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Abstract

We extend Armstrong's [2] result on exclusion in multi-dimensional screening models in two key ways, providing support for the view that this result is generic and applicable to many different markets. First, we relax the strong technical assumptions he imposed on preferences and consumer types. Second, we extend the result beyond the monopolistic market structure to generalized oligopoly settings with entry. We also analyze applications to several quite different settings: credit markets, the automobile industry, research grants, the regulation of a monopolist with unknown demand and cost functions, and involuntary unemployment in the labor market.

JEL Codes: C73, D82

Key words: Multi-dimensional screening, exclusion, regulation of a monopoly, involuntary unemployment.

1   Introduction

When considering the problem of screening, where sellers choose a sales mechanism and buyers have private information about their types, it is well known that the techniques used in the multi-dimensional setting are not as straightforward as those in the one-dimensional setting. As a consequence, while we have a host of successful applications with one-dimensional

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types, to date we have only a few scattered papers that allow for multidimensional types. This is unfortunate because in many, if not most, economic applications multi-dimensional types are needed to capture the basic economics of the environment, and the propositions coming from the one dimensional case do not necessarily generalize to the multi-dimensional case.[1]

One of the most well known results in the theory of multi-dimensional screening, though, comes from Armstrong [2], who shows that a monopolist will find it optimal to not serve some fraction of consumers, even when there is positive surplus associated with those consumers. That is, in settings where consumers vary in at least two different ways, monopolists will choose a sales mechanism that excludes a positive measure of consumers. The intuition behind this result is rather simple: consider a situation where the monopolist serves all consumers; if she increases the price by $\varepsilon > 0$ she earns extra profits of order $O(\varepsilon)$ on the consumers who still buy the product, but will lose only the consumers whose surplus was below ε. If $m > 1$ is the dimension of the vector of consumers' taste characteristics, then the measure of the set of the lost consumers is $O(\varepsilon^m)$. Therefore, it is profitable to increase the price and lose some consumers. In principle, this result has profound implications across a wide range of economic settings. The general belief that heterogeneity of consumer types is likely to be more than one-dimensional in nature, for many different commodities, and that these types are likely to be private information, underlines the importance of this result.[2]

However, the result itself was derived under a relatively strong set of assumptions that could be seen as limiting its applicability, and subsequent research has identified conditions under which the result does not hold. In particular, Armstrong's original analysis assumes that the utility functions of the agents are homogeneous and convex in their types, and that these types belong to a strictly convex and compact body of a finite dimensional space. Basov [7] refers to the latter as the joint convexity assumption and argues that, although convexity of utility in types and convexity of the types set separately are not restrictive and can be seen as a choice of parametrization, the joint convexity assumption is technically restrictive.

The joint convexity assumption has no empirical foundation and is nonstandard. For instance, the benchmark case of independent types fails joint convexity, because the type space is the not strictly convex multi-dimensional box. There is, in general, no theoretical

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[1]See Rochet and Stole [23] and Basov [7] for surveys of the literature.

[2]The type of an economic agent is simply her utility function. If one is agnostic about the preferences and does not want to impose on them any assumptions beyond, perhaps, monotonicity and convexity, then the most natural assumption is that the type is multi-dimensional.

justification for a particular assumption about the curvature of utility functions with respect to types, as opposed to, say, quasi-concavity of utility functions with respect to goods. In the same line, in general, there is no justification, other than analytical tractability, for type spaces to be convex, and for utility functions to be homogeneous in types. Both Armstrong [4] and Rochet and Stole [23] found examples outside of these restrictions where the exclusion set is empty.

We show that these counter-examples are not generic. We also show that exclusion is generic under more general market structures, i.e. the result is actually quite robust. We then provide examples where we believe exclusion is a relevant economic phenomenon.

We start with relaxing the joint convexity and homogeneity assumptions, and show that they are not necessary for the result. Exclusion is generically optimal in the family of models where types belong to sets of locally finite perimeter (which is a class of sets that includes all of the examples the authors are aware of in the literature) and utility functions are smooth and monotone in types. We show that the examples considered in Armstrong[4] and Rochet and Stole [23] are, themselves, very special cases. We then go on to show that the exclusion results can be generalized to the case of an oligopoly and an industry with free entry. Therefore, the inability of some consumers to purchase the good at acceptable terms is solely driven by the multi-dimensional nature of private information rather than by market conditions or the nature of distribution of consumers' tastes.

To illustrate the generality of the results, we apply them in several different settings: credit markets, the automobile industry, and research grants. We also pay particular attention to two applications: the first being the regulation of a monopolist with unknown demand and cost functions, and the second being the existence of equilibrium involuntary unemployment. The former application picks up of the analysis in Armstrong [4], where he reviews Lewis and Sappington [17] and conjectures that exclusion is probably an issue in their analysis. At the time, Armstrong could not prove the point, due to the lack of a more general exclusion result. With our result in hand, we are able to prove Armstrong's conjecture. The latter application is a straightforward way of showing that, when workers have multi-dimensional characteristics, it is generically optimal for employers (with market power in the labor market) to not hire all the workers.

Most generally, we believe that the main result of this paper is that private information leads to exclusion in almost any realistic setting. To avoid it, one must either assume that all allowable preferences lie on a one-dimensional continuum or construct very specific type distributions and preferences.

The remainder of this paper is organized as follows. In Section 2 we present the monopoly problem with consumers that have a type-dependent outside option and then derive conditions under which it is generically optimal to have exclusion. In Section 3 we generalize the results for the case of oligopoly and a market with free entry. Examples and applications are presented in Section 4. The Appendix presents some relevant concepts from geometric measure theory.

2 The Robustness of Exclusion in a Monopolistic Screening Model

Consider a firm with a monopoly over n goods. The tastes of the consumers over these goods are parametrized by a vector $\alpha \in \mathbb{R}^m$. The utility of a type α consumer consuming a bundle $x \in \mathbb{R}^n_+$ and paying $t \in \mathbb{R}$ to the firm is

$$v(\alpha, x, t)$$

where v is strictly increasing and strictly concave in x, and strictly decreasing in t. Our focus is not on relaxing the smoothness assumptions on v, so we will assume that v is twice continuously differentiable, with $v_t(\alpha, x, t) \equiv \frac{\partial v(\alpha, x, t)}{\partial t}$ Lipschitz continuous and bounded away from zero.

The total cost $c(\cdot)$ of producing bundle x is given by

$$c(x) = c \cdot x, \tag{2.1}$$

with $c = (c_1, .., c_n)$. That is, there is constant marginal cost of production. In the Appendix we show that this assumption is without loss of generality, and made simply to conform with the analysis of the oligopoly problem later on.

The firm is not able to observe the consumer's type, but has prior beliefs over the distribution of types, described by the density function $f(\alpha)$, with compact support $\text{supp}(f) = \overline{\Omega}$, where $\Omega \subset \mathbb{R}^m$ is the space of types, and $\overline{\Omega}$ is its closure. We assume that $\Omega \subset U$ is an open set with locally finite perimeter in the open set U, and that f is Lipschitz continuous.[3] Also, we assume that $\nu(\cdot, x, t)$ can be extended by continuity to $\overline{\Omega}$. Consumers have an outside option of value $s_0(\alpha)$, which is assumed to be continuously differentiable, implementable and

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[3]See Evans and Gariepy [13] and Chlebik [11] for the relevant concepts from geometric measure theory. For convenience, a brief summary is presented in the Appendix.

extendable by continuity to $\overline{\Omega}$.[4] Let $x_0(\alpha)$ be the outside option implementing $s_0(\alpha)$ for type α.

The firm looks for a selling mechanism that maximizes its profits. The Taxation Principle (Rochet [20]) implies that one can, without loss of generality, assume that the monopolist simply announces a non-linear tariff $t : \mathbb{R}^n_+ \to \mathbb{R}$.

The above considerations can be summarized by the following model. The firm selects a function $t : \mathbb{R}^n_+ \to \mathbb{R}$ to solve

$$\max_{t(\cdot)} \int_\Omega (t(x(\alpha)) - c(x(\alpha))) f(\alpha) d\alpha, \tag{2.2}$$

where $c(\cdot)$ is defined by (2.1) $x(\alpha)$ satisfies

$$\begin{cases} x(\alpha) \in \arg\max_{x \geq 0} v(\alpha, x, t(x)) & \text{if } \max_{x \geq 0} v(\alpha, x, t(x)) \geq s_0(\alpha) \\ x(\alpha) = x_0(\alpha) & \text{otherwise.} \end{cases} \tag{2.3}$$

Define the net utility as the unique function $u(\alpha, x)$ that solves

$$s_0(\alpha) = v(\alpha, x, u(\alpha, x)) \tag{2.4}$$

The economic meaning of $u(\alpha, x)$ is the maximal amount type α is willing to pay for the bundle x. Note that the optimal tariff paid by type α satisfies

$$t(x(\alpha)) \leq u(\alpha, x(\alpha)). \tag{2.5}$$

Let $s(\alpha)$ denote the surplus obtained by type α:

$$s(\alpha) = \begin{cases} \max_{x \geq 0} v(\alpha, x, t(x)) - s_0(\alpha) & \text{if } \max_{x \geq 0} v(\alpha, x, t(x)) \geq s_0(\alpha) \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

Accordingly, we have the envelope condition

$$\nabla s(\alpha) = \nabla_\alpha v(\alpha, x(\alpha), t(x(\alpha))) - \nabla s_0(\alpha)$$

that holds for almost every α with $x(\alpha) \neq x_0(\alpha)$. From (2.4) we have

$$\nabla s_0(\alpha) = \nabla_\alpha v(\alpha, x(\alpha), u(\alpha, x(\alpha))) + v_t(\alpha, x(\alpha), u(\alpha, x(\alpha))) \nabla_\alpha u(\alpha, x(\alpha)),$$

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[4]For conditions of implementability of a surplus function see Basov [7].

so the envelope condition can be written as

$$\lambda(\alpha)\nabla s(\alpha) = \nabla_\alpha u(\alpha, x(\alpha)) \tag{2.7}$$

for almost every α with $x(\alpha) \neq x_0(\alpha)$, where $\lambda(\alpha) = |v_t(\alpha, x(\alpha), u(\alpha, x(\alpha)))|^{-1}$ is positive and bounded away from zero.

Assumption 2.1. $u(\cdot, x)$ *is strictly increasing in α for each $x \neq x_0(\alpha)$.*

For $a, b \in \mathbb{R}^m$ let $(a \cdot b)$ denote the inner product of a and b.

Assumption 2.2. *There exists $K > 0$ such that $u(\alpha, x) \leq K(\alpha \cdot \nabla_\alpha u(\alpha, x))$ for every $(\alpha, x) \in \overline{\Omega} \times \mathbb{R}_+^n$.*

Assumptions 2.1 and 2.2 are regularity conditions, requiring that the net utility be strictly increasing in α and bounded. Note that $v(\cdot, x, t)$ is allowed to be decreasing in α, as long as Assumptions 2.1 and 2.2 are satisfied.

For any Lebesgue measurable set $E \subset \mathbb{R}^m$ let $\mathcal{L}^m(E)$ denote its Lebesgue measure and $\mathcal{H}^s(E)$ denote its s-dimensional Hausdorff measure. For $s = m$, the Hausdorff measure of a Borel set coincides with the Lebesgue measure, while for $s < m$ it generalizes the notion of the surface area.[5]

Let $\partial_e\Omega$ denote the measure theoretic boundary of Ω. Because Ω has locally finite perimeter, the measure theoretic boundary can be decomposed into countably many smooth pieces and a residual set with measure zero. That is,

$$\partial_e\Omega = \bigcup_{i=1}^\infty K_i \cup N,$$

where K_i is a compact subset of a C^1-hypersurface S_i, for $i \geq 1$, and $\mathcal{H}^{m-1}(N) = 0$.

Assumptions 2.1 and 2.2 hold for any type space Ω considered in this paper. We now describe the underlying *space of all type spaces*. It is given by $(\Omega_\beta)_{\beta \in \mathcal{B}}$, where $\mathcal{B}$ is an index set. For each $\beta \in \mathcal{B}$, Ω_β is an open set with locally finite perimeter in some open set U_β and boundary structure given by

$$\partial_e\Omega_\beta = \bigcup_{i=1}^\infty K_{i,\beta} \cup N_\beta$$

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[5]For a definition of the Hausdorff measure, see Chlebik [11].

where
$$K_{i,\beta} = \{\alpha \in \overline{\Omega}_\beta : g_i(\alpha, \beta) = 0\}$$

for $i > 0$, with $g_i : \mathbb{R}^m \times \mathcal{B} \to \mathbb{R}$ smooth, and N_β is a set of $\mathcal{H}^{m-1}$-measure zero. We make the following assumption about $(\Omega_\beta)_{\beta \in \mathcal{B}}$:

Assumption 2.3. $\mathcal{B}$ *is a smooth finite dimensional manifold and there exist $\beta_1, \beta_2 \in \mathcal{B}$ such that*
$$rank \begin{pmatrix} \nabla_\alpha g_i(\alpha, \beta_1) \\ \nabla_\alpha g_i(\alpha, \beta_2) \end{pmatrix} = 2,$$

for all $\alpha \in \mathbb{R}^m$ and all $i > 0$.

That is, the parameters $\beta \in B$ determine the underlying set of type spaces $(\Omega_\beta)_{\beta \in B}$ that we consider, and we assume that there are at least two type spaces with boundaries that are not parallel shifts of each other. This is obviously a very weak assumption. A seemingly more important requirement is the finite dimensionality of $\mathcal{B}$. But this is just for the a cleaner presentation of our ideas. In Lemma 2.6 below we make use of the standard Transversatiliy Theorem, which is valid in a finite dimensional environment. It is well known that there exist general versions of the Transversality Theorem that allow for infinite dimensions.[6] One can generalize Assumption 2.3 allowing for an infinite dimensional $\mathcal{B}$ and adapt Lemma 2.6 below with a more powerful Transversality Theorem. We leave this task to the interested reader.

Our main result will follow after we establish some technical lemmata. Let $\mathcal{K}(\mathbb{R}^m)$ be the hyperspace of compact sets in $\mathbb{R}^m$, endowed with the topology induced by the Hausdorff distance d_H, given by

$$d_H(E, F) = \inf\{\varepsilon > 0 : E \subset F^\varepsilon, F \subset E^\varepsilon\},$$

where

$$E^\varepsilon = \bigcup_{\alpha \in E} B(\alpha, \varepsilon)$$

and $B(\alpha, \varepsilon)$ is the open ball centered at α and with radius $\varepsilon > 0$. Because

$$\lim_{\varepsilon \to 0+} \mathcal{L}^m(E^\varepsilon) = \mathcal{L}^m(E), \quad \lim_{\varepsilon \to 0+} \mathcal{H}^s(E^\varepsilon) = \mathcal{H}^s(E)$$

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[6] See Golubitsky and Guillemin [14] for the relevant concepts in the theory of transversality.

for all $s \geq 0$, both $\mathcal{L}^m$ and $\mathcal{H}^s$ are upper semicontinuous functions in $\mathcal{K}(\mathbb{R}^m)$ (Beer [9]). Hence the following lemma holds.

Lemma 2.4. *Let $E \in \mathcal{K}(\mathbb{R}^m)$ be such that $\mathcal{L}^m(E) = \mathcal{H}^s(E) = 0$, for some $s \geq 0$, and let $(E_k)_{k \geq 1}$ be a sequence in $\mathcal{K}(\mathbb{R}^m)$ such that $E_k \to E$. Then $\mathcal{L}^m(E_k) \to 0$ and $\mathcal{H}^s(E_k) \to 0$.*

Proof. Because $\mathcal{L}^m$ is a non negative upper semicontinuous set function, we have

$$\lim \inf_{E_k \to E} \mathcal{L}^m(E_k) \geq 0 = \lambda(E) \geq \lim \sup_{E_k \to E} \mathcal{L}^m(E_k),$$

so $\mathcal{L}^m(E_k) \to 0$, and analogously for $\mathcal{H}^s$. □

Lemma 2.4 establishes continuity of Lebesgue and Hausdorff measures at zero. Let us write $\Omega_{0,\beta} = \{\alpha \in \Omega_\beta : s(\alpha; \beta) = 0\}$, where $s(\alpha; \beta)$ is the surplus function obtained by type α when the underlying type space is Ω_β. Likewise, in what follows we make explicit the dependence of the relevant object on the underlying type space indexed by $\beta \in \mathcal{B}$. Extending $s(\cdot; \beta)$ by continuity on $\partial \Omega_\beta$, let $\overline{\Omega}_{0,\beta} = \{\alpha \in \overline{\Omega}_\beta : s(\alpha; \beta) = 0\}$.

Lemma 2.5. *Under Assumption 2.1, $\mathcal{L}^m(\overline{\Omega}_{0,\beta}) = 0$ implies $\overline{\Omega}_{0,\beta} \subset \partial \Omega_\beta$.*

Proof. If $\overline{\Omega}_{0,\beta} \not\subset \partial \Omega_\beta$, there is $\alpha \in \Omega_{0,\beta}$ and an $\varepsilon > 0$ with $B(\alpha, \varepsilon) \subset \Omega$. Then

$$\mathcal{L}^m(\{\hat{\alpha} \in \overline{\Omega}_\beta : \hat{\alpha} \leq \alpha\} \cap B(\alpha, \varepsilon)) > 0.$$

Because of Assumption 2.1, we cannot have $s(\hat{\alpha}; \beta) > 0$ for any $\hat{\alpha} \leq \alpha$, for otherwise $s(\alpha; \beta) > 0$ as well. So

$$\{\hat{\alpha} \in \overline{\Omega}_\beta : \hat{\alpha} \leq \alpha\} \cap B(\alpha, \varepsilon) \subset \Omega_0,$$

contradicting $\mathcal{L}^m(\overline{\Omega}_{0,\beta}) = 0$. □

Lemma 2.5 states that if the exclusion set has Lebesgue measure zero it should be part of the topological boundary of the type set. Assumption 2.1 is crucial for this result. If it does not hold it is easy to come up with counter-examples even in the one-dimensional case. For examples, see Jullien [15].

Lemma 2.6. *Assume $\mathcal{L}^m(\overline{\Omega}_{0,\beta}) = 0$ for all β in some open subset $V \subset \mathcal{B}$, and that Assumption 2.3 holds. Then $\mathcal{H}^{m-1}(\overline{\Omega}_{0,\beta}) = 0$ for an open and dense set of $\beta \in V$.*

Proof. Let $s(\cdot; \beta)$ be the surplus function generated by the optimal tariff via (2.6) when the underlying type space is Ω_β By Lemma 2.5, $\overline{\Omega}_{0,\beta} \subset \partial\Omega_\beta$ for all $\beta \in V$. Because $\mathcal{H}^{m-1}(\partial\Omega_\beta \backslash \partial_e\Omega_\beta) = 0$, consider $\overline{\Omega}_{0,\beta} \cap \partial_e\Omega_\beta$, which is given by

$$\overline{\Omega}_{0,\beta} \cap \partial_e\Omega_\beta = \bigcup_{i=1}^{\infty} \overline{\Omega}_{0i,\beta} \cup (N_\beta \cap \overline{\Omega}_{0,\beta})$$

where

$$\overline{\Omega}_{0i,\beta} = \{\alpha \in \overline{\Omega}_\beta : g_i(\alpha, \beta) = 0, s(\alpha; \beta) = 0\},$$

for $i > 0$. Now Assumption 2.3 ensures that there is, say, $\beta_1 \in \mathcal{B}$ for which $u(\alpha, x; \beta)$ is transversal to $g_i(\alpha, \beta_1)$, at the solution $x(\alpha; \beta)$ for all $i > 0$. The Transversality Theorem then implies that the level sets of $u(\alpha, x; \beta)$ are transversal to the level sets of $g_i(\alpha, \beta')$ for an open and dense subset of β' and all $i > 0$. By continuity of $x(\alpha; \beta)$ in β, for any neighborhood of β, there exists a model β' with the level sets of $u(\alpha, x; \beta')$ transversal to the level sets of $g_i(\alpha, \beta')$ for all $i > 0$, at the solution $x(\alpha, \beta')$. Hence there is a dense subset of V with the level sets of $u(\alpha, x; \beta')$ transversal to those of $g_i(\alpha, \beta')$ for all $i > 0$. By Transversality again, there exists an open and dense set of parameters β'' with the level sets of $u(\alpha, x; \beta')$ transversal to those of $g_i(\alpha, \beta'')$ for all $i > 0$. Now β' is in the interior of this latter set. Hence there exists an open neighborhood of β' with the level sets of $u(\alpha, x; \hat{\beta})$ transversal to those of $g_i(\alpha, \hat{\beta})$ for all $i > 0$ and for all $\hat{\beta}$ in this neighborhood.

Hence, by the Implicit Function Theorem, $\overline{\Omega}_{0i,\beta}$ is a manifold of dimension of $(m-2)$ for all β in an open and dense subset of V. So $\mathcal{H}^{m-1}(\overline{\Omega}_{0i,\beta}) = 0$ in this set. Hence

$$\mathcal{H}^{m-1}(\overline{\Omega}_{0,\beta} \cap \partial_e\Omega_\beta) \leq \sum_{i=1}^{\infty} \mathcal{H}^{m-1}(\overline{\Omega}_{0i,\beta}) + \mathcal{H}^{m-1}(N_\beta \cap \overline{\Omega}_{0,\beta}) = 0$$

for an open and dense set of $\beta \in V$, as we wanted to show. $\qquad\square$

The Generalized Gauss-Green Theorem states that for any Ω with locally finite perimeter in $U \subset \mathbb{R}^m$, and any Lipschitz continuous vector field $\varphi : U \to \mathbb{R}^m$ with compact support in U there is a unique measure theoretic unit outer normal $\tau_\Omega(\alpha)$ such that

$$\int_\Omega \mathrm{div}\varphi d\alpha = \int_U (\varphi \cdot \tau_\Omega) d\mathcal{H}^{m-1}$$

where

$$\mathrm{div}\varphi = \sum_{k=1}^{m} \frac{\partial\varphi_k}{\partial\alpha_k}$$

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is the divergence of the vector field φ.

The main result of this section is Theorem 2.7 below. It is stated without reference to the well known sufficient conditions for implementability and differentiability of $s(\cdot)$ in order to focus on the conditions that highlight the nature of the contribution being made.

Theorem 2.7. *Consider the problem (2.2)-(2.3), and assume that it has a finite solution yielding a continuous allocation $x(\alpha; \beta)$. Then, under Assumptions 2.1, 2.2 and 2.3, the set of consumers with zero surplus at the solution has positive measure for almost all $\beta \in \mathcal{B}$.*

Proof. First we argue that, for each given $\beta \in \mathcal{B}$, there is an equivalent metric in $\mathbb{R}^n$ for which $x(\alpha; \beta)$, and hence $s(\alpha; \beta)$ and $\lambda(\alpha; \beta)$, are Lipschitz continuous functions (for a given $\beta \in \mathcal{B}$.) Let $|| \cdot ||_n$ and $|| \cdot ||_m$ be the Euclidean norms in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $d_1(\alpha, \alpha') = ||\alpha - \alpha'||_n + ||x(\alpha; \beta) - x(\alpha'; \beta)||_m$. Then $||x(\alpha; \beta) - x(\alpha'; \beta)||_m < d_1(\alpha, \alpha')$, so $x(\cdot; \beta)$ is Lipschitz continuous. The metric d_1 is equivalent to the Euclidean metric (Aliprantis and Border [1], Lemma 3.12). Also, any Lipschitz continuous function under the Euclidean metric in $\mathbb{R}^n$ (as the density f) is also Lipschitz continuous under d_1. In fact, $|f(\alpha) - f(\alpha')| \leq c||\alpha - \alpha'||_n = cd_1(\alpha, \alpha') - c||x(\alpha; \beta) - x(\alpha'; \beta)||_m \leq cd_1(\alpha, \alpha')$, for some real number c.

Now, by way of contradiction, assume that $\mathcal{L}^m(\overline{\Omega}_{0,\beta}) = 0$ for all β in some open set $V \subset \mathcal{B}$. For any natural number k, let $\pi_{k,\beta}$ be the profit obtained by selling to the types in

$$\overline{\Omega}_{k,\beta} = \{\alpha \in \overline{\Omega}_\beta : s(\alpha; \beta) \leq \frac{1}{k}\}.$$

Because $c(\cdot)$ is non-negative, we must have

$$\pi_{k,\beta} \leq \int_{\overline{\Omega}_{k,\beta}} t(x(\alpha; \beta))f(\alpha)d\alpha,$$

and from (2.5) we have

$$\pi_{k,\beta} \leq \int_{\overline{\Omega}_{k,\beta}} u(\alpha, x(\alpha; \beta))f(\alpha)d\alpha.$$

Assumption 2.2 and the envelope condition (2.7) (with $\mathcal{L}^m(\overline{\Omega}_{0,\beta}) = 0$, we have $\mathcal{L}^m(\overline{\Omega}_{k,\beta}) = \mathcal{L}^m(\overline{\Omega}_{k,\beta} \backslash \overline{\Omega}_{0,\beta})$, so the envelope condition holds for almost all types in $\overline{\Omega}_{k,\beta}$) yield

$$\pi_{k,\beta} \leq K \int_{\overline{\Omega}_{k,\beta}} (\alpha \cdot \nabla s(\alpha; \beta))\lambda(\alpha; \beta)f(\alpha)d\alpha.$$

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Applying the Generalized Gauss-Green Theorem to the Lipschitz continuous vector field $\varphi(\alpha) = \alpha s(\alpha; \beta)\lambda(\alpha; \beta)f(\alpha)$ we get

$$\pi_{k,\beta} \leq K \int_{U_\beta} s(\alpha; \beta)\lambda(\alpha; \beta)f(\alpha)(\alpha \cdot \tau_\Omega(\alpha))d\mathcal{H}^{m-1}(\alpha)$$
$$- K \int_{\overline{\Omega}_{k,\beta}} s(\alpha; \beta)\mathrm{div}(\alpha\lambda(\alpha; \beta)f(\alpha))d\alpha.$$

The functions $s(\alpha; \beta)$, $\lambda(\alpha; \beta)$, $f(\alpha)$, $(\alpha \cdot \tau_\Omega(\alpha))$ and $\mathrm{div}(\alpha\lambda(\alpha; \beta)f(\alpha; \beta))$ are bounded in $\overline{\Omega}_{k,\beta}$, so we can find a common upper bound B. Because $s(\alpha; \beta) \leq \frac{1}{k}$ in $\overline{\Omega}_{k,\beta}$ and $\mathrm{supp}(f) = \overline{\Omega}_\beta$, we have

$$\pi_{k,\beta} \leq \frac{1}{k}B(\mathcal{H}^{m-1}(\overline{\Omega}_{k,\beta}) + \mathcal{L}^m(\overline{\Omega}_{k,\beta})).$$

Now consider increasing the tariff by $\frac{1}{k}$. The consumers in the set $\overline{\Omega}_{k,\beta}$ will exit, and $\pi_{k,\beta}$ will be lost, but each other consumer will pay $\frac{1}{k}$ more. Because the total number of consumers that exit is bounded by $B\mathcal{L}^m(\overline{\Omega}_{k,\beta})$, the change in profit is at least

$$\Delta\pi_\beta \geq \frac{1}{k}[(1 - B\mathcal{L}^m(\overline{\Omega}_{k,\beta})) - B(\mathcal{H}^{m-1}(\overline{\Omega}_{k,\beta}) + \mathcal{L}^m(\overline{\Omega}_{k,\beta}))].$$

From Lemma 2.6, for almost all $\beta \in V$ we have $\mathcal{H}^{m-1}(\overline{\Omega}_{0,\beta}) = 0$, and hence from Lemma 2.4 we have $\mathcal{L}^m(\overline{\Omega}_{k,\beta}) \to 0$ and $\mathcal{H}^{m-1}(\overline{\Omega}_{k,\beta}) \to 0$ for almost all $\beta \in V$, because, by continuity of $s(\cdot; \beta)$ and the compact support of $f(\cdot)$, each $\overline{\Omega}_{k,\beta}$ is compact. But then for large k, $\Delta\pi_\beta$ must be positive for almost all $\beta \in V$, contradicting the optimality of the tariff in each such model. Therefore, we have $\mathcal{L}^m(\overline{\Omega}_{0,\beta}) > 0$ in a generic set in $\mathcal{B}$. $\qquad\square$

Theorem 2.7 shows that, for a generic $\beta \in \mathcal{B}$, the set $\overline{\Omega}_{0,\beta} = \{\alpha \in \overline{\Omega} : s(\alpha; \beta) = 0\}$ has positive measure. Strictly speaking, $\overline{\Omega}_{0,\beta}$ is the set of consumers who have zero surplus, so it is not necessarily the case that a positive measure of consumers will in fact be excluded. That is, consumers with $v(\alpha, x(\alpha; \beta), t(x(\alpha; \beta))) = s_0(\alpha; \beta)$ have $x(\alpha; \beta) \neq x_0(\alpha; \beta)$, so obtain zero surplus without being excluded. But Assumption 2.1 ensures such consumers represent a zero measure subset of $\Omega_{0,\beta}$ as we now verify.

Corollary 2.8. *Under the assumptions of Theorem 2.7 a positive measure of consumers will be excluded at the solution for almost all $\beta \in \mathcal{B}$.*

Proof. Pick $\alpha \in \overline{\Omega}_{0,\beta}$ and say that there is $\hat{\alpha} \in \overline{\Omega}_{0,\beta}$ with $\hat{\alpha} \leq \alpha$ and $x(\hat{\alpha}; \beta) \neq x_0(\hat{\alpha}; \beta)$. Then $u(\hat{\alpha}, x(\hat{\alpha}; \beta)) = t(x(\hat{\alpha}; \beta))$ because $v(\hat{\alpha}, x(\hat{\alpha}; \beta), t(x(\hat{\alpha}; \beta))) = s_0(\hat{\alpha}; \beta)$ and by Assumption 2.1,

$$u(\alpha, x(\hat{\alpha}; \beta)) > u(\hat{\alpha}, x(\hat{\alpha}; \beta))$$

and, because v is strictly decreasing in t,

$$s_0(\alpha; \beta) = v(\alpha, x(\hat{\alpha}; \beta), u(\alpha, x(\hat{\alpha}; \beta))) < v(\alpha, x(\hat{\alpha}; \beta), t\left(x(\hat{\alpha}; \beta)\right))$$

contradicting the optimality of $x(\alpha; \beta)$ for type α. Therefore we must have $x(\hat{\alpha}; \beta) = x_0(\hat{\alpha}; \beta)$ for all $\hat{\alpha} \leq \alpha$. The same argument shows that if $\alpha \in \overline{\Omega}_{0,\beta}$ and $x(\alpha; \beta) \neq x_0(\alpha; \beta)$, then $\hat{\alpha} \notin \overline{\Omega}_{0,\beta}$ whenever $\hat{\alpha} \geq \alpha$. So for any pair $(\alpha, \hat{\alpha})$ in $\overline{\Omega}_{0,\beta}$ with $x(\alpha; \beta) \neq x_0(\alpha; \beta)$ and $x(\hat{\alpha}; \beta) \neq x_0(\hat{\alpha}; \beta)$ we must have $\alpha \not\leq \hat{\alpha}$ and $\hat{\alpha} \not\leq \alpha$. Let $D \subset \overline{\Omega}_{0,\beta}$ be the set of such points. By what we just verified, for any countable open covering of D made up of cubes there exists another countable open covering made up of cubes whose sum of volumes is strictly smaller than the previous sum of volumes. If follows that the Lebesgue measure of D must be zero. $\qquad\square$

As a consequence, the set of excluded consumers has positive measure.

Let us note that it is standard in the literature to work with a quasilinear framework, where $v(\alpha, x, t) = \upsilon(\alpha, x) - t$ and the net utility is $u(\alpha, x) = \upsilon(\alpha, x) - s_0(\alpha)$. Also, sometimes $s_0(\alpha)$ is assumed to be equal to zero for every α. In fact, this is the setting used by Armstrong [2]. In his setting, Assumptions 2.1 and 2.2 are implied by his assumption that υ is strictly increasing and homogeneous of degree 1 in α, and Assumption 2.3 is implied by his assumption that Ω is strictly convex and υ is strictly convex in α. Clearly, Assumptions 2.1-2.3 are substantially weaker than the standard assumptions in the literature.

3 The Robustness of Exclusion in an Oligopolistic Screening Model

We now extend the framework of Section 2 to the case of a market served by $L > 1$ firms. For simplicity, we assume quasilinearity of the consumer's utility function. The production cost is identical among the firms. The firms simultaneously choose non-linear tariffs, and obtain profits after the consumers make their choices. Consumers choose their optimal bundle after observing the choices of the firms. Consumers' choices may well involve buying goods produced by several firms. A pure strategy of firm ℓ is a non-linear tariff, i.e. a mapping $t^\ell : \mathbb{R}^n_+ \to \mathbb{R}$. Consider a symmetric pure strategy Nash equilibrium at which all firms charge the same tariff. We will argue that at such an equilibrium a positive measure of the consumers is not served.

Firm ℓ's problem is to pick $t^\ell \in T^\ell$, where T^ℓ is the space of allowed tariffs, that solves

$$\max_{t^\ell \in T^\ell} \int_\Omega (t^\ell(x^\ell(\alpha)) - c \cdot x^\ell(\alpha))f(\alpha)d\alpha,$$

subject to:

$$\begin{cases} x(\alpha) \in \arg\max_{x \geq 0} v(\alpha, x) - t(x) & \text{if } \max_{x \geq 0} v(\alpha, x) - t(x) \geq s_{0,\ell}(\alpha) \\ x(\alpha) = 0 & \text{otherwise} \end{cases}$$

where

$$\begin{cases} t(x) = \min \sum_j t^j(x^j) \\ s.t. \sum_j x^j = x, \ x^j \geq 0 \end{cases}, \tag{3.1}$$

and

$$s_{0,\ell}(\alpha) = \max\{s_0^*(\alpha), \max_{x \geq 0, x^\ell = 0} (v(\alpha, x) - t^{-\ell}(x))\} \tag{3.2}$$

and $t^{-\ell}(x)$ solves problem (3.1) subject to the additional constraint $x^\ell = 0$. Equation (3.2) states that the outside option of a consumer seen from the point of view of firm ℓ is determined either by her best opportunity outside the market, $s_0^*(\alpha)$, or by the best bundle she may purchase from the competitors, $\max_{x \geq 0, x^\ell = 0}(v(\alpha, x) - t^{-\ell}(x))$.

Define

$$u(\alpha, x^\ell) = v(\alpha, x^\ell + \sum_{j \neq \ell} x^j(\alpha)) - \sum_{j \neq \ell} t^j(x^j(\alpha)) - s_{0,\ell}(\alpha),$$

where $x^j(\alpha)$ is the equilibrium quantity purchased by the consumer of type α from firm j and $s_{0,\ell}(\alpha)$ is given by (3.2). Then firm ℓ's problem becomes:

$$\max_{t^\ell \in T^\ell} \int_\Omega (t^\ell(x^\ell(\alpha)) - c \cdot x^\ell(\alpha))f(\alpha)d\alpha,$$

subject to:

$$\begin{cases} x^\ell(\alpha) \in \arg\max_{x \geq 0} u(\alpha, x^\ell) - t^\ell(x^\ell) & \text{if} \max_{x \geq 0} u(\alpha, x^\ell) - t^\ell(x^\ell) \geq 0 \\ x^\ell(\alpha) = 0 & \text{otherwise} \end{cases}$$

Now $u(\alpha, x^\ell)$ is endogenous and we cannot impose Assumptions 2.1 and 2.2 on it. Instead, we use

Assumption 3.1. $\frac{\partial^2 v}{\partial \alpha_i \partial x_j} > 0$ *for every* $i = 1, ..., m$, $j = 1, ..., n$ *and* $x \neq x_0(\alpha)$.

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Under Assumption 3.1 $u(\alpha, x^\ell)$ is strictly increasing in α for all $x^\ell \in \mathbb{R}_+^n$. In fact

$$
\begin{aligned}
\frac{\partial u(\alpha, x^\ell)}{\partial \alpha_i} &= \left[\frac{\partial v(\alpha, x)}{\partial \alpha_i} - \frac{\partial s_{0,\ell}(\alpha)}{\partial \alpha_i}\right] + \sum_{j \neq \ell}\sum_l \left(\frac{\partial v}{\partial x_l^j} - \frac{\partial t^j}{\partial x_l^j}\right)\frac{\partial x_l^j}{\partial \alpha_i} \\
&= \left[\frac{\partial v(\alpha, x)}{\partial \alpha_i} - \frac{\partial s_{0,\ell}(\alpha)}{\partial \alpha_i}\right]
\end{aligned}
$$

because the consumer chooses optimally from the other firms. When the relevant alternative from buying from firm ℓ is to buy from other firms, we have

$$
\frac{\partial v(\alpha, x)}{\partial \alpha_i} = \frac{\partial v(\alpha, x^\ell + \sum_{j \neq \ell} x^j(\alpha))}{\partial \alpha_i},
$$

while

$$
\frac{\partial s_{0,\ell}(\alpha)}{\partial \alpha_i} = \frac{\partial v(\alpha, 0 + \sum_{j \neq \ell} x^j(\alpha))}{\partial \alpha_i},
$$

where $x^j(\alpha)$ is the optimal quantities purchased from other firms. So Assumption 3.1 ensures that $\frac{\partial u(\alpha, x^\ell)}{\partial \alpha_i} > 0$ for every $i = 1, ..., n$.

Assumption 3.2. *For each $\ell = 1, ..., L$, $\frac{\partial v(\alpha, x)}{\partial \alpha_i} - \frac{\partial s_{0,\ell}(\alpha)}{\partial \alpha_i}$ is bounded away from zero for all x with $x^\ell \neq 0$.*

Under Assumption 3.2, there exists $B > 0$ such that $u(\alpha, x^\ell) \leq B(\alpha \cdot \nabla_\alpha u(\alpha, x^\ell))$ for every $(\alpha, x) \in \overline{\Omega} \times \mathbb{R}_+^n$ for each $\ell = 1, ..., L$.

Theorem 3.3. *Consider a symmetric equilibrium of the oligopoly game described above, and assume that the resulting allocation for the entire industry is continuous. Then, under Assumptions 2.1, 3.2 and 2.3, the set of consumers with zero surplus has positive measure for almost every type space.*

Proof. Note first that we need only $s(\alpha)$ Lipschitz continuous: under quasilinearity, we need not worry about continuity of λ as we did in Theorem 2.7, since λ is always equal to 1. Because the allocation for the entire industry is continuous, as in Theorem 2.7, the same argument will now yield a Lipschitz continuous $s(\alpha)$. Now consider firm 1. Given the behavior of the competitors, its problem is isomorphic to the problem of a single firm with monopoly power, as described above. Therefore, Theorem 2.7 implies that firm 1 will optimally exclude a positive measure of consumers, for almost every type space. By symmetry, so will every other firm. Finally, by symmetry again, the set of excluded consumers is the same for all firms, so the intersection of the sets of excluded consumers has positive measure. $\square$

Observe that the kind of competition in nonlinear tariffs described above is not necessarily of the undercutting nature of Bertrand price competition. For instance, say that two firms offer the same tariff, and that a type α optimally buys half of her bundle from each firm (because the cost of the total bundle is larger than two times the cost of half of the bundle). Then one firm can increase its profits by charging a bit more for half of the bundle, in such a way that the consumer, while buying a larger fraction from the other firm, will still buy close to half of the bundle from the given firm, but the cost of that smaller fraction may well be significantly smaller than the cost of half of the bundle. If competition was simply done by undercutting, then the result in Theorem 3.3 would not be valid, for firms would always want to serve the entire market.

Let us now assume that the number of producers is not fixed but there is a positive entry cost $F > 0$. It is easy to see that this problem can be reduced to the previous one, since equilibrium number of the producers is always finite. Indeed, with K producers the profits of an oligopolist in a symmetric equilibrium are bounded by π^m/K, where π^m are the profits of a monopolist. Therefore, at equilibrium $K \leq \pi^m/F$ and a positive measure of the consumers will be excluded from the market.

3.1 Existence of Equilibrium in the Oligopolgy Game

Theorem 3.3 is derived under the assumption that a symmetric Nash equilibrium exists for the game played by the firms. Champsuar and Rochet [10] note that the profit functions of the firms might be discontinuous when there are bunching regions. Even though Basov [7] shows that bunching in the multidimensional case is not as typical as suggested by Rochet and Chone [21], existence of an equilibrium has to be established. That's what we do next.

Assume that the space T of allowed tariffs is the space of all bounded monotonic functions from X to $[0, M]$, where $X \subset \mathbb{R}^n_+$ is a compact subset of feasible bundles and M is a bound on the net utility function, hence it is also a bound on the tariffs. The space T is the common strategy space of each producer $\ell = 1, ..., L$. By Helly's theorem, every sequence of tariffs in T has a pointwise convergent subsequence, so T is compact in the topology of a.e. pointwise convergence (where a.e. refers to the Lebesgue measure $\mathcal{L}^m$.) Let $\Delta(T)$ denote the space of Borel probability measures on T, endowed with the weak* topology, so it is a compact, convex space.

Assume that when firms choose a symmetric profile $(t, ..., t)$ of tariffs, they obtain the

same expected profit: $\pi^\ell(t, ..., t) = \pi(t, ..., t)$ for $\ell = 1, ..., L$.[7] Hence the one-shot game $(T \times \cdots \times T, \pi, ..., \pi)$ played by the firms is symmetric, and so is its mixed extension, where firms choose $\sigma \in \Delta(T)$ and payoffs are extended to mixtures by taking expectations. For ease of notation, let $(\hat{\sigma}, \sigma)$ denote the profile $(\sigma, ..., \hat{\sigma}, ..., \sigma)$ of strategies where one firm chooses $\hat{\sigma}$ and the others all choose σ. Use $\pi(\hat{\sigma}, \sigma)$ to denote the expected profit of the firm choosing $\hat{\sigma}$.

Proposition 3.4. *The compact, convex and symmetric game described above has a symmetric mixed strategy Nash equilibrium.*

Proof. We show that the game is diagonally better reply secure (Reny [19]). Let (σ, σ) be a non equilibrium profile, and consider $\pi^* = \lim \pi(\sigma^n, \sigma^n)$ for some sequence with $\sigma^n \to \sigma$. For any $\varepsilon > 0$, there exists a strictly increasing t^ε with $\pi(t^\varepsilon, \sigma) > \pi(\sigma, \sigma)$, as (σ, σ) is not an equilibrium. Because t^ε is strictly increasing, $\pi(t^\varepsilon, \cdot)$ is continuous at σ. If $\pi^* = \pi(\sigma, \sigma)$, then diagonal better reply security is verified. If not, then we have discontinuities at (σ, σ). Along any sequence σ^n converging to σ, there is at least one firm whose profit drops at the limit, and this firm can obtain a profit strictly higher than π^* by using t^ε instead of σ_n, for large n. Hence diagonal better reply security is again verified due to continuity of $\pi(t^\varepsilon, \cdot)$. $\square$

Observe that Theorem 3.3 remains valid at a symmetric mixed strategy equilibrium. As long as the surplus function is Lipschitz continuous, the formulation of the oligopoly game allows us to ascertain that a small increase in every t in the support of σ will be profitable if the measure of excluded types is not positive.

4   Examples and Applications

Let us begin with some examples illustrating Assumptions 2.1, 2.2 and 2.3.

Example 4.1. *Consider a consumer who lives for two periods. Her wealth in the first period is w and in the second period her wealth can take two values, w_H or w_L. Let p be the probability that $w = w_H$, and let $\delta \in (0, 1)$ be the discount factor, so that the private information of the consumer is characterized by a two-dimensional vector $\alpha = (1 - p, 1 - \delta)$.*

---

[7]Either because the consumer chooses optimally to buy a fraction $\frac{1}{L}$ of the optimal bundle from each firm, or because she visits each firm with probability $\frac{1}{L}$, depending on the shape of the commonly offered non-linear tariff t.

The consumer's preferences are given by:

$$V(c_1, c_2) = v(c_1) + \delta E v(c_2)$$

where c_1 and c_2 are the consumption levels in periods 1 and 2 respectively, and $v(\cdot)$ is increasing with v' bounded away from zero. Assume that wealth is not storable between periods. Instead, the consumer can borrow x from a bank in period 1, and repay t in period 2 if $w = w_H$, and to defaut if $w = w_L$ in period 2. If the consumer does not borrow, her expected utility will be:

$$s_0(\alpha) = v(w) + \delta(pv(w_H) + (1-p)v(w_L))$$

which is the type dependent outside option. If she borrows x and repays t, the expected utility will be

$$v(\alpha, x, t) = v(w + x) + \delta(pv(w_H - t) + (1-p)v(w_L))$$

which is strictly increasing in x and strictly decreasing in t. Let $\Omega_1 = (0,1)^2$ be the type space, with boundary captured by $g_i(\alpha, \beta_1)$, $i = 1, ..., 4$, with $\nabla_\alpha g_i(\alpha, \beta_1) = (0,1)$ for $i = 1, 2$ and $\nabla_\alpha g_i(\alpha, \beta_1) = (1,0)$ for $i = 3, 4$. And let Ω_2 be another type space, included in the underlying space of type spaces, with boundary given by $g_i(\alpha, \beta_2)$, $i = 1, ..., 4$, with $\nabla_\alpha g_i(\alpha, \beta_2) = (\varepsilon, 1)$ for $i = 1, 2$ and $\nabla_\alpha g_i(\alpha, \beta_2) = (1, \varepsilon)$ for $i = 3, 4$, for some small $\varepsilon > 0$. Then Assumption 2.3 is clearly met.

As

$$\nabla_\alpha u(\alpha, x) = \left(\frac{\Delta v}{pv'}, \frac{\Delta v}{\delta v'} \right)$$

where $\Delta v = v(w_H) - v(w_H - u(\alpha, x)) > 0$, Assumptions 2.1 and 2.2 are met. And note that the use of Assumption 2.3 in Lemma 2.6 is that it ensures the existence of a model, say β_1, with

$$rank \begin{pmatrix} \nabla_\alpha u(\alpha, x) \\ \nabla_\alpha g_i(\alpha, \beta_1) \end{pmatrix} = rank \begin{pmatrix} \frac{\Delta v}{pv'} & \frac{\Delta v}{\delta v'} \\ 0 & 1 \end{pmatrix} = 2, \text{ for } i = 1, 2$$

$$rank \begin{pmatrix} \nabla_\alpha u(\alpha, x) \\ \nabla_\alpha g_j(\alpha, \beta_1) \end{pmatrix} = rank \begin{pmatrix} \frac{\Delta v}{pv'} & \frac{\Delta v}{\delta v'} \\ 1 & 0 \end{pmatrix} = 2, \text{ for } i = 3, 4.$$

Example 4.1 is a natural setting to discuss unavailability of credit to some individuals, which is important to justify monetary equilibria in the search theoretic models of money.[8] The next example comes from the theory of industrial organization.

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[8]See, for example, Lagos and Wright [16].

Example 4.2. *Suppose a monopolist produces cars of high quality. The utility of a consumer is quasilinear, $v(\alpha, x, t) = v(\alpha, x) - t$, with*

$$v(\alpha, x) = A + \sum_{i=1}^{n} \alpha_i x_i \qquad (4.1)$$

where $A > 0$ can be interpreted as utility of driving a car, and the second term in (4.1) is a quality premium. Suppose a consumer has three choices: to buy a car from the monopolist, to buy a car from a competitive fringe, and to buy no car at all. We will normalize the utility of buying no car at all to be zero. Assume the competitive fringe serves cars of quality $-x_0$, where $x_0 \in \mathbb{R}^n_{++}$ at price p. That is, the consumers experience disutility from the quality of the cars of the competitive fringe, and the higher their type, the higher the disutility. The utility of the outside option in this case is given by:

$$s_0(\alpha) = \max(0, A - p - \sum_{i=1}^{n} \alpha_i x_{0i})$$

and is decreasing in α. Therefore, Assumptions 2.1 and 2.2 hold, because in a quasilinear setting the net utility is $u(\alpha, x) = v(\alpha, x) - s_0(\alpha)$. As for Assumption 2.3, say that we start off with Ω being the unit square in $\mathbb{R}^n$. Parametrize each of the edges, and consider models obtained by small perturbations of the parameters, hence the edges (like for instance small rotations of the unit square). When such type spaces are included in $\mathcal{B}$, Assumption 2.3 is met.

Now let us turn to models that do not satisfy Assumptions 2.1, 2.2 and 2.3. First, consider any model that yields an excluded set Ω_0 with positive measure, and modify the problem considering only the types in $\Omega \backslash \Omega'$, where $\Omega_0 \subset \Omega'$. Would the modified problem have no exclusion? Though this will indeed be the case if $\Omega' = \Omega_0$,[9] it will not hold for a generic superset Ω'. This would only be the case if the shape of Ω_0 stood in a tight relation with the shape of Ω', a non generic situation. That is, even if Ω_0 stood in the particular tight relation with Ω', a slight change in the boundary structure of Ω' would suffice for us to have exclusion in the modified model.

In the same vein, Rochet and Stole [23] provided an example where the exclusion set is empty.[10] In their quasilinear example $v : \Omega \times R_+ \to \mathbb{R}$ has the form

$$v(\alpha, x) = (\alpha_1 + \alpha_2)x$$

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[9] We are grateful to an anonymous referee for this observation.

[10] Another example along similar lines is provided by Deneckere and Severinov [12]. Though it is a bit more intricate and the authors provide sufficient conditions that ensure full participation in the case of one quality dimension and two-dimensional characteristics, their condition also does not hold generically.

and Ω is a rectangle with sides parallel to the 45 degrees and -45 degrees lines. They argued that one can shift the rectangle sufficiently far to the right to have an empty exclusion region. Their result is driven by the fact that they allow only very special collection of type spaces, rectangles with parallel sides. Formally, the model used in this case cannot be used in Lemma 2.6 because $\nabla_\alpha g_i(\alpha, \beta) = (1, 1)$ for $i = 1, 3$ and $\nabla_\alpha g_i(\alpha, \beta) = (-1, 1)$ for $i = 2, 4$, so that (using $u(\alpha, x) = v(\alpha, x)$ because $s_0(\alpha) = 0$)

$$\left(\begin{array}{c} \nabla_\alpha u(\alpha, x) \\ \nabla_\alpha g_i(\alpha, \beta) \end{array} \right) = \left(\begin{array}{cc} x & x \\ 1 & 1 \end{array} \right) \Rightarrow rank \left(\begin{array}{c} \nabla_\alpha u(\alpha, x) \\ \nabla_\alpha g_i(\alpha, \beta) \end{array} \right) = 1$$

for $i = 1, 3$.

Observe that a very small change in the type set changes that result. Consider, for example, a slightly perturbed type space, with $\nabla_\alpha g_i(\alpha, \beta_0) = (1, 1 + \varepsilon)$, for $i = 1, 3$, where ε is a small positive real number. Then, for all $x \neq 0$ and $i = 1, ..., 4$

$$rank \left(\begin{array}{c} \nabla_\alpha u(\alpha, x) \\ \nabla_\alpha g_i(\alpha, \beta_0) \end{array} \right) = 2$$

as required in Lemma 2.6.

We stress that our result does not guarantee a non-empty exclusion region for every multi-dimensional screening problem. Rather, it asserts that problems for which the exclusion region is empty can be slightly perturbed and transformed into problems with a positive measure of excluded consumers. To understand the results intuitively, assume first that, in equilibrium, all consumers are served. First, note that at least one consumer should be indifferent between participating and not participating, since otherwise the tariffs can be uniformly increased for everyone by a small amount, increasing the monopolist's profits. Now, consider increasing the tariff by $\varepsilon > 0$. The consumers who used to obtain surplus below ε will drop out. The measure of such consumers is $o(\varepsilon)$, unless the iso-surplus hyper-surfaces happen to be parallel to the boundary of the type space. Under Assumption 2.3, there will be a model where the iso-surplus hyper-surfaces will not only not be parallel to the boundary of Ω, it will be transversal. Such situations may still occur endogenously, which is the reason why our result holds for *almost* all, rather then for all, screening problems. One class of problems, for which full participation may occur are models with random outside options. They were first considered by Rochet and Stole [22] for both monopolistic and oligopolistic settings and generalized by Basov and Yin [8] for the case of risk averse principal(s). Armstrong and Vickers [5] considered another generalization, allowing for multidimensional vertical types. In this type of models, the type consists of a vector of vertical

characteristics, $\alpha \in \Omega \subset \mathbb{R}^m$, and a parameter $\gamma \in [0,1]$ capturing horizontal preferences. The type space is given by the Cartesian product $\Omega \times [0,1]$ and γ is assumed to be distributed independently of α. The utility of a consumer is given by:

$$v(\alpha, x; \gamma) = v(\alpha, x) - t\gamma,$$

where t is a commonly known parameter. Let $v(\alpha, 0) = 0$ so that the iso-surplus hypersurface corresponding to zero quality is $t\gamma = $ constant, which is parallel to the vertical boundary of type space, $\gamma = 0$. Therefore, in such model there is the possibility of full participation. The model was also investigated in an oligopolistic setting, where t was interpreted as a transportation cost for the Hotelling model. Conditions for full participation under different assumptions on the dimensionality of α and the monopolist's risk preferences were obtained by Armstrong and Vickers [5], Rochet and Stole [22], and Basov and Yin [8]. Let us assume that the boundary of set Ω is described by the equation

$$g_0(\alpha) = 0$$

and embed our problem into a family of problems, for which boundary of the type space is described by the equation

$$g(\alpha, \gamma; \beta) = 0,$$

where $g(\cdot, \beta) : \Omega \times [0,1] \to \mathbb{R}$ is a smooth function with

$$g(\alpha, \gamma; 0) = g_0(\alpha)(g_0(\alpha) - b)\gamma(\gamma - 1),$$

for some constant b. For instance, when $\beta = 0$ the type space becomes the cylinder over the set Ω considered by Armstrong and Vickers [5]. Our result is that for almost all β the exclusion region is non-empty. However, as we saw above, for $\beta = 0$ exclusion region may be empty.

We now consider another class of models, where full participation is possible. The example will also be interesting, since it will allow us to investigate how the relative measure of excluded consumers changes with the dimension of Ω.

Example 4.3. *Let consumer's preferences be given be quasilinear with:*

$$v(\alpha, x) = \sum_{i=1}^{n} \alpha_i \sqrt{x_i},$$

and the monopolist's cost be given by

$$c(x) = \frac{1}{2} \sum_{i=1}^{n} x_i.$$

The type space is intersection of the region between balls with radii a and $a + 1$ with $\mathbb{R}^n_+$, i.e.

$$\Omega = \{\alpha \in R^n_+ : a \leq \|\alpha\| \leq a + 1\}, \tag{4.2}$$

where $\|\cdot\|$ denotes the Euclidean norm

$$\|\beta\| = \sqrt{\sum_{i=1}^n \beta_i^2}.$$

To solve for the optimal nonlinear tariff with a fixed number of characteristics, consider the consumer surplus:

$$s(\alpha) = \max_x (\sum_{i=1}^n \alpha_i \sqrt{x_i} - t(x)).$$

By symmetry, we look for a solution of the form

$$s = s(\|\alpha\|)$$

In the "separation region"' it solves

$$\begin{cases} \frac{1}{r^{n-1}} \frac{\partial}{\partial r}(r^{n-1} s'(r)) + \frac{s'(r) f'(r)}{f(r)} = n + 1 + \frac{r f'(r)}{f(r)} \\ s'(a+1) = a + 1 \end{cases} \tag{4.3}$$

where $r \equiv \|\alpha\|$ and we used the envelope theorem to obtain

$$\sqrt{x_i} = \frac{\partial s}{\partial \alpha_i}.$$

The monopolist's problem can now be written as

$$\max_s \int [\alpha \cdot \nabla s(\alpha) - c(\nabla s(\alpha)) - s(\alpha)] d\alpha$$

$$s.t. \ s(\cdot)\text{-convex}, \ s \geq 0.$$

(see Rochet and Chone [21]. Ignoring for now the convexity constraint, we have a standard calculus of variations problem with free boundary. Therefore, in the participation region (i.e., the points with $s > 0$) we have:

$$\sum_{i=1}^n \frac{\partial}{\partial \alpha_i} \frac{\partial L}{\partial s_i} = \frac{\partial L}{\partial s}$$

$$\sum_{i=1}^n \alpha_i \frac{\partial L}{\partial s_i} = 0 \tag{4.4}$$

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(see Basov [7]), where s_i denotes the ith partial derivative of s and

$$L = \alpha \cdot \nabla s(\alpha) - c(\nabla s(\alpha)) - s(\alpha)$$

Observe that this is exactly the system (4.3). Assume that types are distributed uniformly on Ω, so the derivative of the type distribution vanishes. Then, solving (4.3) we get:

$$x_i(\alpha) = [\max(0, \frac{\alpha_i}{n}(n + 1 - (\frac{a + 1}{r})^n))]^2.$$

The corresponding iso-surplus hyper-surfaces are given by the intersection of a sphere of appropriate dimension with $\mathbb{R}^n_+$. They are parallel to the boundary, hence it is possible that the exclusion region is empty. Note that the exclusion region is given by

$$\Omega_0 = \{\alpha \in \Omega : \|\alpha\| \leq \frac{a + 1}{\sqrt[n]{1 + n}}\},$$

so it is non-empty if

$$\frac{a + 1}{\sqrt[n]{1 + n}} > a.$$

Observe that if $n = 1$ the exclusion region is empty if and only if $a > 1$, if $n = 2$ it is empty if and only if $a > 1/(\sqrt{3} - 1) \approx 1.36$, and since

$$\lim_{n \to \infty} \frac{1}{\sqrt[n]{1 + n}} = 1,$$

the exclusion region is non-empty for any $a > 0$ for sufficiently large n. The relative measure of the excluded consumer's (the measure of excluded consumers if we normalize the total measure of consumers to be one for all n) is:

$$\zeta = \frac{(a + 1)^n/(n + 1) - a^n}{(a + 1)^n - a^n}.$$

It is easy to see that as $n \to \infty$ the measure of excluded consumers converges to zero as $1/n$ goes to zero, i.e. as exclusion becomes asymptotically less important. This accords with results obtained by Armstrong [3]. The convergence, however, is not monotone. For example, if $a = 1.3$ the measure of excluded customers first rises from zero for $n = 1$ to 11.6% for $n = 5$, and falls slowly thereafter. For $a = 2$ maximal exclusion of 8.3% obtains when $n = 11$ and for $a = 0.7$ maximal exclusion of 19.7% obtains when $n = 2$.

Also observe that, although an asymptotically higher fraction of consumers gets served as $n \to \infty$, this does not mean that the consumers become better off. Indeed, as $n \to \infty$ the radius of the exclusion region converges to $(a + 1)$. That is, almost all served consumers are located near the upper boundary. This means that the trade-off between the efficient provision of quality and minimization of information rents disappears. Asymptotically, the monopolist provides the efficient quality but is able to appropriate almost the entire surplus.

4.1 An Application to the Regulation of a Monopolist with Unknown Demand and Cost Functions

Armstrong [4] reviews Lewis and Sappington's [17] analysis of optimal regulation of a monopolist firm when the firm's private information is two dimensional. In this analysis, a single product monopolist faces a stochastic demand function given by $q(x) = a + \theta - x$, where x is the product's price, a is a fixed parameter and θ is a stochastic component to demand, taking values in an interval $\left[\underline{\theta}, \overline{\theta}\right] \subset \mathbb{R}_+$. The firm's cost is represented by the function $C(q) = (c_0 - c)q + K$, where q is the quantity produced, c_0 and K are fixed parameters and c is a stochastic component to the cost, taking values in an interval[11] $[-\overline{c}, -\underline{c}] \subset \mathbb{R}_-$. The firm observes both the demand and the cost functions, but the regulator only knows that $\alpha = (\theta, c)$ is distributed according to the strictly positive continuous density function $f(\theta, c)$ on the rectangle $\Omega = \left[\underline{\theta}, \overline{\theta}\right] \times [-\overline{c}, -\underline{c}]$. For the sake of feasibility we assume that $a + \theta > c_0 - c$ for all $\alpha = (\theta, c) \in \Omega$, i.e., the highest demand exceeds marginal costs, for all possible realizations of the stochastic components of demand and costs.

The regulator wants to maximize social welfare and presents to the monopolist a menu of contracts $\{(x, t(x))\}$. If the firm chooses contract $(x, t(x))$ it sells its product at price x and pays a tax $t(x)$ from the regulator.

Therefore, the regulator's problem is to select a continuous subsidy schedule $t(\cdot) : R_+ \to \mathbb{R}$ to solve:

$$\max_{t(\cdot)} \int_{\Omega} (t(x(\alpha)) - c(x(\alpha)))f(\alpha)d\alpha,$$

where $x(\alpha)$ satisfies

$$\begin{cases} x(\alpha) \in \arg\max_{x \geq 0} u(\alpha, x) - t(x) & \text{if } \max_{x \geq 0} u(\alpha, x) - t(x) \geq 0 \\ x(\alpha) = a + \theta & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} u(\alpha, x) &= (a + \theta - x)(x - c_0 + c) - K \\ c(x) &= -\frac{1}{2}(a + \theta - x)^2, \end{aligned}$$

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[11] In the original model $C(q) = (c_0 + c)q + K$ with $c \in [\underline{c}, \overline{c}] \subset \mathbb{R}_+$. We substitute c by its negative for convenience.

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so that "cost" is the negative of the consumer's surplus. The choice of $x(\alpha)$ by the monopolist depends on whether she can derive nonnegative returns when producing. If that is not possible, she will choose $x(\alpha) = a + \theta$ and there will be zero demand, i.e., the firm shuts down.

A fundamental hypothesis in Lewis and Sappington's [17] analysis is that the parameter a can be chosen sufficiently large relative to parameters K and c_0 so that a firm will always find it in its interest to produce, even for the very small values of θ. However, Armstrong [4] shows that such a hypothesis cannot be made when Ω is the square $\Omega = [\underline{\theta}, \overline{\theta}] \times [-\overline{c}, -\underline{c}] = [0, 1] \times [-1, 0]$. Furthermore, when Ω is a strictly convex subset of that square, Armstrong [4] uses the optimality of exclusion theorem in Armstrong [2] to show that some firms will necessarily shut down under the optimal regulatory policy, in equilibrium. Armstrong [4] then adds "... I believe that the condition that the support be convex is *strongly* sufficient and that it will be the usual case that exclusion is optimal, even if a is much larger than the maximum possible marginal cost." That insight could not be pursued further due to a lack of a more general result, and Armstrong [4] switched to a discrete-type model in order to check the robustness of the main conclusions in Lewis and Sappington [17].

Note that the regulator's problem is essentially the standard problem solved in Section 2 of this paper. In order to apply Theorem 2.7, first note that it is sufficient that Assumptions 2.1-2.3 hold at the relevant ranges of the choice variables.

Now notice that $u(\alpha, x)$ is strictly increasing in c, as long as $a + \theta - x > 0$. But this is always the case for $x(\alpha)$, since $a + \theta - x(\alpha)$ is a demand curve. Moreover, $u(\alpha, x)$ is strictly increasing in θ, as long as $x - c_0 + c > 0$. This is again the case for $x(\alpha)$ since this is the difference between price and marginal cost. Therefore, $u(\alpha, x)$ is strictly increasing in α and bounded for the relevant choice of price x. Assumption 2.3 is also met, as long as we include type spaces that are not parallel shifts of $[0, 1] \times [-1, 0]$, which we can clearly do.

In fact, let g be given by:

$$g_1(\alpha, 0) = \theta, g_2(\alpha, 1) = \theta - 1, g_3(\alpha, -1) = c + 1, g_4(\alpha, 0) = c.$$

Then we can define

$$\Sigma_1 = \{\alpha \in \Omega : \ g_1(\alpha, \underline{\theta}) = 0\}$$
$$\Sigma_2 = \{\alpha \in \Omega : \ g_2(\alpha, \overline{\theta}) = 0\}$$
$$\Sigma_3 = \{\alpha \in \Omega : \ g_3(\alpha, -\underline{c}) = 0\}$$
$$\Sigma_4 = \{\alpha \in \Omega : \ g_4(\alpha, -\overline{c}) = 0\}$$

Therefore, the boundary of Ω can be expressed as:

$$\partial\Omega = \overset{4}{\underset{i=1}{\cup}} \Sigma_i.$$

Moreover, the gradient of function u is

$$\nabla_\alpha u(\alpha, x) = (x - c_0 + c, a + \theta - x)$$

and

$$\nabla_\alpha g_i(\alpha, \beta) = (1, 0), i = 1, 2, \nabla_\alpha g_j(\alpha, \beta) = (0, 1), j = 1, 2.$$

Therefore, for $i = 1, 2$ and for $j = 1, 2$,

$$\begin{pmatrix} \nabla_\alpha u(\alpha, x) \\ \nabla_\alpha g_i(\alpha, \beta) \end{pmatrix} = \begin{pmatrix} x - c_0 + c & a + \theta - x \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \nabla_\alpha u(\alpha, x) \\ \nabla_\alpha g_j(\alpha, \beta) \end{pmatrix} = \begin{pmatrix} x - c_0 + c & a + \theta - x \\ 0 & 1 \end{pmatrix}$$

In particular, the rank of these matrices is 2, and we illustrate once again the use of Assumption 2.3 in Lemma 2.6.[12] All the hypothesis of Theorem 2.7 are satisfied, so we may conclude that a set of positive firms will generically be excluded from the regulated market, i.e., will not produce at all. Armstrong's [4] conjecture is therefore confirmed.

4.2 An Application to Involuntary Unemployment

Consider a firm in an industry that produces n goods captured by a vector $x \in \mathbb{R}^n_+$. The firm hires workers to produce these goods. A worker is characterized by the cost she bears in order to produce goods $x \in \mathbb{R}^n_+$, which is given by the effort cost function $e(\alpha, x)$. The parameter $\alpha \in \Omega \subset \mathbb{R}^m$ is the worker's unobservable type distributed on an open, bounded, set $\Omega \subset \mathbb{R}^m$ according to a strictly positive, continuous density function $f(\cdot)$.

Therefore, if a worker of type α is hired to produce output x and receives wage $\omega(x)$, her utility is $\omega(x) - e(\alpha, x)$, where $c(\alpha, \cdot)$ is cost of effort, which depends on the type of the worker. If the worker is not hired by the firm, she will receive a net utility $s_0(\alpha)$, either by working on a different firm, or by receiving unemployment compensation.

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[12] Indeed, it cannot be the case that $x = p = c_0 - c = a + \theta$ since the price cannot be, at the same time, the marginal cost (prefect competitive price) and the price that makes demand vanish.

Suppose the firm sells its products for competitive international prices, $p(x)$. Then, the firm's problem is to select a wage schedule $\omega(\cdot) : R_+^n \to \mathbb{R}$ to solve:

$$\max_{\omega(\cdot)} \int_\Omega \left[p(x(\alpha)) x(\alpha) - \omega(x(\alpha)) \right] f(\alpha) d\alpha$$

where $x(\alpha)$ satisfies

$$\begin{cases} x(\alpha) \in \arg\max_{x \geq 0} \omega(x) - e(\alpha, x) & \text{if } \max_{x \geq 0} \omega(x) - e(\alpha, x) \geq s_0(x) \\ x(\alpha) = 0 & \text{otherwise} \end{cases}$$

Consider the following change in variables: $t(x) = -\omega(x)$, $v(\alpha, x) = -e(\alpha, x)$, $c(x) = -p(x) x$, then the firm's problem can be rewritten as:

$$\max_{t(\cdot)} \int_\Omega (t(x(\alpha)) - c(x(\alpha))) f(\alpha) d\alpha,$$

where $x(\alpha)$ satisfies:

$$\begin{cases} x(\alpha) \in \arg\max_{x \geq 0} v(\alpha, x) - t(x) & \text{if } \max_{x \geq 0}(v(\alpha, x) - t(x)) \geq s_0(x) \\ x(\alpha) = 0 & \text{otherwise} \end{cases}$$

Therefore, the same arguments that have been presented for the monopolist can also be extended for the hiring decision of the firm. In particular, the firm will generically find it optimal not to hire a set of positive measure. If the firm is a monopsonist in the sense that agents can work only at that firm, then Theorem 2.7 provides a rationale for involuntary unemployment. Note that, according to Theorem 3.3, the result can be extended to environments with several firms hiring for the production of goods $x \in \mathbb{R}_+^n$, so that there is an oligopsony for workers, as long as the corresponding industry is the only source of formal work. This is true even in the case of free entry in that industry, according to the comment following Theorem 3.3. Finally, if one includes the category of informal work (underemployment) as unemployment, the present model suggests that an informal sector will generically exist in equilibrium.[13]

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[13]This application is, to the knowledge of the authors, the first explanation of involuntary unemployment based on the adverse selection problem, whereby firms decide to offer a wage schedule that excludes some less productive workers so they can require higher output levels from the more productive ones.

5 Conclusions

We showed that Armstrong's [2] exclusion result holds generically under weak assumptions on the underlying economic model. And in particular it holds beyond the monopoly case. So it is a robust result. Because it applies to a diverse set of markets in the economy, it offers a deep insight into the workings of market economies. In general, outside of the very special cases of perfect competition, complete and perfect information, or one-dimensional private information, we should expect to see exclusion operating in markets. We have explored, in this paper, five diverse settings where we believe this result applies: credit markets, automobiles, research grants, monopoly regulation, and labor markets. Further applications, and further depth on these applications, seem warranted for future research. [14]

A Appendix

The Constant Marginal Costs Assumption

Constant marginal costs can be justified as follows. The firm first produces some quantity Q of goods at cost $C(Q)$, and then distributes it to the consumers. Hence the optimization problem is with respect to $t(\cdot)$ and Q:

$$\max_{t(\cdot),Q} \int_\Omega t(x(\alpha))f(\alpha)d\alpha - C(Q)$$
$$s.t. \int_\Omega x(\alpha)f(\alpha)d\alpha = Q$$
$$\begin{cases} x(\alpha) \in \arg\max_{x \geq 0} v(\alpha, x, t(x)) & \text{if } \max_{x \geq 0} v(\alpha, x, t(x)) \geq s_0(\alpha) \\ x(\alpha) = x_0(\alpha) & \text{otherwise.} \end{cases}$$

The problem can be solved in two steps:

a) Fix Q and solve

$$\max_{t(\cdot)} \int_\Omega t(x(\alpha))f(\alpha)d\alpha$$
$$s.t. \int_\Omega x(\alpha)f(\alpha)d\alpha = Q$$
$$\begin{cases} x(\alpha) \in \arg\max_{x \geq 0} v(\alpha, x, t(x)) & \text{if } \max_{x \geq 0} v(\alpha, x, t(x)) \geq s_0(\alpha) \\ x(\alpha) = x_0(\alpha) & \text{otherwise.} \end{cases}$$

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[14] Another interesting extension is the auction-theoretic setting considered in Monteiro, Svaiter, and Page, [18].

Let $V(Q)$ denote a solution for this problem.

b) Find Q from

$$Q \in \arg\max(V(Q) - C(Q)).$$

Part (b) is a standard multiproduct monopoly problem. To solve part (a) let λ be the Lagrange multiplier on the production constraint. Then the firm solves:

$$\max_{t(\cdot)} \int_\Omega (t(x(\alpha)) - \lambda x(\alpha)) f(\alpha) d\alpha$$

$$s.t. \begin{cases} x(\alpha) \in \arg\max_{x \geq 0} v(\alpha, x, t(x)) & \text{if } \max_{x \geq 0} v(\alpha, x, t(x)) \geq s_0(\alpha) \\ x(\alpha) = x_0(\alpha) & \text{otherwise.} \end{cases}$$

Therefore, part (a) is equivalent to the problem of a monopolist with a fixed marginal cost, λ. Of course, this cost will in general depend on Q, but since the second stage is straightforward, to simplify the issues we simply assume constant marginal cost from the start. In any event, with additively separable costs, linearity can always be obtained by an appropriate change of variables. What is crucial is that the social surplus be strictly convex.

Some Geometric Measure Theory Concepts

Let $U \subset \mathbb{R}^m$ be a domain, i.e. an open, simple connected set. A set $\Omega \subset \mathbb{R}^m$ has a has *finite perimeter* in U if $\Omega \cap U$ is measurable and there exists a finite Borel measure μ on U and a Borel function $v : U \to S^{m-1} \cup \{0\} \subset \mathbb{R}^m$ with

$$\int_\Omega \mathrm{div}\varphi dx = \int_U \varphi \cdot v d\mu$$

for every Lipschitz continuous vector field $\varphi : U \to \mathbb{R}^m$ with compact support, where S^{m-1} is the $m-1$ dimensional unit sphere. The perimeter of Ω in U is defined as:

$$P(\Omega, U) = \sup \int_U \mathrm{div}\varphi dx,$$

where supremum is taken over all Lipschitz continuous vector fields with compact support and such that $\|\varphi\|_{L^\infty} \leq 1$. A set $\Omega \subset \mathbb{R}^m$ is of *locally finite perimeter* if $P(\Omega, V) < \infty$ for every open proper subset of U.

The *measure theoretic boundary* of Ω is given by

$$\partial_e(\Omega) = \{x \in \mathbb{R}^m : 0 < \mathcal{L}^m(\Omega \cap B_\varepsilon(x)) < \mathcal{L}^m(B_\varepsilon(x)), \ \forall \varepsilon > 0\}$$

where $\mathcal{L}^m$ is the $m$dimensional Lebesgue measure and $B_\varepsilon(x)$ is the open ball centered at x with radius $\varepsilon > 0$. When Ω has locally finite perimeter we have $\partial_e\Omega = \bigcup_{i=1}^{\infty} K_i \cup N$, where K_i is a compact subset of a C^1 hypersurface S_i, for $i = 1, 2, ...$, and $\mathcal{H}^{m-1}(N) = 0$ where $\mathcal{H}^{m-1}$ is the $m-1$ dimensional Hausdorff measure, and a C^1 hypersurface $S \subset \mathbb{R}^m$ is a set for which ∂S is the graph of a smooth function near each $x \in \partial S$. The *measure theoretic unit outer normal* $v_\Omega(x)$ of Ω at x is the unique point $u \in S^{m-1}$ such that $\theta^m(O, x) = \theta^m(I, x) = 0$, where $O = \{y \in \Omega : (y - x) \cdot u > 0\}$ and $I = \{y \notin \Omega : (y - x) \cdot u < 0\}$, and $\theta^m(A, x)$ is the m-dimensional density at x. The reduced boundary $\partial^*\Omega$ is the set of points x for which Ω has a measure theoretic unit outer normal at x. For a set of locally finite perimeter Ω the three boundaries $\partial\Omega$, $\partial_e\Omega$ and $\partial^*\Omega$ are up to $\mathcal{H}^{m-1}$ null-sets the same.

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