreme points, i.e., $\text{ext}\psi(s) = \text{ext}\varphi(s)$. Thus, it suffices to show that the correspondence $s \mapsto \overline{\text{ext}}\psi(s)$ is lower measurable. Let $\{x^m\}$ be a countable, dense subset of $\Re^d$, and for each $m$, define the continuous mapping $d_m : \Re^d \to \Re$ by $d_m(x) = ||x^m - x||$. By a measurable version of the maximum theorem (see Theorem 18.19 of AB), the correspondence $\Phi^m : S \rightrightarrows \Re^d$ defined by

$$\Phi^m(s) \quad = \quad \arg\max\{d_m(x) : x \in \psi(s)\}$$

is lower measurable. By Corollary 7.87 of AB, $\Phi^m(s)$ is contained among the exposed points of $\psi(s)$, and therefore $\Phi^m(s) \subseteq \text{ext}\psi(s)$.[6] By Lemma 18.4 of AB, it follows that the correspondence $\Phi : S \rightrightarrows \Re^d$ defined by $\Phi(s) = \bigcup_{m=1}^{\infty} \Phi^m(s)$ is lower measurable.

Given any $s \in S$, we claim that $\Phi(s)$ is dense among the exposed points of $\psi(s)$. Let $y$ be any exposed point of $\psi(s)$, and let $f : \Re^d \to \Re$ be a linear function such that $\arg\max\{f(x) : x \in \psi(s)\} = \{y\}$, i.e., letting $p$ be the gradient of $f$ normalized so that $||p|| = 1$, we have $p \cdot x < p \cdot y$ for all $x \in \psi(s)$ with $x \neq y$. Consider any $\epsilon > 0$, and define $z_n = y - np$. We will prove that for $n > 0$ large enough, we have $\arg\max\{||z_n - x|| : x \in \psi(s)\} \subseteq B_\epsilon(y)$. If not, then for arbitrarily large $n$, there exists $v_n \in \psi(s) \setminus B_\epsilon(y)$ such that $||z_n - v_n|| \geq ||z_n - y|| = n$. By compactness of $\psi(s)$, we may assume $v_n \to v \in \psi(s)$. Since $y$ uniquely maximizes $f$ on $\psi(s)$ and $y \neq v$, there exists $a > 0$ such that $p \cdot v + a < p \cdot y$. Setting $w = y - ap$, we have $p \cdot v = p \cdot w$, and in particular, the vectors $v - w$ and $w - z_n$ are orthogonal. It follows that

$$||z_n - v|| \;=\; \sqrt{||z_n - w||^2 + ||v - w||^2} \;=\; \sqrt{(n-a)^2 + ||v - w||^2},$$

which is strictly less than $n$ for $n$ great enough. This implies $v \in B_n(z_n)$ for high enough $n$. Furthermore, the sequence $\{B_n(z_n)\}$ is increasing in the sense of set inclusion, for given any $x \in B_n(z_n)$, we have $||z_{n+1} - x|| \leq ||z_{n+1} - z_n|| + ||z_n - x|| \leq n + 1$, implying $B_n(z_n) \subseteq B_{n+1}(z_{n+1})$. We conclude that $||z_n - v_n|| < n$ for high enough $n$, a contradiction. Thus, $\arg\max\{d||z_n - x|| : x \in \psi(s)\} \subseteq B_\epsilon(y)$ for some $n$. Since $\{x^m\}$ is dense in $\Re^d$, we may approximate $z_n$ to an arbitrary degree by elements $x^m$, and then the theorem of the maximum (Theorem 17.31 of AB) implies that $\Phi^m(s) \subseteq B_\epsilon(y)$ for some $m$, and therefore $\Phi(s) \cap B_\epsilon(s) \neq \varnothing$. We conclude that $\Phi(s)$ is dense among the exposed points of $\psi(s)$, as claimed.

Finally, Theorem 7.89 of AB implies that the exposed points of $\psi(s)$ are dense among the extreme points of $\psi(s)$, and therefore $\overline{\Phi}(s) = \overline{\text{ext}}\psi(s)$ for all $s \in S$. Then

[6]Given a set $A \subseteq \Re^d$, we say $x \in \Re^d$ is a *strongly exposed point* of $A$ if it is the unique maximizer over $A$ of a linear function.

lower hemicontinuity of $s \mapsto \overline{\text{ext}}\psi(s)$ follows from Lemma 18.3 of AB. This completes the proof of the lemma.

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DEPT. OF ECONOMICS, UNIVERSITY OF ROCHESTER.

DEPT. OF POLITICAL SCIENCE AND DEPT. OF ECONOMICS, UNIVERSITY OF ROCHESTER.