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Abstract

We extend Armstrong's [2] result on exclusion in multi-dimensional screening models in two key ways, providing support for the view that this result holds true in a large class of models and is applicable to many different markets. First, we relax the strong technical assumptions he imposed on preferences and consumer types. Second, we extend the result beyond the monopolistic market structure to some oligopoly settings. We illustrate the results with several examples and applications.

JEL Codes: C72, D42, D43, D82

Key words: Multi-dimensional screening, exclusion, regulation of a monopoly, involuntary unemployment.

1 Introduction

When considering the problem of screening, where sellers choose a sales mechanism and buyers have private information about their types, it is well known that the techniques used in the multi-dimensional setting are not as straightforward as those in the one-dimensional setting. As a consequence, while we have a host of successful applications with one-dimensional types, to date we have only a few scattered papers that allow for multi-dimensional types.

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This is unfortunate because in many, if not most, economic applications multi-dimensional types are needed to capture the basic economics of the environment, and the propositions coming from the one-dimensional case do not necessarily generalize to the multi-dimensional case.[1]

One intriguing result in the theory of multi-dimensional screening comes from Armstrong [2], who shows that a monopolist will find it optimal to not serve some fraction of consumers, even when there is positive surplus associated with those consumers. That is, in settings where consumers vary in at least two different ways, monopolists will choose a sales mechanism that excludes a positive measure of consumers. The intuition behind this result is rather simple: consider a situation where the monopolist serves all consumers; if she increases the price by $\varepsilon > 0$ she earns extra profits of order $O(\varepsilon)$ on the consumers who still buy the product, but will lose only the consumers whose surplus was below ε. If $m > 1$ is the dimension of the vector of consumers' taste characteristics, then the measure of the set of the lost consumers is $O(\varepsilon^m)$. Therefore, it is profitable to increase the price and lose some consumers. In principle, this result has profound implications across a wide range of economic settings. The general belief that heterogeneity of consumer types is likely to be more than one-dimensional in nature, for many different commodities, and that these types are likely to be private information, underlines the significance of Armstrong's result.[2]

However, the result itself was derived under a relatively strong set of assumptions that could be seen as limiting its applicability, and subsequent research has identified conditions under which the result does not hold. In particular, Armstrong's original analysis assumes that the utility functions of the agents are homogeneous and convex in their types, and that these types belong to a strictly convex and compact body of a finite dimensional space. Basov [7] refers to the latter as the joint convexity assumption and argues that, although convexity of utility in types and convexity of the types set separately are not restrictive and can be seen as a choice of parametrization, the joint convexity assumption is technically restrictive.

The joint convexity assumption has no empirical foundation and is nonstandard. For instance, the benchmark case of independent types fails joint convexity because the type space is the not strictly convex multi-dimensional box. There is, in general, no theoretical justification for a particular assumption about the curvature of utility functions with respect

[1] See Rochet and Stole [24] and Basov [7] for surveys of the literature.

[2] The type of an economic agent is simply her utility function. If one is agnostic about the preferences and does not want to impose on them any assumptions beyond, perhaps, monotonicity and convexity, then the most natural assumption is that the type is multi-dimensional.

to types, as opposed to, say, quasi-concavity of utility functions with respect to goods. In the same line, in general, there is no justification, other than analytical tractability, for type spaces to be convex, and for utility functions to be homogeneous in types. Both Armstrong [4] and Rochet and Stole [24] found examples outside of these restrictions where the exclusion set is empty.

We show that these counter-examples are knife-edge cases. Exclusion is generically optimal for a monopolist in the family of models where utility functions are smooth and monotone in types, and types belong to sets of locally finite perimeter. The class of sets of locally finite perimeter is a class of sets that includes all of the examples the authors are aware of in the literature, and we stress, includes type spaces that are nowhere close to being convex. That is, exclusion is generically optimal in a large class of models.

Once this is established, a natural question to address is whether market power is crucial for the result. That is, whether the result hinges on the seller being a monopolist. We show that a similar exclusion result holds in a symmetric equilibrium in an industry composed of finitely many firms, provided that in such an equilibrium each firm retains some market power. So, yes, the result hinges on market power, but it is compatible with forms of competition that do not mitigate an individual firm's power. We show our result by first formulating the problem of a firm in an oligopoly setting in an analogous way to the problem of a monopolist firm. This is done by including the option of purchasing from other firms as one of the outside options of a consumer, from the perspective of a given firm. Second, we note that the exclusion result in the monopoly setting can be interpreted as follows: the consumers that are excluded are those who do not benefit from the presence of the monopolist, as they are better off sticking to their outside option. In an oligopolist setting, from the perspective of one firm, the excluded consumers are those that do not benefit from the presence of that one firm, as they are better off sticking to their outside option, which may well be to purchase from other firms. We then show that, under the additional assumption of strict supermodularity of utility functions, the exclusion result extends to the oligopoly case where individual firms retain some market power in equilibrium. The particular formulation that we choose has firms choosing their capacities first, and then competing in non-linear tariffs. Other formulations that preserve individual market power would produce similar results. We conclude that in a large class of models, generically a positive set of consumers will be excluded from a given firm in an oligopoly, in a symmetric equilibrium of the game played by the firms.

We illustrate the generality of the results with a few examples and two applications, namely the regulation of a monopolist with unknown demand and cost functions, and the emergence of involuntary unemployment as a result of screening by employers. The former application picks up of the analysis in Armstrong [4], where he reviews Lewis and Sappington [18] and conjectures that exclusion is probably an issue in their analysis. At the time, Armstrong could not prove the point, due to the lack of a more general exclusion result. With our results in hand, we are able to prove Armstrong's conjecture. The latter application is a straightforward way of showing that, when workers have multi-dimensional characteristics, it is generically optimal for employers (with market power in the labor market) to not hire all the workers.

In sum, the paper provides evidence to the proposition that private information leads to exclusion in many realistic settings. To avoid it, one must either assume that all allowable preferences lie on a one-dimensional continuum, or construct very specific type distributions and preferences, or have very strong forms of competition among firms.

The remainder of this paper is organized as follows. In Section 2 we present the monopoly problem with consumers that have a type-dependent outside option and then derive conditions under which it is generically optimal to have exclusion. In Section 3 we generalize the results for the case of oligopoly and a market with free entry. Examples and applications are presented in Section 4. The Appendix presents some relevant concepts from geometric measure theory.

2 Exclusion in a Monopolistic Screening Model

Consider a firm with a monopoly over n goods. The tastes of the consumers over these goods are parametrized by a vector $\alpha \in \mathbb{R}^m$. The utility of a type α consumer consuming a bundle $x \in \mathbb{R}^n_+$ and paying $t \in \mathbb{R}$ to the firm is

$$v(\alpha, x, t)$$

where v is strictly increasing and strictly concave in x, and strictly decreasing in t. Our focus is not on relaxing the smoothness assumptions on v, so we will assume that v is twice continuously differentiable, with $v_t(\alpha, x, t) \equiv \frac{\partial v(\alpha,x,t)}{\partial t}$ Lipschitz continuous and bounded away from zero.

The total cost $c(\cdot)$ of producing bundle x is given by $c(x)$, where $c(\cdot)$ is a convex function (possibly linear). The firm is not able to observe the consumer's type, but has prior beliefs over the distribution of types, described by the density function $f(\alpha)$, with compact support $\operatorname{supp}(f) = \overline{\Omega}$, where $\Omega \subset \mathbb{R}^m$ is the space of types, and $\overline{\Omega}$ is its closure. We assume that $\Omega \subset U$ is an open set with *locally finite perimeter* in the open set U, and that f is Lipschitz continuous.[3] Intuitively, a set has locally finite perimeter if its characteristic function is a function of bounded variation, hence it is a large class of open sets that includes the class of open convex sets as a very small subclass.[4] Also, we assume that $\nu(\cdot, x, t)$ can be extended by continuity to $\overline{\Omega}$. Consumers have an outside option of value $s_0(\alpha)$, which is assumed to be continuously differentiable, implementable and extendable by continuity to $\overline{\Omega}$.[5] Let $x_0(\alpha)$ be the outside option implementing $s_0(\alpha)$ for type α.

The firm looks for a selling mechanism that maximizes its profits. The Taxation Principle (Rochet [21]) implies that one can, without loss of generality, assume that the monopolist simply announces a non-linear tariff $t : \mathbb{R}^n_+ \to \mathbb{R}$.

The above considerations can be summarized by the following model. The firm selects a function $t : \mathbb{R}^n_+ \to \mathbb{R}$ to solve

$$\max_{t(\cdot)} \int_\Omega (t(x(\alpha)) - c(x(\alpha)))f(\alpha)d\alpha, \tag{2.1}$$

where $x(\alpha)$ satisfies

$$\begin{cases} x(\alpha) \in \arg\max_{x\geq 0} v(\alpha, x, t(x)) & \text{if } \max_{x\geq 0} v(\alpha, x, t(x)) \geq s_0(\alpha) \\ x(\alpha) = x_0(\alpha) & \text{otherwise.} \end{cases} \tag{2.2}$$

Define the net utility as the unique function $u(\alpha, x)$ that solves

$$s_0(\alpha) = v(\alpha, x, u(\alpha, x)) \tag{2.3}$$

The economic meaning of $u(\alpha, x)$ is the maximal amount type α is willing to pay for the bundle x. Note that the optimal tariff paid by type α satisfies

$$t(x(\alpha)) \leq u(\alpha, x(\alpha)). \tag{2.4}$$

Let $s(\alpha)$ denote the surplus obtained by type α:

[3] See Evans and Gariepy [13] and Chlebik [11] for the relevant concepts from geometric measure theory. For convenience, a brief summary is presented in the Appendix.

[4] A set of finite perimeter can have many "holes" and its boundary can be quite "rough".

[5] For conditions of implementability of a surplus function see Basov [7].

$$s(\alpha) = \begin{cases} \max_{x \geq 0} v(\alpha, x, t(x)) - s_0(\alpha) & \text{if } \max_{x \geq 0} v(\alpha, x, t(x)) \geq s_0(\alpha) \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

Accordingly, we have the envelope condition

$$\nabla s(\alpha) = \nabla_\alpha v(\alpha, x(\alpha), t(x(\alpha))) - \nabla s_0(\alpha)$$

that holds for almost every α with $x(\alpha) \neq x_0(\alpha)$. From (2.3) we have

$$\nabla s_0(\alpha) = \nabla_\alpha v(\alpha, x(\alpha), u(\alpha, x(\alpha))) + v_t(\alpha, x(\alpha), u(\alpha, x(\alpha)))\nabla_\alpha u(\alpha, x(\alpha)),$$

so the envelope condition can be written as

$$\lambda(\alpha)\nabla s(\alpha) = \nabla_\alpha u(\alpha, x(\alpha)) \tag{2.6}$$

for almost every α with $x(\alpha) \neq x_0(\alpha)$, where $\lambda(\alpha) = |v_t(\alpha, x(\alpha), u(\alpha, x(\alpha)))|^{-1}$ is positive and bounded away from zero.

We are interested in the the set of *excluded consumers*, given by

$$\{\alpha \in \Omega : x(\alpha) = x_0(\alpha)\},$$

that is, the set of types that optimally choose to not participate.

Assumption 2.1. *$u(\cdot, x)$ is strictly increasing in α for each $x \neq x_0(\alpha)$.*

For $a, b \in \mathbb{R}^m$ let $(a \cdot b)$ denote the inner product of a and b.

Assumption 2.2. *There exists $K > 0$ such that $u(\alpha, x) \leq K(\alpha \cdot \nabla_\alpha u(\alpha, x))$ for every $(\alpha, x) \in \overline{\Omega} \times \mathbb{R}^n_+$.*

Assumptions 2.1 and 2.2 are regularity conditions, requiring that the net utility be strictly increasing in α and bounded. Note that $v(\cdot, x, t)$ is allowed to be decreasing in α, as long as Assumptions 2.1 and 2.2 are satisfied.

For any Lebesgue measurable set $E \subset \mathbb{R}^m$ let $\mathcal{L}^m(E)$ denote its Lebesgue measure and $\mathcal{H}^s(E)$ denote its s-dimensional Hausdorff measure. For $s = m$, the Hausdorff measure of a Borel set coincides with the Lebesgue measure, while for $s < m$ it generalizes the notion of the surface area.[6]

[6]For a definition of the Hausdorff measure, see Chlebik [11].

Let $\partial_e\Omega$ denote the measure theoretic boundary of Ω. Because Ω has locally finite perimeter, the measure theoretic boundary can be decomposed into countably many smooth pieces and a residual set with measure zero. That is,

$$\partial_e\Omega = \bigcup_{i=1}^{\infty} K_i \cup N,$$

where K_i is a compact subset of a C^1-hypersurface S_i, for $i \geq 1$, and $\mathcal{H}^{m-1}(N) = 0$.

We now describe the underlying *space of all type spaces.* It is given by $(\Omega_\beta)_{\beta\in\mathcal{B}}$, where $\mathcal{B}$ is an index set. For each $\beta \in \mathcal{B}$, Ω_β is an open set with locally finite perimeter in some open set U_β and its boundary structure is given by

$$\partial_e\Omega_\beta = \bigcup_{i=1}^{\infty} K_{i,\beta} \cup N_\beta$$

where

$$K_{i,\beta} = \{\alpha \in \overline{\Omega}_\beta : g_i(\alpha, \beta) = 0\}$$

for $i > 0$, with $g_i : \mathbb{R}^m \times \mathcal{B} \to \mathbb{R}$ smooth, and N_β is a set of $\mathcal{H}^{m-1}$-measure zero. We make the following assumption about $(\Omega_\beta)_{\beta\in\mathcal{B}}$:

Assumption 2.3. *(i) $\mathcal{B}$ is a smooth finite dimensional open manifold; (ii) the correspondence $\varphi : \mathcal{B} \rightrightarrows \mathbb{R}^m$, given by $\varphi(\beta) = \Omega_\beta$, has open graph; (iii) there exist $\hat{\beta} \in \mathcal{B}$ such that*

$$\nabla_\alpha g_i(\alpha, \hat{\beta}) \ngeq 0$$

for all $\alpha \in \mathbb{R}^m$ and all $i > 0$.

That is, the parameters $\beta \in B$ determine the underlying set of type spaces $(\Omega_\beta)_{\beta\in\mathcal{B}}$ that we consider. Requirements (i) and (ii) are mild technical requirements so that we can apply transversality ideas. In fact, one can argue that the correspondence φ must be a continuous correspondence, so that proximity in $\mathcal{B}$ implies proximity of the corresponding type spaces. Nevertheless, all that is need for the main argument is the weaker requirement (ii). Requirement (iii) is also quite mild: in two dimensions ($m = 2$), it is satisfied as long as a "slanted right-ward diamond" is included as a member of the allowed types spaces. A seemingly more important requirement is the finite dimensionality of $\mathcal{B}$. But this is just for the a cleaner presentation of our ideas. In Lemma 2.7 below we make use of the standard Transversatiliy Theorem, which is valid in a finite dimensional environment. It is well known that there exist general versions of the Transversality Theorem that allow for infinite dimensions.[7] One can generalize Assumption 2.3 allowing for an infinite dimensional $\mathcal{B}$ and adapt Lemma 2.7 below with a more powerful Transversality Theorem. We leave this task to the interested reader.

Let $\mathcal{K}(\mathbb{R}^m)$ be the hyperspace of compact sets in $\mathbb{R}^m$, endowed with the topology induced by the Hausdorff distance d_H, given by

$$d_H(E, F) = \inf\{\varepsilon > 0 : E \subset F^\varepsilon, F \subset E^\varepsilon\},$$

where

$$E^\varepsilon = \bigcup_{\alpha \in E} B(\alpha, \varepsilon)$$

and $B(\alpha, \varepsilon)$ is the open ball centered at α and with radius $\varepsilon > 0$. Because

$$\lim_{\varepsilon \to 0+} \mathcal{L}^m(E^\varepsilon) = \mathcal{L}^m(E), \quad \lim_{\varepsilon \to 0+} \mathcal{H}^s(E^\varepsilon) = \mathcal{H}^s(E)$$

for all $s \geq 0$, both $\mathcal{L}^m$ and $\mathcal{H}^s$ are upper semicontinuous functions in $\mathcal{K}(\mathbb{R}^m)$ (Beer [9]).

The Generalized Gauss-Green Theorem states that for any Ω with locally finite perimeter in $U \subset \mathbb{R}^m$, and any Lipschitz continuous vector field $\varphi : U \to \mathbb{R}^m$ with compact support in U there is a unique measure theoretic unit outer normal $\tau_\Omega(\alpha)$ such that

$$\int_\Omega \text{div}\varphi d\alpha = \int_U (\varphi \cdot \tau_\Omega) d\mathcal{H}^{m-1}$$

where

$$\text{div}\varphi = \sum_{k=1}^{m} \frac{\partial \varphi_k}{\partial \alpha_k}$$

is the divergence of the vector field φ.

The main result of this section is Theorem 2.4 below. It is stated without reference to the well known sufficient conditions for implementability and differentiability of $s(\cdot)$ in order to focus on the conditions that highlight the nature of the contribution being made.

Let us write $\Omega_{0,\beta} = \{\alpha \in \Omega_\beta : s(\alpha; \beta) = 0\}$, where $s(\alpha; \beta)$ is the surplus function obtained by type α when the underlying type space is Ω_β. Likewise, we shall make explicit the dependence of the relevant object on the underlying type space indexed by $\beta \in \mathcal{B}$, viz.

[7] See Golubitsky and Guillemin [14] for the relevant concepts in the theory of transversality.

$x(\alpha;\beta)$, $x_0(\alpha;\beta)$, etc. Extending $s(\cdot;\beta)$ by continuity to $\partial\Omega_\beta$, let $\overline{\Omega}_{0,\beta} = \{\alpha \in \overline{\Omega}_\beta : s(\alpha;\beta) = 0\}$.

Theorem 2.4. *Consider the problem (2.1)-(2.2), and assume that it has a finite solution yielding an allocation $x(\alpha;\beta)$ and surplus $s(\alpha;\beta)$ which are continuous at each (α,β) in the graph of φ. Then, under Assumptions 2.1, 2.2 and 2.3, for each model β in an open and dense subset of $\mathcal{B}$, the set of excluded consumers at the solution has positive measure.*

Proof. We divide the proof into several intermediate steps.

Lemma 2.5. *Let $E \in \mathcal{K}(\mathbb{R}^m)$ be such that $\mathcal{L}^m(E) = \mathcal{H}^s(E) = 0$, for some $s \geq 0$, and let $(E_k)_{k\geq 1}$ be a sequence in $\mathcal{K}(\mathbb{R}^m)$ such that $E_k \to E$. Then $\mathcal{L}^m(E_k) \to 0$ and $\mathcal{H}^s(E_k) \to 0$.*

Proof of Lemma. Because $\mathcal{L}^m$ is a non negative upper semicontinuous set function, we have

$$\liminf_{E_k\to E} \mathcal{L}^m(E_k) \geq 0 = \mathcal{L}^m(E) \geq \limsup_{E_k\to E} \mathcal{L}^m(E_k),$$

so $\mathcal{L}^m(E_k) \to 0$, and analogously for $\mathcal{H}^s$. □

Lemma 2.5 establishes continuity of Lebesgue and Hausdorff measures at zero.

Lemma 2.6. *Under Assumption 2.1, $\mathcal{L}^m(\overline{\Omega}_{0,\beta}) = 0$ implies $\overline{\Omega}_{0,\beta} \subset \partial\Omega_\beta$.*

Proof of Lemma. If $\overline{\Omega}_{0,\beta} \not\subseteq \partial\Omega_\beta$, there is $\alpha \in \Omega_{0,\beta}$ and an $\varepsilon > 0$ with $B(\alpha,\varepsilon) \subset \Omega$. Then

$$\mathcal{L}^m(\{\hat{\alpha} \in \overline{\Omega}_\beta : \hat{\alpha} \leq \alpha\} \cap B(\alpha,\varepsilon)) > 0.$$

Because of Assumption 2.1, we cannot have $s(\hat{\alpha};\beta) > 0$ for any $\hat{\alpha} \leq \alpha$, for otherwise $s(\alpha;\beta) > 0$ as well. So

$$\{\hat{\alpha} \in \overline{\Omega}_\beta : \hat{\alpha} \leq \alpha\} \cap B(\alpha,\varepsilon) \subset \overline{\Omega}_{0,\beta},$$

contradicting $\mathcal{L}^m(\overline{\Omega}_{0,\beta}) = 0$. □

Lemma 2.6 states that if the exclusion set has Lebesgue measure zero it should be part of the topological boundary of the type set. Assumption 2.1 is crucial for this result. If it does not hold it is easy to come up with counter-examples even in the one-dimensional case. For examples, see Jullien [15].

Lemma 2.7. *Under Assumption 2.3, if $\mathcal{L}^m(\overline{\Omega}_{0,\beta}) = 0$ for all β in some open subset $V \subset \mathcal{B}$, then there exists $\beta' \in V$ such that $\mathcal{H}^{m-1}(\overline{\Omega}_{0,\beta'}) = 0$.*

Proof of Lemma. By Lemma 2.6, $\overline{\Omega}_{0,\beta} \subset \partial\Omega_\beta$ for all $\beta \in V$. Because $\mathcal{H}^{m-1}(\partial\Omega_\beta \backslash \partial_e\Omega_\beta) = 0$, consider $\overline{\Omega}_{0,\beta} \cap \partial_e\Omega_\beta$, which is given by

$$\overline{\Omega}_{0,\beta} \cap \partial_e\Omega_\beta = \bigcup_{i=1}^{\infty} \overline{\Omega}_{0i,\beta} \cup (N_\beta \cap \overline{\Omega}_{0,\beta})$$

where

$$\overline{\Omega}_{0i,\beta} = \{\alpha \in \overline{\Omega}_\beta : g_i(\alpha, \beta) = 0, s(\alpha; \beta) = 0\},$$

for $i > 0$. Now Assumptions 2.1 and 2.3(ii) ensure that there is $\hat{\beta} \in \mathcal{B}$ for which the level sets of $u(\alpha, x; \beta)$ are transversal to $g_i(\alpha, \hat{\beta})$, at the solution $x(\alpha; \beta)$ for all $i > 0$. The Transversality Theorem then implies that the level sets of $u(\alpha, x; \beta)$ are transversal to the level sets of $g_i(\alpha, \beta')$ for an open and dense subset of β' and all $i > 0$. By continuity of $x(\alpha; \beta)$ in β, for any neighborhood of β, there exists a model β' with the level sets of $u(\alpha, x; \beta')$ transversal to the level sets of $g_i(\alpha, \beta')$ for all $i > 0$, at the solution $x(\alpha, \beta')$. Note that $\beta' \in V$.

By the Implicit Function Theorem, $\overline{\Omega}_{0i,\beta'}$ is a manifold of dimension of $(m - 2)$. So $\mathcal{H}^{m-1}(\overline{\Omega}_{0i,\beta'}) = 0$. Hence

$$\mathcal{H}^{m-1}(\overline{\Omega}_{0,\beta'} \cap \partial_e\Omega_{\beta'}) \leq \sum_{i=1}^{\infty} \mathcal{H}^{m-1}(\overline{\Omega}_{0i,\beta'}) + \mathcal{H}^{m-1}(N_{\beta'} \cap \overline{\Omega}_{0,\beta'}) = 0,$$

as we wanted to show. □

Lemma 2.7 provides the basic step in establishing denseness of the set of models where exclusion occurs with positive probability. It is a straightforward application of the standard Transversality Theorem. Nevertheless, it shows that Assumptions 2.1 and 2.3 are potent, albeit being quite weak.

Lemma 2.8. *For any $\beta \in \mathcal{B}$, the Lebesgue measure of the set $D_\beta = \{\alpha \in \overline{\Omega}_{0,\beta} : x(\alpha; \beta) \neq x_0(\alpha; \beta)\}$ is zero.*

Proof of Lemma. If $\overline{\Omega}_{0,\beta} = \emptyset$ then there is nothing to prove. So assume it is not empty, pick $\alpha \in \overline{\Omega}_{0,\beta}$ and say that there is $\hat{\alpha} \in \overline{\Omega}_{0,\beta}$ with $\hat{\alpha} \leq \alpha$ and $x(\hat{\alpha}; \beta) \neq x_0(\hat{\alpha}; \beta)$. Then $u\left(\hat{\alpha}, x\left(\hat{\alpha}; \beta\right)\right) = t\left(x(\hat{\alpha}; \beta)\right)$ because $v(\hat{\alpha}, x(\hat{\alpha}; \beta), t\left(x(\hat{\alpha}; \beta)\right)) = s_0(\hat{\alpha}; \beta)$ and by Assumption 2.1,

$$u\left(\alpha, x\left(\hat{\alpha}; \beta\right)\right) > u\left(\hat{\alpha}, x\left(\hat{\alpha}; \beta\right)\right)$$

and, because v is strictly decreasing in t,

$$s_0(\alpha; \beta) = v(\alpha, x(\hat{\alpha}; \beta), u(\alpha, x(\hat{\alpha}; \beta))) < v(\alpha, x(\hat{\alpha}; \beta), t\left(x(\hat{\alpha}; \beta)\right))$$

contradicting the optimality of $x(\alpha;\beta)$ for type α. Therefore we must have $x(\hat{\alpha};\beta) = x_0(\hat{\alpha};\beta)$ for all $\hat{\alpha} \leq \alpha$. The same argument shows that if $\alpha \in \overline{\Omega}_{0,\beta}$ and $x(\alpha;\beta) \neq x_0(\alpha;\beta)$, then $\hat{\alpha} \notin \overline{\Omega}_{0,\beta}$ whenever $\hat{\alpha} \geq \alpha$. So for any pair $(\alpha, \hat{\alpha})$ in $\overline{\Omega}_{0,\beta}$ with $x(\alpha;\beta) \neq x_0(\alpha;\beta)$ and $x(\hat{\alpha};\beta) \neq x_0(\hat{\alpha};\beta)$ we must have $\alpha \not\leq \hat{\alpha}$ and $\hat{\alpha} \not\leq \alpha$. Now consider any countable family of rectangles $\{R_k\}$ covering D_β, i.e, with $D_\beta \subset \bigcup_k R_k$. By picking a point α_k in $D_\beta \cap R_k$ for each k, and excluding from R_k the points that are strictly greater and strictly smaller than α_k, we can construct another countable cover of D_β by rectangles $\{R'_j\}$ with $\sum_j I(R'_j) \leq \frac{1}{2} \sum_k I(R_k)$, where $I(R)$ is the volume of a rectangle R in $\mathbb{R}^m$. It follows that $\mathcal{L}^m(D_\beta) = 0$. □

Let $\mathcal{E} \subset \mathcal{B}$ be the set of models where the set of excluded consumers has positive measure:

$$\mathcal{E} = \{\beta \in \mathcal{B} : \mathcal{L}^m(\overline{\Omega}_{0,\beta}) > 0\}.$$

Lemma 2.9. *$\mathcal{E}$ is open in $\mathcal{B}$.*

Proof of Lemma. Note that $\overline{\Omega}_{0,\beta}$ can be expressed as

$$\overline{\Omega}_{0,\beta} = \{\alpha \in \overline{\Omega} : \hat{s}(\alpha;\beta) \leq s_0(\alpha;\beta)\}$$

where $\hat{s}(\alpha;\beta)$ solves the monopolist problem without the participation constraint and coincides with the optimal solution in the participation region. Now decompose it as

$$\overline{\Omega}_{0,\beta} = \overline{\Omega}^1_{0,\beta} \cup \overline{\Omega}^2_{0,\beta}$$

where $\overline{\Omega}^1_{0,\beta} = \{\alpha \in \overline{\Omega}_\beta : \hat{s}(\alpha;\beta) < s_0(\alpha;\beta)\}$ and $\overline{\Omega}^2_{0,\beta} = \{\alpha \in \overline{\Omega}_\beta : \hat{s}(\alpha;\beta) = s_0(\alpha;\beta)\}$.

By continuity of $\hat{s}(\cdot;\beta)$ and $s_0(\cdot;\beta)$, $\overline{\Omega}^1_{0,\beta}$ is an open set. By Lemma 2.8, $\overline{\Omega}^2_{0,\beta} = \{\alpha \in \overline{\Omega}_{0,\beta} : x(\alpha;\beta) \neq x_0(\alpha;\beta)\}$ has zero Lebesgue measure. If $\mathcal{E} = \emptyset$ there is nothing to prove, so assume it is not empty. Take $\beta \in \mathcal{E}$, and note that $\overline{\Omega}^1_{0,\beta} \neq \emptyset$. Pick $\delta > 0$ such that the set $C_\beta = \{\alpha \in \overline{\Omega}_\beta : \hat{s}(\alpha;\beta) - s_0(\alpha;\beta) < -\delta\} \subset \overline{\Omega}^1_{0,\beta}$ has positive Lebesgue measure. We claim that there must exist $\varepsilon > 0$ such that whenever β' is ε-close to β, we have $\hat{s}(\alpha;\beta') - s_0(\alpha;\beta') < 0$ for all $\alpha \in C_\beta$. If not, then there would exist a net $\beta_\varepsilon \to \beta$ as $\varepsilon \to 0$, and (by assumption 2.3(ii)) an $\alpha(\beta_\varepsilon) \in C_\beta$ for each β_ε with $\hat{s}(\alpha(\beta_\varepsilon);\beta_\varepsilon) - s_0(\alpha(\beta_\varepsilon);\beta_\varepsilon) \geq 0$, whereas $\hat{s}(\alpha(\beta_\varepsilon);\beta) - s_0(\alpha(\beta_\varepsilon);\beta) < -\delta$, which is impossible because $\hat{s}$ and s_0 are continuous functions. It follows that $C_\beta \subset \overline{\Omega}_{0,\beta'}$, and hence $\mathcal{L}^m(\Omega_{0,\beta'}) > 0$, for every β' ε-close to β, as we wanted to show. □

Lemma 2.10. *For each given $\beta \in \mathcal{B}$, there is an equivalent metric in $\mathbb{R}^n$ for which $x(\alpha;\beta)$, and hence $s(\alpha;\beta)$ and $\lambda(\alpha;\beta)$, are Lipschitz continuous functions.*

Proof of Lemma. Let $||\cdot||_n$ and $||\cdot||_m$ be the Euclidean norms in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $d_1(\alpha,\alpha') = ||\alpha-\alpha'||_n + ||x(\alpha;\beta)-x(\alpha';\beta)||_m$. Then $||x(\alpha;\beta)-x(\alpha';\beta)||_m < d_1(\alpha,\alpha')$, so $x(\cdot;\beta)$ is Lipschitz continuous. The metric d_1 is equivalent to the Euclidean metric (Aliprantis and Border [1], Lemma 3.12). Also, any Lipschitz continuous function under the Euclidean metric in $\mathbb{R}^n$ (as the density f) is also Lipschitz continuous under d_1. In fact, $|f(\alpha)-f(\alpha')| \leq c||\alpha-\alpha'||_n = cd_1(\alpha,\alpha') - c||x(\alpha;\beta)-x(\alpha';\beta)||_m \leq cd_1(\alpha,\alpha')$, for some real number c. □

We are now ready to conclude the argument. We shall show that $\mathcal{E}$ is a dense subset of $\mathcal{B}$, which, in light of Lemma 2.9, establishes the result.

By way of contradiction, assume that $\mathcal{L}^m(\overline{\Omega}_{0,\beta}) = 0$ for all β in some open set $V \subset \mathcal{B}$. For any natural number k, let $\pi_{k,\beta}$ be the profit obtained by selling to the types in

$$\overline{\Omega}_{k,\beta} = \{\alpha \in \overline{\Omega}_\beta : s(\alpha;\beta) \leq \frac{1}{k}\}.$$

Because $c(\cdot)$ is non-negative, we must have

$$\pi_{k,\beta} \leq \int_{\overline{\Omega}_{k,\beta}} t(x(\alpha;\beta))f(\alpha)d\alpha,$$

and from (2.4) we have

$$\pi_{k,\beta} \leq \int_{\overline{\Omega}_{k,\beta}} u(\alpha, x(\alpha;\beta))f(\alpha)d\alpha.$$

Assumption 2.2 and the envelope condition (2.6) (with $\mathcal{L}^m(\overline{\Omega}_{0,\beta}) = 0$, we have $\mathcal{L}^m(\overline{\Omega}_{k,\beta}) = \mathcal{L}^m(\overline{\Omega}_{k,\beta}\backslash\overline{\Omega}_{0,\beta})$, so the envelope condition holds for almost all types in $\overline{\Omega}_{k,\beta}$) yield

$$\pi_{k,\beta} \leq K \int_{\overline{\Omega}_{k,\beta}} (\alpha \cdot \nabla s(\alpha;\beta))\lambda(\alpha;\beta)f(\alpha)d\alpha.$$

Applying the Generalized Gauss-Green Theorem to the Lipschitz continuous vector field $\varphi(\alpha) = \alpha s(\alpha;\beta)\lambda(\alpha;\beta)f(\alpha)$ we get

$$\pi_{k,\beta} \leq K \int_{U_\beta} s(\alpha;\beta)\lambda(\alpha;\beta)f(\alpha)(\alpha\cdot\tau_\Omega(\alpha))d\mathcal{H}^{m-1}(\alpha)$$
$$- K \int_{\overline{\Omega}_{k,\beta}} s(\alpha;\beta)\mathrm{div}(\alpha\lambda(\alpha;\beta)f(\alpha))d\alpha.$$

The functions $s(\alpha;\beta)$, $\lambda(\alpha;\beta)$, $f(\alpha)$, $(\alpha\cdot\tau_\Omega(\alpha))$ and $\mathrm{div}(\alpha\lambda(\alpha;\beta)f(\alpha;\beta))$ are bounded in $\overline{\Omega}_{k,\beta}$, so we can find a common upper bound B. Because $s(\alpha;\beta) \leq \frac{1}{k}$ in $\overline{\Omega}_{k,\beta}$ and $\mathrm{supp}(f) = \overline{\Omega}_\beta$, we have

$$\pi_{k,\beta} \leq \frac{1}{k}B(\mathcal{H}^{m-1}(\overline{\Omega}_{k,\beta}) + \mathcal{L}^m(\overline{\Omega}_{k,\beta})).$$

Now consider increasing the tariff by $\frac{1}{k}$. The consumers in the set $\overline{\Omega}_{k,\beta}$ will exit, and $\pi_{k,\beta}$ will be lost, but each other consumer will pay $\frac{1}{k}$ more. Because the total number of consumers that exit is bounded by $B\mathcal{L}^m(\overline{\Omega}_{k,\beta})$, the change in profit is

$$\Delta\pi_\beta \geq \frac{1}{k}[(1 - B\mathcal{L}^m(\overline{\Omega}_{k,\beta})) - B(\mathcal{H}^{m-1}(\overline{\Omega}_{k,\beta}) + \mathcal{L}^m(\overline{\Omega}_{k,\beta}))].$$

From Lemma 2.7, there exists $\beta' \in V$ with $\mathcal{H}^{m-1}(\overline{\Omega}_{0,\beta'}) = 0$, and hence from Lemma 2.5 we have $\mathcal{L}^m(\overline{\Omega}_{k,\beta'}) \to 0$ and $\mathcal{H}^{m-1}(\overline{\Omega}_{k,\beta'}) \to 0$, because, by continuity of $s(\cdot;\beta')$ and the compact support of $f(\cdot)$, each $\overline{\Omega}_{k,\beta'}$ is compact. But then for large k, $\Delta\pi_{\beta'}$ must be positive, contradicting the optimality of the tariff. Therefore, we have $\mathcal{L}^m(\overline{\Omega}_{0,\beta'}) > 0$. As V was arbitrary, $\mathcal{E}$ is dense, as we wanted to show. □

Remark that Lemma 2.8 ensures that the set of positive measure is the set of types that choose the outside option, that is, the set of excluded types. The types that obtain zero surplus and yet choose to participate form a neglible set.

Let us note that it is standard in the literature to work with a quasilinear framework, where $v(\alpha, x, t) = v(\alpha, x) - t$ and the net utility is $u(\alpha, x) = v(\alpha, x) - s_0(\alpha)$. Also, sometimes $s_0(\alpha)$ is assumed to be equal to zero for every α. In fact, this is the setting used by Armstrong [2]. In his setting, Assumptions 2.1 and 2.2 are implied by his assumption that v is strictly increasing and homogeneous of degree 1 in α, and Assumption 2.3 is implied by his assumption that Ω is strictly convex and v is strictly convex in α. Clearly, Assumptions 2.1-2.3 are substantially weaker than the standard assumptions in the literature.

3 Exclusion in an Oligopolistic Screening Model

We now extend the framework of Section 2 to the case of a market served by $L > 1$ firms. For simplicity, we assume quasilinearity of the consumer's utility function and a type-independent outside option normalized to 0. The production cost is identical among the firms. The firms simultaneously choose non-linear tariffs, and obtain profits after the consumers make their choices. Consumers choose their optimal bundle after observing the choices of the firms. Consumers' choices may well involve buying goods produced by several firms. A pure strategy of firm ℓ is a non-linear tariff, i.e. a mapping $t_\ell : \mathbb{R}^n_+ \to \mathbb{R}$. Consider a symmetric Nash equilibrium at which all firms charge the same tariff. We shall show that, generically, at any such equilibrium where firms retain some market power a positive measure of the consumers is not served.

For concreteness, we will present a formulation where individual firms retain some market power in a symmetric equilibrium. Each firm ℓ first produces a quantity $\bar{x}_\ell$ at cost $c(\bar{x}_\ell)$, and then competes with the other firms by choosing a non-linear tariff to sell $\bar{x}_\ell$ to the consumers.[8] Hence it solves

$$\max_{t_\ell(\cdot),\bar{x}_\ell} \int t_\ell(x_\ell(\alpha))f(\alpha)d\alpha - c(\bar{x}_\ell)$$

subject to:

$$\begin{cases} \int x_\ell(\alpha)f(\alpha)d\alpha = \bar{x}_\ell & \\ x(\alpha) \in \arg\max_{x\geq 0} \upsilon(\alpha,x) - t(x) & \text{if } \max_{x\geq 0} \upsilon(\alpha,x) - t(x) \geq s_{0,\ell}(\alpha) \\ x(\alpha) = 0 & \text{otherwise} \end{cases}$$

where

$$\begin{cases} t(x) = \min \sum_\mu t_\mu(x_\mu) \\ s.t. \sum_\mu x_\mu = x, \ x_\mu \geq 0 \end{cases}, \tag{3.1}$$

and

$$s_{0,\ell}(\alpha) = \max\{s_0^*(\alpha), \max_{x\geq 0, x_\ell = 0}(\upsilon(\alpha,x) - t_{-\ell}(x))\} \tag{3.2}$$

and $t_{-\ell}(x)$ solves problem (3.1) subject to the additional constraint $x_\ell = 0$. Equation (3.2) states that the outside option of a consumer seen from the point of view of firm ℓ is determined either by her best opportunity outside the market, $s_0^*(\alpha)$, or by the best bundle she may purchase from the competitors, $\max_{x\geq 0, x_\ell=0}(\upsilon(\alpha,x) - t_{-\ell}(x))$. We assume that the capacity of firm ℓ, $\bar{x}_\ell$, is decided first, and after that the firm picks the best non-linear tariff t_ℓ to distribute $\bar{x}_\ell$ to consumers.

Define

$$u(\alpha, x_\ell) = \upsilon(\alpha, x_\ell + \sum_{\mu\neq\ell} x_\mu(\alpha)) - \sum_{\mu\neq\ell} t_\mu(x_\mu(\alpha)) - s_{0,\ell}(\alpha),$$

where $x_\mu(\alpha)$ is the equilibrium quantity purchased by the consumer of type α from firm $\mu \neq \ell$ and $s_{0,\ell}(\alpha)$ is given by (3.2). Then firm ℓ's problem becomes:

$$\max_{t_\ell(\cdot),\bar{x}_\ell} \int t_\ell(x_\ell(\alpha))f(\alpha)d\alpha - c(\bar{x}_\ell),$$

subject to:

[8] It is well-known that such a formulation preserves the market power of individual firms. See Kreps and Sheinkman [16].

$$\begin{cases} \int x_\ell(\alpha) f(\alpha) d\alpha = \bar{x}_\ell & \\ x_\ell(\alpha) \in \arg\max_{x\geq 0} u(\alpha, x_\ell) - t_\ell(x_\ell) & \text{if} \max_{x\geq 0} u(\alpha, x_\ell) - t_\ell(x_\ell) \geq 0 \\ x_\ell(\alpha) = 0 & \text{otherwise.} \end{cases}$$

In the formulation above, $u(\alpha, x_\ell)$ is endogenous, so we cannot impose Assumptions 2.1 and 2.2 on it. It turns out that restricting $\upsilon(\alpha, x)$ to the class of functions satisfying strict "supermodularity" in (α, x) is enough to ensure the required properties of $u(\alpha, x_\ell)$.

Assumption 3.1. *For every $i = 1, ..., m$, $p = 1, ..., n$ and $x \neq 0$ we have*

$$\frac{\partial^2 \upsilon}{\partial \alpha_i \partial x_p} > 0.$$

Proposition 3.2. *Under Assumption 3.1 we have: (a) $u(\alpha, x_\ell)$ is strictly increasing in α for all $x_\ell \in \mathbb{R}^n_+$; (b) There exists $B > 0$ such that $u(\alpha, x_\ell) \leq B(\alpha \cdot \nabla_\alpha u(\alpha, x_\ell))$ for every $(\alpha, x_\ell) \in \overline{\Omega} \times \mathbb{R}^n_+$, and each $\ell = 1, ..., L$.*

Proof. For (a), note that

$$\begin{aligned} \frac{\partial u(\alpha, x_\ell)}{\partial \alpha_i} &= [\frac{\partial \upsilon(\alpha, x)}{\partial \alpha_i} - \frac{\partial s_{0,\ell}(\alpha)}{\partial \alpha_i}] + \sum_{\mu \neq \ell} \sum_p (\frac{\partial \upsilon}{\partial x_{\mu,p}} - \frac{\partial t_\mu}{\partial x_{\mu,p}}) \frac{\partial x_{\mu,p}}{\partial \alpha_i} \\ &= [\frac{\partial \upsilon(\alpha, x)}{\partial \alpha_i} - \frac{\partial s_{0,\ell}(\alpha)}{\partial \alpha_i}] \end{aligned}$$

because the consumer chooses optimally from the other firms. When the relevant alternative from buying from firm ℓ is to buy from other firms, we have

$$\frac{\partial \upsilon(\alpha, x)}{\partial \alpha_i} = \frac{\partial \upsilon(\alpha, x_\ell + \sum\limits_{\mu \neq \ell} x_\mu(\alpha))}{\partial \alpha_i},$$

while

$$\frac{\partial s_{0,\ell}(\alpha)}{\partial \alpha_i} = \frac{\partial \upsilon(\alpha, 0 + \sum\limits_{\mu \neq \ell} x_\mu(\alpha))}{\partial \alpha_i},$$

where $x_\mu(\alpha)$ is the optimal quantities purchased by firm $\mu \neq \ell$. So $\frac{\partial u(\alpha, x_\ell)}{\partial \alpha_i} > 0$ for every $i = 1, ..., m$ and every $x \neq 0$.

Turn now to (b), and define

$$\begin{aligned} h(\alpha, x_\ell) &= \frac{u(\alpha, x_\ell)}{\alpha \cdot \nabla_\alpha u(\alpha, x_\ell)}, \text{ if } x_\ell \neq 0 \\ h(\alpha, x_\ell) &= \limsup_{x_\ell \to 0} \frac{u(\alpha, x_\ell)}{\alpha \cdot \nabla_\alpha u(\alpha, x_\ell)}, \text{ if } x_\ell = 0. \end{aligned}$$

Observe that

$$\lim_{x_{\ell,p}\to 0, x_{\ell,q}=0} \sup \frac{u(\alpha, x_\ell)}{\alpha \cdot \nabla_\alpha u(\alpha, x_\ell)} = \frac{\partial u/\partial x_\ell}{\sum_{q=1}^{n} \partial^2 v/\partial x_{\ell,q}\partial\alpha_i}.$$

Let $X = \prod_{k=1}^{n}[0, x_k^*]$, where x_k^* is the efficient (i.e. the outcome of a perfectly competitive market) value of x_k and define

$$B = \max_{X\times\overline{\Omega}} h(\alpha, x_\ell),$$

which is well defined because the set $X \times \overline{\Omega}$ is compact and h is a continuous function. Then

$$u(\alpha, x_\ell) \leq B(\alpha \cdot \nabla_\alpha u(\alpha, x_\ell)),$$

for all $(\alpha, x) \in \overline{\Omega} \times X$, for each $\ell = 1, ..., L$. □

Theorem 3.3. *Consider a symmetric equilibrium of the oligopoly game described above, and assume that the resulting allocation for the entire industry is continuous at each (α, β) in the graph of φ. Then, under Assumptions 3.1 and 2.3, the set of excluded consumers has positive measure for an open and dense set in $\mathcal{B}$.*

Proof. Note first that we need only $s(\alpha; \beta)$ Lipschitz continuous: under quasilinearity, we need not worry about continuity of λ as we did in Theorem 2.4, since λ is always equal to 1. Because the allocation for the entire industry is continuous, as in Theorem 2.4, the same argument will now yield a Lipschitz continuous $s(\alpha; \beta)$. Now consider firm ℓ. Given the behavior of the competitors and their fixed capacity $\bar{x}_{-\ell}$, the problem described in the formulation above is isomorphic to the problem of a single firm with monopoly power. In fact, if firm ℓ is to increase its tariff by $1/k$ as in the last part of Theorem 2.4, then types with surplus for firm ℓ, s_ℓ, greater than $1/k$ will not move to firm ℓ's competitors, as these are already selling their capacity. Such types will keep purchasing from firm ℓ, so the deviation will be profitable if the set of excluded types has measure zero, contradicting the assumption of an equilibrium. That is, the argument in Theorem 2.4 applies and each firm ℓ will optimally exclude a positive measure of consumers, for an open and dense subset of $\mathcal{B}$. By symmetry, so will every other firm. Finally, by symmetry again, the set of excluded consumers is the same for all firms, so the intersection of the sets of excluded consumers has positive measure. □

We remark that our formulation with firms choosing their capacities first and then competing in non-linear tariffs is one such formulation that ensures that each firm retains some market power in a symmetric equilibrium. Another such formulation would be to assume that the common cost function is strictly convex. In all, what we really want to avoid is linear costs with no capacity constraint, which can lead to an equilibrium with marginal cost pricing, where no exclusion occurs.

Let us also remark that, in the formulation above, we can allow or entry, as long as there is a positive entry cost $F > 0$. It is easy to see that this problem can be reduced to the previous one, since equilibrium number of the producers is always finite. Indeed, with K producers the profits of an oligopolist in a symmetric equilibrium are bounded by π^m/K, where π^m are the profits of a monopolist. Therefore, at equilibrium $K \leq \pi^m/F$ and a positive measure of the consumers will be excluded from the market.

3.1 Existence of Equilibrium in the Oligopoly Game

Theorem 3.3 is derived under the assumption that a symmetric Nash equilibrium exists for the game played by the firms. Champsuar and Rochet [10] note that the profit functions of the firms might be discontinuous when there are bunching regions. Even though Basov [7] shows that bunching in the multi-dimensional case is not as typical as suggested by Rochet and Chone [22], existence of an equilibrium has to be established. That's what we do next.

We focus on the continuation game after the capacity choices $\{\bar{x}_\ell\}_{\ell=1}^L$ are made. This is the game where discontinuities might be a problem. Assume that the space T of allowed tariffs is the space of all bounded monotonic functions from X to $[0, M]$, where $X = [0, \bar{x}_\ell] \subset \mathbb{R}^n_+$ is the compact subset of feasible bundles and M is a bound on the net utility function, hence it is also a bound on the tariffs. The space T is the common strategy space of each producer $\ell = 1, ..., L$ (by symmetry, $\bar{x}_\ell = \bar{x}_\mu$ for all $\ell, \mu = 1, ..., L$.) By Helly's theorem, every sequence of tariffs in T has a pointwise convergent subsequence, so T is compact in the topology of a.e. pointwise convergence (where a.e. refers to the Lebesgue measure $\mathcal{L}^m$.) Let $\Delta(T)$ denote the space of Borel probability measures on T, endowed with the weak* topology, so it is a compact, convex space.

Assume that when firms choose a symmetric profile $(t, ..., t)$ of tariffs, they obtain the same expected profit: $\pi_\ell(t, ..., t) = \pi(t, ..., t)$ for $\ell = 1, ..., L$.[9] Hence the one-shot game $(T \times \cdots \times T, \pi, ..., \pi)$ played by the firms is symmetric, and so is its mixed extension, where

[9] Either because the consumer chooses optimally to buy a fraction $\frac{1}{L}$ of the optimal bundle from each firm, or because she visits each firm with probability $\frac{1}{L}$, depending on the shape of the commonly offered non-linear tariff t.

firms choose $\sigma \in \Delta(T)$ and payoffs are extended to mixtures by taking expectations. For ease of notation, let $(\hat{\sigma}, \sigma)$ denote the profile $(\sigma, ..., \hat{\sigma}, ..., \sigma)$ of strategies where one firm chooses $\hat{\sigma}$ and the others all choose σ. Use $\pi(\hat{\sigma}, \sigma)$ to denote the expected profit of the firm choosing $\hat{\sigma}$.

Proposition 3.4. *The compact, convex and symmetric game described above has a symmetric mixed strategy Nash equilibrium.*

Proof. We show that the game is diagonally better reply secure (Reny [20]). Let (σ, σ) be a non equilibrium profile, and consider $\pi^* = \lim \pi(\sigma^n, \sigma^n)$ for some sequence with $\sigma^n \to \sigma$. For any $\varepsilon > 0$, there exists a strictly increasing t^ε with $\pi(t^\varepsilon, \sigma) > \pi(\sigma, \sigma)$, as (σ, σ) is not an equilibrium. Because t^ε is strictly increasing, $\pi(t^\varepsilon, \cdot)$ is continuous at σ. If $\pi^* = \pi(\sigma, \sigma)$, then diagonal better reply security is verified. If not, then we have discontinuities at (σ, σ). Along any sequence σ^n converging to σ, there is at least one firm whose profit drops at the limit, and this firm can obtain a profit strictly higher than π^* by using t^ε instead of σ_n, for large n. Hence diagonal better reply security is again verified due to continuity of $\pi(t^\varepsilon, \cdot)$. □

Observe that Theorem 3.3 remains valid at a symmetric mixed strategy equilibrium. As long as the surplus function is Lipschitz continuous, the formulation of the oligopoly game allows us to ascertain that a small increase in every t in the support of σ will be profitable if the capacities are held fixed and the measure of excluded types is not positive.

4 Examples and Applications

Let us begin with some examples illustrating Assumptions 2.1, 2.2 and 2.3.

Example 4.1. *Consider a consumer who lives for two periods. Her wealth in the first period is w and in the second period her wealth can take two values, w_H or w_L. Let p be the probability that $w = w_H$, and let $\delta \in (0, 1)$ be the discount factor, so that the private information of the consumer is characterized by a two-dimensional vector $\alpha = (1-p, 1-\delta)$. The consumer's preferences are given by:*

$$V(c_1, c_2) = v(c_1) + \delta E v(c_2)$$

where c_1 and c_2 are the consumption levels in periods 1 and 2 respectively, and $v(\cdot)$ is increasing with its derivative v' bounded away from zero. Assume that wealth is not storable *between periods. Instead, the consumer can borrow x from a bank in period 1, and repay t in period 2 if $w = w_H$, and to defaut if $w = w_L$ in period 2. If the consumer does not borrow, her expected utility will be:*

$$s_0(\alpha) = \upsilon(w) + \delta(p\upsilon(w_H) + (1-p)\upsilon(w_L))$$

which is the type dependent outside option. If she borrows x and repays t, the expected utility will be

$$v(\alpha, x, t) = \upsilon(w+x) + \delta(p\upsilon(w_H - t) + (1-p)\upsilon(w_L))$$

which is strictly increasing in x and strictly decreasing in t. Let $\Omega_1 = (0,1)^2$ be the type space, with boundary captured by $g_i(\alpha, \beta_1)$, $i = 1, ..., 4$, with $\nabla_\alpha g_i(\alpha, \beta_1) = (0,1)$ for $i = 1,2$ and $\nabla_\alpha g_i(\alpha, \beta_1) = (1,0)$ for $i = 3,4$. Let Ω_2 be another type space, included in the underlying space of type spaces, with boundary given by $g_i(\alpha, \beta_2)$, $i = 1, ..., 4$, with $\nabla_\alpha g_i(\alpha, \beta_2) = (-\varepsilon, 1)$ for $i = 1,2$ and $\nabla_\alpha g_i(\alpha, \beta_2) = (1, -\varepsilon)$ for $i = 3,4$, for some $\varepsilon > 0$. Assumption 2.3 is thus met.

As

$$\nabla_\alpha u(\alpha, x) = \left(\frac{\Delta\upsilon}{p\upsilon'}, \frac{\Delta\upsilon}{\delta\upsilon'}\right)$$

where $\Delta\upsilon = \upsilon(w_H) - \upsilon(w_H - u(\alpha, x)) > 0$, Assumptions 2.1 and 2.2 are met as well.

Example 4.1 is a natural setting to discuss unavailability of credit to some individuals, which is important to justify monetary equilibria in the search theoretic models of money.[10] The next example comes from the theory of industrial organization.

Example 4.2. *Suppose a monopolist produces cars of high quality. The utility of a consumer is quasilinear, $v(\alpha, x, t) = \upsilon(\alpha, x) - t$, with*

$$\upsilon(\alpha, x) = A + \sum_{i=1}^{n} \alpha_i x_i \tag{4.1}$$

where $A > 0$ can be interpreted as utility of driving a car, and the second term in (4.1) is a quality premium. Suppose a consumer has three choices: to buy a car from the monopolist, to buy a car from a competitive fringe, and to buy no car at all. We will normalize the utility of buying no car at all to be zero. Assume the competitive fringe serves cars of quality $-x_0$, where $x_0 \in \mathbb{R}^n_{++}$ at price p. That is, the consumers experience disutility from the quality of

[10] See, for example, Lagos and Wright [17].

the cars of the competitive fringe, and the higher their type, the higher the disutility. The utility of the outside option in this case is given by:

$$s_0(\alpha) = \max(0, A - p - \sum_{i=1}^{n} \alpha_i x_{0i})$$

and is decreasing in α*. Therefore, Assumptions 2.1 and 2.2 hold, because in a quasilinear setting the net utility is* $u(\alpha, x) = v(\alpha, x) - s_0(\alpha)$*. As for Assumption 2.3, a type space of the kind of* Ω_2 *in Example 4.1 above suffices. Observe that all that is required is the following. Say that we start off with* Ω *being the unit square in* $\mathbb{R}^n$*. We then parametrize each of the edges, and consider models obtained by small perturbations of the parameters, hence the edges (like for instance small rotations of the unit square). When such type spaces are included in* $\mathcal{B}$*, Assumption 2.3 is met.*

Observe that Examples 4.1 and 4.2 can easily be extended to the oligopoly case considered in Section 3. In the former, banks would first earmark a fixed amount of consumer credit and then compete with non-linear lending schedules, and in the latter producers would first determine the quantity of high quality cars and then compete in non-linear tariffs.

Now let us turn to models that do not satisfy Assumptions 2.1, 2.2 and 2.3. First, consider any model that yields an excluded set Ω_0 with positive measure, and modify the problem considering only the types in $\Omega \backslash \Omega'$, where $\Omega_0 \subset \Omega'$. Would the modified problem have no exclusion? Though this will indeed be the case if $\Omega' = \Omega_0$,[11] it will not hold for a generic superset Ω'. This would only be the case if the shape of Ω_0 stood in a tight relation with the shape of Ω', a non generic situation. That is, even if Ω_0 stood in the particular tight relation with Ω', a slight change in the boundary structure of Ω' would suffice for us to have exclusion in the modified model.

In the same vein, Rochet and Stole [24] provided an example where the exclusion set is empty.[12] In their quasilinear example $v : \Omega \times R_+ \to \mathbb{R}$ has the form

$$v(\alpha, x) = (\alpha_1 + \alpha_2)x$$

and Ω is a rectangle with sides parallel to the 45 degrees and -45 degrees lines. They argued that one can shift the rectangle sufficiently far to the right to have an empty exclusion region.

[11] We are grateful to an anonymous referee for this observation.

[12] Another example along similar lines is provided by Deneckere and Severinov [12]. Though it is a bit more intricate and the authors provide sufficient conditions that ensure full participation in the case of one quality dimension and two-dimensional characteristics, their condition also does not hold generically.

Their result is driven by the fact that they allow only very special collections of type spaces, rectangles with parallel sides. Formally, the model used in this case cannot be used in Lemma 2.7 because $\nabla_\alpha g_i(\alpha, \beta) = (1, 1)$ for $i = 1, 3$ and $\nabla_\alpha g_i(\alpha, \beta) = (-1, 1)$ for $i = 2, 4$, so that (using $u(\alpha, x) = v(\alpha, x)$ because $s_0(\alpha) = 0$)

$$\begin{pmatrix} \nabla_\alpha u(\alpha, x) \\ \nabla_\alpha g_i(\alpha, \beta) \end{pmatrix} = \begin{pmatrix} x & x \\ 1 & 1 \end{pmatrix} \Rightarrow rank \begin{pmatrix} \nabla_\alpha u(\alpha, x) \\ \nabla_\alpha g_i(\alpha, \beta) \end{pmatrix} = 1$$

for $i = 1, 3$.

Observe that a very small change in the type set changes that result. Consider, for example, a slightly perturbed type space, with $\nabla_\alpha g_i(\alpha, \beta_0) = (1, 1 + \varepsilon)$, for $i = 1, 3$, where ε is a small positive real number. Then, for all $x \neq 0$ and $i = 1, ..., 4$

$$rank \begin{pmatrix} \nabla_\alpha u(\alpha, x) \\ \nabla_\alpha g_i(\alpha, \beta_0) \end{pmatrix} = 2$$

as required in Lemma 2.7.

We stress that our result does not guarantee a non-empty exclusion region for every multi-dimensional screening problem. Rather, it asserts that problems for which the exclusion region is empty can be slightly perturbed and transformed into problems with a positive measure of excluded consumers. To understand the results intuitively, assume first that, in equilibrium, all consumers are served. First, note that at least one consumer should be indifferent between participating and not participating, since otherwise the tariffs can be uniformly increased for everyone by a small amount, increasing the monopolist's profits. Now, consider increasing the tariff by $\varepsilon > 0$. The consumers who used to obtain surplus below ε will drop out. The measure of such consumers is $O(\varepsilon)$, unless the iso-surplus hyper-surfaces happen to be parallel to the boundary of the type space. Under Assumption 2.3, there will be a model where the iso-surplus hyper-surfaces will not only not be parallel to the boundary of Ω, they will be transversal. The knife-edge cases of iso-surplus hyper-surfaces parallel to the boundary of Ω may still occur endogenously, which is the reason why our result holds for *almost* all, rather then for all, screening problems. One class of problems, for which full participation may occur are models with random outside options. They were first considered by Rochet and Stole [23] for both monopolistic and oligopolistic settings and generalized by Basov and Yin [8] for the case of risk averse principal(s). Armstrong and Vickers [5] considered another generalization, allowing for multidimensional vertical types. In this type of models, the type consists of a vector of vertical characteristics, $\alpha \in \Omega \subset \mathbb{R}^m$, and a parameter $\gamma \in [0, 1]$ capturing horizontal preferences. The type space is given by the

Cartesian product $\Omega \times [0,1]$ and γ is assumed to be distributed independently of α. The utility of a consumer is given by:

$$v(\alpha, x; \gamma) = v(\alpha, x) - t\gamma,$$

where t is a commonly known parameter. Let $v(\alpha, 0) = 0$ so that the iso-surplus hypersurface corresponding to zero quality is $t\gamma$ = constant, which is parallel to the vertical boundary of type space, $\gamma = 0$. Therefore, in such a model there is the possibility of full participation. The model was also investigated in an oligopolistic setting, where t was interpreted as a transportation cost for the Hotelling model. Conditions for full participation under different assumptions on the dimensionality of α and the monopolist's risk preferences were obtained by Armstrong and Vickers [5], Rochet and Stole [23], and Basov and Yin [8]. Let us assume that the boundary of set Ω is described by the equation

$$g_0(\alpha) = 0$$

and embed our problem into a family of problems, for which boundary of the type space is described by the equation

$$g(\alpha, \gamma; \beta) = 0,$$

where $g(\cdot, \beta) : \Omega \times [0,1] \to \mathbb{R}$ is a smooth function with

$$g(\alpha, \gamma; 0) = g_0(\alpha)(g_0(\alpha) - b)\gamma(\gamma - 1),$$

for some constant b. For instance, when $\beta = 0$ the type space becomes the cylinder over the set Ω considered by Armstrong and Vickers [5]. Our result is that for almost all β the exclusion region is non-empty. However, as we saw above, for $\beta = 0$ the exclusion region may be empty.

We now consider another class of models, where full participation is possible. The example will also be interesting, since it will allow us to investigate how the relative measure of excluded consumers changes with the dimension of Ω.

Example 4.3. *Let consumer's preferences be given be quasilinear with:*

$$v(\alpha, x) = \sum_{i=1}^{n} \alpha_i \sqrt{x_i},$$

and the monopolist's cost be given by

$$c(x) = \frac{1}{2} \sum_{i=1}^{n} x_i.$$

The type space is intersection of the region between balls with radii a and $a+1$with $\mathbb{R}^n_+$, i.e.

$$\Omega = \{\alpha \in R^n_+ : a \leq \|\alpha\| \leq a+1\}, \tag{4.2}$$

where $\|\cdot\|$ denotes the Euclidean norm

$$\|\beta\| = \sqrt{\sum_{i=1}^{n} \beta_i^2}.$$

To solve for the optimal nonlinear tariff with a fixed number of characteristics, consider the consumer surplus:

$$s(\alpha) = \max_x (\sum_{i=1}^{n} \alpha_i \sqrt{x_i} - t(x)).$$

By symmetry, we look for a solution of the form

$$s = s(\|\alpha\|)$$

In the "separation region"' it solves

$$\begin{cases} \frac{1}{r^{n-1}} \frac{\partial}{\partial r}(r^{n-1} s'(r)) + \frac{s'(r) f'(r)}{f(r)} = n + 1 + \frac{r f'(r)}{f(r)} \\ s'(a+1) = a+1 \end{cases} \tag{4.3}$$

where $r \equiv \|\alpha\|$ and we used the envelope theorem to obtain

$$\sqrt{x_i} = \frac{\partial s}{\partial \alpha_i}.$$

The monopolist's problem can now be written as

$$\max_s \int [\alpha \cdot \nabla s(\alpha) - c(\nabla s(\alpha)) - s(\alpha)] d\alpha$$

$$s.t.\ s(\cdot)\text{-convex},\ s \geq 0.$$

(see Rochet and Chone [22].) Ignoring for now the convexity constraint, we have a standard calculus of variations problem with free boundary. Therefore, in the participation region (i.e., the points with $s > 0$) we have:

$$\begin{aligned} \sum_{i=1}^{n} \frac{\partial}{\partial \alpha_i} \frac{\partial L}{\partial s_i} &= \frac{\partial L}{\partial s} \\ \sum_{i=1}^{n} \alpha_i \frac{\partial L}{\partial s_i} &= 0 \end{aligned} \tag{4.4}$$

(see Basov [7]), where s_i denotes the i^{th} partial derivative of s and

$$L = \alpha \cdot \nabla s(\alpha) - c(\nabla s(\alpha)) - s(\alpha)$$

Observe that this is exactly the system (4.3). Assume that types are distributed uniformly on Ω, so the derivative of the type distribution vanishes. Then, solving (4.3) we get:

$$x_i(\alpha) = [\max(0, \frac{\alpha_i}{n}(n+1-(\frac{a+1}{r})^n))]^2.$$

The corresponding iso-surplus hyper-surfaces are given by the intersection of a sphere of appropriate dimension with $\mathbb{R}^n_+$. They are parallel to the boundary, hence it is possible that the exclusion region is empty. Note that the exclusion region is given by

$$\Omega_0 = \{\alpha \in \Omega : \|\alpha\| \leq \frac{a+1}{\sqrt[n]{1+n}}\},$$

so it is non-empty if

$$\frac{a+1}{\sqrt[n]{1+n}} > a.$$

Observe that if $n = 1$ the exclusion region is empty if and only if $a > 1$, if $n = 2$ it is empty if and only if $a > 1/(\sqrt{3}-1) \approx 1.36$, and since

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{1+n}} = 1,$$

the exclusion region is non-empty for any $a > 0$ for sufficiently large n. The relative measure of the excluded consumer's (the measure of excluded consumers if we normalize the total measure of consumers to be one for all n) is:

$$\zeta = \frac{(a+1)^n/(n+1) - a^n}{(a+1)^n - a^n}.$$

It is easy to see that as $n \to \infty$ the measure of excluded consumers converges to zero as $1/n$ goes to zero, i.e. as exclusion becomes asymptotically less important. This accords with results obtained by Armstrong [3]. The convergence, however, is not monotone. For example, if $a = 1.3$ the measure of excluded customers first rises from zero for $n = 1$ to 11.6% for $n = 5$, and falls slowly thereafter. For $a = 2$ maximal exclusion of 8.3% obtains when $n = 11$ and for $a = 0.7$ maximal exclusion of 19.7% obtains when $n = 2$.

Also observe that, although an asymptotically higher fraction of consumers gets served as $n \to \infty$, this does not mean that the consumers become better off. Indeed, as $n \to \infty$ the radius of the exclusion region converges to $(a+1)$. That is, almost all served consumers are located near the upper boundary. This means that the trade-off between the efficient provision of quality and minimization of information rents disappears. Asymptotically, the monopolist provides the efficient quality but is able to appropriate almost the entire surplus.

4.1 An Application to the Regulation of a Monopolist with Unknown Demand and Cost Functions

Armstrong [4] reviews Lewis and Sappington's [18] analysis of optimal regulation of a monopolist firm when the firm's private information is two dimensional. In this analysis, a single product monopolist faces a stochastic demand function given by $q(x) = a + \theta - x$, where x is the product's price, a is a fixed parameter and θ is a stochastic component to demand, taking values in an interval $\left[\underline{\theta}, \overline{\theta}\right] \subset \mathbb{R}_+$. The firm's cost is represented by the function $C(q) = (c_0 - c)\, q + K$, where q is the quantity produced, c_0 and K are fixed parameters and c is a stochastic component to the cost, taking values in an interval[13] $[-\overline{c}, -\underline{c}] \subset \mathbb{R}_-$. The firm observes both the demand and the cost functions, but the regulator only knows that $\alpha = (\theta, c)$ is distributed according to the strictly positive continuous density function $f(\theta, c)$ on the rectangle $\Omega = \left[\underline{\theta}, \overline{\theta}\right] \times [-\overline{c}, -\underline{c}]$. For the sake of feasibility we assume that $a + \theta > c_0 - c$ for all $\alpha = (\theta, c) \in \Omega$, i.e., the highest demand exceeds marginal costs, for all possible realizations of the stochastic components of demand and costs.

The regulator wants to maximize social welfare and presents to the monopolist a menu of contracts $\{(x, t(x))\}$. If the firm chooses contract $(x, t(x))$ it sells its product at price x and pays a tax $t(x)$ from the regulator.

Therefore, the regulator's problem is to select a continuous subsidy schedule $t(\cdot) : R_+ \to \mathbb{R}$ to solve:

$$\max_{t(\cdot)} \int_{\Omega} (t(x(\alpha)) - c(x(\alpha))) f(\alpha) d\alpha,$$

where $x(\alpha)$ satisfies

$$\begin{cases} x(\alpha) \in \arg\max_{x \geq 0} u(\alpha, x) - t(x) & \text{if } \max_{x \geq 0} u(\alpha, x) - t(x) \geq 0 \\ x(\alpha) = a + \theta & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} u(\alpha, x) &= (a + \theta - x)(x - c_0 + c) - K \\ c(x) &= -\frac{1}{2}(a + \theta - x)^2, \end{aligned}$$

[13] In the original model $C(q) = (c_0 + c)\, q + K$ with $c \in [\underline{c}, \overline{c}] \subset \mathbb{R}_+$. We substitute c by its negative for convenience.

so that "cost" is the negative of the consumer's surplus. The choice of $x(\alpha)$ by the monopolist depends on whether she can derive nonnegative returns when producing. If that is not possible, she will choose $x(\alpha) = a + \theta$ and there will be zero demand, i.e., the firm shuts down.

A fundamental hypothesis in Lewis and Sappington's [18] analysis is that the parameter a can be chosen sufficiently large relative to parameters K and c_0 so that a firm will always find it in its interest to produce, even for the very small values of θ. However, Armstrong [4] shows that such a hypothesis cannot be made when Ω is the square $\Omega = \left[\underline{\theta}, \overline{\theta}\right] \times [-\overline{c}, -\underline{c}] = [0,1] \times [-1,0]$. Furthermore, when Ω is a strictly convex subset of that square, Armstrong [4] uses the optimality of exclusion theorem in Armstrong [2] to show that some firms will necessarily shut down under the optimal regulatory policy, in equilibrium. Armstrong [4] then adds "... I believe that the condition that the support be convex is *strongly* sufficient and that it will be the usual case that exclusion is optimal, even if a is much larger than the maximum possible marginal cost." That insight could not be pursued further due to a lack of a more general result, and Armstrong [4] switched to a discrete-type model in order to check the robustness of the main conclusions in Lewis and Sappington [18].

Note that the regulator's problem is essentially the standard problem solved in Section 2 of this paper. In order to apply Theorem 2.4, first note that it is sufficient that Assumptions 2.1–2.3 hold at the relevant ranges of the choice variables.

Now notice that $u(\alpha, x)$ is strictly increasing in c, as long as $a + \theta - x > 0$. But this is always the case for $x(\alpha)$, since $a + \theta - x(\alpha)$ is a demand curve. Moreover, $u(\alpha, x)$ is strictly increasing in θ, as long as $x - c_0 + c > 0$. This is again the case for $x(\alpha)$ since this is the difference between price and marginal cost. Therefore, $u(\alpha, x)$ is strictly increasing in α and bounded for the relevant choice of price x. Assumption 2.3 is also met, as long as we include type spaces that are not parallel shifts of $[0,1] \times [-1,0]$, which we can clearly do. In fact, it suffices to include the simple rotation of the unit square Ω_2 presented in Example 4.1 as one of the allowed type spaces. All the hypothesis of Theorem 2.4 are satisfied, so we may conclude that a set of positive firms will generically be excluded from the regulated market, i.e., will not produce at all. Armstrong's [4] conjecture is therefore confirmed.

4.2 An Application to Involuntary Unemployment

Consider a firm in an industry that produces n goods captured by a vector $x \in \mathbb{R}^n_+$. The firm hires workers to produce these goods. A worker is characterized by the cost she bears in order to produce goods $x \in \mathbb{R}_+^n$, which is given by the effort cost function $e(\alpha, x)$. The parameter $\alpha \in \Omega \subset \mathbb{R}^m$ is the worker's unobservable type distributed on an open, bounded set $\Omega \subset \mathbb{R}^m$ according to a strictly positive, continuous density function $f(\cdot)$.

Therefore, if a worker of type α is hired to produce output x and receives wage $\omega(x)$, her utility is $\omega(x) - e(\alpha, x)$, where $e(\alpha, \cdot)$ is cost of effort, which depends on the type of the worker. If the worker is not hired by the firm, she will receive a net utility $s_0(\alpha)$, either by working on a different firm, or by receiving unemployment compensation.

Suppose the firm sells its products for competitive international prices, $p(x)$. Then, the firm's problem is to select a wage schedule $\omega(\cdot) : R_+^n \to \mathbb{R}$ to solve:

$$\max_{\omega(\cdot)} \int_{\Omega} \left[p\left(x\left(\alpha\right)\right) x\left(\alpha\right) - \omega\left(x\left(\alpha\right)\right)\right] f(\alpha) d\alpha$$

where $x(\alpha)$ satisfies

$$\begin{cases} x(\alpha) \in \arg\max_{x \geq 0} \omega(x) - e(\alpha, x) & \text{if } \max_{x \geq 0} \omega(x) - e(\alpha, x) \geq s_0(x) \\ x(\alpha) = 0 & \text{otherwise} \end{cases}$$

Consider the following change in variables: $t(x) = -\omega(x)$, $v(\alpha, x) = -e(\alpha, x)$, $c(x) = -p(x)x$, then the firm's problem can be rewritten as:

$$\max_{t(\cdot)} \int_{\Omega} (t(x(\alpha)) - c(x(\alpha))) f(\alpha) d\alpha,$$

where $x(\alpha)$ satisfies:

$$\begin{cases} x(\alpha) \in \arg\max_{x \geq 0} v(\alpha, x) - t(x) & \text{if } \max_{x \geq 0}(v(\alpha, x) - t(x)) \geq s_0(x) \\ x(\alpha) = 0 & \text{otherwise} \end{cases}$$

Therefore, the same arguments that have been presented for the monopolist can also be extended for the hiring decision of the firm. In particular, the firm will generically find it optimal not to hire a set of positive measure. If the firm is a monopsonist in the sense that agents can work only at that firm, then Theorem 2.4 provides a rationale for involuntary unemployment. Note that, according to Theorem 3.3, the result can be extended to environments with several firms hiring for the production of goods $x \in \mathbb{R}_+^n$, so that there is an oligopsony for workers, as long as the corresponding industry is the only source of formal work.[14] This is true even in the case of free entry in that industry, according to the comment following Theorem 3.3. Finally, if one includes the category of informal work (underemployment) as unemployment, the present model suggests that an informal sector will generically exist in equilibrium.[15]

5 Conclusions

We showed that Armstrong's [2] exclusion result holds generically under weak assumptions on the underlying economic model. And in particular it holds beyond the monopoly case. So one can say that it is a robust result. Because it applies to a diverse set of markets in the economy, it offers a deep insight into the workings of market economies. In general, outside of very special cases of strong competition that mitigates the market power of individual firms, complete or one-dimensional private information, we should expect that a positive measure of types will be excluded. We have explored some settings to illustrate this finding. Further applications, and further depth on these applications, seem warranted for future research.[16]

A Appendix

Some Geometric Measure Theory Concepts

Let $U \subset \mathbb{R}^m$ be a domain, i.e. an open, simple connected set. A set $\Omega \subset \mathbb{R}^m$ has a has *finite perimeter* in U if $\Omega \cap U$ is measurable and there exists a finite Borel measure μ on U and a Borel function $v : U \to S^{m-1} \cup \{0\} \subset \mathbb{R}^m$ with

$$\int_{\Omega} \text{div}\varphi dx = \int_{U} \varphi \cdot v d\mu$$

for every Lipschitz continuous vector field $\varphi : U \to \mathbb{R}^m$ with compact support, where S^{m-1}

[14] The formulation with capacity constraints will then represent firms first opening a certain number of vacancies and then competing for workers using non-linear wage schedules.

[15] This application is, to the knowledge of the authors, the first explanation of involuntary unemployment based on the adverse selection problem, whereby firms decide to offer a wage schedule that excludes some less productive workers so they can require higher output levels from the more productive ones.

[16] Another interesting extension is the auction-theoretic setting considered in Monteiro, Svaiter, and Page, [19].

is the $m-1$ dimensional unit sphere. The perimeter of Ω in U is defined as:

$$P(\Omega, U) = \sup \int_U \text{div}\varphi dx,$$

where supremum is taken over all Lipschitz continuous vector fields with compact support and such that $\|\varphi\|_{L^\infty} \leq 1$. A set $\Omega \subset \mathbb{R}^m$ is of *locally finite perimeter* if $P(\Omega, V) < \infty$ for every open proper subset of U.

The *measure theoretic boundary* of Ω is given by

$$\partial_e(\Omega) = \{x \in \mathbb{R}^m : 0 < \mathcal{L}^m(\Omega \cap B_\varepsilon(x)) < \mathcal{L}^m(B_\varepsilon(x)),\ \forall \varepsilon > 0\}$$

where $\mathcal{L}^m$ is the $m$dimensional Lebesgue measure and $B_\varepsilon(x)$ is the open ball centered at x with radius $\varepsilon > 0$. When Ω has locally finite perimeter we have $\partial_e \Omega = \bigcup_{i=1}^\infty K_i \cup N$, where K_i is a compact subset of a C^1 hypersurface S_i, for $i = 1, 2, ...,$ and $\mathcal{H}^{m-1}(N) = 0$ where $\mathcal{H}^{m-1}$ is the $m-1$ dimensional Hausdorff measure, and a C^1 hypersurface $S \subset \mathbb{R}^m$ is a set for which ∂S is the graph of a smooth function near each $x \in \partial S$. The *measure theoretic unit outer normal* $v_\Omega(x)$ of Ω at x is the unique point $u \in S^{m-1}$ such that $\theta^m(O, x) = \theta^m(I, x) = 0$, where $O = \{y \in \Omega : (y-x) \cdot u > 0\}$ and $I = \{y \notin \Omega : (y-x) \cdot u < 0\}$, and $\theta^m(A, x)$ is the m-dimensional density at x. The reduced boundary $\partial^*\Omega$ is the set of points x for which Ω has a measure theoretic unit outer normal at x. For a set of locally finite perimeter Ω the three boundaries $\partial\Omega$, $\partial_e\Omega$ and $\partial^*\Omega$ are up to $\mathcal{H}^{m-1}$ null-sets the same.

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