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Choice-theoretic Solutions for Strategic Form Games

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Abstract

We model a player's decision as a choice set and analyze equilibria in which each player's choice set is a best response to the other players' sets. We formalize the notion of best response by an abstract "choice structure," which captures iteratively undominated strategies (for several definitions of dominance), rationalizability, and a number of formulations of choice sets. We investigate properties of choice structures and provide a general existence result for choice-theoretic solutions. We give sufficient conditions for uniqueness of a maximal solution, we provide a robust iterative procedure for computing this solution, and we show that it encompasses the strategy profiles possible under common knowledge of the choice structure. We also give sufficient conditions for uniqueness of a minimal solution for a class of games that includes two-player games with Pareto optimal payoffs and n-player games with a unique mixed strategy equilibrium. Our uniqueness results for maximal solutions explain many known features of iterative elimination of strictly dominated strategies, as well as regularities observed in the literature on rationalizability, and they apply to a number of new choice structures. Our uniqueness result for minimal solutions generalizes Shapley's (1964) uniqueness result for the saddle of a two-player, zero-sum game, and it provides conditions under which there is a unique minimal rationalizable set.

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Contents

1 Introduction

The focus of game theory, and the source of its richness, is the strategic indeterminacy inherent in many social situations: players may have incentives to engender strategic uncertainty, and in such cases, it is not possible to make a consistent point prediction in terms of pure strategies. The traditional approach to resolving this indeterminacy is to view the players' decisions (or beliefs) as probabilistic, and to model behavior in terms of mixed strategies (cf. von Neumann and Morgenstern (1944) and Nash (1951)). The standard approach yields generalized point predictions in the extended space of mixed strategy profiles, but this benefit comes at the cost of assuming a particular model of beliefs and preferences over lotteries. Moreover, the concept of mixed strategy equilibrium generally relies not only on common knowledge of rationality, but on mutual knowledge of the players' mixed strategies themselves.[1]

We propose an alternative approach to the resolution of strategic indeterminacy based on the classical theory of choice (cf. Arrow (1959), Richter (1966), and Sen (1971)), in which behavior is formalized by a choice function that specifies the plausible choices for an agent as a function of the feasible set. We model uncertainty regarding a player's decision by a *set* of plausible pure strategies, rather than using mixed strategies, that may depend on the choice sets of other players, and the objects of analysis are equilibria in terms of the players' choice sets. More precisely, we define a "solution" as a profile $(Y_1, \ldots, Y_n)$ of subsets of strategies, one set for each player, such that each player's choice set is a best response to the choice sets of the other players. We take an abstract perspective on the meaning of "best response," augmenting the standard primitives of a strategic form game with a mapping $\mathscr{C}_i(Y_1, \ldots, Y_n)$ for each player that specifies the viable choice sets (i.e., best responses) for player i given the choice sets of others. We refer to $\mathscr{C} = (\mathscr{C}_i)$ as a "choice structure." The specification of a choice structure for each player is analogous to adding preferences over mixed strategy profiles in the standard approach; indeed, a choice structure may be based on the notion of expected utility, but it need not be. Then a solution, given the choice structure $\mathscr{C}$, is a profile $(Y_1, \ldots, Y_n)$ of choice sets such that $Y_i \in \mathscr{C}_i(Y_1, \ldots, Y_n)$ for all players i.

We consider several properties of abstract choice structures and give conditions

[1] See Aumann and Brandenburger (1995) for work on the epistemic foundations of Nash equilibrium.

under which: (i) at least one solution exists, (ii) there is a unique solution that is maximal with respect to set inclusion, and this can be obtained via a robust iterative procedure, and (iii) there is a unique minimal solution. We refer to a maximal solution as a "$\mathscr{C}$-tract" and to a minimal solution as a "$\mathscr{C}$-core." Furthermore, we show that under conditions for (ii), the unique $\mathscr{C}$-tract corresponds to the possible strategy profiles when common knowledge of the choice structure is assumed (but not common or mutual knowledge of the choice sets themselves). The conditions imposed for these results are fairly permissive, and we generate a family of results as the choice structure is varied, providing insights into some existing game-theoretic concepts and some new ones. We can easily specialize the framework to capture players who eliminate strictly dominated strategies, given the other players' choice sets. Here, given $(Y_1, \ldots, Y_n)$, a player i's choice set consists of any strategy such that no strategy always yields a strictly higher payoff as the other players' strategies vary across Y_{-i}. This generates the "Shapley solutions," the coarsest of the solutions we consider. From our uniqueness results for $\mathscr{C}$-tracts, we obtain the well-known fact that iterative elimination of strictly dominated strategies determines a unique set, regardless of the order of elimination. Under this specification, a $\mathscr{C}$-core extends Shapley's (1964) notion of the saddle of a two-player, zero-sum game, and our uniqueness result for $\mathscr{C}$-cores generalizes his result to a class of multi-player games that we call "equilibrium safe." In Appendix B, we show that every two-player, zero-sum game is equilibrium safe, as are all games with a unique mixed strategy equilibrium.

The analysis captures ideas of rationalizability by defining the viable choice set of a player to consist of all pure-strategy best responses to the plausible strategies of other players, or to mixtures over the plausible strategies of others; a solution in the first sense is a point rationalizable set, and a solution in the second sense is a rationalizable set. We then obtain well-known characterizations of the (point) rationalizable strategy profiles, due to Bernheim (1984) and Pearce (1984), including the fact that the rationalizable strategy profiles are obtained by iteratively deleting all strategies that are never best responses. An implication of our results is that this outcome is independent of the precise order in which these strategies are deleted, generalizing (for the case of finite games) Bernheim's Propositions 3.1 and 3.2. We also obtain results on minimal rationalizable and point rationalizable sets, the former coinciding with the minimal CURB sets of

Basu and Weibull (1991); these solutions are axiomatized by Voorneveld, Kets, and Norde (2005) and given epistemic foundations by Asheim, Voorneveld, and Weibull (2009). Existence of such sets is relatively straightforward,[2] but we also show that in equilibrium safe games, there is a *unique* minimal rationalizable set, a set that contains all best responses to all mixed strategy profiles supported by it and that is included in all other sets with that property. This generalizes Theorem 6 of Duggan and Le Breton (1999), which holds for a special class of two-player, zero-sum games generated by an underlying tournament relation (in the sequel, we refer to these as "tournament games"), and it complements Proposition 2 of Pruzhansky (2003), which establishes a unique minimal CURB in finite extensive form games of perfect information. An implication is that there is a uniquely tightest prediction consistent with common knowledge of rationality, an observation that can significantly simplify the computation of minimal rationalizable sets in such games.[3]

Our general approach subsumes several other choice structures, some considered in the literature and some novel. Börgers (1993) characterizes the strategies that are rational (i.e., best responses to some profile of mixed strategies for the other players) for some expected utility preferences compatible with the player's ordinal payoffs. He shows that these strategies are exactly those surviving one round of elimination of an intermediate form of dominance that we call "Börgers dominance," but he does not consider the implications of common knowledge in his setting. Our results imply that: there is a unique maximal Börgers solution, that this can be obtained by iterative elimination of Börgers dominated strategies, and that these are exactly the strategy profiles implied by common knowledge of the players' ordinal preferences and rationality for some compatible von Neumann-Morgenstern preferences.[4] Moreover, we show that there is a unique minimal Börgers solution in equilibrium safe games, with an interpretation similar to that for the minimal rationalizable set, weakening common knowledge of von Neumann-Morgenstern preferences to common knowledge of ordinal preferences.

We also consider a version of the Shapley choice structure that permits mixing, while

[2] The existence result of Basu and Weibull (1991) for minimal CURB sets applies to infinite games and is more general than ours in this respect.

[3] For computational work on minimal rationalizable sets, see, e.g., Benisch, Davis, and Sandholm (2006) and Klimm, Sandholm, and Weibull (2010).

[4] See also Fishburn (1978) for an equilibrium point approach to this problem.

maintaining the focus on choice sets that consist of pure strategies: given choice sets of the other players, a player's choice set consists of any pure strategy such that there is no mixed strategy that always yields a strictly higher payoff as the strategies of other players are varied within their choice sets. This choice structure is investigated by Duggan and Le Breton (1999, 2001) for the class of tournament games, and the minimal solution corresponding to this choice structure is termed the "mixed saddle" in that work. In this paper, our results on uniqueness of minimal solutions generalize their result to equilibrium safe games. Furthermore, we prove uniqueness of maximal mixed Shapley solutions for general strategic form games, provide an iterative procedure for calculating it, and establish the common knowledge foundations of the maximal solution. We also consider new choice structures, "monotonic maximin" and "monotonic leximin," based on pessimistic conjectures that possess unique maximal solutions that can be computed by iteratively rejecting unsatisfactory strategies.

Alternative forms of rationality and non-expected utility preferences have been considered in the game-theoretic literature,[5] and equilibria in terms of sets have been considered elsewhere. Along with the saddle, Shapley (1964) defines a refinement, the weak saddle, using weak, rather than strict, dominance, and these correspond to the minimal "weak Shapley solution" in the current framework. Although existence and uniqueness of the weak saddle do not hold generally, Duggan and Le Breton (1996) establish these properties for two-player, zero-sum games with non-zero off-diagonal payoffs. What we call "point rational solutions" and "rational solutions" are the fixed points of Bernheim's (1984) best response operators, λ and Λ, respectively. Samuelson's (1992) "consistent pairs" correspond to weak Shapley solutions. Closely related is the weakly admissible set of McKelvey and Ordeshook (1976) in the context of spatial models of politics.

Kalai and Samet (1984) define the notion of persistent retract as a minimal subset of mixed strategy profiles possessing a particular stability property, and they show that when players do not have redundant strategies, each such solution is defined by a product of choice sets of pure strategies. Voorneveld (2004,2005) defines a preparation set directly as a product of choice sets satisfying the weaker stability property that each

[5]Perfect and cautious rationalizability are defined by Bernheim and Pearce in an attempt to refine away less credible rationalizable strategies. Brandenburger (1992) and Stahl (1995) explore the bounds on rational play when players evaluate lotteries using lexicographic probability systems, as in Blume, Brandenburger, and Dekel (1991).

player's choice set contain a best response to all mixed strategy profiles with probability one on the others' sets, and he proves that minimal preparation sets exist in a general class of games and that they coincide with the persistent retracts in generic games.[6] Tercieux (2006) defines a p-best response set as a product of choice sets such that each player's set contains all best responses to strategy profiles that place at least probability p on the other players' choice sets; for two-player games, he shows existence and, if $p \leq \frac{1}{2}$, uniqueness of a minimal p-best response set. Goemans, Mirrokni, and Vetta (2005) define a sink equilibrium as a subset of strategy profiles that is an irreducible, absorbing set for the "better response" graph; these subsets need not have a product structure, however, so they cannot be reduced to a product of choice sets.

The next section presents the choice-theoretic framework for the analysis of strategic form games, and it provides a basic existence result for choice structures satisfying a weak monotonicity condition. Section 3 specializes the framework to choice structures that have a binary representation and provides numerous results on the properties of such binary structures. Section 4 focusses on specific examples of choice structures and the logical relationships among them. Section 5 contains the uniqueness results of the paper. Section 6 concludes. Appendix A contains proofs omitted from Section 3, and Appendix B contains auxiliary results on equilibrium safe games.

2 Choice-theoretic Framework

In this section, we lay out the choice-theoretic framework for the analysis of strategic form games, and we provide a basic existence result. We consider a non-cooperative strategic form game $\Gamma = (I, (X_i)_{i \in I}, (u_i)_{i \in I})$, where I is a non-empty, finite set of players, denoted i or j; each X_i is a non-empty, finite set of strategies, denoted x_i, y_i, etc.; and u_i is a payoff function defined on $X \equiv \Pi_{i \in I} X_i$, with elements denoted x, y, etc.[7] Subsets of X_i are denoted Y_i or Z_i, and we write $\mathbf{X} = \left\{ \prod_{i \in I} Y_i \mid \text{for all } i, \emptyset \neq Y_i \subseteq X_i \right\}$ for the collection of products of nonempty sets of strategies. Typical elements of $\mathbf{X}$ will be denoted $Y = \prod_{i \in I} Y_i$ or $Z = \prod_{i \in I} Z_i$. Given a player j and a collection $(Y_i)_{i \neq j}$, we

[6] See Lavi and Nisan (2005) for the related concept of set-Nash equilibria in the pure sense, which demands that each player's choice set contain a best response to all profiles of pure strategies from the others' choice sets.

[7] We allow the possibility that Γ is the strategic form of a finite extensive game, or a finite Bayesian game. We do not, however, exploit the additional structure that is present in these applications.

write Y_{-j} for $\Pi_{i \neq j} Y_i$ and x_{-j}, y_{-j}, etc., for typical elements of Y_{-j}. A strategy profile x is a *Nash equilibrium* (in pure strategies) if for all i and all $y_i \in X_i$, $u_i(x) \geq u_i(y_i, x_{-i})$; and it is a *strict Nash equilibrium* if the latter inequality holds strictly for all i and all y_i distinct from x_i.

Following the classical theory of choice (Arrow (1959), Richter (1966), Sen (1971)), we model a player i's decision as a choice set, a set Y_i of strategies acceptable to i according to some criterion. Whereas the standard approach to equilibrium analysis requires players' preferences over lotteries, we must instead specify how player i formulates her choice set Y_i. This set should be allowed to vary with the choice sets of other players, Y_{-i}, and a general theory should take an abstract view of the formulation of a choice set. It is common to model a player's choice set as the maximal elements of a binary relation representing her strategic incentives, but we can more generally allow for choice sets defined by consistency properties, such as the internal and external consistency properties that characterize the stable sets of von Neumann and Morgenstern; in this case, it is important to allow for multiple choice sets consistent with a player's strategic preferences, given the choice sets of other players.[8] We develop the general model in this section, without committing to a particular mechanism for the construction of choice sets, and we impose greater structure as the analysis proceeds.

Thus, given the product set Y of strategy profiles, we will write $\mathscr{C}_i(Y)$ for the collection of viable choice sets $Y_i \subseteq X_i$ given the choice sets Y_{-i} of the other players. Formally, the correspondence $\mathscr{C}_i \colon \mathbf{X} \rightrightarrows 2^{X_i} \setminus \{\emptyset\}$ is a *choice structure for i* if for all $Y \in \mathbf{X}$,

(i) for all $Y' \in \mathbf{X}$, $Y_{-i} = Y'_{-i}$ implies $\mathscr{C}_i(Y) = \mathscr{C}_i(Y')$,

(ii) for all distinct $Y'_i, Y''_i \in \mathscr{C}_i(Y)$, neither $Y'_i \subseteq Y''_i$ nor $Y''_i \subseteq Y'_i$,

(iii) for all $Y'_i \in \mathscr{C}_i(Y)$, all $y_i \in Y'_i$ and all $x_i \in X_i$, there exists $y_{-i} \in Y_{-i}$ such that $u_i(x_i, y_{-i}) \leq u_i(y_i, y_{-i})$.

Given i, $\mathscr{C}_i$, and $Y \in \mathbf{X}$, we say Y'_i is *viable for i* if $Y'_i \in \mathscr{C}_i(Y)$. Condition (i) merely formalizes the idea that the viable choice sets for i depend on the choice sets of the other

[8] Another example of a framework that relies on multiplicity of choice sets is the theory of preparation sets, due to Voorneveld (2004,2005), in which a set Y_i is a viable choice set if for every belief with support on Y_{-i}, the set Y_i contains at least one best response. Of the special cases considered in the sequel, only Nash dominance and mixed Nash dominance allow multiple viable choice sets.

players. Condition (ii) requires that the player's viable choice sets are non-nested and implicitly excludes the use of redundant strategies, and it is consistent with our focus on maximal and minimal solution concepts in the sequel.[9] Condition (iii), which connects the choice structure to player i's payoffs in the underlying game, requires that no strategy belonging to a viable choice set is strictly dominated over profiles of other players' strategies in their choice sets. In particular, if Y_{-i} is a singleton, e.g., $Y_{-i} = \{y_{-i}\}$, then the viable choice sets $Y_i' \in \mathscr{C}_i(Y)$ for player i are subsets of best responses to y_{-i}; and if y is a strict Nash equilibrium, then it is a $\mathscr{C}$-solution. Then $\mathscr{C} = (\mathscr{C}_i)_{i \in I}$ is a *choice structure* if $\mathscr{C}_i$ is a choice structure for each i. The preceding definition is abstract, and in particular it does not impose a specific relationship between choice structure for i and the player's payoff function; of course, particular choice structures of interest will rely on further properties of the players' payoff functions.

Given the choice structure $\mathscr{C}$, a product set $Y \in \mathbf{X}$ is a *$\mathscr{C}$-solution* if for all i, we have $Y_i \in \mathscr{C}_i(Y)$; that is, the choice sets are in equilibrium, in the sense that each player's choice set is viable given correct expectations of the other players' choice sets. A product set Y is an *outer $\mathscr{C}$-solution* if for all i, there exists $Z_i \in \mathscr{C}_i(Y)$ such that $Z_i \subseteq Y_i$, and Y is an *inner $\mathscr{C}$-solution* if for all i, there exists $Z_i \in \mathscr{C}_i(Y)$ such that $Y_i \subseteq Z_i$. Note that by the assumption that choice sets are non-nested in condition (ii), Y is a $\mathscr{C}$-solution if and only if it is both an outer and inner $\mathscr{C}$-solution. Indeed, for the "if" direction, if Y is an outer and inner $\mathscr{C}$-solution, then for each i, there exist $Z_i, Z_i' \in \mathscr{C}_i(Y)$ such that $Z_i \subseteq Y_i \subseteq Z_i'$, and then the definition of choice structure implies $Y_i = Z_i = Z_i'$. A product set $Y \in 2^X$ is a *$\mathscr{C}$-base* if for all i, we have $Y_i \subseteq \bigcap \mathscr{C}_i(Y)$; note that the empty set is trivially a $\mathscr{C}$-base. We refer to a $\mathscr{C}$-solution that is maximal with respect to set inclusion as a *$\mathscr{C}$-tract*, and to one that is minimal with respect to set inclusion as a *$\mathscr{C}$-core.*

The analysis will focus on several key properties of choice structures.

Definition 1 $\mathscr{C}$ *is* monotonic *if for all $Y, Z \in \mathbf{X}$ with $Y \subseteq Z$, all i, and all $Z_i' \in \mathscr{C}_i(Z)$, there exists $Y_i' \in \mathscr{C}(Y)$ such that $Y_i' \subseteq Z_i'$.*

[9] We could define the concept of *extended choice structure* by imposing only conditions (i) and (iii). In this case, we can define a corresponding choice structure by selecting the viable sets that are minimal with respect to set inclusion. If the extended choice structure is monotonic, in the sense of Definition 1, then this selection is also monotonic, and Proposition 1 on existence and Proposition 26 on uniqueness of minimal solutions can be applied.

Definition 2 *$\mathscr{C}$ is* closed *if for all $Y, Z \in \mathbf{X}$ and all collections $(Y_i')_{i\in I}$ and $(Z_i')_{i\in I}$ such that $Y_i' \in \mathscr{C}_i(Y)$ and $Z_i' \in \mathscr{C}_i(Z)$ for all i, the set $\prod_{i\in I}(Y_i' \cap Z_i')$, if nonempty, is an outer $\mathscr{C}$-solution.*

Definition 3 *$\mathscr{C}$ is* hard *if for all $Y, Z \in \mathbf{X}$ and all collections $(Y_i')_{i\in I}$ and $(Z_i')_{i\in I}$ such that $Y_i' \in \mathscr{C}_i(Y)$ and $Z_i' \in \mathscr{C}_i(Z)$ for all i, we have $Y_i' \cup Z_i' \subseteq \bigcap \mathscr{C}_i\Big(\prod_{j\in I}(Y_j \cup Z_j)\Big)$ for all i.*

In addition, say $\mathscr{C}$ is *univalent* if the viable choice set is always unique, i.e., for all i and all $Y \in \mathbf{X}$, we have $|\mathscr{C}_i(Y)| = 1$. To understand monotonicity at an intuitive level, suppose Z_i' is a viable choice set for player i given Z_{-i}. Then, in a sense, for every expectation about the choices of other players from Z_{-i}, there is a plausible response $y_i \in Z_i'$. So, when choice sets of other players are reduced to Y_{-i}, Z_i' will still contain plausible responses to choices of other players from Y_{-i}. After collecting these responses and possibly removing some redundancies, we have a subset $Y_i' \subseteq Z_i'$ that is viable against Y_{-i}. To understand closed choice structures, we consider two products, Y and Z, and for each player i, viable sets Y_i' against Y_{-i} and Z_i' against Z_{-i}. If Y_i' contains all plausible response to expectations of other players' choices from Y_{-i}, and similarly for Z_i', then the intersection $Y_i' \cap Z_i'$ should contain plausible responses to expectations in the reduced set $Y_{-i} \cap Z_{-i}$. If this is true for every player, then possibly after removing some redundancies, we should find a $\mathscr{C}$-solution contained in $Y \cap Z$. For hard choice structures, consider choice sets Y_i' viable against Y and Z_i' viable against Z. Then each $y_i \in Y_i'$ is, in a sense, a plausible choice given some expectation over Y_{-i}; if this is so, then y_i is also a plausible choice given that all other players choose strategies in $Y_j \cup Z_j$, and thus we should have $y_{-i} \in \mathscr{C}_i(\prod_{j\neq i}(Y_j \cup Z_j))$. Note that if $\mathscr{C}$ is hard, then it is univalent: for all i, all Y, and all $Y_i' \in \mathscr{C}_i(Y)$, we have $Y_i' \subseteq \bigcap \mathscr{C}_i(Y)$ because $\mathscr{C}$ is hard, and since Y_i' is an arbitrary viable set, this implies $\mathscr{C}_i(Y) = \{Y_i'\}$.

The solutions of one choice structure, $\mathscr{C}$, may be finer than the solutions of another, $\mathscr{C}'$, in two ways: first, it may be that every $\mathscr{C}'$-solution includes some $\mathscr{C}$-solution; and second, it may be that every $\mathscr{C}$-solution is included in some $\mathscr{C}'$-solution. If both statements are true, then we say that $\mathscr{C}$ is finer than $\mathscr{C}'$ (and $\mathscr{C}'$ is coarser than $\mathscr{C}$) in the *full sense.*

A *dominance structure for i* is any mapping $\mathscr{D}_i \colon \mathbf{X} \to 2^{X_i \times X_i}$ such that for all

$Y \in \mathbf{X}$,

(1) for all $Y' \in \mathbf{X}$, $Y_{-i} = Y'_{-i}$ implies $\mathscr{D}_i(Y) = \mathscr{D}_i(Y')$,

(2) there is at least one set $Y'_i \subseteq X_i$ satisfying

(2a) there do not exist distinct $x_i, y_i \in Y'_i$ such that $x_i \mathscr{D}_i(Y) y_i$,

(2b) for all $x_i \in X_i \setminus Y'_i$, there exists $y_i \in Y'_i$ such that $y_i \mathscr{D}_i(Y) x_i$,

(3) for all $x_i, y_i \in X_i$, if $u_i(x) > u_i(y_i, x_{-i})$ for all $x_{-i} \in Y_{-i}$, then $x_i \mathscr{D}_i(Y) y_i$.

Here, we interpret the binary relation $\mathscr{D}_i(Y)$ as representing player i's strategic preferences over Y_i given the other players' choice sets Y_{-i}. The set Y'_i satisfies *internal* (resp. *external*) *stability* with respect to $\mathscr{D}_i(Y)$ if the condition (2a) (resp. condition (2b)) holds. We say $\mathscr{D} = (\mathscr{D}_i)_{i \in I}$ is a *dominance structure* if $\mathscr{D}_i$ is a dominance structure for each i. A dominance structure $\mathscr{D}$ is a *binary representation* of a choice structure $\mathscr{C}$ if for all i, all $Y \in \mathbf{X}$, and all $Y'_i \subseteq X_i$, we have $Y'_i \in \mathscr{C}_i(Y)$ if and only if conditions (2a) and (2b) hold, in which case we say $\mathscr{D}$ *generates* $\mathscr{C}$, and $\mathscr{C}$ is *binary.* In this case, we may refer to a "$\mathscr{C}$-solution" as a "$\mathscr{D}$-solution" (or an "outer $\mathscr{C}$-solution" as an "outer $\mathscr{D}$-solution," or a $\mathscr{C}$-base as a "$\mathscr{D}$-base") to bring out the dependence on the underlying dominance structure. Note that (2) is satisfied if, for example, $\mathscr{D}_i(Y)$ is acyclic for all i and all Y, but acyclicity is not necessary; Theorem 1 of Richardson (1953) establishes that a stable set satisfying internal and external consistency exists if the relation $\mathscr{D}_i(Y)$ has no cycles of odd length.

A choice structure $\mathscr{C}$ is *maximally binary* if it is generated by a dominance structure $\mathscr{D}$ such that for all i and all Y, $\mathscr{C}_i(Y)$ consists of the maximal set of $\mathscr{D}_i(Y)$, i.e.,

$$\mathscr{C}_i(Y) \quad = \quad \Big\{ \{ x_i \in Y_i \mid \text{for all } y_i \in Y_i, \text{ not } y_i \mathscr{D}_i(Y) x_i \ \} \Big\}.$$

Note that a maximally binary choice structure is by construction univalent. Moreover, every univalent choice structure is maximally binary: given any univalent choice structure $\mathscr{C}$, we can define a dominance structure $\mathscr{D}$ so that for all i, all $Y \in \mathbf{X}$, and all $x_i, y_i \in X_i$, $x_i \mathscr{D}_i(Y) y_i$ holds if and only if y_i is not a member of the unique set belonging to $\mathscr{C}_i(Y)$. This dominance structure trivially generates $\mathscr{C}$, so the maximally binary choice structures coincides with the univalent ones. Of course, some choice structures have a more natural (or useful) binary representation, while others are less naturally formulated this way.

For a monotonic choice structure, $\mathscr{C}$-solutions always exist: if an outer $\mathscr{C}$-solution Y is minimal with respect to set inclusion, in the sense that there is no other outer $\mathscr{C}$-solution Z such that $Z \subsetneqq Y$, then Y is a $\mathscr{C}$-solution. In fact, it is a simple matter to extend this result to provide a way of constructing $\mathscr{C}$-solutions containing any given $\mathscr{C}$-base, as stated in the next proposition. Because the empty set is a $\mathscr{C}$-base, the proposition does indeed imply that every monotonic choice structure $\mathscr{C}$ yields at least one $\mathscr{C}$-solution; and in fact, it implies that every outer $\mathscr{C}$-solution contains at least one $\mathscr{C}$-solution.

Proposition 1 *Assume the choice structure $\mathscr{C}$ is monotonic. If Z is a $\mathscr{C}$-base, and if Y is an outer $\mathscr{C}$-solution that is minimal among the collection*

$$\left\{ Y' \in \mathbf{X} \mid \; Y' \textit{ is an outer } \mathscr{C}\textit{-solution and } Z \subseteq Y' \; \right\},$$

then Y is a $\mathscr{C}$-solution.

For later use, we define the *mixed extension* of a strategic form game Γ as the strategic form game $\tilde{\Gamma} = (I, (\tilde{X}_i)_{i \in I}, (\tilde{u}_i)_{i \in I})$, where $\tilde{X}_i$ is the set of probability distributions (mixed strategies), denoted p_i or q_i, over X_i; the set $\tilde{X} = \Pi_{i \in I} \tilde{X}_i$ consists of strategy profiles, denoted p or q, in the mixed extension; and $\tilde{u}_i$ is the real-valued function on $\tilde{X}$ defined by

$$\tilde{u}_i(p) \quad = \quad \sum_{x \in X} \Pi_{j \in N} p_j(x_j) u_i(x).$$

A *mixed strategy Nash equilibrium* of Γ is a Nash equilibrium of the mixed extension. That is, it is a profile p of mixed strategies such that for all i and all $q_i \in \tilde{X}_i$, $\tilde{u}_i(p) \geq \tilde{u}_i(q_i, p_{-i})$. For $p_i \in \tilde{X}_i$, let $\sigma_i(p_i) = \{x_i \in X_i | p_i(x_i) > 0\}$ denote the support of p_i, and for $p \in \tilde{X}$, let $\sigma(p) = \Pi_{i \in I} \sigma_i(p_i)$. As usual, $\sigma_{-i}(p_{-i}) = \Pi_{j \neq i} \sigma_j(p_j)$. We will simply write $x_i \in \tilde{X}_i$ for the mixed strategy that places probability one on x_i, and given a set $Y_i \subseteq X_i$, we write $\tilde{Y}_i$ for the mixed strategies with support contained in Y_i.

3 Binary Choice Structures

In this section, we specialize the framework to binary choice structures, which admit a rich class of choice structures that have been considered in the literature. Specifically, we provide an elementary characterization of $\mathscr{D}$-solutions in terms of minimal externally stable choice sets; we give sufficient conditions for the properties introduced in the previous section; we propose the monotonic kernel of a dominance structure as a way of improving a poorly behaved dominance structure; and we provide a basis for comparison of solutions generated by specific dominance structures introduced in the next section.

3.1 Properties of Dominance Structures

A dominance structure $\mathscr{D}$ is *transitive* (resp. *irreflexive*) if for all i and all $Y \in \mathbf{X}$, $\mathscr{D}(Y)$ is a transitive (resp. irreflexive) relation on X_i. It is *monotonic* if for all i, all $Y, Z \in \mathbf{X}$ with $Y \subseteq Z$, and all $x_i, y_i \in X_i$, $x_i \mathscr{D}_i(Z) y_i$ implies $x_i \mathscr{D}(Y) y_i$. In words, monotonicity means that if x_i dominates y_i over a set of profiles of strategies of other players, then x_i dominates y_i over any smaller set. Before proceeding, we establish an elementary fact that will be useful in the subsequent analysis: for a monotonic and transitive dominance structure, a choice set is viable if and only if it is a minimal externally stable set.

Proposition 2 *Let $\mathscr{D}$ be a dominance structure and $\mathscr{C}$ the choice structure generated by $\mathscr{D}$. For all i, all $Y \in \mathbf{X}$, and all $Y_i' \subseteq X_i$, if $Y_i' \in \mathscr{C}_i(Y)$, then Y_i' is minimal with respect to set inclusion among the sets that are externally stable with respect to $\mathscr{D}_i(Y)$; furthermore, the converse holds if $\mathscr{D}$ is monotonic and transitive.*

As the next proposition shows, the transitive and monotonic dominance structures generate monotonic choice structures. Thus, with Proposition 1, we can easily generate a family of choice structures for which $\mathscr{C}$-solutions generally exist; see the next section for a number of examples. The proof of this result and the next rely critically on Proposition 2.

Proposition 3 *Assume the dominance structure $\mathscr{D}$ is transitive and monotonic. Then the choice structure $\mathscr{C}$ generated by $\mathscr{D}$ is monotonic.*

The dominance structure $\mathscr{D}$ is *weakly irreflexive* if for all i, all $x_i, y_i \in X_i$, and all $Y \in \mathbf{X}$, $x_i \mathscr{D}_i(Y) x_i$ implies $y_i \mathscr{D}_i(Y) x_i$. That is, viewing $\mathscr{D}_i(Y)$ intuitively as a ranking, irreflexivities can occur only among bottom-ranked strategies. The next proposition establishes that the choice structure generated by a transitive, monotonic, weakly irreflexive dominance structure is closed.

Proposition 4 *Assume the dominance structure $\mathscr{D}$ is transitive, monotonic, and weakly irreflexive. Then the choice structure $\mathscr{C}$ generated by $\mathscr{D}$ is closed.*

A dominance structure $\mathscr{D}$ is *non-trivial* if for all i and all $Y \in \mathbf{X}$, there exist $x_i, y_i \in X_i$ such that not $x_i \mathscr{D}_i(Y) y_i$. Clearly, every irreflexive dominance structure is non-trivial. The next proposition shows that the above properties of dominance structures yield hard choice structures.

Proposition 5 *Assume the dominance structure $\mathscr{D}$ is transitive, monotonic, weakly irreflexive, and non-trivial. Then the choice structure $\mathscr{C}$ generated by $\mathscr{D}$ is hard.*

3.2 Monotonic Kernels

Although not all dominance structures are monotonic, we can give a general method for transforming a non-monotonic dominance structure $\mathscr{D}$ into a monotonic one, the *monotonic kernel* of $\mathscr{D}$, denoted $\mathscr{D}^\bullet$ and defined as follows: $x_i \mathscr{D}_i^\bullet(Y) y_i$ if and only if for all $Z \in \mathbf{X}$ with $Z \subseteq Y$, there exists $z_i \in X_i$ such that $z_i \mathscr{D}_i(Z) y_i$. Note that the relation $x_i \mathscr{D}_i^\bullet(Y) y_i$ does not indicate a relationship that is specific to the pair (x_i, y_i), but rather it indicates the status of y_i relative to a selection of strategies, one for each product subset $Z \subseteq Y$. By construction, the monotonic kernel of a dominance structure is not only monotonic, but transitive as well, so $\mathscr{D}^\bullet$-solutions exist generally, even if the properties of the original dominance structure are poor. These observations are formalized in the next proposition, which also establishes that monotonic kernels are weakly irreflexive and that they generate closed and hard choice structures quite generally.

Proposition 6 *Let $\mathscr{D}$ be a dominance structure. Then the monotonic kernel $\mathscr{D}^\bullet$ is transitive, monotonic, and weakly irreflexive, and the choice structure generated by $\mathscr{D}^\bullet$ is closed. Furthermore, if $\mathscr{D}$ is transitive, weakly irreflexive, and non-trivial, then $\mathscr{D}^\bullet$ is non-trivial, and the choice structure it generates is hard.*

A potential deficiency of the monotonic kernel is that the notions of internal and external stability characterizing the $\mathscr{D}^\bullet$-solutions are not the natural ones: if $\mathscr{C}^\bullet$ is the choice structure generated by $\mathscr{D}^\bullet$ and $Y_i' \in \mathscr{C}_i(Y)$, for example, then external stability of Y_i' with respect to $\mathscr{D}_i^\bullet(Y)$ means that for every $x_i \in X_i \setminus Y_i'$, there is some $y_i \in Y_i'$ such that $y_i \mathscr{D}_i^\bullet(Y) x_i$, but this just means that for every product set $Z \subseteq Y$, we have $z_i \mathscr{D}_i(Z) x_i$ for some z_i (not necessarily for z_i in Y_i). Thus, external stability with respect to the monotonic kernel does not by definition imply an advantage for the elements of the player's choice set. The next proposition establishes that when the original dominance structure is transitive and weakly irreflexive, the issue is moot: the solutions generated by the monotonic kernel are characterized by stability conditions that, as desired, confer an advantage for chosen strategies over unchosen ones.

Proposition 7 *Let $\mathscr{D}$ be a dominance structure, and let $\mathscr{C}^\bullet$ be the choice structure generated by its monotonic kernel. If $\mathscr{D}$ is transitive and weakly irreflexive, then for all i, all $Y \in \mathbf{X}$, and all $Y_i' \subseteq X_i$, $Y_i' \in \mathscr{C}_i^\bullet(Y)$ if and only if*

(i) for all $x_i \in Y_i'$, there exists $Z \in \mathbf{X}$ with $Z \subseteq Y$ such that for all $y_i \in Y_i'$, not $y_i \mathscr{D}_i(Z) x_i$,

(ii) for all $x_i \in X_i \setminus Y_i'$ and all $Z \in \mathbf{X}$ with $Z \subseteq Y$, there exists $y_i \in Y_i'$ such that $y_i \mathscr{D}_i(Z) x_i$.

3.3 Comparing Dominance Structures

A dominance structure $\mathscr{D}$ is *stronger* than another $\mathscr{D}'$ if for all i and all $Y \in \mathbf{X}$, $\mathscr{D}_i(Y) \subseteq \mathscr{D}_i'(Y)$, i.e., $x_i \mathscr{D}_i(Y) y_i$ implies $x_i \mathscr{D}_i'(Y) y_i$ for all $x_i, y_i \in X_i$; in this case, we say that $\mathscr{D}'$ is *weaker* than $\mathscr{D}$. For a related notion, we say $\mathscr{D}$ *subjugates* $\mathscr{D}'$ if for all i, all $Y \in \mathbf{X}$, and all $x_i, y_i \in X_i$, if $y_i \mathscr{D}_i(Y) x_i$, then there exists $z_i \in X_i \setminus \{x_i\}$ such that $z_i \mathscr{D}_i'(Y) x_i$. In words, this means that if y_i dominates x_i according to $\mathscr{D}_i(Y)$, then x_i is also dominated by a strategy (though not necessarily y_i) according to $\mathscr{D}_i(Y)$. Note that if $\mathscr{D}$ is stronger than $\mathscr{D}'$ and $\mathscr{D}'$ is irreflexive, then $\mathscr{D}$ subjugates $\mathscr{D}'$.

Proposition 8 *Let $\mathscr{D}$ and $\mathscr{D}'$ be dominance structures. (i) If $\mathscr{D}$ is stronger than $\mathscr{D}'$, then every $\mathscr{D}$-solution is an outer $\mathscr{D}'$-solution. (ii) If $\mathscr{D}$ subjugates $\mathscr{D}'$ and $\mathscr{D}'$ is transitive, then every $\mathscr{D}'$-solution is a $\mathscr{D}$-base.*

Proposition 8 immediately provides a means for comparing the solutions of different dominance structures. It follows that if $\mathscr{D}$ is stronger than and subjugates $\mathscr{D}'$, and if both dominance structures are transitive and monotonic, then the solutions of $\mathscr{D}'$ are finer than the solutions of $\mathscr{D}$ in the full sense.

Proposition 9 *Let $\mathscr{D}$ and $\mathscr{D}'$ be dominance structures. (i) If $\mathscr{D}$ is stronger than $\mathscr{D}'$, and if $\mathscr{D}'$ is transitive and monotonic, then every $\mathscr{D}$-solution includes some $\mathscr{D}'$-solution. (ii) If $\mathscr{D}$ subjugates $\mathscr{D}'$, if $\mathscr{D}'$ is transitive, and if $\mathscr{D}$ is transitive and monotonic, then every $\mathscr{D}'$-solution is included in some $\mathscr{D}$-solution.*

It is apparent that if the original $\mathscr{D}$ is irreflexive, then the monotonic kernel $\mathscr{D}^\bullet$ subjugates $\mathscr{D}$. Thus, Proposition 9 immediately implies that if $\mathscr{D}$ is transitive, then each $\mathscr{D}$-solution (if any exist) is included in some $\mathscr{D}^\bullet$-solution.

4 Special Choice Structures

This section explores a number of specific choice structures of interest. In the first subsection, we focus on binary choice structures generated by three dominance structures: Shapley dominance, weak Shapley dominance, and Nash dominance. The first two of these dominance structures extend concepts defined by Shapley (1964) for two-player, zero-sum games: whereas Shapley analyzes the minimal Shapley solution in this smaller class of games, here we extend the analysis to arbitrary (finite) strategic form games and to general (not necessarily minimal) solutions. The third dominance structure provides a choice-theoretic version of pure strategy Nash equilibrium. We note that Shapley and Nash solutions exist in general, but weak Shapley solutions may not exist in some games.[10] We then define a closely related dominance structure, due to Börgers (1993), for which solutions do generally exist. The solutions generated by the above dominance structures are typically weak, but Nash solutions are not unambiguously weaker than mixed strategy Nash equilibrium: we give an example in which there is a mixed strategy Nash equilibrium that is disjoint from every Nash solution.

In the second subsection, we define additional dominance structures based on pessimistic conjectures, maximin dominance and leximin dominance. Although these dominance structures are not monotonic, we consider their monotonic kernels, which are

[10] Duggan and Le Breton (1996) show that weak Shapley solutions exist in two-player, zero-sum games with non-zero off-diagonal payoffs; for these games, there is a unique minimal $\mathscr{W}$-solution, which is referred to as the "weak saddle."

well-behaved. Finally, we consider the logical relationships among the solutions entailed by these dominance structures, and in particular we note that the monotonic leximin solutions provide a refinement of the Börgers solutions.

In the third subsection, we examine choice structures based on rationalizability and obtain the point rationalizable strategies of Bernheim (1984) and the rationalizable strategies of Bernheim (1984) and Pearce (1984). In fact, we show subsequently that the point rationalizable (resp. rationalizable) strategy profiles correspond to the maximal $\mathscr{P}$-solution (resp. $\mathscr{R}$-solution), which is unique. We discuss the related idea of cautious rationalizability, due to Pearce (1984), and technical difficulties of that solution. Finally, we summarize the logical relationships between the monotonic choice structures considered, and in particular we note that the rational solutions refine the Börgers solutions.

In the last subsection, we consider the choice structures induced by mixed versions of the above dominance structures, in which a player's choice set is defined by internal and external stability conditions that permit the player to use mixed strategies. In particular, the mixed version of Shapley dominance generalizes Duggan and Le Breton's (1999, 2001) concept of mixed saddle: whereas that work analyzes the minimal mixed Shapley saddle of a tournament game, here we extend the analysis to arbitrary (finite) strategic form games and to general (not necessarily minimal) solutions. Finally, we explore the logical relationships among the solutions entailed by these choice structures, and in particular we note that the mixed Shapley solutions refine the Shapley solutions and coarsen the rational solutions.

4.1 Shapley, Weak Shapley, and Nash Dominance

Three dominance structures of interest are defined next. The first two, Shapley and weak Shapley dominance, are quite demanding — a dominated strategy *must,* in a sense, be rejected — while the third, Nash dominance, is only suggestive — a dominated strategy *can* be rejected without harm. In effect, Nash dominance eliminates redundancies from a player's choice set.

(1) **Nash** for all i and all Y, $x_i \ \mathscr{N}_i(Y)\ y_i$ if and only if for all $x_{-i} \in Y_{-i}$, $u_i(x) \geq u_i(y_i, x_{-i})$.

(2) **Weak Shapley** for all i and all Y, $x_i \mathscr{W}_i(Y) y_i$ if and only if $x_i \mathscr{N}_i(Y) y_i$ and not $y_i \mathscr{N}_i(Y) x_i$.

(3) **Shapley** for all i and all Y, $x_i \mathscr{S}_i(Y) y_i$ if and only if for all $x_{-i} \in Y_{-i}$, $u_i(x) > u_i(y_i, x_{-i})$.

Note the connection between $\mathscr{N}$-solutions and pure strategy Nash equilibria: $\{x\}$ is a $\mathscr{N}$-solution if and only if x is a pure strategy Nash equilibrium; of course, not all $\mathscr{N}$-solutions are singletons. Also note that $\{x\}$ is a $\mathscr{S}$-solution (or $\mathscr{W}$-solution) if and only if x is a strict Nash equilibrium. Clearly, the three dominance structures coincide for generic games, where $u_i(x) \neq u_i(y)$ for all i, x, and y, but in general they are distinct; see Example 3 below for illustrations of the solutions of these different dominance structures.

All three dominance structures are transitive, and $\mathscr{W}$ and $\mathscr{S}$ are irreflexive as well. Thus, the choice structures generated by Shapley and weak Shapley dominance are maximally binary, and therefore univalent. Because $\mathscr{N}$ violates irreflexivity, it need not be univalent. The Shapley and Nash dominance structures are monotonic, as their definitions invoke either all weak or all strict inequalities, so Proposition 3 implies that the corresponding choice structures are monotonic; in particular, $\mathscr{S}$-solutions and $\mathscr{N}$-solutions exist.

The next proposition establishes that $\mathscr{S}$-solutions are the weakest possible in our framework, as every $\mathscr{C}$-solution is contained in an $\mathscr{S}$-solution. The proof is immediate: if Y is a $\mathscr{C}$-solution, then part (3) in the definition of choice structure implies that Y is an $\mathscr{S}$-base, and thus Proposition 1 yields an $\mathscr{S}$-solution $Z \supseteq Y$. Moreover, we show later that there is a unique $\mathscr{S}$-tract, which is therefore the largest possible solution.

Proposition 10 *For every choice structure $\mathscr{C}$, if Y is a $\mathscr{C}$-solution, then there is an $\mathscr{S}$-solution Z such that $Y \subseteq Z$.*

As well, Propositions 4 and 5 imply that the choice structure generated by $\mathscr{S}$ is closed and hard, but $\mathscr{N}$ does not deliver either property generally, as demonstrated in the following example.

Example 1 Let $|I| = 2$ and $X_1 = X_2 = \{a, b, c\}$, with payoffs given below.

	a	b	c
a	(1,1)	(1,2)	(1,1)
b	(2,1)	(0,0)	(2,1)
c	(1,1)	(1,2)	(1,1)

Here, $\{a, b\} \times \{a, b\}$ and $\{b, c\} \times \{b, c\}$ are $\mathscr{N}$-solutions, but the product of the intersection of choice sets, $\{b\} \times \{b\}$, is not an outer $\mathscr{N}$-solution. In addition, letting $\mathscr{C}$ be the choice structure generated by $\mathscr{N}$, we have $\{a, b\} \in \mathscr{C}_i(\{a, b\})$ and $\{b, c\} \in \mathscr{C}_i(\{b, c\})$ for both players, but $X_i \notin \mathscr{C}_i(X)$. Therefore, $\mathscr{C}$ is neither closed nor hard.

Next is a counter-example to a general result for $\mathscr{W}$-solutions. It is quite similar to Samuelson's (1992) Example 8, where he illustrates the possible inconsistency of common knowledge of "admissibility" (a generalization of our $\mathscr{W}$-solutions). Following the example, we will see how existence can be obtained for a dominance structure closely related to $\mathscr{W}$.

Example 2 Let $|I| = 2$ and $X_1 = X_2 = \{a, b\}$, with payoffs specified below.

	a	b
a	(2,1)	(1,2)
b	(1,2)	(1,1)

Suppose there is a $\mathscr{W}$-solution $Y = Y_1 \times Y_2$. If $Y_2 = \{a, b\}$ then $Y_1 = \{a\}$, but then $Y_2 = \{b\}$. If $Y_2 = \{a\}$ then $Y_1 = \{a\}$, but then $Y_2 = \{b\}$. Finally, if $Y_2 = \{b\}$ then $Y_1 = \{a, b\}$, and then $Y_2 = \{a, b\}$, a contradiction.

The $\mathscr{W}$-solution sometimes fails to exist because $\mathscr{W}$ is not generally monotonic. Börgers (1993) defines a closely related notion of dominance in his investigation of rationality in games when players know only the ordinal preferences of their opponents. Unlike $\mathscr{W}$, this dominance structure is monotonic.

(4) **Börgers** for all i and all Y, $x_i \ \mathscr{B}_i(Y) \ y_i$ if and only if for all $Z \subseteq Y$, there exists $z_i \in X_i$ such that $z_i \ \mathscr{W}_i(Z) \ y_i$.

Note that $x_i \ \mathscr{B}_i(Y) \ y_i$ offers grounds that y_i be rejected, but not necessarily because x_i should be chosen instead — this relation indicates not an advantage of x_i, but only a deficiency of y_i. Nonetheless, $\mathscr{B}$ is a well-behaved dominance structure by our criteria: it is transitive and evidently monotonic, so Propositions 1 and 3 establish existence of $\mathscr{B}$-solutions. In fact, Börgers dominance is the monotonic kernel of weak Shapley, i.e., $\mathscr{B} = \mathscr{W}^\bullet$, so we see that the dominance notion proposed by Börgers is an example of the more general monotonic kernel operation applied to a specific notion of dominance.[11]

The next example illustrates the solutions for the dominance structures of this subsection. It suggests a particular nesting of solutions that is confirmed for the general case in the subsequent discussion.

Example 3 Let $I = 2$, $X_1 = \{a, b, c\}$, and $X_2 = \{a, b, c, d, e\}$, with payoffs specified below.

	a	b	c	d	e
a	(1,-1)	(-1,1)	(0,0)	(1,-1)	(0,-1)
b	(-1,1)	(1,-1)	(0,0)	(0,0)	(0,0)
c	(-1,1)	(1,-1)	(0,0)	(0,1)	(0,0)

Here $\{a, b\} \times \{a, b\}$ is not a $\mathscr{N}$-solution, a $\mathscr{W}$-solution, a $\mathscr{B}$-solution, or an $\mathscr{S}$-solution; $\{a, b\} \times \{a, b, c\}$ is a $\mathscr{N}$-solution but not a $\mathscr{W}$-solution, a $\mathscr{B}$-solution, or an $\mathscr{S}$-solution; $\{a, b, c\} \times \{a, b, c\}$ is a $\mathscr{W}$-solution but not a solution for the other dominance structures;

[11] An alternative approach to rectifying weak Shapley dominance is to define player i's choice set to consist of x_i such that the following does not hold: for each $x_{-i} \in Y_{-i}$, there exists y_i with $y_i \mathscr{N}_i(Y) x_i$ and $u_i(y_i, x_{-i}) > u_i(x)$. This choice structure is monotonic, closed, and hard. For reasons of space, we leave it for future investigation.

$$
\begin{tikzcd}
\mathscr{S} \arrow[r] & \mathscr{B} \arrow[r] \arrow[d, Rightarrow] \arrow[dr] & \text{mse} \\
& \mathscr{W} \arrow[r, dashed] & \mathscr{N} \arrow[u, dashed]
\end{tikzcd}
$$

Figure 1: Logical relationships

$\{a,b,c\}\times\{a,b,c,d\}$ is a $\mathscr{B}$-solution but not a solution for the other dominance structures; and X is an $\mathscr{S}$-solution but not a solution for the other dominance structures.

To compare the solutions generated by these dominance structures, note that Shapley dominance is stronger than weak Shapley, which is stronger than Nash. With Proposition 9, this immediately yields the logical relationships depicted in Figure 1. Here, a thick arrow from one dominance structure to another indicates that the second is finer than the first in the full sense: e.g., every Shapley solution contains a Börgers solution, and every Börgers solution is contained in a Shapley solution. The double arrow indicates that every $\mathscr{W}$-solution is included in some $\mathscr{B}$-solution. Indeed, the latter follows because $\mathscr{W}$ is irreflexive, so its monotonic kernel $\mathscr{B} = \mathscr{W}^{\bullet}$ subjugates it; since $\mathscr{W}$ is transitive and $\mathscr{B}$ is transitive and monotonic, the claim follows from Proposition 9. A dashed arrow indicates that the solutions of the second dominance structure are finer than the solutions of the first in the sense that, e.g., every $\mathscr{W}$-solution includes some $\mathscr{N}$-solution; because $\mathscr{N}$ is transitive and monotonic, this follows directly from Proposition 9. Further relationships that follow by transitivity are obvious and omitted from the figure for simplicity.

In addition to the relationships described above, Figure 1 indicates that every $\mathscr{N}$-solution includes the support $\sigma(p)$ of at least one mixed strategy equilibrium; and it indicates that for every mixed strategy equilibrium p, there is at least one $\mathscr{B}$-solution that contains the support set $\sigma(p)$. These connections are established formally in the following proposition.

Proposition 11 *(i) If Y is an outer $\mathscr{N}$-solution, then there is a mixed strategy Nash equilibrium p such that $\sigma(p) \subseteq Y$. (ii) If p is a mixed strategy Nash equilibrium, then there is a $\mathscr{B}$-solution $Y \supseteq \sigma(p)$.*

Proof: (i) Let Y be an outer $\mathscr{N}$-solution, and consider the restricted game in which each player's strategy set is Y_i and payoffs are given by u_i restricted to Y, and let p be a mixed strategy Nash equilibrium of this game. If it is not an equilibrium of the unrestricted game, there is some i and some $y_i \in X_i \setminus Y_i$ such that $\tilde{u}_i(y_i, p_{-i}) > \tilde{u}_i(p)$. Since Y is an outer $\mathscr{N}$-solution, there exists some $x_i \in Y_i$ such that $x_i \mathscr{N}_i(Y_{-i}) y_i$, but then $\tilde{u}_i(x_i, p_{-i}) > \tilde{u}_i(p)$, and p is not an equilibrium of the restricted game, a contradiction. (ii) We claim that $\sigma(p)$ is a $\mathscr{B}$-base, for otherwise there exist i and $Y_i \in \mathscr{C}_i(\sigma(p))$ such that $\sigma_i(p_i) \setminus Y_i \neq \emptyset$, where $\mathscr{C}$ is the choice structure generated by $\mathscr{B}$. Note that because $\mathscr{B}$ is transitive and monotonic, Proposition 3 implies that $\mathscr{C}$ is monotonic. Letting $z_i \in \sigma_i(p_i) \setminus Y_i$, external stability yields $y_i \in Y_i$ such that $y_i \mathscr{B}_i(\sigma(p)) z_i$. In particular, there exists $x_i \in X_i$ such that $x_i \mathscr{W}_i(\sigma(p)) y_i$, but then $\tilde{u}_i(x_i, p_{-i}) > \tilde{u}_i(z_i, p_{-i})$, contradicting the fact that z_i is a best response to p_{-i}. Thus, $\sigma(p)$ is a $\mathscr{B}$-base, and Proposition 1 yields a $\mathscr{B}$-solution $Y \supseteq \sigma(p)$. ∎

The next example shows it is not generally the case that the support of every mixed strategy equilibrium is included in an $\mathscr{N}$-solution. In fact, in the example there is a mixed strategy equilibrium with support disjoint from every $\mathscr{N}$-solution. Thus, we can use $\mathscr{N}$-solutions to refine mixed strategy Nash equilibria, effectively eliminating those in which redundant strategies are used: every game has at least one $\mathscr{N}$-solution and every $\mathscr{N}$-solution includes the support of at least one Nash equilibrium, so we can retain these equilibria and discard the others.

Example 4 Let $|I| = 2$, $X_1 = \{a, b\}$, and $X_2 = \{a, b, c, d\}$, with payoffs given below.

	a	b	c	d
a	(1,0)	(1,10)	(1,11)	(1,-1)
b	(1,10)	(1,0)	(1,-1)	(1,11)

Here, the mixed strategies that place probability one half each on a and b constitute a mixed strategy Nash equilibrium with minimal support, $\{a, b\} \times \{a, b\}$. However, there is no $\mathscr{N}$-solution intersecting $\{a, b\} \times \{a, b\}$: the only $\mathscr{N}$-solutions are $\{a\} \times \{c\}$ and $\{b\} \times \{d\}$.

4.2 Maximin and Leximin Dominance

We next define a dominance structure that reflects pessimistic assessments: one strategy is strategically advantaged over another if it does better in the worst case scenario, subject to the constraint that other players' strategies are selected from their choice sets.

(5) **maximin** for all i and all Y, $x_i \mathscr{M}_i(Y) y_i$ if and only if

$$\min_{y_{-i}\in Y_{-i}} u_i(x_i, y_{-i}) \quad > \quad \min_{y_{-i}\in Y_{-i}} u_i(y_i, y_{-i}).$$

A refinement of the latter dominance structure is defined is next. Given two real-valued functions v and v' defined on any subset $Z_{-i} \subseteq X_{-i}$, let $Z_{-i}(v, v') = \{y_{-i} \in Z_{-i} | v(y_{-i}) \neq v'(y_{-i})\}$. We write $vL_i(Z_{-i})v'$ if $Z_{-i}(v, v') \neq \emptyset$ and $\min\{v(z_{-i})|z_{-i} \in Z_{-i}(v, v')\} > \min\{v'(z_{-i})|z_{-i} \in Z_{-i}(v, v')\}$.

(6) **leximin** for all i and all Y, $x_i \mathscr{L}_i(Y) y_i$ if and only if

$$u_i(x_i, \cdot) L_i(Y_{-i}) u_i(y_i, \cdot).$$

This dominance structure is similar to $\mathscr{M}$, but in case the minimum payoffs of z_i and y_i over Y_{-i} are tied, we compare their next lowest payoffs, or if those are tied, the next lowest, and so on. These dominance structures are irreflexive and transitive, so the choice structures generated by them are rational and univalent, but the dominance structures are not generally monotonic, and their corresponding solutions may fail to exist, as illustrated in the next example.

Example 5 Let $|I| = 2$, $X_1 = \{a, b\}$, and $X_2 = \{a, b\}$, with payoffs given below.

	a	b
a	(1,2)	(1,1)
b	(1,1)	(0,2)

Here, in any $\mathscr{M}$- or $\mathscr{L}$-solution, if row's choice set contains b, and then column's contains b, and then row's contains a, and then column's contains a, but then row cannot choose

b; and if row's choice set contains a, then column's contains a, and then row's contains b. Thus, there is no solution for either dominance structure.

In response to this failure, we can consider the monotonic kernels, $\mathscr{M}^\bullet$ and $\mathscr{L}^\bullet$, which by Proposition 1 and 6 rectify the existence problem. These dominance structures are defined explicitly as follows.

(7) **monotonic maximin** for all i and all Y, $x_i \mathscr{M}_i^\bullet(Y) y_i$ if and only if for all $Z \in \mathbf{X}$ with $Z \subseteq Y$, there exists $z_i \in X_i$ such that

$$\min_{y_{-i} \in Z_{-i}} u_i(z_i, y_{-i}) \quad > \quad \min_{y_{-i} \in Z_{-i}} u_i(y_i, y_{-i}).$$

(8) **monotonic leximin** for all i and all Y, $x_i \mathscr{L}_i^\bullet(Y) y_i$ if and only if for all $Z \in \mathbf{X}$ with $Z \subseteq Y$, there exists $z_i \in X_i$ such that

$$u_i(z_i, \cdot) L_i(Z_{-i}) u_i(y_i, \cdot).$$

Because $\mathscr{M}$ is transitive and irreflexive, Proposition 7 implies that the choice structure generated by $\mathscr{M}^\bullet$ is characterized by appropriate internal and external stability conditions, and likewise for $\mathscr{L}^\bullet$. By Proposition 6, these monotonic kernels are transitive, monotonic, weakly irreflexive, and non-trivial; and the choice structures they generate are closed and hard.[12] Thus, because $\mathscr{M}$ and $\mathscr{L}$ are irreflexive, these dominance structures are subjugated by their respective monotonic kernels, and Proposition 9 implies that every $\mathscr{M}$-solution (if any) is included in some $\mathscr{M}^\bullet$-solution, and every $\mathscr{L}$-solution (if any) is included in some $\mathscr{L}^\bullet$-solution.

Note that Shapley dominance is stronger than (and subjugates) monotonic maximin, so Proposition 9 implies that $\mathscr{M}^\bullet$ is finer than $\mathscr{S}$ in the full sense. As well, Börgers dominance is stronger than (and subjugates) monotonic leximin, so that $\mathscr{L}^\bullet$ is finer than $\mathscr{B}$ in the full sense. Figure 2 indicates the logical relationships among the transitive, monotonic dominance structures defined above. The next example illustrates that an

[12] The definitions of monotonic maximin and monotonic leximin suggest a family of well-behaved structures. For example, we could modify monotonic maximin by replacing "min" with "max" in the definition of $\mathscr{M}^\bullet$. Or we could require that the maximum *and* minimum payoff of z_i over Z_{-i} are greater than the maximum and minimum payoff, respectively, of y_i over Z_{-i}. Or we could require that for each Z_{-i}, there exist z_i and z_i' such that the minimum payoff of z_i over Z_i is higher than the minimum of y_i, and the maximum of z_i' over Z_{-i} is higher than the maximum of y_i.

$$
\begin{tikzcd}
\mathscr{S} \arrow[r] \arrow[dr] & \mathscr{B} \arrow[r] \arrow[dr] & \mathscr{N} \arrow[r, dashed] & \text{mse} \\
 & \mathscr{M}^{\bullet} \arrow[r] & \mathscr{L}^{\bullet} &
\end{tikzcd}
$$

Figure 2: Logical relationships

$\mathscr{M}^{\bullet}$-solution, in contrast to an $\mathscr{L}^{\bullet}$-solution, need not be included in a $\mathscr{B}$-solution; indeed, it suggests that the internal stability requirement imposed by monotonic maximin dominance may be unreasonably weak.

Example 6 Let $|I| = 2$, $X_1 = \{a, b\}$, and $X_2 = \{a, b, c\}$, with payoffs given below.

	a	b	c
a	(0,1)	(0,0)	(0,0)
b	(0,0)	(0,1)	(0,0)

Here, $\{a, b\} \times \{a, b, c\}$ is an $\mathscr{M}^{\bullet}$–solution, but $\{a, b\} \times \{a, b\}$ is the unique $\mathscr{B}$-solution.

The next example shows that $\mathscr{N}$-solutions and $\mathscr{B}$-solutions are not generally included within $\mathscr{L}^{\bullet}$- or even $\mathscr{M}^{\bullet}$-solutions.

Example 7 Let $|I| = 2$, $X_1 = \{a, b\}$, and $X_2 = \{a, b, c\}$, with payoffs given below.

	a	b	c
a	(1,-1)	(0,0)	(0,-2)
b	(0,1)	(1,0)	(0,2)

Here, $\{a, b\} \times \{a, b, c\}$ is an $\mathscr{N}$-solution and a $\mathscr{B}$-solution, but it is not an $\mathscr{M}^{\bullet}$- or $\mathscr{L}^{\bullet}$-solution, because the lowest payoff to column player from a is less than the lowest from b when row player's choice set is $\{a\}$ or $\{a, b\}$, and it is less than the payoff from c when row player's choice set is $\{b\}$.

Clearly, a singleton $Y = \{y\}$ is an $\mathscr{M}^\bullet$-solution if and only if it is an $\mathscr{L}^\bullet$-solution, which holds if and only if y is a strict Nash equilibrium. Furthermore, it is easily seen that if y is a Nash equilibrium, then it is contained in an $\mathscr{M}^\bullet$-solution and in an $\mathscr{L}$-solution: if y is a Nash equilibrium, then $\{y\}$ is an $\mathscr{M}^\bullet$-base and an $\mathscr{L}^\bullet$-base, so the claim follows from Proposition 1. In general, however, it is not possible to state a close relationship between mixed strategy equilibria and the solutions generated by these choice structures. In the next example, there is an $\mathscr{M}^\bullet$-solution and $\mathscr{L}^\bullet$-solution disjoint from the support of the unique mixed strategy equilibrium. Furthermore, this equilibrium is in pure strategies, so it comprises an $\mathscr{N}$-solution.

Example 8 Let $|I| = 2$ and $X_1 = X_2 = \{a, b, c\}$, with payoffs given below.

	a	b	c
a	(0,0)	(-1,0)	(3,0)
b	(0,-1)	(0,-6)	(0,0)
c	(0,3)	(-6,4)	(4,0)

Here, (a, a) is a Nash equilibrium, and it is unique among all mixed strategy equilibria, yet $\{b, c\} \times \{b, c\}$ is an $\mathscr{M}^\bullet$- and $\mathscr{L}^\bullet$-solution.

In the latter example, the set X it itself an $\mathscr{M}^\bullet$- and $\mathscr{L}^\bullet$-solution, and it contains the unique Nash equilibrium. The next example shows that there can be a mixed strategy equilibrium with support set disjoint from the unique $\mathscr{M}^\bullet$- and $\mathscr{L}^\bullet$-solution.

Example 9 Let $I = 2$, $X_1 = \{a, b\}$, and $X_2 = \{a, b, c, d, e\}$, with payoffs specified below.

	a	b	c	d	e
a	(0,1)	(0,-1)	(0,2)	(0,-3)	(0,0)
b	(0,-1)	(0,1)	(0,-3)	(0,2)	(0,0)

Here, the mixed strategies that place probability one half each on a and b form an equilibrium. But $\{a, b\} \times \{a, b\}$ is not contained in any $\mathscr{M}^{\bullet}$-solution (or any $\mathscr{L}^{\bullet}$-solution), because the lowest payoff for column player from a is less than the lowest payoff from e when row player's choice set is $\{a, b\}$; it is less than the payoff from c when row player's choice set is $\{a\}$; and it is less than the payoff from d when row player's choice set is $\{b\}$. In fact, the unique $\mathscr{M}^{\bullet}$- and $\mathscr{L}^{\bullet}$-solution is $\{a, b\} \times \{c, d, e\}$.

4.3 Choice via Rationalizability

We initially focus on pure strategies only. Given Y, let $BP_i(Y)$ denote the set of i's pure strategy best responses to pure strategy profiles $y_{-i} \in Y_{-i}$. That is, $x_i \in BP_i(Y)$ if and only if there exists $y_{-i} \in Y_{-i}$ such that

$$u_i(x_i, y_{-i}) \quad = \quad \max_{y_i \in X_i} u_i(y_i, y_{-i}).$$

We say Y is a *point rationalizable set* if $Y_i = BP_i(Y)$ for all i, i.e., it is a fixed point of Bernheim's (1984) operator λ. A strategy x_i is then *point rationalizable* if $x_i \in Y_i$ for some point rationalizable set Y. Consider the following choice structure.

(9) **Point Rational** for all i and all Y, $\mathscr{P}_i(Y) = \{BP_i(Y)\}$.

Clearly, a set Y is point rationalizable if and only if $Y_i \in \mathscr{P}_i(Y)$ for all i, i.e., Y is a $\mathscr{P}$-solution. Bernheim's (1984) Proposition 3.1 shows that there is a unique $\mathscr{P}$-solution that is maximal with respect to set inclusion, i.e., there is a unique $\mathscr{P}$-tract, and it follows that the point rationalizable strategy profiles coincide with this unique $\mathscr{P}$-tract; we obtain this uniqueness result as a special case of the analysis in the next section.

The point rational choice structure $\mathscr{P}$ is obviously univalent. The next proposition establishes that it is monotonic, closed, and hard. An implication is the well-known property that point rational solutions generally exist.

Proposition 12 *The point rational choice structure $\mathscr{P}$ is monotonic, closed, and hard.*

While straightforward to prove directly, this proposition can be easily proven by defining the dominance structure $\mathscr{D}^p$ as follows: for all i, all $Y \in \mathbf{X}$, and all $x_i, y_i \in X_i$, $x_i \mathscr{D}_i^p(Y) y_i$ if and only if $y_i \notin BP_i(Y)$. Since Y is a $\mathscr{P}$-solution if and only if it is a $\mathscr{D}^p$-solution, this shows that, in a trivial sense, the point rational choice structure (as with any univalent choice structure) is maximally binary. The usefulness of this observation in the present context is that the dominance structure $\mathscr{D}^p$ is obviously transitive, monotonic, weakly irreflexive, and non-trivial. Thus, Propositions 3–5 immediately deliver the above proposition.

The previous choice structure focussed on best responses to pure strategy profiles, but we can easily extend the concept to allow for best responses to mixed strategy profiles, delivering the rationalizable strategies of Bernheim (1984) and Pearce (1984). Given $Y \in \mathbf{X}$, let $BR_i(Y)$ denote the set of i's pure strategy best responses to mixed strategy profiles $p_{-i} \in \tilde{Y}_{-i}$. That is, $x_i \in BR_i(Y)$ if and only if there exists $p \in \tilde{Y}$ such that

$$\tilde{u}_i(x_i, p_{-i}) \quad = \quad \max_{y_i \in X_i} \tilde{u}_i(y_i, p_{-i}).$$

We say Y is a *rationalizable set* if $Y_i = BR_i(Y)$ for all i, i.e., it is a fixed point of Bernheim's (1984) operator Λ. In the terminology of Basu and Weibull (1991), the rationalizable sets are precisely the tight CURBS. A strategy x_i is then *rationalizable* if $x_i \in Y_i$ for some rationalizable set Y. We can consider choice based on rationalizability using the following choice structure.

(10) **Rational** for all i and all Y, $\mathscr{R}_i(Y) = \{BR_i(Y)\}$.

A set Y is rationalizable if and only if $Y_i \in \mathscr{R}_i(Y)$ for all i, i.e., Y is an $\mathscr{R}$-solution. Using the fact, proved in Bernheim's (1984) Proposition 3.2 or (essentially) Pearce's (1984) Proposition 2, that there is a unique $\mathscr{R}$-tract, it follows that the rationalizable strategy profiles coincide with this unique $\mathscr{R}$-tract; we obtain this uniqueness result as a special case of the analysis in the next section.

The rational choice structure is univalent, and as the next proposition establishes, it is monotonic, closed, and hard. An implication is the well-known property that rational solutions generally exist.

Proposition 13 *The rational choice structure $\mathscr{R}$ is monotonic, closed, and hard.*

As with Proposition 12, the result is easily proven by defining the dominance structure $\mathscr{D}^r$ as follows: for all i, all $Y \in \mathbf{X}$, and all $x_i, y_i \in X_i$, $x_i \mathscr{D}_i^r(Y) y_i$ if and only if $y_i \notin BR_i(Y)$. A set Y is an $\mathscr{R}$-solution if and only if it is a $\mathscr{D}^r$-solution, and $\mathscr{D}^r$ is transitive, monotonic, weakly irreflexive, and non-trivial. Thus, Propositions 3–5 deliver the above proposition.

Pearce (1984) defines a related type of rationalizability that roughly combines rationalizability with iterative elimination of weakly dominated strategies. He defines the "cautious rationalizable" strategy profiles as those remaining after iterative deletion of any strategies that are not a best response to a strategy profile that is completely mixed over the possible strategies of the other players. More precisely, define the sequence $Y^1, Y^2, \ldots$ such that $Y^1 = X$ and for all k, $Y^k = \prod_{i \in I} Y_i^k$, where Y_i^k consists of all strategies in Y_i^{k-1} that are best responses to a mixed strategy profile p_{-i} satisfying $\sigma_{-i}(p_{-i}) = Y_i^{k-1}$. For high enough k, we have $Y^k = Y^{k+1}$, and this fixed point consists of all *cautious rationalizable* strategy profiles. We approximate these notions using the following choice structure, where given $Y \in \mathbf{X}$, we denote by $\tilde{Y}^\circ$ the mixed strategy profiles p with support equal to Y, i.e., $\sigma(p) = Y$.

(11) **Cautious Rational** for all i and all Y, $\mathscr{R}_i^\circ(Y) = \{BR_i(\tilde{Y}^\circ)\}$.

Thus, $x_i \in \mathscr{R}_i^\circ(Y)$ if x_i is a best response to some strategy profile completely mixed on Y_{-i}. Pearce's (1984) Lemma 4 and Myerson's (1991) Theorem 1.7 show that when $|I| = 2$, $x_i \notin BR_i(\tilde{Y}^\circ)$ if and only if $y_i \mathscr{W}_i(Y) x_i$ for some y_i, and therefore $\mathscr{R}^\circ$-solutions are equivalent to $\mathscr{W}$-solutions in two-player games. It follows that the $\mathscr{R}^\circ$-solutions inherit the difficulties of $\mathscr{W}$-solutions. Indeed, $\mathscr{R}^\circ$ is transitive but not monotonic, and Example 2 demonstrates that $\mathscr{R}^\circ$-solutions need not exist. This points to an important restriction in Pearce's (1984) definition of cautious rationalizability: at each step in the sequence $Y^1, Y^2, \ldots$, it is critical that the best response set Y^k is limited to strategies in Y^{k-1}, rather than allowing all best responses to profiles completely mixed over Y^{k-1}. This is illustrated in Example 2, where the first round of deletion removes strategy b for row player, and the second round of deletion removes strategy a for column player, leaving the single profile (a, b); now, b is also a best response for row player, but this strategy is discarded in Pearce's algorithm.

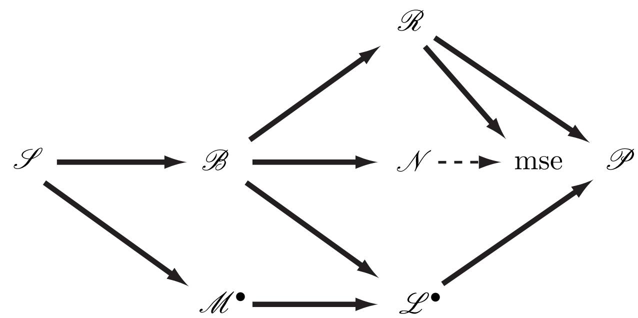

Figure 3: Logical relationships

Before moving to the logical relationships between $\mathscr{P}$- and $\mathscr{R}$-solutions and the solutions introduced in the previous subsections, we note that an $\mathscr{R}^\circ$-solution is always an $\mathscr{R}$-base; thus, Proposition 1 implies that if an $\mathscr{R}^\circ$-solution exists, then it is included in an $\mathscr{R}$-solution. The next example shows, however, that an $\mathscr{R}^\circ$-solution need not include a $\mathscr{P}$-solution.

Example 10 Let $|I| = 2$, $X_1 = \{a, b\}$, and $X_2 = \{a, b\}$, with payoffs specified below.

	a	b
a	(2,0)	(1,0)
b	(1,0)	(1,0)

Here, $\{a\} \times \{a, b\}$ is an $\mathscr{R}^\circ$-solution, but there is no $\mathscr{P}$-solution included in it: in any $\mathscr{P}$-solution Y, a and b must both belong to column player's choice set, but then, since b is a best response for row player to b, a and b must belong to row player's choice set, and therefore, $Y = \{a, b\} \times \{a, b\}$.

Figure 3 depicts the relationships among the transitive, monotonic dominance structures and the rational and point rational choice structures. First, note that $BP_i(Y) \subseteq BR_i(Y)$ in general, which implies that every $\mathscr{R}$-solution is an outer $\mathscr{P}$-solution, and every $\mathscr{P}$-solution is an $\mathscr{R}$-base. Thus, Proposition 1 implies that the $\mathscr{P}$-solutions are finer than the $\mathscr{R}$-solutions in the full sense. Next, note that every $\mathscr{B}$-solution is an outer

$\mathscr{R}$-solution. Indeed, let Y be a $\mathscr{B}$-solution, and note every $x_i \in BR_i(Y)$ is a best response to some mixed strategy profile p with $\sigma(p) \subseteq Y$, so there is no $y_i \in X_i$ such that $y_i \mathscr{W}_i(\sigma(p)) x_i$. It follows that $x_i \in Y_i$, and therefore $BR_i(Y) \in \mathscr{R}_i(Y)$ and $BR_i(Y) \subseteq Y_i$, as claimed. By a similar argument, every $\mathscr{R}$-solution is a $\mathscr{B}$-base, and then Proposition 1 implies that the $\mathscr{R}$-solutions are finer than the $\mathscr{B}$-solutions in the full sense. To see that every $\mathscr{L}^{\bullet}$-solution is an outer $\mathscr{P}$-solution, let Y be a $\mathscr{L}^{\bullet}$-solution, and note that every $x_i \in BP_i(Y)$ is a best response to some $z \in Y$, and setting $Z_{-i} = \{z_{-i}\}$, there is no $y_i \in X_i$ such that $u_i(y_i, \cdot) L_i(Z_{-i}) u_i(x_i, \cdot)$. It follows that $x_i \in Y_i$, and therefore $BP_i(Y) \in \mathscr{P}_i(Y)$ and $BP_i(Y) \subseteq Y_i$, as required. By a similar argument, every $\mathscr{P}$-solution is an $\mathscr{L}^{\bullet}$-base, and then Proposition 1 implies that the $\mathscr{P}$-solutions are finer than the $\mathscr{L}^{\bullet}$-solutions in the full sense.

Like $\mathscr{S}$-solutions and $\mathscr{B}$-solutions, $\mathscr{R}$-solutions are coarser than mixed strategy Nash equilibrium in the full sense, as the next proposition states.

Proposition 14 *(i) If Y is an outer $\mathscr{R}$-solution, then there is a mixed strategy Nash equilibrium p such that $\sigma(p) \subseteq Y$. (ii) If p is a mixed strategy Nash equilibrium, then there is an $\mathscr{R}$-solution $Y \supseteq \sigma(p)$.*

Proof: To prove (i), let Y be an outer $\mathscr{R}$-solution, so for each i, there exists $Z_i \in \mathscr{R}_i(Y)$ with $Z_i \subseteq Y_i$, or equivalently, $BR_i(Y) \subseteq Y_i$. Consider the restricted game in which each player's strategy set is Z_i and payoffs are given by u_i restricted to Z. Let p be a mixed strategy Nash equilibrium of this game. If p is not an equilibrium of the unrestricted game, then there exist i and $x_i \in X_i \setminus Z_i$ such that $\tilde{u}_i(x_i, p_{-i}) > \tilde{u}_i(p)$. Since x_i is not a best response to any mixed strategy profile with support Z, it follows that any best response to p, denoted y_i, satisfies $\tilde{u}_i(y_i, p_{-i}) > \tilde{u}_i(x_i, p_{-i}) > \tilde{u}_i(p)$. Since $y_i \in Z_i$, however, this contradicts the assumption that p is an equilibrium of the restricted game. We conclude that p is an equilibrium of the unrestricted game, and then we have an equilibrium such that $\sigma(p) \subseteq Y$, as required. To prove (ii), it suffices to note that the support $\sigma(p)$ of a mixed strategy equilibrium p is a $\mathscr{R}$-base. Indeed, because each strategy in the support of p_i is a best response to p, it follows that $\sigma_i(p_i) \subseteq BR_i(\sigma(p)) = \bigcap \mathscr{R}_i(\sigma(p))$. Then Proposition 1 implies that there is a $\mathscr{R}$-solution $Y \supseteq \sigma(p)$, as required. ∎

That the $\mathscr{R}$-solutions and $\mathscr{N}$-solutions are not generally nested follows from two examples. The first demonstrates that an $\mathscr{R}$-solution can contain all $\mathscr{N}$-solutions as proper subsets.

Example 11 Let $|I| = 2$ and $X_1 = X_2 = \{a, b, c, d\}$, with payoffs given below.

	a	b	c	d
a	(0,0)	(1,-1)	(1,-1)	(-1,1)
b	(-1,1)	(0,0)	(1,-1)	(-1,1)
c	(-1,1)	(-1,1)	(0,0)	(1,-1)
d	(1,-1)	(1,-1)	(-1,1)	(0,0)

Here, $\{a, b, c, d\}\times\{a, b, c, d\}$ is an $\mathscr{R}$-solution, but it is not a $\mathscr{N}$-solution, since $a\mathscr{N}_1(X_2)b$. In fact, $a\mathscr{W}_1(X_2)b$, and it is not a $\mathscr{W}$-solution either.

The second example demonstrates that an $\mathscr{N}$-solution can contain the unique $\mathscr{R}$-solution as a proper subset; it exploits the fact that a pure strategy may fail to be a best response to any mixed strategy, yet not be weakly dominated by any other pure strategy.

Example 12 Let $|I| = 2$, $X_1 = \{a, b\}$, and $X_2 = \{a, b, c\}$, with payoffs given below.

	a	b	c
a	(0,4)	(1,1)	(0,2)
b	(1,1)	(0,4)	(0,2)

Here, $\{a, b\} \times \{a, b, c\}$ is the unique $\mathscr{N}$-solution, but strategy c is strictly dominated for column player by the mixed strategy that places probability one half each on a and b, so the unique $\mathscr{R}$-solution is $\{a, b\} \times \{a, b\}$.

Finally, note that in the preceding example, $X = \{a, b\} \times \{a, b, c\}$ is the unique $\mathscr{L}^\bullet$-solution; in particular, the lowest payoff to column player from strategy c is higher than the lowest payoff from a and b when row player's choice set is $\{a, b\}$. And in Example 9, all strategies are rationalizable, so X is a $\mathscr{R}$-solution, while the unique $\mathscr{M}^\bullet$-solution is the proper subset $\{a, b\} \times \{c, d, e\}$. Thus, no general inclusion relationship holds among the pessimistic solutions and the rational solutions.

4.4 Mixed Choice Structures

We now extend the Shapley, weak Shapley, and Nash dominance structures to allow for the possibility that players use mixed strategies in constructing their choice sets. A convenient way of constructing these extensions is to define a notion of binary representation to allow for strategic preferences between mixed and pure strategies, and to modify internal and external stability to account for the possibility that a player may mix over her choice set. A *mixed dominance structure for* i is any mapping $\tilde{\mathscr{D}}_i \colon \mathbf{X} \to 2^{\tilde{X}_i \times X_i}$ such that for all $Y \in \mathbf{X}$,

(1) for all $Y' \in \mathbf{X}$, $Y_{-i} = Y'_{-i}$ implies $\tilde{\mathscr{D}}_i(Y) = \tilde{\mathscr{D}}_i(Y')$,

(2) there is at least one set $Y'_i \subseteq X$ such that

   (2a) there do not exist $x_i \in Y'_i$ and $p_i \in \tilde{Y}'_i$ (not degenerate on x_i) such that $p_i \tilde{\mathscr{D}}_i(Y) x_i$,

   (2b) for all $x_i \in X_i \setminus Y'_i$, there exists $p_i \in \tilde{Y}'_i$ such that $p_i \tilde{\mathscr{D}}_i(Y) x_i$,

(3) for all $x_i, y_i \in X_i$, if $u_i(x) > u_i(y_i, x_{-i})$ for all $x_{-i} \in Y_{-i}$, then $x_i \tilde{\mathscr{D}}_i(Y) y_i$.

Then $\tilde{\mathscr{D}} = (\tilde{\mathscr{D}}_i)_{i \in I}$ is a *mixed dominance structure* if $\tilde{\mathscr{D}}_i$ is a mixed dominance structure for each i, and $\tilde{\mathscr{D}}$ is a *mixed binary representation* for a choice structure $\mathscr{C}$ if for all i, all $Y \in \mathbf{X}$, and all $Y'_i \subseteq X_i$, we have $Y'_i \in \mathscr{C}_i(Y)$ if and only if conditions (2a) and (2b) hold, in which case $\tilde{\mathscr{D}}$ *generates* $\mathscr{C}$, and $\mathscr{C}$ is *mixed binary.*

We define three such choice structures by taking mixed versions of Nash, weak Shapley, and Shapley dominance.

(12) **Mixed Nash** for all i and all Y, $p_i \; \tilde{\mathscr{N}_i}(Y) \; x_i$ if and only if for all $x_{-i} \in Y_{-i}$, $u_i(p_i, x_{-i}) \geq u_i(x)$.

(13) **Mixed Weak Shapley** for all i and all Y, $p_i\ \tilde{\mathscr{W}}_i(Y)\ x_i$ if and only if for all $x_{-i} \in Y_{-i}$, $u_i(p_i, x_{-i}) \geq u_i(x)$, with strict inequality for at least one x_{-i}.

(14) **Mixed Shapley** for all i and all Y, $p_i\ \tilde{\mathscr{S}}_i(Y)\ x_i$ if and only if for all $x_{-i} \in Y_{-i}$, $u_i(p_i, x_{-i}) > u_i(x)$.

Of these mixed dominance structures, Shapley and Nash are monotonic in the sense that for all i, all $Y, Z \in \mathbf{X}$ with $Y \subseteq Z$, all $p_i \in \tilde{X}_i$, and all $y_i \in Y_i$, $p_i \tilde{\mathscr{N}}_i(Z) y_i$ implies $p_i \tilde{\mathscr{N}}_i(Y) y_i$, and $p_i \tilde{\mathscr{S}}_i(Z) y_i$ implies $p_i \tilde{\mathscr{S}}_i(Y) y_i$. In contrast, mixed weak Shapley inherits the difficulties of weak Shapley, i.e., the choice structure it generates may violate monotonicity. In contrast to the pure strategy setting, the monotonic kernel is not an interesting response to this difficulty, as the mixed version of Börgers dominance (using the conventions above) would be equivalent to mixed Shapley.

The next result establishes that mixed Shapley dominance is especially well-behaved. In particular, monotonicity implies that mixed Shapley solutions generally exist.

Proposition 15 *The choice structure generated by mixed Shapley dominance is monotonic, closed, and hard.*

Proof: Define the dominance structure $\mathscr{S}'$ as follows: for all i, all $Y \in \mathbf{X}$, and all $x_i, y_i \in X_i$, $x_i \mathscr{S}'_i(Y) y_i$ if and only if there exists $p_i \in \tilde{X}_i$ such that $p_i \tilde{\mathscr{S}}_i(Y) y_i$. Note that the mixed strategy p_i in the latter condition can put positive probability on strategies outside Y_i and cannot be degenerate on y_i. Let $\tilde{\mathscr{C}}$ be the choice structure generated by $\tilde{\mathscr{S}}$, and let $\mathscr{C}'$ be the choice structure generated by $\mathscr{S}'$. We claim that $\tilde{\mathscr{C}} = \mathscr{C}'$. Fix any i and any $Y \in \mathbf{X}$. Consider any $Z_i \in \tilde{\mathscr{C}}_i(Y)$, and note that external stability with respect to $\mathscr{S}'_i(Y)$ follows immediately: for all $x_i \in X_i \setminus Z_i$, there exists $p_i \in \tilde{Z}_i \subseteq \tilde{X}_i$ such that $p_i \tilde{\mathscr{S}}_i(Y) x_i$, and therefore $p_i \mathscr{S}'_i(Y) x_i$. To prove internal stability with respect to $\mathscr{S}'_i(Y)$, suppose there exist $x_i \in Z_i$ and $p'_i \in \tilde{X}_i$ such that $p'_i \tilde{\mathscr{S}}_i(Y) x_i$. Since p'_i is not degenerate on x_i, we assume without loss of generality that $p'_i(x_i) = 0$. Index the elements of $X_i \setminus Z_i$ as $y_i^1, \ldots, y_i^K$. Because Z_i is externally stable with respect to $\tilde{\mathscr{S}}_i(Y)$, it follows that for each k, there exists $p_i^k \in \tilde{Z}_i$ such that $p_i^k \tilde{\mathscr{S}}_i(Y) y_i^k$. Now define p_i to be the same as p'_i but modified so that if p'_i places positive probability on some y_i^k, then that probability is transferred to p_i^k, i.e., for all $z_i \in X_i$,

$$p_i(z_i) \;=\; p_i'(z_i) + \sum_{k=1}^{K} p_i'(y_i^k) p_i^k(z_i).$$

Then we have $p_i \in \tilde{Z}_i$ and $p_i \tilde{\mathscr{S}}_i(Y) x_i$, contradicting internal stability of Z_i with respect to $\tilde{\mathscr{S}}_i(Y)$. Therefore, Z_i is internally stable with respect to $\mathscr{S}_i'(Y)$, and we conclude that $Z_i \in \mathscr{C}_i'(Y)$. Now consider any $Z_i' \in \mathscr{C}_i'(Y)$, and note that internal stability with respect to $\tilde{\mathscr{S}}_i(Y)$ follows immediately: there do not exist $x_i \in Z_i'$ and $p_i \in \tilde{X}_i$, nor therefore $p_i \in \tilde{Z}_i$, such that $p_i \tilde{\mathscr{S}}_i(Y) x_i$. To prove external stability of Z_i' with respect to $\tilde{\mathscr{S}}_i(Y)$, consider any $x_i \in X_i \backslash Z_i'$. Because Z_i' is externally stable with respect to $\mathscr{S}_i'(Y)$, there exists $p_i' \in \tilde{X}_i$ such that $p_i' \tilde{\mathscr{S}}_i(Y) x_i$. By a construction similar to that above, we can transform p_i' to a mixed strategy $p_i \in \tilde{Z}_i'$ such that $p_i \tilde{\mathscr{S}}_i(Y) x_i$, as required. We conclude that $Z_i' \in \tilde{\mathscr{C}}_i(Y)$, and therefore $\tilde{\mathscr{C}} = \mathscr{C}'$, as claimed. Finally, we observe that $\mathscr{S}'$ is monotonic, transitive, weakly irreflexive, and non-trivial. Then Propositions 3–5 imply that $\tilde{\mathscr{C}}$ is monotonic, closed, and hard. ∎

The choice structure generated by the mixed Nash dominance structure is monotonic, as shown next.

Proposition 16 *The choice structure generated by mixed Nash dominance is monotonic.*

Proof: Let $\tilde{\mathscr{C}}$ be the choice structure generated by $\tilde{\mathscr{N}}$. To prove that $\tilde{\mathscr{C}}$ is monotonic, consider any $Y, Z \in \mathbf{X}$ with $Y \subseteq Z$, any i, and any $Z_i' \in \tilde{\mathscr{C}}_i(Z)$. Because Z_i' is externally stable with respect to $\tilde{\mathscr{N}}_i(Z)$, monotonicity of $\tilde{\mathscr{N}}$ implies that Z_i' is externally stable with respect to $\tilde{\mathscr{N}}_i(Y)$. Let Y_i' be a set that is maximal with respect to set inclusion among the subsets of Y_i that are externally stable with respect to $\tilde{\mathscr{N}}_i(Y)$. Suppose that Y_i' is not internally stable with respect to $\tilde{\mathscr{N}}_i(Y)$, so there exist $x_i \in Y_i'$ and $p_i \in \tilde{Y}_i'$ (not degenerate on x_i) such that $p_i \tilde{\mathscr{N}}_i(Y) x_i$. Assume without loss of generality that $p_i(x_i) = 0$. We claim that $Y_i'' = Y_i' \setminus \{x_i\}$ is externally stable. Index the elements of Y_i' as $y_i^1, \ldots, y_i^K$. Consider any $y_i \in X_i \setminus Y_i''$. If $y_i = x_i$, then we have $p_i \in \tilde{Y}_i''$ such that $p_i \tilde{\mathscr{N}}_i(Y) y_i$. If $y_i \neq x_i$, then $y_i \notin Y_i'$, so there exists $q_i \in \tilde{Y}_i'$ such that $q_i \tilde{\mathscr{D}}_i(Y) y_i$. Now define p_i' to be the same as q_i but modified so that if q_i places any probability on x_i, then that probability is transferred to p_i, i.e., for all $z_i \in X_i$,

$$p'_i(z_i) \quad = \quad \begin{cases} q_i(z_i) + q_i(x_i)p_i(z_i) & \text{if } z_i \neq x_i \\ 0 & \text{if } z_i = x_i. \end{cases}$$

Then we have $p'_i \in \tilde{Y}''_i$ and $p'_i \tilde{\mathscr{N}}_i(Y)y$, as claimed. Thus, Y''_i is a proper subset of Y'_i that is external stable, a contradiction. We conclude that Y'_i is indeed internally stable, which implies that $Y'_i \in \mathscr{C}_i(Y)$. Therefore, $\mathscr{C}$ is monotonic. ∎

An immediate implication of Proposition 16, with Proposition 1, is that mixed Nash solutions exist generally. That mixed Nash dominance does not generally yield a choice structure that is closed or hard is demonstrated in Example 1, where $\{a, b\} \times \{a, b\}$ and $\{b, c\} \times \{b, c\}$ are $\tilde{\mathscr{N}}$-solutions, as well as $\mathscr{N}$-solutions.

For this reason, we focus on mixed Shapley dominance in the sequel. It is straightforward to see that every $\mathscr{S}$-solution is an outer $\tilde{\mathscr{S}}$-solution. Indeed, let $\tilde{\mathscr{C}}$ be the choice structure generated by $\tilde{\mathscr{S}}$, and let Y be an $\mathscr{S}$-solution, so each Y_i satisfies external stability with respect to $\mathscr{S}_i(Y)$ and therefore with respect to $\tilde{\mathscr{S}}_i(Y)$. We then let Y'_i be minimal with respect to set inclusion among the subsets of Y_i that are externally stable with respect to $\tilde{\mathscr{S}}_i(Y)$, and following the procedure in the proof of Proposition 15, we arrive at $Y'_i \in \tilde{\mathscr{C}}_i(Y)$. Furthermore, every $\tilde{\mathscr{S}}$-solution is an $\mathscr{S}$-base, and Proposition 1 implies that the mixed Shapley solutions are finer than the Shapley solutions in the full sense.

Proposition 17 *The $\tilde{\mathscr{S}}$-solutions are finer than the $\mathscr{S}$-solutions in the full sense.*

The choice structure generated by mixed Shapley dominance is coarser than the rational choice structure, as the next proposition states.

Proposition 18 *The $\mathscr{R}$-solutions are finer than the $\tilde{\mathscr{S}}$-solutions in the full sense.*

Proof: We first claim that every $\tilde{\mathscr{S}}$-solution is an outer $\mathscr{R}$-solution. Let Y be a $\tilde{\mathscr{S}}$-solution, let $\mathscr{C}$ be the choice structure generated by $\mathscr{R}$, and consider any i. Note that $\mathscr{C}_i(Y) = \{BR_i(Y)\}$. For all $x_i \in X_i \setminus Y_i$, external stability of Y_i with respect to $\tilde{\mathscr{S}}_i(Y)$ yields $p_i \in \tilde{Y}_i$ such that $p_i \tilde{\mathscr{S}}_i(Y)x_i$, so x_i is not a best response to any mixed strategy profile with support in Y. Therefore, $BR_i(Y) \subseteq Y_i$, and Y is an outer $\mathscr{R}$-solution, as claimed. Next, we claim that every $\mathscr{R}$-solution is a $\tilde{\mathscr{S}}$-base. Let Y be an $\mathscr{R}$-solution, let

$\tilde{\mathscr{C}}$ be the choice structure generated by $\tilde{\mathscr{S}}$, and consider any i and any $Y_i' \in \tilde{\mathscr{C}}_i(Y)$. Note that $Y_i = BR_i(Y)$. For all $x_i \in X_i \setminus Y_i'$, external stability of Y_i' with respect to $\tilde{\mathscr{S}}_i(Y)$ yields $p_i \in \tilde{Y}_i'$ such that $p_i \tilde{\mathscr{S}}_i(Y) x_i$, which implies $x_i \notin BR_i(Y)$. Therefore, $Y_i \subseteq Y_i'$, and Y is an $\tilde{\mathscr{S}}$-base, as claimed. Then the result follows from Proposition 1. ∎

The close correlation between the rational choice structure and the choice structure generated by mixed Shapley dominance becomes exact in two-player games. The proof of the next proposition is straightforward, using Pearce's (1984) Lemma 2 or Myerson's (1991) Theorem 1.6, and is omitted. Duggan and Le Breton (2001) show that in tournament games, there is a unique $\tilde{\mathscr{S}}$-solution, i.e., the mixed saddle, and that the choice set pinned down by this solution is the top cycle set of the tournament; thus, we find the same correspondence between the top cycle and the minimal rationalizable set of a tournament game.

Proposition 19 *Let $\tilde{\mathscr{C}}$ be the choice structure generated by mixed Shapley dominance. If $|I| = 2$, then $\tilde{\mathscr{C}} = \mathscr{R}$.*

Less obvious is that when there are just two players, the Börgers solutions refine the mixed Shapley solutions. The proof proceeds by defining the dominance structure $\mathscr{D}^s$ as follows: for all i, all $Y \in \mathbf{X}$, and all $x_i, y_i \in X_i$, $x_i \mathscr{D}_i^s(Y) y_i$ if and only if there exists $p_i \in \tilde{X}_i$ such that $p_i \tilde{\mathscr{S}}_i(Y) y_i$. Note that Y is an $\tilde{\mathscr{S}}$-solution if and only if it is a $\mathscr{D}^s$-solution. Moreover, $\mathscr{D}^s$ is transitive and monotonic. Then the result follows from Proposition 9 if we can show that Börgers dominance is stronger than $\mathscr{D}^s$-dominance.

Proposition 20 *If $|I| = 2$, then the $\tilde{\mathscr{S}}$-solutions are finer than the $\mathscr{B}$-solutions in the full sense.*

Proof: In line with the above discussion, it suffices to show that $\mathscr{B}$ is stronger than $\mathscr{D}^s$. Consider player 1, without loss of generality, and any $x_1, y_1 \in X_1$ such that $x_1 \mathscr{B}_1(Y) y_1$. Then there exists $z_1^0 \in X_1$ such that $z_1^0 \mathscr{W}_1(Y) y_1$. So there is a set $Y_2^0 \subsetneqq Y_2$ such that for all $y_2 \in Y_2 \setminus Y_2^0$, we have $u_1(z_1^0, y_2) > u_1(y_1, y_2)$, and for all $y_2 \in Y_2^0$, we have $u_1(z_1^0, y_2) = u_1(y_1, y_2)$. If Y_2^0 is nonempty, then there also exists $z_1^1 \in X_1$ such that $z_1^1 \mathscr{W}_1(Y_1 \times Y_2^0) y_1$. So there is a set $Y_2^1 \subsetneqq Y_2^0$ such that for all $y_2 \in Y_2^0 \setminus Y_2^1$, we have $u_1(z_1^1, y_2) > u_1(y_1, y_2)$, and for all $y_2 \in Y_1^2$, we have $u_1(z_1^1, y_2) = u_1(y_1, y_2)$. Define the mixed strategy p_1^1 for player 1 by placing sufficiently small probability $\epsilon_1 > 0$ on z_1^1 and the remaining probability $1 - \epsilon_1$ on z_1^0, so that for all $y_2 \in Y_2 \setminus Y_2^1$, we have $u_1(p_1^1, y_2) > u_1(y_1, y_2)$, and for all $y_2 \in Y_2^1$, we have $u_1(p_1^1, y_2) = u_1(y_1, y_2)$.

Proceeding inductively, suppose we are given $Y_2^0 \supsetneqq Y_2^1 \supsetneqq \cdots \supsetneqq Y_2^{k-1}$ with Y_2^{k-1} nonempty and p_1^{k-1} such that for all $y_2 \in Y_2 \setminus Y_2^{k-1}$, we have $u_1(p_1^{k-1}, y_2) > u_1(y_1, y_2)$, and for all $y_2 \in Y_2^{k-1}$, we have $u_1(p_1^{k-1}, y_2) = u_1(y_1, y_2)$. Select $z_1^k \in X_1$ such that $z_1^k \mathscr{W}_1(Y_1 \times Y_2^{k-1}) y_1$. Then there is a set $Y_2^k \subsetneqq Y_2^{k-1}$ such that for all $y_2 \in Y_2^{k-1} \setminus Y_2^k$, we have $u_1(z_1^k, y_2) > u_1(y_1, y_2)$, and for all $y_2 \in Y_2^{k-1}$, we have $u_1(z_1^k, y_2) = u_1(y_1, y_2)$. Again, choose $\epsilon_k > 0$ sufficiently small, and let p_1^k be the mixture $p_1^k = \epsilon_k z_1^k + (1-\epsilon_k) p_1^{k-1}$ with weight ϵ_k on z_1^k and the remaining weight on p_1^{k-1}. Then for all $y_2 \in Y_2 \setminus Y_2^k$, we have $u_1(p_1^k, y_2) > u_1(y_1, y_2)$, and for all $y_2 \in Y_2^k$, we have $u_1(p_1^k, y_2) = u_1(y_1, y_2)$.

Since X_2 is finite, there is a k such that $Y_2^{k+1} = \emptyset$, and then $p_1^k \tilde{\mathscr{S}}_1(Y) y_1$, and we conclude that $x_1 \mathscr{D}_1^s(Y) y_1$, as required. ∎

The next example shows that in games with three or more players, the preceding result does not hold.

Example 13 Let $|I| = 3$, $X_1 = \{a, b, c, d\}$, $X_2 = \{U, D\}$, and $X_3 = \{L, R\}$. We let the payoffs of players 2 and 3 be constant, and we specify the payoffs of player 1 in the following four matrices, which correspond to strategies a, b, c, and d, respectively.

$$\begin{array}{c|c|c|}
 & L & R \\ \hline
U & 1 & 0 \\ \hline
D & 0 & 1 \\ \hline
\end{array}
\qquad
\begin{array}{c|c|c|}
 & L & R \\ \hline
U & 0 & -2 \\ \hline
D & 1 & 0 \\ \hline
\end{array}
\qquad
\begin{array}{c|c|c|}
 & L & R \\ \hline
U & 0 & 1 \\ \hline
D & -2 & 0 \\ \hline
\end{array}
\qquad
\begin{array}{c|c|c|}
 & L & R \\ \hline
U & 0 & 0 \\ \hline
D & 0 & 0 \\ \hline
\end{array}$$

Note that d is Börgers dominated: a weakly dominates d over $X_2 \times X_3$, $\{U\} \times X_3$, $\{D\} \times X_3$, $X_2 \times \{L\}$, and $X_2 \times \{R\}$, and it is a better response than d to (U, L), and (D, R), while b is a better response to (D, L), and c is a better response to (U, R). Thus, the unique $\mathscr{B}$-solution is $\{a, b, c\} \times \{U, D\} \times \{L, R\}$. But X is an $\tilde{\mathscr{S}}$-solution, and in particular, there is no mixed strategy p_1 such that $p_1 \tilde{\mathscr{S}}_1(X) d$. To see this, let $p_{1,b}$ and $p_{1,c}$ be the probabilities on strategies b and c, and simply note that $p_{1,b}(1) + p_{1,c}(-2) > 0$ and $p_{1,b}(-2) + p_{1,c}(1) > 0$ are inconsistent.

The relationships between mixed Shapley and the monotonic choice structures defined in previous sections are summarized in Figure 4. Note that a mixed Shapley

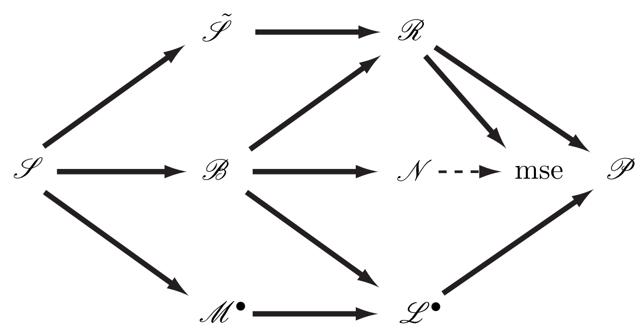

Figure 4: Logical relationships

solution can be a proper superset of all $\mathscr{B}$-solutions, as illustrated in Example 13, and of all $\mathscr{M}^\bullet$-solutions, as illustrated in Example 9 (where X is a $\tilde{\mathscr{S}}$-solution). Furthermore, the unique mixed Shapley solution can itself be a proper subset of a unique $\mathscr{N}$-solution and a unique $\mathscr{L}^\bullet$-solution, as illustrated in Example 12 (where X is the unique $\mathscr{N}$-solution and the unique $\mathscr{L}^\bullet$-solution).

5 Uniqueness of Solutions

In this section, we turn to uniqueness of $\mathscr{C}$-solutions for various choice structures. Clearly, uniqueness will not obtain generally: if x and y are distinct strict Nash equilibria, then $\{x\}$ and $\{y\}$ are $\mathscr{C}$-solutions for every choice structure. Even when multiplicity occurs, however, it may yet be that there is a unique solution that is maximal with respect to set inclusion, i.e., the $\mathscr{C}$-tract may be unique. We show that every monotonic, hard choice structure admits a unique maximal solution, a result that applies to the Shapley, Börgers, monotonic maximin, monotonic leximin, point rational, rational, and mixed Shapley choice structures. We provide an iterative procedure to calculate these sets and demonstrate invariance with respect to the details of the iterative procedure, and we show that the unique $\mathscr{C}$-tract corresponds to the possible strategy profiles under common knowledge of the choice structure. These results imply the well-known fact that iterative elimination of strictly dominated strategies is independent of the order of elimination, and they yield characterizations of the rationalizable strategy profiles due to Bernheim (1984) and Pearce (1984). Moreover, relying on Börgers (1993), they provide a characterization of the strategy profiles implied by common knowledge of the players' ordinal preferences and rationality for some compatible von Neumann-Morgenstern preferences over lotteries. The applications to mixed Shapley and to monotonic maximin and leximin provide novel results for these dominance structures.

For a special class of games possessing a "safe" equilibrium (including all two-player, zero-sum games), we moreover establish uniqueness of the minimal $\mathscr{C}$-solution, or $\mathscr{C}$-core, for coarsenings of the rational choice structure: an implication is that in such games, there is a unique minimal Shapley solution, Börgers solution, mixed Shapley solution, and rational solution. Uniqueness of the Shapley set generalizes Shapley's (1964) result for the saddle of a two-player, zero-sum game, while uniqueness of the mixed Shapley set generalizes the results of Duggan and Le Breton (1999, 2001) for the mixed saddle of a tournament game. Uniqueness of the rational set shows that for the class of equilibrium safe games, there is a uniquely tightest prediction consistent with common knowledge of the players' preferences over lotteries, of rationality given those preferences, and of the players' choice sets themselves. The result for the Börgers set has similar implications, weakening common knowledge of von Neumann-Morgenstern preferences to common knowledge of ordinal preferences.

5.1 Maximal Solutions

We first focus on uniqueness of $\mathscr{C}$-tracts. The importance of our uniqueness results, as shown in Proposition 24 at the end of this subsection, stems from common knowledge considerations: under broad conditions, the unique $\mathscr{C}$-tract describes the possible outcomes of a game when it is common knowledge among the players that each player i's choice set is viable relative to $\mathscr{C}$, but the players' choice sets themselves are not common knowledge. Thus, although the assumption that the players' choice sets are common knowledge is implicit in the definition of $\mathscr{C}$-solution, we can drop this assumption by analyzing the properties of the $\mathscr{C}$-tract.

Proposition 21 *Assume the choice structure $\mathscr{C}$ is monotonic and hard. Then there is a unique maximal $\mathscr{C}$-solution.*

Proof: Existence follows from Proposition 1. For uniqueness, suppose there are distinct $\mathscr{C}$-tracts Y and Z. Let $W_i = Y_i \cup Z_i$ for all i, and define $W = \prod_{i \in I} W_i$. Since $\mathscr{C}$ is hard, and $Y_i \in \mathscr{C}_i(Y)$ and $Z_i \in \mathscr{C}_i(Z)$ for all i, it follows that $W_i \subseteq \bigcap \mathscr{C}_i(W)$ for all i. Thus,

W is a $\mathscr{C}$-base, and then Proposition 1 yields a $\mathscr{C}$-solution $W' \supseteq W \supsetneqq Y$, contradicting maximality of Y. ∎

An implication of Proposition 21 is that uniqueness of maximal solutions in finite strategic form games holds for the Shapley, Börgers, monotonic maximin, monotonic leximin, point rational, rational, and mixed Shapley choice structures. Furthermore, it holds for any choice structure generated by transitive, monotonic, weakly irreflexive, and non-trivial dominance structure. Of the monotonic choice structures defined in previous sections, the only one that is not generally hard is Nash dominance, and as a consequence, some games admit multiple Nash-tracts: for example, in any game with constant payoffs, every singleton $\{y\}$ is a Nash-tract.

The next proposition provides an iterative procedure for computing the $\mathscr{C}$-tract of any monotonic, hard choice structure. In general, a sequence $Y^1, Y^2, \ldots \in \mathbf{X}$ of product sets of strategies is a *$\mathscr{C}$-sequence* if $Y^1 = X$, and for all $k \geq 2$, $Y^k = \prod_{i \in I} Y_i^k$, where $Y_i^k \in \mathscr{C}_i(Y^{k-1})$ for all i. Note that if $\mathscr{C}$ is univalent, then it admits a unique $\mathscr{C}$-sequence, but our construction is more general. The sequence is a *proper* $\mathscr{C}$-sequence if there exists k such that $Y^k = Y^{k+1}$, and for all $\ell < k$, we have $Y^\ell \subsetneqq Y^{\ell+1}$. The next proposition establishes that if $\mathscr{C}$ is monotonic, then it admits at least one proper $\mathscr{C}$-sequence; and if $\mathscr{C}$ is also hard, then the sequence terminates with the unique maximal $\mathscr{C}$-solution. Since every hard choice structure is univalent, an implication is that for a monotonic, hard choice structure, the unique $\mathscr{C}$-sequence produces the unique $\mathscr{C}$-tract.

Proposition 22 *Let $\mathscr{C}$ be a choice structure. If $\mathscr{C}$ is monotonic, then it admits a proper $\mathscr{C}$-sequence $Y^1, Y^2, \ldots$ with $Y^{k-1} \neq Y^k = Y^{k+1}$, and Y^k is a $\mathscr{C}$-solution. Furthermore, if $\mathscr{C}$ is hard, then Y^k is the unique $\mathscr{C}$-tract.*

Proof: Assume $\mathscr{C}$ is monotonic, and consider any ℓ such that $Y^\ell \subsetneqq Y^{\ell+1}$. Note that for each i, $Y_i^{\ell+1} \in \mathscr{C}_i(Y^\ell)$, so by monotonicity there exists $Y_i^{\ell+2} \in \mathscr{C}_i(Y^{\ell+1})$ such that $Y_i^{\ell+2} \subseteq Y_i^{\ell+1}$. Since $Y^2 \subseteq Y^1$, this yields a weakly decreasing sequence $Y^1, Y^2, \ldots$, and since X is finite, there must exist k such that $Y^k = Y^{k+1}$. This set Y^k is evidently a $\mathscr{C}$-solution. Therefore, the sequence so-defined is a proper $\mathscr{C}$-sequence. Now also assume that $\mathscr{C}$ is hard, and suppose that Y^k is not a maximal $\mathscr{C}$-solution. Then there is a $\mathscr{C}$-solution $Z \supsetneqq Y^k$. Note that $Z \notin \{Y^1, \ldots, Y^k\}$, for otherwise the sequence would terminate with Z, rather than Y^k. Moreover, $Z \subseteq Y^1$, so there is a smallest index ℓ such that $Z \subseteq Y^\ell$ but $Z \nsubseteq Y^{\ell+1}$. Let $W_i = Y_i^\ell \cup Z_i$ for each i, and define $W = \prod_{i\in I} W_i$. Note that, in fact, $W = Y^\ell$. Since $\mathscr{C}$ is hard and $Y_i^{\ell+1} \in \mathscr{C}_i(Y^\ell)$ and $Z_i \in \mathscr{C}_i(Z)$ for all i, it follows that $Y_i^{\ell+1} \cup Z_i \subseteq \bigcap \mathscr{C}_i(W)$, and since $Y_i^{\ell+1} \in \mathscr{C}_i(Y^\ell)$, that $Z_i \subseteq Y_i^{\ell+1}$ for all i, but then $Z \subseteq Y^{\ell+1}$, a contradiction. We conclude that Y^k is maximal, as required. By Proposition 21, Y^k is the unique $\mathscr{C}$-tract. ∎

The algorithm provided in Proposition 22 is robust. A sequence $Y^1, Y^2, \ldots \in \mathbf{X}$ is an *outer $\mathscr{C}$-sequence* if: (i) $Y^1 = X$, (ii) for all $k \geq 2$, $Y^k = \prod_{i\in I} Y_i^k$, where Y_i^k contains a viable set $Z_i^k \in \mathscr{C}_i(Y^{k-1})$ for all i, and (iii) there exists k such that Y^k is a $\mathscr{C}$-solution and $Y^\ell \subsetneqq Y^{\ell+1}$ for all $\ell < k$. Obviously, every proper $\mathscr{C}$-sequence is an outer $\mathscr{C}$-sequence. The next result shows that the unique $\mathscr{C}$-tract for a monotonic, hard choice structure can be reached by computing any outer $\mathscr{C}$-solution.

Proposition 23 *Let $\mathscr{C}$ be a choice structure. If $\mathscr{C}$ is monotonic and hard, then for every outer $\mathscr{C}$-sequence $Y^1, Y^2, \ldots$ such that Y^k is a $\mathscr{C}$-solution, Y^k is the unique $\mathscr{C}$-tract.*

Proof: Assume $\mathscr{C}$ is monotonic and hard, let $Y^1, Y^2, \ldots$ be an outer $\mathscr{C}$-sequence as in the proposition, and suppose that Y^k is not maximal. Let Y be a $\mathscr{C}$-solution such that $Y \supsetneqq Y^k$, so $k \geq 2$. Let $Z^2, \ldots, Z^{k+1}$ satisfy $Z_i^\ell \in \mathscr{C}_i(Y^{\ell-1})$ for all i and $Z^\ell \subseteq Y^\ell$, $\ell = 2, \ldots, k+1$, and set $Z^1 = X$. Since $Y^2 \supseteq \cdots \supseteq Y^{k+1}$, monotonicity implies that $\mathscr{C}_i(Y^\ell)$ contains a subset of Z^ℓ for all $\ell = 2, \ldots, k$, and we may therefore assume without loss of generality that $Z^1 \supseteq Z^2 \supseteq \cdots \supseteq Z^{k+1}$. Note that $Z^{k+1} \in \mathscr{C}_i(Y^k) = \mathscr{C}_i(Y^{k+1})$, and with $Z^{k+1} \subseteq Y^{k+1}$, this implies $Z^{k+1} = Y^{k+1} = Y^k$. Thus, $Y \supsetneqq Z^{k+1}$. Let $\ell \geq 2$ be the smallest index such that $Y \nsubseteq Z^\ell$, so that $Y \subseteq Z^{\ell-1}$. Let $W_i = Y_i \cup Y_i^{\ell-1}$ for each i, define $W = \prod_{i\in I} W_i$, and note that, in fact, $W = Y^{\ell-1}$. Since $\mathscr{C}$ is hard and $Y_i \in \mathscr{C}_i(Y)$ and $Z_i^\ell \in \mathscr{C}_i(Y^{\ell-1})$ for all i, it follows that $Y_i \cup Z_i^\ell \subseteq \bigcap \mathscr{C}_i(W)$ for all i. This implies that $Y_i \subseteq \bigcap \mathscr{C}_i(Y^{\ell-1})$, and since $Z_i^\ell \in \mathscr{C}_i(Y^{\ell-1})$, that $Y_i \subseteq Z_i^\ell$ for all i. But then $Y \subseteq Z^\ell$, a contradiction. By Proposition 21, Y^k is the unique $\mathscr{C}$-tract. ∎

Propositions 22 and 23 have useful implications for a dominance structure $\mathscr{D}$ that is irreflexive, transitive, monotonic, and non-trivial. Letting $\mathscr{C}$ be the choice structure generated by $\mathscr{D}$, the unique proper $\mathscr{C}$-sequence is calculated by removing all dominated strategies at each step: for $k \geq 2$,

$$Y_i^k \;=\; \Big\{ x_i \in X_i \mid \text{for all } y_i \in X_i, \text{ not } y_i \mathscr{D}_i(Y^{k-1}) x_i \Big\}.$$

By Proposition 22, the iterated exhaustive elimination of dominated strategies for all players leads to the unique maximal $\mathscr{C}$-solution. This yields the equivalences $P'(G) = P''(G)$ and $R'(G) = R''(G)$ in Bernheim's (1984) Propositions 3.1 and 3.2 (for the special case of finite games). Proposition 23 implies that the outcome of this algorithm is invariant with respect to order of elimination. An outer $\mathscr{C}$-sequence can be calculated by removing some dominated strategies at each step: for $k \geq 2$,

$$Y_i^{k-1} \setminus Y_i^k \;=\; \Big\{ x_i \in Y_i^{k-1} \mid \text{for some } y_i \in X_i,\ y_i \mathscr{D}_i(Y^{k-1}) x_i \Big\}.$$

Applied to Shapley dominance, this yields the well-known result that the iterated removal of strictly dominated strategies is invariant with respect to the order of elimination. It generalizes Bernheim's (1984) Propositions 3.1 and 3.2, showing that the point rationalizable (resp. rationalizable) strategies can be reached by the arbitrary elimination of strategies that are not best responses to the possible pure (resp. mixed) strategy profiles for other players. Our propositions apply as well to the Börgers, monotonic maximin, monotonic leximin, and the mixed Shapley choice structures, providing new results for their corresponding solutions. For example, the unique $\mathscr{B}$-tract can be obtained by iteratively removing, for each player i, any strategy x_i that is dominated in the sense of Börgers. Our propositions do not apply to weak Shapley dominance, and Börgers and Samuelson's (1992) Example 4 demonstrates that there may be multiple maximal $\mathscr{W}$-solutions.

Finally, we examine the epistemic foundations of the maximal $\mathscr{C}$-solution. A *belief system for i* is a mapping $\beta^i \colon \bigcup_{n=1}^{\infty} I^n \to \mathbf{X}$, where we interpret $\beta^i(i)$ as player i's beliefs about the choice sets of other players, with $\beta^i_i(i)$ being i's own choice set; $\beta^i(j)$ for $j \neq i$ is i's beliefs about j's beliefs about the choice sets of other players, with $\beta^i_j(j)$ being the choice set of j anticipated by i; $\beta^i(j_1, j_2)$ for $j_1 \neq i$ is i's beliefs about j_1's beliefs about j_2's beliefs about the choice sets of other players, with $\beta^i_{j_2}(j_1, j_2)$ being equal to $\beta^i_{j_2}(j_1)$; and so on. A sequence $(j_1, \ldots, j_n) \in I^n$ is *admissible for i* if $j_1 \neq i$, and (when $n \geq 2$) for all $k = 1, \ldots, n-1$, $j_k \neq j_{k+1}$.[13] A set Y_i is *$\mathscr{C}$-rationalizable for i* if there is a belief system β^i for i such that for all sequences $j_1, \ldots, j_n$ admissible for i, we have (i) $Y_i = \beta^i_i(i) \in \mathscr{C}_i(\beta^i(i))$, (ii) $\beta^i_{j_n}(j_1, \ldots, j_n) = \beta^i_{j_n}(j_1, \ldots, j_{n-1})$, and (iii) $\beta^i_{j_n}(j_1, \ldots, j_{n-1}) \in \mathscr{C}_{j_n}(\beta^i(j_1, \ldots, j_n))$, where we identify j_0 with i. In particular, (iii) implies that the conjectured beliefs of j_{n-1} specify that j_n chooses a viable set, given j_{n-1}'s conjectured beliefs about j_n's beliefs. In this case, we say Y_i is *$\mathscr{C}$-rationalized* by β^i. Say $Y \in \mathbf{X}$ is *$\mathscr{C}$-rationalizable* if Y_i is $\mathscr{C}$-rationalizable for all i.

Proposition 24 *Let $\mathscr{C}$ be a choice structure. Then every $\mathscr{C}$-solution is $\mathscr{C}$-rationalizable. Furthermore, if $\mathscr{C}$ is monotonic and hard, and if Y is $\mathscr{C}$-rationalizable, then Y is contained in the unique $\mathscr{C}$-tract.*

Proof: If Y is a $\mathscr{C}$-solution, then we can specify that $\beta^i \equiv Y$ for all i to establish that Y is $\mathscr{C}$-rationalizable. Now assume $\mathscr{C}$ is monotonic and hard, let Y be $\mathscr{C}$-rationalizable, and let Y_i be $\mathscr{C}$-rationalizable by β^i for each i. Fix player i. Define the collection

$$\mathscr{Y}^i_j = \left\{ \beta^i_j(j_1, \ldots, j_n) \mid \begin{array}{c} \text{either both } n = 1 \text{ and } j_1 = i, \text{ or } n \geq 2 \\ \text{and } (j_1, \ldots, j_n) \text{ is admissible for } i \end{array} \right\}$$

of choice sets conjectured by i for j. Note that by condition (i) in the definition of rationalizability, we have $Y_j \in \mathscr{Y}^j_j$. Let $\mathscr{Y}_j = \bigcup_{i \in I} \mathscr{Y}^i_j$ be the collection of choice sets ascribed to j, and let $\mathscr{Y} = \prod_{j \in I} \mathscr{Y}_j$ be the collection of products of these sets. Since X is finite, so is $\mathscr{Y}$, and we may enumerate it as $Y^1, Y^2, \ldots, Y^L$. Note that the definition of rationality implies that for all Y^m, there exists Y^ℓ such that for all j, $Y^m_j \in \mathscr{C}_j(Y^\ell)$. Repeated application of the assumption that $\mathscr{C}$ is hard implies that for all j, $\bigcup_{\ell=1}^L Y^\ell_j \subseteq \bigcap \mathscr{C}_j(\bigcup_{\ell=1}^L Y^\ell)$, so $\bigcup_{\ell=1}^L Y^\ell$ is a $\mathscr{C}$-base. Since $\mathscr{C}$ is monotonic, Proposition 1 yields a $\mathscr{C}$-solution $Z \supseteq \bigcup_{\ell=1}^L Y^\ell$, and by Proposition 21, we may take Z to be the unique maximal $\mathscr{C}$-solution. Since $Y \subseteq \bigcup_{\ell=1}^L Y^\ell$, we conclude that $Y \subseteq Z$, as required. ∎

It follows that a strategy profile x is point rationalizable if and only if it is contained in the unique maximal $\mathscr{P}$-solution, and it is rationalizable if and only if it is contained in the unique maximal $\mathscr{R}$-solution, giving us the equivalences $P(G) = P'(G)$ and $R(G) = R'(G)$ in Bernheim's Propositions 3.1 and 3.2 (for the special case of finite games).

[13] For simplicity, we define the belief system β^i on non-admissible sequences, but that information is not used in the definition of $\mathscr{C}$-rationalizability.

With the results of Börgers (1993), we conclude that the maximal $\mathscr{B}$-solution consists of just the strategy profiles possible when it is common knowledge among the players that each knows the others' ordinal preferences and each maximizes expected utility for compatible some von Neumann-Morgenstern preferences. Similarly, we obtain the unique $\mathscr{S}$-, $\tilde{\mathscr{S}}$-, $\mathscr{M}^\bullet$-, and $\mathscr{L}^\bullet$-tracts by removal of dominated strategies in any order.

5.2 Minimal Solutions

We turn now to uniqueness of minimal solutions, or $\mathscr{C}$-cores, which provide the tightest possible predictions in our framework. Because a strict Nash equilibrium is a minimal solution for every choice structure, uniqueness of minimal solutions cannot hold generally, but uniqueness may hold for special classes of strategic form game. In fact, Shapley (1964) proves uniqueness of the $\mathscr{S}$-core for two-player, zero-sum games, and Duggan and Le Breton (1999, 2001) prove uniqueness of the $\tilde{\mathscr{S}}$-core for the smaller class of tournament games. We extend these results to a wider class of strategic form games, called "equilibrium safe," and to other dominance structures: we show that every equilibrium safe game possesses a unique $\mathscr{S}$-, $\mathscr{B}$-, $\tilde{\mathscr{S}}$-, and $\mathscr{R}$-core. Every two-player, zero-sum game is equilibrium safe, but this class of games also includes multi-player games characterized, broadly speaking, by the absence of equilibrium coordination problems.

The next proposition establishes that for a wide class of choice structures, if there are multiple minimal solutions, then they must be pairwise disjoint.

Proposition 25 *Let $\mathscr{C}$ be a monotonic, closed choice structure. If Y and Z are distinct $\mathscr{C}$-cores, then $Y \cap Z = \emptyset$.*

Proof: Let $\mathscr{C}$ be monotonic and closed, and let Y and Z be distinct $\mathscr{C}$-cores. If $Y \cap Z \neq \emptyset$, then since $\mathscr{C}$ is closed, it follows that $Y \cap Z$ is an outer $\mathscr{C}$-solution. Then Proposition 1 yields a $\mathscr{C}$-solution $W \subseteq Y \cap Z \subsetneq Y$, contradicting minimality of Y. ∎

Thus, to prove uniqueness of the $\mathscr{C}$-core, it is sufficient to show that any two minimal $\mathscr{C}$-solutions have nonempty intersection. We proceed by focussing on the $\mathscr{R}$-solutions in a special class of games: a strategic form game Γ is *equilibrium safe* if there exists a mixed strategy Nash equilibrium p^* of its mixed extension $\tilde{\Gamma}$ such that for all mixed strategy equilibria p and all i, $\tilde{u}_i(p_i^*, p_{-i}) \geq \tilde{u}_i(p)$; in this case, the equilibrium p^* is called *safe.* Thus, if player i anticipates that the other players will play some equilibrium p, i's expected payoff is no worse playing a safe equilibrium strategy. Several sufficient conditions for equilibrium safety are immediate: games with a unique mixed strategy Nash equilibrium, games with any dominant strategy equilibria, and two-player, zero-sum games are equilibrium safe; Appendix B contains more general sufficient conditions for equilibrium safety. The next proposition establishes that every equilibrium safe game possesses a unique $\mathscr{R}$-core, and it extends this result to a family of choice structures that, in a sense, encompass the rational choice structure. Given two choice structures $\mathscr{C}$ and $\mathscr{C}'$, we say $\mathscr{C}$ is as *heavy* as $\mathscr{C}'$ if every $\mathscr{C}$-solution is an outer $\mathscr{C}'$-solution. As usual, we extend this concept to dominance structures (or mixed dominance structures) by comparing the choice structures they generate. Obviously, a choice structure is as heavy as itself.

Proposition 26 *Assume Γ is equilibrium safe, and let $\mathscr{C}$ be a monotonic, closed choice structure. If $\mathscr{C}$ is as heavy as $\mathscr{R}$, then there is a unique $\mathscr{C}$-core.*

Proof: Assume $\mathscr{C}$ is monotonic, closed, and as heavy as $\mathscr{R}$. Suppose that an equilibrium safe game admits distinct $\mathscr{C}$-cores Y and Z. By Proposition 25, $Y \cap Z = \emptyset$. Since $\mathscr{C}$ is heavier than $\mathscr{R}$, it follows that Y and Z are outer $\mathscr{R}$-solutions, and Proposition 1 yields $\mathscr{R}$-solutions $Y' \subseteq Y$ and $Z' \subseteq Z$. Letting p^* denote a safe equilibrium, we claim that $\sigma(p^*) \subseteq Y$. To see this, let $p \in \tilde{Y}$ be a mixed strategy Nash equilibrium of the restricted game with strategy sets Y_i' and payoffs given by the restriction of u_i to Y' for each i. That p is a mixed strategy Nash equilibrium of the original game follows since $Y_i' = BR_i(Y')$ contains all best responses to p for each i. By the definition of equilibrium safety, p_i^* is a best response to p_{-i}, and so therefore is each strategy in the support of p_i^*, which implies $\sigma_i(p_i^*) \subseteq BR_i(Y') = Y_i'$, as claimed. Similarly, $\sigma(p) \subseteq Z'$, contradicting $Y' \cap Z' = \emptyset$. Thus, $\mathscr{C}$ admits a unique minimal $\mathscr{C}$-solution. ∎

The analysis in foregoing sections has revealed that $\mathscr{S}$ is as heavy as $\mathscr{B}$ and $\tilde{\mathscr{S}}$, which are both as heavy as $\mathscr{R}$. We conclude that equilibrium safe games possess a unique minimal solution for the Shapley, Börgers, mixed Shapley, and rational choice structures. This extends Shapley's (1964) result from two-player, zero-sum games to the class of multi-player, equilibrium safe games. It complements Bernheim's (1984)

Proposition 3.2 by establishing existence of a unique minimal (rather than maximal) rationalizable set, providing the tightest possible prediction consistent with rationalizability. It generalizes the uniqueness result of Duggan and Le Breton (1999,2001) for the mixed saddle in tournament games to equilibrium safe games. Finally, it provides a new result on uniqueness of the minimal Börgers solution. Appendix B contains straightforward extensions of Proposition 26 to classes of games that are equivalent (under certain transformations) to equilibrium safe games.

Equilibrium safety is not necessary for uniqueness of $\mathscr{S}$-, $\mathscr{B}$-, $\tilde{\mathscr{S}}$-, or $\mathscr{R}$-cores, as the next example shows.

Example 14 Let $|I| = 2$, $X_1 = X_2 = \{a, b\}$, with payoffs as below.

	a	b
a	(2,1)	(1,1)
b	(1,1)	(2,1)

Here, $\{a, b\} \times \{a, b\}$ is the unique $\mathscr{R}$-solution, and therefore the unique $\tilde{\mathscr{S}}$-, $\mathscr{B}$-, and $\mathscr{S}$-solution as well. However, this game is not equilibrium safe: no other strategy gives row player as high a payoff as a when column player picks a, but, when column player picks b, b gives row player a strictly higher payoff than a.

The next example shows that we focus on maximal and minimal solutions out of necessity: multiple $\mathscr{C}$-solutions exist for the choice structures considered above, even in a restricted class of two-player, symmetric, zero-sum games. Of course, consistent with Propositions 21 and 26, there is a unique $\mathscr{C}$-tract and unique $\mathscr{C}$-core for these choice structures. Because the example is highly structured, there appear to be no reasonable conditions on games that would ensure uniqueness of solutions in general.

Example 15 Let $|I| = 2$, $X_1 = X_2 = \{a, b, c, d\}$, with zero-sum payoffs as below.

	a	b	c	d
a	(0,0)	(1,-1)	(-1,1)	(-1,1)
b	(-1,1)	(0,0)	(-1,1)	(1,-1)
c	(1,-1)	(1,-1)	(0,0)	(1,-1)
d	(1,-1)	(-1,1)	(-1,1)	(0,0)

Here, $\{a, b, c, d\} \times \{a, b, c, d\}$ is the unique $\mathscr{S}$-, $\mathscr{B}$-, $\tilde{\mathscr{S}}$-, and $\mathscr{R}$-tract, but the $\mathscr{C}$-core for all of these dominance structures is $\{c\} \times \{c\}$.

6 Conclusion

We provide a framework for the choice-theoretic analysis of strategic form games for a broad spectrum of choice structures, and we establish a number of results: existence of solutions, uniqueness and epistemic foundations of maximal solutions (and an iterative procedure for their computation), and uniqueness of minimal solutions in equilibrium safe games. Because we state the results abstractly, they can be applied to generate insights into existing concepts, such as iterative elimination of strictly dominated strategies and Shapley's saddles, rationalizable and point rationalizable sets, and Börgers dominance; and they have implications for new choice structures based on pessimistic conjectures (e.g., monotonic maximin and leximin) and permitting the use of mixed strategies (e.g., mixed Shapley). An advantage of the framework is that it resolves strategic indeterminacy in a parsimonious way — modeling the players' decisions via choice sets, rather than mixed strategies — and it can accommodate choice structures that reflect the strategic capabilities of players that are most realistic in a given environment. A further advantage is that for many choice structures, maximal solutions are easily calculated; see Brandt and Brill (2012) for recent work on the computation of solutions in our framework.

A problem not addressed in the current paper, following Sprumont's (2000) analysis of testable implications of Nash behavior, is to understand conditions under which observed choice sets can be "rationalized" by certain classes of choice structure. Given player set I and sets $(X_i)_{i \in I}$ of conceivable strategies, observed behavior can be represented by a correspondence $\mathscr{B}\colon \mathbf{X} \rightrightarrows 2^{\mathbf{X}}$ satisfying $\bigcup \mathscr{B}(Y) \subseteq Y$ for all $Y \in \mathbf{X}$, where $\mathscr{B}(Y)$ gives the products of viable choice sets when strategy sets are $(Y_i)_{i \in I}$. One problem of interest then is to find axioms on $\mathscr{B}$ under which there exist payoff functions $(u_i)_{i \in I}$ and monotonic choice structures $(\mathscr{C}_i)_{i \in I}$ such that for all $Y \in \mathbf{X}$, the collection $\mathscr{B}(Y)$ consists of all $\mathscr{C}$-solutions in the game $(I, (Y_i)_{i \in I}, (u_i|_Y)_{i \in I})$. Of course, there are abundant variations of this problem, as we could impose further properties on choice structures or focus on a particular choice structure of interest, or we could restrict attention to maximal or minimal $\mathscr{C}$-solutions.

A hurdle to empirical applications of the framework lies in the structural estimation of games that are solved via the choice-theoretic approach, a question that is outside the scope of this paper; but an interesting aspect of the approach is that the choice structure describing the players' behavior could itself conceivably be considered as a parameter to be estimated. We take it as an empirical question whether the strategic behavior of real-world actors is more accurately described, e.g., by monotonic leximin solution or by Börgers solutions.

A Proofs from Section 3

Proposition 1 *Assume the choice structure $\mathscr{C}$ is monotonic. If Z is a $\mathscr{C}$-base, and if Y is an outer $\mathscr{C}$-solution that is minimal among the collection*

$$\left\{ Y' \in \mathbf{X} \mid \ Y' \text{ is an outer } \mathscr{C}\text{-solution and } Z \subseteq Y' \ \right\},$$

then Y is a $\mathscr{C}$-solution.

Proof: Let Y be an outer $\mathscr{C}$-solution that is minimal among the outer $\mathscr{C}$-solutions containing the $\mathscr{C}$-base Z. Since Y is an outer $\mathscr{C}$-solution, it follows that for each i, there exists $Y_i' \in \mathscr{C}_i(Y)$ such that $Y_i' \subseteq Y_i$. Let $Y' = \prod_{i \in I} Y_i'$. If $Y' = Y$, then Y is a $\mathscr{C}$-solution. Otherwise, $Y' \subsetneq Y$. We claim that Y' is an outer $\mathscr{C}$-solution: indeed, note that $Y'_i \in \mathscr{C}_i(Y)$ for all i and $Y' \subseteq Y$, so monotonicity yields $Y''_i \in \mathscr{C}_i(Y')$ with $Y''_i \subseteq Y_i$ for all i, as claimed. Since $Y'_i \in \mathscr{C}_i(Y)$ for all i and $Z \subseteq Y$, monotonicity yields $Z'_i \in \mathscr{C}_i(Z)$ with $Z'_i \subseteq Y'_i$ for all i. Since Z is a $\mathscr{C}$-base, it follows that $Z_i \subseteq Z'_i \subseteq Y'_i$ for all i. But then Y' is an outer $\mathscr{C}$-solution containing Z and is a proper subset of Y, contradicting minimality of Y. ∎

Proposition 2 *Let $\mathscr{D}$ be a dominance structure and $\mathscr{C}$ the choice structure generated by $\mathscr{D}$. For all i, all $Y \in \mathbf{X}$, and all $Y'_i \subseteq X_i$, if $Y'_i \in \mathscr{C}_i(Y)$, then Y'_i is minimal with respect to set inclusion among the sets that are externally stable with respect to $\mathscr{D}_i(Y)$; furthermore, the converse holds if $\mathscr{D}$ is monotonic and transitive.*

Proof: Consider any $Y \in \mathbf{X}$, any i, and any $Y'_i \subseteq X_i$. If $Y'_i \in \mathscr{C}_i(Y)$, then the set is externally stable with respect to $\mathscr{D}_i(Y)$. If it is not minimal among those sets, then there exists $Y''_i \subsetneqq Y'_i$ that is also externally stable. Then there exists $x_i \in Y'_i \setminus Y''_i$, and external stability yields $y_i \in Y''_i$ such that $y_i \mathscr{D}_i(Y) x_i$, but then x_i and y_i are distinct elements of Y'_i such that $y_i \mathscr{D}_i(Y) x_i$, contradicting internal stability. For the converse direction, assume $\mathscr{D}$ is monotonic and transitive, and let Y'_i be minimal among the externally stable sets. It suffices to show that Y'_i is internally stable with respect to $\mathscr{D}_i(Y)$. Otherwise, there exist distinct $x_i, y_i \in Y'_i$ such that $x_i \mathscr{D}_i(Y) y_i$, so we can define $Y''_i = Y'_i \setminus \{y_i\}$. Consider any $z_i \in X_i \setminus Y''_i$. Then either $z_i = y_i$, in which case $x_i \mathscr{D}_i(Y) z_i$, or $z_i \in X_i \setminus Y'_i$. In the latter case, there exists $w_i \in Y'_i$ such that $w_i \mathscr{D}_i(Y) z_i$. If $w_i \neq y_i$, then we have $w_i \in Y''_i$ with $w_i \mathscr{D}_i(Y) z_i$; and if $w_i = y_i$, then we have $x_i \mathscr{D}_i(Y) y_i \mathscr{D}_i(Y) z_i$, and transitivity of $\mathscr{D}$ implies $x_i \mathscr{D}_i(Y) z_i$. Thus, Y''_i is externally stable with respect to $\mathscr{D}_i(Y)$ and is a proper subset of Y'_i, contradicting minimality. We conclude that Y'_i is indeed externally stable, and therefore $Y'_i \in \mathscr{C}_i(Y)$. ∎

Proposition 3 *Assume the dominance structure $\mathscr{D}$ is transitive and monotonic. Then the choice structure $\mathscr{C}$ generated by $\mathscr{D}$ is monotonic.*

Proof: Consider any $Y, Z \in \mathbf{X}$ with $Y \subseteq Z$ and any player i. Let $Z'_i \in \mathscr{C}_i(Z)$. Note that Z'_i satisfies external stability with respect to $\mathscr{D}_i(Z)$, and therefore, by monotonicity of $\mathscr{D}$, with respect to $\mathscr{D}_i(Y)$. Then we can define Y'_i as any set that is minimal with respect to set inclusion among the subsets of Z_i' that are externally stable with respect to $\mathscr{D}_i(Y)$. By Proposition 2, it follows that $Y_i' \in \mathscr{C}_i(Y)$, as required. ∎

Proposition 4 *Assume the dominance structure $\mathscr{D}$ is transitive, monotonic, and weakly irreflexive. Then the choice structure $\mathscr{C}$ generated by $\mathscr{D}$ is closed.*

Proof: Consider any $Y, Z \in \mathbf{X}$ and any collections $\{Y_i'\}$ and $\{Z_i'\}$ such that for all i, $Y_i' \in \mathscr{C}_i(Y)$, $Z_i' \in \mathscr{C}_i(Z)$, and $Y_i' \cap Z_i' \neq \emptyset$. Letting $W_i = Y_i' \cap Z_i'$ and $W = \prod_{i \in I} W_i$, we must show that W is an outer $\mathscr{C}$-solution. We claim that for all i, W_i is externally stable with respect to $\mathscr{D}_i(W)$. Consider any i and any $x_i \in X_i \backslash W_i$, and assume without loss of generality that $x_i \notin Y_i'$. We identify an element $w_i \in W_i$ as follows. Because Y_i' is externally stable with respect to $\mathscr{D}_i(Y)$, there exists $w_i^1 \in Y_i'$ such that $w_i^1 \mathscr{D}_i(Y) x_i$. If $w_i^1 \in Z_i'$, so that $w_i^1 \in W_i$, then set $w_i = w_i^1$, and note that monotonicity of $\mathcal{Q}$ implies $w_i \mathscr{D}_i(W) x_i$. Otherwise, because Z_i' is externally stable with respect to $\mathscr{D}_i(Z)$, there exists $w_i^2 \in Z_i'$ such that $w_i^2 \mathscr{D}_i(Z) w_i^1$. If $w_i^2 \in Y_i'$, then set $w_i = w_i^2$. Otherwise, we again invoke external stability of Y_i', and so on. In case this procedure terminates with $w_i = w_i^k \in W_i$, we apply monotonicity of $\mathscr{D}$ to deduce a sequence

$$w_i = w_i^k \mathscr{D}_i(W) w_i^{k-1} \cdots w_i^1 \mathscr{D}_i(W) x_i,$$

and then transitivity of $\mathscr{D}$ implies $w_i \mathscr{D}_i(W) x_i$, establishing the external stability claim. In case the procedure does not terminate, then because X_i is finite, there exist integers $k < \ell$ such that $w_i^k = w_i^\ell$. Applying monotonicity of $\mathscr{D}$, we then deduce a sequence

$$w_i^\ell \mathscr{D}_i(W) w_i^{\ell-1} \cdots w_i^{k+1} \mathscr{D}_i(W) w_i^k,$$

and transitivity of $\mathscr{D}$ implies $w_i^k \mathscr{D}_i(W) w_i^k$. Now letting w_i be any element of W_i, weak irreflexivity yields a sequence

$$w_i \mathscr{D}_i(W) w_i^k \mathscr{D}_i(W) w_i^{k-1} \cdots w_i^1 \mathscr{D}_i(W) x_i,$$

and transitivity implies $w_i \mathscr{D}_i(W) x_i$. Thus, we have established the external stability claim. Finally, for each i, let W_i' be minimal among the subsets of W_i that are externally stable with respect to $\mathscr{D}_i(W)$. By Proposition 2, it follows that $W_i' \in \mathscr{C}_i(W)$ for all i, and therefore W is an outer $\mathscr{C}$-solution. ∎

Proposition 5 *Assume the dominance structure $\mathscr{D}$ is transitive, monotonic, weakly irreflexive, and non-trivial. Then the choice structure $\mathscr{C}$ generated by $\mathscr{D}$ is hard.*

Proof: Consider any $Y, Z \in \mathbf{X}$ and any collections $\{Y_i'\}$ and $\{Z_i'\}$ such that $Y_i' \in \mathscr{C}_i(Y)$ and $Z_i' \in \mathscr{C}_i(Z)$ for all i. Letting $W = \prod_{i \in I}(Y_i \cup Z_i)$, we must show that $Y_i' \cup Z_i' \subseteq \mathscr{C}_i(W)$ for all i. Consider any i, any $x_i \in Y_i \cup Z_i$, and any $W_i' \in \mathscr{C}_i(W)$, and assume without loss of generality that $x_i \in Y_i$. We claim that $x_i \in W_i'$; otherwise, because W_i' is externally stable with respect to $\mathscr{D}_i(W)$, there exists $y_i \in W_i'$ such that $y_i \mathscr{D}_i(W) x_i$. Since $Y \subseteq W$, monotonicity of $\mathscr{D}$ implies $y_i \mathscr{D}_i(Y) x_i$. Then, because $y_i \neq x_i$, internal stability of Y_i with respect to $\mathscr{D}_i(Y)$ implies $y_i \notin Y_i$, and external stability yields $z_i \in Y_i$ such that $z_i \mathscr{D}_i(Y) y_i$. By transitivity of $\mathscr{D}$ and $z_i \mathscr{D}_i(Y) y_i \mathscr{D}_i(Y) x_i$, we have $z_i \mathscr{D}_i(Y) x_i$, and then internal stability implies $x_i = z_i$, which implies $x_i \mathscr{D}_i(Y) x_i$. By weak irreflexivity, it follows that for all $w_i \in X_i$, $w_i \mathscr{D}_i(Y) x_i$. Applying internal stability again, we have $Y_i = \{x_i\}$, and then external stability implies that for all $w_i \in X_i$, $x_i \mathscr{D}_i(Y) w_i$. With transitivity, this implies that $\mathscr{D}_i(Y) = X_i \times X_i$, contradicting the assumption that $\mathscr{D}$ is non-trivial. Therefore, $x_i \in W_i'$, and we conclude that $Y_i \cup Z_i \subseteq \bigcap \mathscr{C}_i(W)$. ∎

Proposition 6 *Let $\mathscr{D}$ be a dominance structure. Then the monotonic kernel $\mathscr{D}^\bullet$ is transitive, monotonic, and weakly irreflexive, and the choice structure generated by $\mathscr{D}^\bullet$ is closed. Furthermore, if $\mathscr{D}$ is transitive, weakly irreflexive, and non-trivial, then $\mathscr{D}^\bullet$ is non-trivial, and the choice structure it generates is hard.*

Proof: Transitivity, monotonicity, and weak irreflexivity are evident. That the choice structure generated by $\mathscr{D}^\bullet$ is closed then follows from Proposition 4. Now assume $\mathscr{D}$ is transitive, weakly irreflexive, and non-trivial, and to show that $\mathscr{D}^\bullet$ is non-trivial, suppose to the contrary that there exist i and $Y \in \mathbf{X}$ such that for all $x_i, y_i \in X_i$, $x_i \mathscr{D}_i^\bullet(Y) y_i$. By transitivity of $\mathscr{D}_i(Y)$, there exists $x_i^* \in X_i$ that is maximal in the sense that for all $y_i \in X_i$, if $y_i \mathscr{D}_i(Y) x_i^*$, then $x_i^* \mathscr{D}_i(Y) y_i$. By supposition, there exists $y_i^* \in X_i$ such that $y_i^* \mathscr{D}_i(Y) x_i^*$, so $x_i^* \mathscr{D}_i(Y) y_i^* \mathscr{D}_i(Y) x_i^*$ and transitivity yield $x_i^* \mathscr{D}_i(Y) x_i^*$. By weak irreflexivity, we then have $y_i \mathscr{D}_i(Y) x_i^*$ for all $y_i \in X_i$, and then by an application of transitivity, we have $x_i \mathscr{D}_i(Y) y_i$ for all $x_i, y_i \in X_i$, contradicting the assumption that $\mathscr{D}$ is non-trivial. Thus, $\mathscr{D}^\bullet$ is non-trivial, and it follows from Proposition 5 that the choice structure it generates is hard. ∎

Proposition 7 *Let $\mathscr{D}$ be a dominance structure, and let $\mathscr{C}^\bullet$ be the choice structure generated by its monotonic kernel. If $\mathscr{D}$ is transitive and weakly irreflexive, then for all i, all $Y \in \mathbf{X}$, and all $Y_i' \subseteq X_i$, $Y_i' \in \mathscr{C}_i^\bullet(Y)$ if and only if*

(i) for all $x_i \in Y_i'$, there exists $Z \in \mathbf{X}$ with $Z \subseteq Y$ such that for all $y_i \in Y_i'$, not $y_i \mathscr{D}_i(Z) x_i$,

(ii) for all $x_i \in X_i \setminus Y_i'$ and all $Z \in \mathbf{X}$ with $Z \subseteq Y$, there exists $y_i \in Y_i'$ such that $y_i \mathscr{D}_i(Z) x_i$.

Proof: Consider any i, any $Y \in \mathbf{X}$, and any $Y_i' \subseteq X_i$. First, assume $Y_i' \in \mathscr{C}_i^\bullet(Y)$. If (i) is violated, then there exists $x_i \in X_i$ such that for all $Z \subseteq Y$, there exists $y_i \in Y_i'$ such that $y_i \mathscr{D}_i(Z) x_i$. In particular, there exists $z_i \in Y_i'$, and the foregoing implies $z_i \mathscr{D}_i^\bullet(Y) x_i$, contradicting internal stability of Y_i'. Thus, (i) holds. To verify (ii), consider any $x_i \in X_i \setminus Y_i'$ and any $Z \subseteq Y$. By external stability of Y_i', there exists $z_i \in Y_i'$ such that $z_i \mathscr{D}_i^\bullet(Y) x_i$, and in particular, there exists $y_i \in X_i$ such that $y_i \mathscr{D}_i(Z) x_i$. By transitivity of $\mathscr{D}$, we can further specify that y_i is maximal, in the sense that for all $z_i \in X_i$, if $z_i \mathscr{D}_i(Z) y_i$, then $y_i \mathscr{D}_i(Z) x_i$. If $y_i \in Y_i'$, then (ii) is verified. Otherwise, $y_i \in X_i \setminus Y_i'$, and there exists $z_i \in X_i$ such that $z_i \mathscr{D}_i(Z) y_i$. Then $y_i \mathscr{D}_i(Z) z_i \mathscr{D}_i(Z) y_i$ and transitivity yield $y_i \mathscr{D}_i(Z) y_i$. Then, selecting any $w_i \in Y_i'$, weak irreflexivity implies that $w_i \mathscr{D}_i(Z) y_i$, as required. Thus, (ii) holds.

Next, assume (i) and (ii) hold. To show that $Y_i' \in \mathscr{C}_i^\bullet(Y)$, we must establish internal and external stability with respect to $\mathscr{D}_i^\bullet(Y)$. Consider any $x_i \in Y_i'$. Then (i) yields $Z \subseteq Y$ such that for all $z_i \in Y_i'$, not $z_i \mathscr{D}_i(Z) x_i$. Suppose that there exists $z_i \in X_i \setminus Y_i'$ such that $z_i \mathscr{D}_i(Z) x_i$. By transitivity of $\mathscr{D}$, we can specify that z_i is maximal in the sense defined above. By (ii), however, there exists $y_i \in Y_i'$ such that $y_i \mathscr{D}_i(Z) z_i \mathscr{D}_i(Z) x_i$, and transitivity implies $y_i \mathscr{D}_i(Z) x_i$, a contradiction. Thus, $z_i \mathscr{D}_i(Z) x_i$ holds for no $z_i \in X_i$, and it follows that $y_i \mathscr{D}_i^\bullet(Y) x_i$ holds for no $y_i \in X_i$, which delivers internal stability. Now consider any $x_i \in X_i \setminus Y_i'$. Then (ii) implies that for all $Z \subseteq Y$, there exists $z_i \in Y_i'$ such that $z_i \mathscr{D}_i(Z) x_i$. In particular, selecting any $y_i \in Y_i'$, we have $y_i \mathscr{D}_i^\bullet(Y) x_i$, delivering external stability. ∎

Proposition 8 *Let $\mathscr{D}$ and $\mathscr{D}'$ be dominance structures. (i) If $\mathscr{D}$ is stronger than $\mathscr{D}'$,* *then every $\mathscr{D}$-solution is an outer $\mathscr{D}'$-solution. (ii) If $\mathscr{D}$ subjugates $\mathscr{D}'$ and $\mathscr{D}'$ is transitive, then every $\mathscr{D}'$-solution is a $\mathscr{D}$-base.*

Proof: Let $\mathscr{C}$ and $\mathscr{C}'$ be the choice structures generated, respectively, by $\mathscr{D}$ and $\mathscr{D}'$. To prove (i), assume $\mathscr{D}$ is stronger than $\mathscr{D}'$, and let Y be a $\mathscr{D}$-solution. Then for all i, Y_i is externally stable with respect to $\mathscr{D}_i(Y)$, which implies that it is externally stable with respect to $\mathscr{D}'_i(Y)$. Letting Y'_i be minimal with respect to set inclusion among the subsets of Y_i that are externally stable with respect to $\mathscr{D}'_i(Y)$, Proposition 2 implies that $Y'_i \in \mathscr{C}'_i(Y)$. Thus, Y is an outer $\mathscr{D}'$-solution. To prove (ii), assume that $\mathscr{D}$ subjugates $\mathscr{D}'$ and that $\mathscr{D}'$ is transitive. Consider any $\mathscr{D}'$-solution Y', any $Y_i \in \mathscr{C}_i(Y')$, and suppose that there exists $x_i \in Y'_i \setminus Y_i$. Because Y_i is externally stable, there exists $y_i \in Y_i$ such that $y_i \mathscr{D}_i(Y) x_i$. Because $\mathscr{D}$ subjugates $\mathscr{D}'$, there exists $z_i \in X_i \setminus \{x_i\}$ such that $z_i \mathscr{D}'_i(Y') x_i$. Because Y'_i is internally stable, it follows that $z_i \notin Y'_i$. Because Y'_i is externally stable, there exists $w_i \in Y'_i$ such that $w_i \mathscr{D}'_i(Y') z_i \mathscr{D}'_i(Y') x_i$, and transitivity of $\mathscr{D}'$ implies $w_i \mathscr{D}'_i(Y') x_i$, contradicting internal stability. Thus, $Y'_i \subseteq Y_i$, and we conclude that Y' is a $\mathscr{D}$-base. ∎

Proposition 9 *Let $\mathscr{D}$ and $\mathscr{D}'$ be dominance structures. (i) If $\mathscr{D}$ is stronger than $\mathscr{D}'$, and if $\mathscr{D}'$ is transitive and monotonic, then every $\mathscr{D}$-solution includes some $\mathscr{D}'$-solution. (ii) If $\mathscr{D}$ subjugates $\mathscr{D}'$, if $\mathscr{D}'$ is transitive, and if $\mathscr{D}$ is transitive and monotonic, then every $\mathscr{D}'$-solution is included in some $\mathscr{D}$-solution.*

Proof: To prove (i), consider any $\mathscr{D}$-solution Y. Since $\mathscr{D}$ is stronger than $\mathscr{D}'$, Proposition 8 implies that Y is an outer $\mathscr{D}'$-solution. Since $\mathscr{D}'$ is transitive and monotonic, Proposition 3 implies that the choice structure generated by $\mathscr{D}'$ is monotonic, and then Proposition 1 yields a $\mathscr{D}'$-solution $Y' \subseteq Y$. To prove (ii), consider any $\mathscr{D}'$-solution Y'. Since $\mathscr{D}$ subjugates $\mathscr{D}'$ and $\mathscr{D}'$ is transitive, Proposition 8 implies that Y' is a $\mathscr{D}$-base. Since $\mathscr{D}$ is transitive and monotonic, Proposition 3 implies that the choice structure generated by $\mathscr{D}$ is monotonic, and then Proposition 1 yields a $\mathscr{D}$-solution $Y \supseteq Y'$. ∎

B Sufficient Conditions for Equilibrium Safety

In this appendix, we augment the sufficient conditions provided above for equilibrium safety. We now focus on two classes of game: a strategic form game Γ is *equilibrium* *interchangeable* if for all mixed strategy Nash equilibria p and p' of the mixed extension $\tilde{\Gamma}$ and all i, (p_i, p'_{-i}) is an equilibrium; and Γ is *safe* if (a) there is a unique equilibrium payoff vector $(\overline{u}_i)_{i\in I}$, and (b) there is a mixed strategy Nash equilibrium $\tilde{p}$ such that, for all i and all $q_{-i} \in \tilde{X}_{-i}$, $\tilde{u}_i(\tilde{p}_i, q_{-i}) \geq \overline{u}_i$. In other words, a game is safe if each player has a "good" equilibrium strategy. Clearly, any game with a unique mixed strategy Nash equilibrium is both equilibrium interchangeable and safe. Every two-player, zero-sum game is both equilibrium interchangeable and safe. Aumann (1961) defines *almost strictly competitive* games, a class of two-player strategic form games generalizing two-player, zero-sum games, that are both equilibrium interchangeable and safe.[14] And any game with a dominant strategy equilibrium is safe (but not necessarily equilibrium interchangeable). The next proposition, the proof of which is self-evident, establishes the sufficiency of these conditions for equilibrium safety.

Proposition 25 *If a strategic form game Γ is equilibrium interchangeable or safe, then it is equilibrium safe.*

It is clear that a safe game need not be equilibrium interchangeable, and the next example establishes the converse. Thus, the conditions are logically independent.

[14] For any bimatrix game (A_1, A_2), where A_1 gives player 1's payoffs and A_2 player 2's, a *twisted equilibrium* is an equilibrium of the "twisted game" $(-A_2, -A_1)$. Aumann defines a bimatrix game as *almost strictly competitive* if (i) the set of equilibrium payoffs coincides with the set of twisted equilibrium payoffs, and (ii) the sets of equilibria and twisted equilibria have non-empty intersection.

Example 16 Let $|I| = 2$, $X_1 = \{x_1, x_2, x_3, x_4, x_5\}$, and $X_2 = \{y_1, y_2, y_3, y_4, y_5\}$, with payoffs as below.

	y_1	y_2	y_3	y_4	y_5
x_1	(0,0)	(1,-1)	(-1,1)	(-2,0)	(2,0)
x_2	(-1,1)	(0,0)	(1,-1)	(1,0)	(-1,0)
x_3	(1,-1)	(-1,1)	(0,0)	(1,0)	(-1,0)
x_4	(0,-2)	(0,1)	(0,1)	(-2,-1)	(-1,-1)
x_5	(0,2)	(0,-1)	(0,-1)	(-1,-1)	(-1,-2)

Note that the pair $p = (p_1, p_2)$, where $p_1 = p_2 = (1/3, 1/3, 1/3, 0, 0)$ is an equilibrium with zero payoffs for the players. Moreover, these strategies guarantee the players at least this payoff. Now define $p'_1 = p'_2 = (0, 0, 0, 1/2, 1/2)$, and note that (p_1, p'_2) and (p'_1, p_2) are equilibria but that (p'_1, p'_2) is not. Thus, this game is not equilibrium interchangeable. It remains only to be checked that zero is the unique equilibrium payoff. Consider any equilibrium $q = (q_1, q_2)$, and define $r_1 = q_1(x_1) + q_1(x_2) + q_1(x_3)$ and $r_2 = q_2(y_1) + q_2(y_2) + q_2(y_3)$. If $r_1 = r_2 = 1$, the players are essentially playing a symmetric zero-sum game, so their payoff must be zero. If $r_1 < 1$ and $r_2 = 1$, player 1's expected payoff from x_4 and x_5 is zero, so his payoff from q is zero. Thus, $q_2 = p_2$ (otherwise player 1 could deviate profitably) and 2's payoff is zero. Similarly if $r_1 = 1$ and $r_2 < 1$. If $r_1 < 1$ and $r_2 < 1$, then player 1's payoff from x_4 and x_5 is negative, worse than p_1, contradicting our assumption that q is an equilibrium.

Proposition 25 allows us to apply the results of Kats and Thisse (1992) on equilibrium interchangeability. Their analysis uses the following more primitive conditions, defined for strategic form games with possibly infinite strategy sets. The first condition is defined only for two-player games and the second extends it to multi-player games.

(i) A two-player game is *strictly competitive* if for all i and $j \neq i$ and for all $x, y \in X$,

$$u_i(y_i, x_j) > u_i(x) \Leftrightarrow u_j(y_i, x_j) < u_j(x).$$

(ii) A strategic form game is *unilaterally competitive* if for all i, all $x_i, y_i \in X_i$, and all $x_{-i} \in X_{-i}$,

$$(u_i(y_i, x_{-i}) > u_i(x)) \Leftrightarrow (\forall j \neq i)(u_j(y_i, x_{-i}) < u_j(y_i, x_{-i})).$$

(iii) A strategic form game is *weakly unilaterally competitive* if for all i, all $x_i, y_i \in X_i$, and all $x_{-i} \in X_{-i}$,

$$(u_i(y_i, x_{-i}) > u_i(x)) \Rightarrow (\forall j \neq i)(u_j(y_i, x_{-i}) \leq u_j(x))$$

and

$$(u_i(y_i, x_{-i}) = u_i(x)) \Rightarrow (\forall j \neq i)(u_j(y_i, x_{-i}) = u_j(x)).$$

Thus, a two-player game is strictly competitive if its payoffs are strictly Pareto optimal. The idea of unilateral competitiveness extends this concept, but only applies the Pareto optimality criterion to unilateral changes in strategies. The third condition weakens unilateral competitiveness, now allowing for one player to improve his payoff with a unilateral move, as long as no other player is made better off. The next theorem follows directly from Theorem 25 and Kats and Thisse's Theorem 2, where they prove the sufficiency of their conditions for equilibrium interchangeability.

Proposition 26 *Let Γ be a finite strategic form game. (i) If $|I| = 2$ and the mixed extension $\tilde{\Gamma}$ is weakly unilaterally competitive, then Γ is equilibrium safe. (ii) If the mixed extension $\tilde{\Gamma}$ is unilaterally competitive, then Γ is equilibrium safe.*

Unfortunately, this result uses conditions on the mixed extension of Γ, which may be difficult to verify. The next example shows that, even in two-player games, requiring strict competitiveness of Γ itself is not sufficient for existence of the $\mathscr{R}$-core or $\tilde{\mathscr{S}}$-core.

Example 17 Let $|I| = 2$ and $X_1 = X_2 = \{a, b, c, d\}$, with payoffs below.

	a	b	c	d
a	(5,1)	(1,5)	(2,2)	(2,2)
b	(1,5)	(5,1)	(2,2)	(2,2)
c	(2,2)	(2,2)	(5,1)	(1,5)
d	(2,2)	(2,2)	(1,5)	(5,1)

This game is strictly competitive, but not equilibrium safe, as evidenced by the fact that it has two $\mathscr{R}$-cores (and $\tilde{\mathscr{S}}$-cores), $\{a, b\} \times \{a, b\}$ and $\{c, d\} \times \{c, d\}$.

Two games, $\Gamma = (I, (X_i)_{i \in I}, (u_i)_{i \in I})$ and $\Gamma' = (I', (X_i')_{i \in I}, (u_i')_{i \in I})$, are *order equivalent* if $I = I'$; for all i, $X_i = X_i'$; and for all i, all $x_i, y_i \in X_i$, and all $x_{-i} \in X_{-i}$,

$$u_i(x_i, x_{-i}) \geq u_i(y_i, x_{-i}) \Leftrightarrow u_i'(x_i, x_{-i}) \geq u_i'(y_i, x_{-i}).$$

In a two-player matrix game, for example, order equivalence means that the row player's ordering of cells in any given column is the same, and the column player's ordering of cells in any given row is the same. Relationships between payoffs in cells that do not lie on the same row or column are unrestricted. The game in Example 17 does have a unique $\mathscr{S}$-core and $\mathscr{B}$-core, as we will see is true of all two-player, strictly competitive games, which are known to be order equivalent to two-player, zero-sum games.

Two games, Γ and Γ', are *best response equivalent* if $I = I'$; for all i, $X_i = X_i'$; and for all i and all $p_{-i} \in \tilde{X}_{-i}$, i's pure strategy best responses to p_{-i} in Γ and Γ' coincide. Abusing notation slightly, $BR_i(p_{-i}) = BR_i'(p_{-i})$. Two games may be order equivalent but not best response equivalent, and they may be best response equivalent but not order equivalent.[15] It is clear that $\mathscr{R}$-solutions are invariant under best response equivalent transformations: if Γ and Γ' are best response equivalent, then Y is an $\mathscr{R}$-

[15]See Rosenthal's (1974) Examples 5 and 6.

solution of Γ if and only if it is an $\mathscr{R}$-solution of Γ'. This gives us the following extension of Proposition 26.

Proposition 27 *Assume Γ' is best response equivalent to a finite, equilibrium safe game Γ, and let $\mathscr{C}$ be a monotonic, hard choice structure. If $\mathscr{C}$ is as heavy as $\mathscr{R}$, then Γ' has a unique $\mathscr{C}$-core.*

Note that $\mathscr{S}$-solutions are invariant with respect to order equivalent transformations — if Γ and Γ' are order equivalent, then Y is an $\mathscr{S}$-solution of Γ if and only if it is an $\mathscr{S}$-solution of Γ' — and that the same is true of $\mathscr{B}$-solutions. Thus, we can extend Proposition 26 even further for $\mathscr{S}$-cores and $\mathscr{B}$-cores.

Proposition 28 *Assume that Γ'' is order equivalent to Γ', and that Γ' is best response equivalent to an equilibrium safe game Γ. Then Γ'' has a unique $\mathscr{S}$-core and a unique $\mathscr{B}$-core.*

The preceding result immediately yields uniqueness of the $\mathscr{S}$-core and $\mathscr{B}$-core in two-player strictly competitive games, which are order equivalent to zero-sum games. The next example further illustrates the scope of the proposition.

Example 18 Let $|I| = 2$ and $X_1 = X_2 = \{a, b, c, d\}$, with payoffs below.

	a	b	c	d
a	(5,1)	(1,5)	(3,3)	(3,3)
b	(1,5)	(5,1)	(3,3)	(3,3)
c	(3,3)	(3,3)	(5,1)	(1,5)
d	(3,3)	(3,3)	(1,5)	(5,1)

This game is order equivalent (but not best response equivalent) to the game in Example 17. It is equilibrium safe, since $p_1 = p_2 = (1/2, 1/2, 0, 0)$ is a safe equilibrium, and has a unique $\mathscr{R}$-core and unique $\mathscr{S}$-core. Thus, as noted above, the game in Example 17 has a unique $\mathscr{S}$-core and $\mathscr{B}$-core as well.

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