Rochester Center for Economic Research

Compromising between the proportional and constrained equal awards rules

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Working Paper No. 584

August 2014

UNIVERSITY OF ROCHESTER

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August 27, 2014

Abstract

For the problem of adjudicating conflicting claims, we define a family of two-claimant rules that offer a compromise between the proportional and constrained equal awards rules. We identify the members of the family that satisfy particular properties. We generalize the rules to general populations by requiring "consistency": the recommendation made for each problem should be "in agreement" with the recommendation made for each reduced problem that results when some claimants receive their awards and leave. We identify which members of the two-claimant family have consistent extensions, and we characterize these extensions. Here too, we identify which extensions satisfy particular properties. Finally, we propose and study a "dual" family.

JEL classification number: C79; D63; D74
Key-words: claims problems; proportional rule; constrained equal awards rule; consistency; consistent extension.

1 Introduction

Imagine a group of people having claims on a resource but there is not enough of the resource to honor all of the claims. A "rule" specifies for each such "claims problem" a division of the amount available, the "endowment" (O'Neill, 1982). Bankruptcy, but also taxation, are two applications of the formal model that we will write down, but we will use language that best fits its application to the adjudication of conflicting claims. Two central rules in the literature (for surveys, see Thomson, 2003, 2006, 2014a) are the "proportional rule", for which awards are proportional to claims, and the "constrained equal awards rule", for which awards are as equal as possible subject to no one receiving more than his claim.

Proportionality and equality are indeed fundamental principles around which the debate about how to allocate resources is often organized. We introduce here a family of two-claimant rules that offer a simple yet flexible compromise between these principles. Our proposal is simple because, as the endowment grows from zero, it is first divided between the two claimants in fixed proportions, these proportions lying between the proportions according to which it is first divided by the proportional and constrained equal awards rules, and this occurs until the smaller claimant is fully compensated; at that point of course, each increment has to be assigned to the larger claimant. It is flexible because we allow the direction of the initial segment in the path followed by the awards vector to depend on the claims held by the two agents.

Our first task is to identify which members of the family satisfy particular properties. We review all of the properties that have been central in the literature. Some of them are satisfied by all of our rules, but for others, restrictions are needed. For each property, we spell out these restrictions and describe the resulting rules.

The next question is what to do for more claimants, where geometric intuition is much less of a guide than in the two-claimant case. How should our definition be generalized to arbitrarily many claimants? Our strategy here is to proceed by requiring "consistency". Informally, this says that the manner in which the total amount assigned to any group of claimants is distributed among them should pass the same test as the allocation chosen for the entire population; thus, the choice made for each problem is "reinforced" or "confirmed" for subpopulations. Somewhat more precisely, consider the "reduced problem" that results after some claimants have received their awards and left the scene. Then, to each of the agents involved in this problem, the rule should assign the same amount as it did in the initial problem. The consistency principle has played an important role in the axiomatics of resource allocation, in the context of a great variety of classes of problems. Motivations and applications are discussed in Thomson (2014b), and a fairness interpretation is developed in Thomson (2012b).

Of course, not all two-claimant rules have consistent extensions to arbitrary populations. Thus, we are led to searching for the conditions that a two-claimant rule in the family we start with should satisfy for it to have such an extension, and we ask what the extensions look like. The proportional and constrained equal awards rules, when applied to two-claimant problems, are members of our family, and when applied to problems involving populations of arbitrary size, they are consistent, as is well-known and easily checked. But are there others? The answer is yes, and we identify these conditions and describe these rules. The rules constitute a family that is new to the literature.

As we did in the two-claimant case, we also identify which ones of the rules satisfy additional properties of interest.

We also propose a way of compromising in the two-claimant case between the proportional rule and the "constrained equal losses rule", the rule that awards amounts such that the losses incurred by all claimants are as equal as possible subject to no one receiving a negative amount. Again, we identify the conditions under which consistent extensions to arbitrary populations exist, and we describe the resulting rules. These rules too are new to the literature.

Finally, we compare how "evenly" two rules in our families distribute the endowment. We do so by invoking the Lorenz criterion. We identify when two rules in a family can be Lorenz ranked, and we also compare rules across the families.

2 The model and our two-claimant proposal

There is an infinite set of potential "claimants", indexed by the natural numbers. Let $\mathcal{N}$ be the family of all finite subsets of $\mathbb{N}$. A **claims problem with claimant set $\boldsymbol{N} \in \boldsymbol{\mathcal{N}}$** is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum c_i \geq E$. An **awards vector of $\boldsymbol{(c, E)}$** is a vector $x \in \mathbb{R}^N_+$ such that $\sum x_i = E$. Let $\boldsymbol{\mathcal{C}^N}$ be the class of all problems with claimant set N. A **rule on $\boldsymbol{\mathcal{C}^N}$** is a function that associates with each $(c, E) \in \mathcal{C}^N$ a unique awards vector of

(c, E). The **path of awards of $\boldsymbol{S}$ for $\boldsymbol{c} \in \mathbb{R}^{\boldsymbol{N}}_+$** is the locus of the awards vector S selects for (c, E) as E varies from 0 to $\sum c_i$. A **rule** is a function that associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$ a unique awards vector of (c, E). Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. For the **proportional rule, $\boldsymbol{P}$**, for each $i \in N$, claimant i's award is λc_i, λ being chosen, as in the next two definitions, so that awards add up to E; for the **constrained equal awards rule, $\boldsymbol{CEA}$**, claimant i's award is $\min\{c_i, \lambda\}$; for the **constrained equal losses rule, $\boldsymbol{CEL}$**, it is $\max\{c_i - \lambda, 0\}$. (Historical references are in O'Neill, 1982.)

Given $a \in \mathbb{R}^N_+$, **box$[\boldsymbol{0}, \boldsymbol{a}]$** is the set $\{x \in \mathbb{R}^N_+ \colon 0 \leqq x \leqq a\}$. Given $a, b, c \in \mathbb{R}^N$, **seg$[\boldsymbol{a}, \boldsymbol{b}]$** is the segment connecting a and b, and **bro.seg$[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$** is the broken segment $\text{seg}[a, b] \cup \text{seg}[b, c]$.

Next is the compromise we propose between the proportional and constrained equal awards rules. Let $N \in \mathcal{N}$ be given with $|N| = 2$. For each claims vector, we choose the path of awards to consist of (i) the segment connecting the origin to a weighted average of the maximal vector of equal coordinates in $\text{box}[0, c]$ and c itself, and (ii) the segment from this weighted average to c. Formally, let i, j be the two members of N and $g^N \colon \mathbb{R}^N_+ \to \mathbb{R}_+$ be a function such that for each $c \in \mathbb{R}^N_+$, $\min c_k \leq g^N(c) \leq \max c_k$. If $c_i \leq c_j$, then the path of awards for c is the broken segment connecting the following three points: the origin, the point whose i-th coordinate is c_i and j-th coordinate is $g^N(c)$, and c (in Figure 1a, $i = 1$ and $j = 2$). (If c has equal coordinates, the path for c is simply $\text{seg}[(0, 0), c]$.) Let $\boldsymbol{\mathcal{G}^N}$ be the family of rules so defined. If for each $c \in \mathbb{R}^N_+$, $g^N(c) = \min c_k$, then $S = CEA$, and if for each $c \in \mathbb{R}^N_+$, $g^N(c) = \max c_k$, then $S = P$.

We impose no restriction on the weights placed on the constrained equal awards and proportional rules: it is up to the user of the theory to choose them so as to express his or her relative preference for one or the other of the equality and proportionality principles. Given $c \in \mathcal{C}^N$ with $c_1 < c_2$, let us fix $x_1 \in]0, c_1[$ and ask the following question. Supposing that claimant 1 has been assigned x_1, how much should agent 2 be assigned for the pair (x_1, x_2) to "look right" in relation to c? (For the pair to be feasible, the endowment should be $x_1 + x_2$.) For a strict believer in egalitarianism, the answer is $x_2 \equiv x_1$. For an adherent to proportionality, the answer is $x_2 \equiv x_1 \frac{c_2}{c_1}$. Our proposal allows positions that are intermediate between these two amounts. It is in this sense that it can be seen as a compromise between the proportional and constrained equal awards rules.

Our superscript to g^N indicates that the identity of the two claimants may

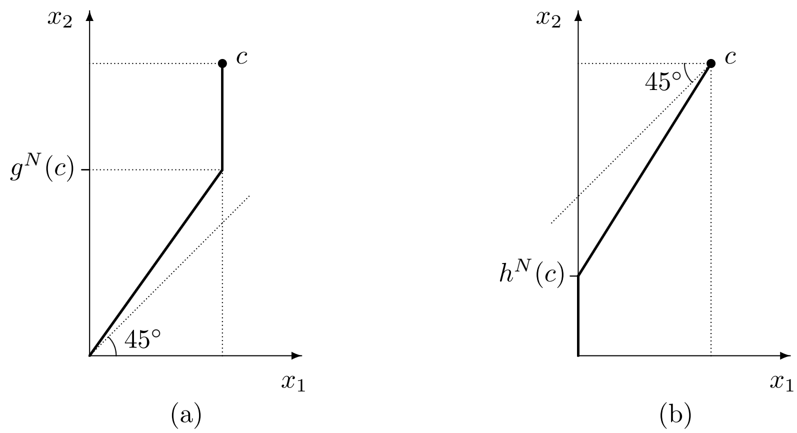

Figure 1: Compromising between the proportional rule and the constrained equal awards (or constrained equal losses) rules. Here, $N \equiv \{1,2\}$ and $c \in \mathbb{R}^N_+$ is such that $c_1 < c_2$. (a) A function $g^N \colon \mathbb{R}^N_+ \to \mathbb{R}_+$ such that $\min c_k \leq g^N(c) \leq \max c_k$ is given. For the member of family $\mathcal{G}^N$ associated with g^N, the path of awards for c is bro.seg$[(0,0),(c_1, g^N(c)), c]$. (b) Here, a function $h^N \colon \mathbb{R}^N_+ \to \mathbb{R}_+$ such that $h^N(c) \leq \max c_k - \min c_k$ is given. For the member of family $\mathcal{H}^N$ associated with h^N, the path for c is bro.seg$[(0,0),(0,h^N(c)),c]$.

be taken into account in specifying the extent to which the path of awards is inflected towards that of the constrained equal awards rule or towards that of the proportional rule. This contributes to the flexibility of our proposal: it allows considerations other than claims to enter into the resolution of problems. Whether a claimant represents a financial institution or a household for example, can thereby be accommodated. Altogether, our starting point is a collection of two-claimant rules, one for each two-claimant population. Let $\mathcal{G}^2 \equiv (\mathcal{G}^N)_{N \in \mathcal{N}, |N|=2}$ be the set of these collections.

We also offer a symmetric way of compromising between the proportional and constrained equal losses rules. Again, let $N \in \mathcal{N}$ be such that $|N| = 2$. Let i and j be the two members of N. Let $h^N \colon \mathbb{R}^N_+ \to \mathbb{R}_+$ be a function such that for each $c \in \mathbb{R}^N_+$, $h^N(c) \leq \max c_k - \min c_k$. Then, consider the rule on $\mathcal{C}^N$ whose path of awards for each $c \in \mathbb{R}^N_+$, and again assuming $c_i \leq c_j$, is the broken segment connecting the following three points: the origin, the point whose i-th coordinate is 0 and whose j-th coordinate is $h^N(c)$, and c (in Figure 1b, $i = 1$ and $j = 2$). (Here too, if c has equal coordinates, the resulting path is seg$[(0,0),c]$.) Let $\boldsymbol{\mathcal{H}^N}$ be the family of rules so defined. If, for each $c \in \mathbb{R}^N_+$, $h^N(c) \equiv \max c_k - \min c_k$, then $S \equiv CEL$; also, if for each $c \in \mathbb{R}^N_+$, $h^N(c) = 0$, then $S = P$. Let $\mathcal{H}^2 \equiv (\mathcal{H}^N)_{N \in \mathcal{N}, |N|=2}$.

An alternative way to reach this second definition is through the concept of duality (Aumann and Maschler, 1985). For each problem, the dual of a rule S is the rule that divides the endowment in the manner in which S divides the deficit (the difference between the sum of the claims and the endowment).

Dual of rule S, S^d: For each $(c, E) \in \mathcal{C}^N$,

$$S^d(c, E) \equiv c - S(c, \sum c_i - E).$$

Given $g^N\colon \mathbb{R}^N_+ \to \mathbb{R}_+$, let $h^N\colon \mathbb{R}^N_+ \to \mathbb{R}_+$ be defined by setting for each $c \in \mathbb{R}^N_+$, $h^N(c) \equiv \max c_k - g^N(c)$. Then, the member of the family $\mathcal{G}^N$ associated with g^N is the dual of the member of the family $\mathcal{H}^N$ associated with h^N.

Geometrically, the path of awards of a member of $\mathcal{G}^N$ is a weighted average of the paths of the proportional and constrained equal awards rules parallel to the axis along which the larger claim is measured, and for a member of $\mathcal{H}^N$, the average is of the paths of the proportional and constrained equal losses rules, also parallel to the axis along which the larger claim is measured.

It will also be useful to have the concept of duality for properties of rules: two **properties are dual** if whenever a rule satisfies the first property, the dual rule satisfies the other property.

We can certainly think of other ways of compromising between proportionality and equality (of awards or of losses). Another proposal is to take a weighted average of the paths of the constrained equal awards and constrained equal losses rules parallel to the 45° line. We obtain then the path of a member of a family that links a number of other rules, the ICI family (Thomson, 2008), a subfamily of which is studied by Moreno-Ternero and Villar (2006a,b), under the name of TAL family.

Yet another way of compromising between proportionality and equality is obtained by choosing, for each claims vector, the path of awards to consist of a segment contained in the 45° line and a segment to the claims vector (Thomson, 2007). Giménez-Gómez and Peris (2014)'s proposal is along the same lines. By duality, we obtain a compromise between proportionality of losses to claims and equality of losses.

Finally is a family of rules whose definition involves partitioning awards space into cones and for each claims vector in each cone, choosing as path of awards what can be seen as a "compressed" version of the path of either the constrained equal awards rule or the constrained equal losses rule, the two segments that the path consists of being parallel to the two boundary rays of the cone (Moulin, 2000). These rules are not required to treat two claimants with equal claims equally, which in some situations is a useful, even necessary, option to achieve fairness, as we already noted, but here we insist on this property.

3 Properties

Although the rules that we proposed have geometrically simple paths of awards, the family they constitute is nevertheless rather large because so far, we have imposed no restriction on the function giving the kinks in paths of awards. In this section, we investigate what is required of this function for the resulting rule to satisfy various properties of interest: given a property of rules, we ask whether rules in $\mathcal{G}^N$ for $N \in \mathcal{N}$ with $|N| = 2$, or rules in $\mathcal{G}^2$, satisfy it. For some properties, the answer is always positive; for others, it is always negative; for some, it depends on the rule; for each property in that last category, we identify the subset of rules in $\mathcal{G}^N$ that do satisfy it. (Some properties apply non-trivially only when the number of claimants is greater than 2; then, there is nothing to say about $\mathcal{G}^2$ concerning them.)[1]

For each property except for one (*anonymity*, defined later), a rule in $\mathcal{G}^2$—let $(g^N)_{N\in\mathcal{N}}$ be the family of functions with which it is associated—satisfies the property if and only if for each $N \in \mathcal{N}$ with $|N| = 2$, the component of the rule pertaining to population N satisfies it. Thus, it is enough to understand the issue for some $N \in \mathcal{N}$ with $|N| = 2$.

• **Equal treatment of equals** says that if two agents have equal claims, their awards should be equal. All rules in $\mathcal{G}^2$ satisfy the property. This follows directly from their definition.

• **Anonymity** says that an exchange of the names of two agents should be accompanied by an exchange of their awards. Let $N \in \mathcal{N}$ be given with $|N| = 2$. Obviously, a rule $S \in \mathcal{G}^N$—let g^N be the function with which it is associated—is *anonymous* if and only if g^N itself is invariant under renamings of agents.

[1]The terminology concerning properties of rules is not uniform in the literature. Here is how the terms we use correspond to the terms that are most common to designate the properties we define below: we use the "$\frac{1}{2}$-truncated claims lower bound" instead of "securement", "order preservation under endowment variation" instead of "super-modularity", "homogeneity" instead of "scale invariance", "minimal rights first" instead of "composition from minimal rights", "composition down" instead of "path independence", and "composition up" instead of "composition".

In a variable-population context, the coverage of *anonymity* is wider than in a fixed-population context: it means invariance not only with respect to exchanges of the names of the members of a given population, but also with respect to *replacements* of these agents by others. A rule in $\mathcal{G}^2$—let $(g^N)_{N\in\mathcal{N},|N|=2}$ be the list of functions with which it is associated—is *anonymous* if this list satisfies the following requirement. Let $N, N' \in \mathcal{N}$ be such that $|N| = |N'| = 2$ and $r\colon N \to N'$ be a "renaming function". Let $c \in \mathbb{R}^N_+$ and $c' \in \mathbb{R}^{N'}_+$ be such that, abusing notation slightly, $r(c) = c'$. Then, for each $E \in \mathbb{R}_+$ such that $(c, E) \in \mathcal{C}^N$, it should be the case that $r(S(c, E)) = S(c', E)$. (Note that $N = N'$ is allowed: then, we obtain the fixed-population version of the property.)

• **Order preservation** (Aumann and Maschler, 1985) is in two parts: (i) awards should be ordered as claims are (*order preservation in awards*); (ii) so should losses (*order preservation in losses*). Let $N \in \mathcal{N}$ be given with $|N| = 2$. Geometrically, a rule *preserves order* if for each $c \in \mathbb{R}^N_+$, its path of awards for c lies above the 45° line (for (i)) and below the line of slope 1 passing through c (for (ii)). All rules in $\mathcal{G}^2$ *preserve order.* This follows directly from their definition.

• **The $\frac{1}{|N|}$-truncated claims lower bound** (Moreno-Ternero and Villar, 2004) says that each claimant should receive at least $\frac{1}{|N|}$ of his claim truncated at the endowment. For $|N| = 2$ this implies that paths of awards should include a segment of slope 1 emanating from the origin. Thus, the constrained equal awards rule is the only rule in $\mathcal{G}^2$ to pass this test.

• **Conditional full compensation** (Herrero and Villar, 2002) says that if an agent's claim is such that by substituting it for the claim of each agent whose claim is greater, there is now enough to compensate everyone, the agent should be fully compensated. For $|N| = 2$, this implies that the path of awards should contain the segment from the greatest point of equal coordinates that is dominated by the claims vector to the claims vector. Thus, the constrained equal awards rule is the only rule in $\mathcal{G}^2$ to pass this test.

• **Endowment monotonicity** says that if the endowment increases, each claimant's award should be at least as large as it was initially. Geometrically, this means that paths of awards are monotone curves. All rules in $\mathcal{G}^2$ are *endowment monotone.* This follows directly from their definition.

• **Endowment continuity, claims continuity**, and **full continuity** say that that $(*)$ for each claims vector, small changes in the endowment should not lead to large changes in the chosen awards vector; $(**)$ for each endowment, small changes in the claims vector should not lead to large changes in the chosen awards vector; and $(***)$ small changes in the data of the problem should not lead to large changes in the chosen awards vector.

Let $N \in \mathcal{N}$ be given with $|N| = 2$. All rules in $\mathcal{G}^N$ are *endowment continuous*. This follows directly from their definition.

A rule in $\mathcal{G}^N$—let g^N be the function with with it is associated—is *claims continuous*, or *fully continuous*, if and only if g^N is continuous.

• **Order preservation under endowment variation** (Dagan, Serrano, and Volij, 1997) says that as the endowment increases, changes in awards should be ordered as claims are. For simplicity, let $N \equiv \{1, 2\}$ and $c \in \mathbb{R}^N_+$ be such that $c_1 < c_2$. Geometrically, the property means that the slope of the paths of awards for c (when well defined) is at least 1. All rules in $\mathcal{G}^2$ trivially satisfy the property.

• **Claims monotonicity** says that if an agent's claim increases, his award should be at least as large as it was initially. For simplicity, let $N \equiv \{1, 2\}$. The property has the following geometric interpretation: let $c, c' \in \mathbb{R}^N_+$ be such that $c_2 = c'_2$ and $c_1 < c'_1$. Then, the path of awards for c' should lie to the south-east of the path for c, a parallel statement holding when agent 1's claim is held fixed and agent 1's claim increases (the two properties, monotonicity of agent 1's award with respect to his claim and monotonicity of agent 2's award with respect to his claim, are independent).

Let $N \in \mathcal{N}$ be given with $|N| = 2$. The following example shows that a rule in $\mathcal{G}^N$ is not necessarily *claims monotonic*:

Example 1 *Let $N \equiv \{1, 2\}$ and let $S \in \mathcal{G}^N$ be associated with a function g^N such that for $c \equiv (1, 4)$, $g^N(c) = 1$ (this implies that its path of awards for c is that of the constrained equal awards rule), and for $c' \equiv (2, 4)$, $g^N(c') = 4$ (this implies that its path of awards for c' is that of the proportional rule).*

Let $E \equiv 2$. Then, $S(c, E) = (1, 1)$ and $S(c', E) = (\frac{2}{3}, \frac{4}{3})$: as agent 1's claim increases from 1 to 2, his award decreases from 1 to $\frac{2}{3}$. (The definition of S can easily be completed so that S is continuous.)

A rule in $\mathcal{G}^N$ is *claims monotonic* if (i) for each $c_2 \in \mathbb{R}_+$, the function $g^N(., c_2)$ is such that for each pair $c_1, c'_1 \in \mathbb{R}_+$ with $0 < c_1 < c'_1 \leq c_2$, we have $\frac{g^N(c_1,c_2)}{c_1} \geq \frac{g^N(c'_1,c_2)}{c'_1}$ (Figure 2a), and (ii) for each each pair $c_2, c'_2 \in \mathbb{R}_+$ with

$0 < c_2 < c_2'$, over the interval $[0, c_2]$, the graph of $g^N(., c_2')$ is bounded below by the graph of $g^N(., c_2)$ (Figure 2b).

Two statements parallel to (i) and (ii), obtained by exchanging the roles of the two coordinates, should also hold.

Statement (i) is necessary and sufficient for the path for (c_1', c_2) to indeed lie to the south-east of the path for (c_1, c_2). Geometrically, it means that the graph of $g^N(., c_2)$ for $c_1 \leq c_2$ is **visible from the origin**: for an observer standing at the origin, and thinking of the graph as opaque, no part of it would hide any other part of it. The limit case when there is a non-degenerate segment in the graph that is lined up with the origin is allowed. In Figure 2a, seg$[p, q]$ illustrates the possibility. A curve is **strictly visible from the origin** if there is no such segment. This is the case for the graph of $g^N(., c_2')$ in Figure 2b.

Statement (ii) guarantees that for each $c_1 \in \mathbb{R}_+$ and each pair $c_2, c_2' \in \mathbb{R}_+$ with $c_1 \leq c_2 < c_2'$, agent 2's award when his claim is c_2' is at least as large as his award when his claim is c_2.

Figures 2a,b show that the graphs referred to in (i) need not be upward-sloping curves: in Figure 2a, the graph of $g^N(., c_2)$ has a downward-sloping part whose right endpoint has abscissa c_1', and in Figure 2b, the graph of $g^N(., c_2')$ has a downward-sloping part whose left endpoint has abscissa 0.

For the proportional rule, for each $c_2 \geq 0$, the graph of $g^N(., c_2)$ is the horizontal half-line of ordinate c_2 (visibility is strict) and for the constrained equal awards rule, it is seg$[(0, 0), (c_2, c_2)]$ (visibility is nowhere strict).

• **Homogeneity** says that for each problem and each $\alpha > 0$, multiplying the data defining the problem by α results in a new problem whose chosen awards vector should be obtained by multiplying the chosen awards vector of the initial problem by α.

Let $N \in \mathcal{N}$ be given with $|N| = 2$. Let $S \in \mathcal{G}^N$ and g^N be the function with which it is associated. Let $\alpha > 0$. For S to be *homogeneous*, it is necessary and sufficient that its path for αc be obtained by subjecting its path for c to a scale expansion of factor α. This holds if and only if $g^N(\alpha c) = \alpha g^N(c)$. For a rule in $\mathcal{G}^2$ to be *homogeneous*, this statement should hold for each component g^N.

• **Claims truncation invariance** (Dagan and Volij, 1993) says that replacing all claims that are greater than the endowment by the endowment should not affect the chosen awards vector.

Let $N \in \mathcal{N}$ be given with $|N| = 2$. *Equal treatment of equals*, which all

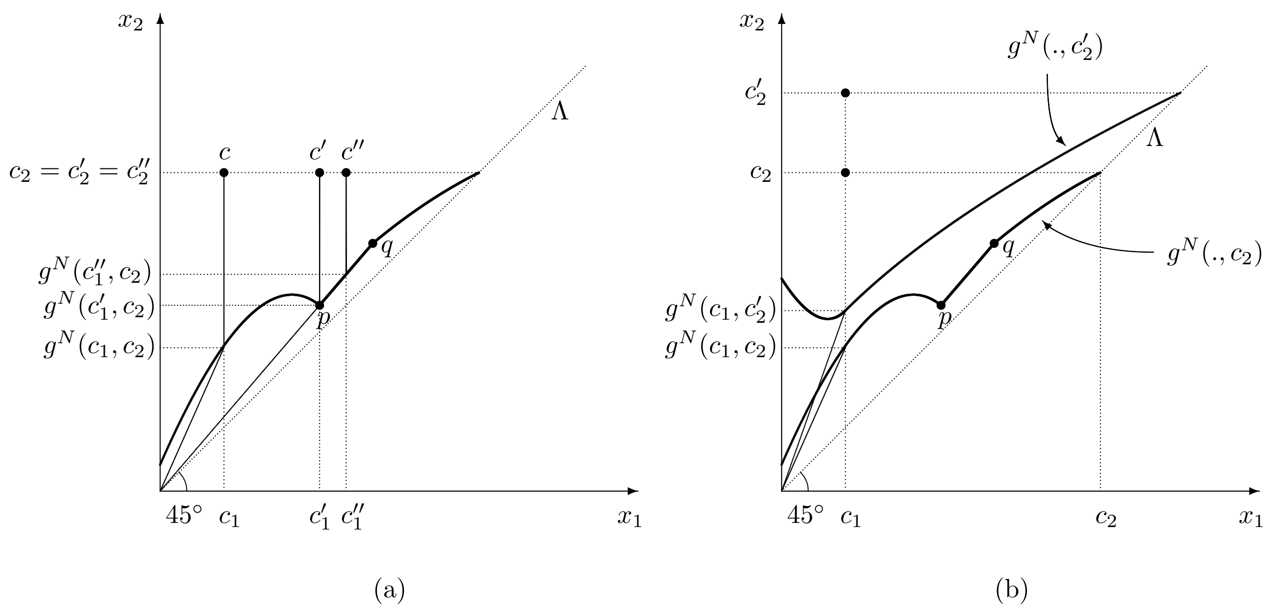

Figure 2: Identifying two-claimant rules that are claims monotonic. Here, $N \equiv \{1, 2\}$. (a) Keeping c_2 fixed, the locus of the kink in the path of awards for (c_1, c_2) as c_1 varies in $[0, c_2]$ (this is part of the graph of $g^N(., c_2)$) should be visible from the origin. (b) Given c_2 and $c'_2 \in \mathbb{R}_+$ with $c_2 < c'_2$, the locus of the kinks for c_2 (the graph of $g^N(., c_2)$ when c_1 varies in $[0, c_2]$) should lie everywhere on or below the locus of kinks for c'_2 (the graph of $g^N(., c'_2)$ when c_1 varies in $[0, c'_2]$).

rules $S \in \mathcal{G}^N$ satisfy (see above), and *claims truncation invariance* together imply that for each $c \in \mathbb{R}^N_+$,

(†) the path of awards of S for c contains seg$[0, (\frac{\min c_i}{2}, \frac{\min c_i}{2})]$.[2]

Indeed, for an endowment no greater than the smaller claim, both truncated claims are equal, and by *equal treatment of equals*, awards are equal then.

Now, observe that when $0 < c_1, c_2$ and $c_1 \neq c_2$, the constrained equal awards rule is the only rule in $\mathcal{G}^N$ satisfying (†). Moreover, this rule is *claims truncation invariant*. Thus, it is the only rule in $\mathcal{G}^N$ that is *claims truncation invariant*.

• **Minimal rights first** (Curiel, Maschler, and Tijs, 1987) says that we should be able to solve each problem in either one of the following two ways: (a) directly, that is, ignoring the initial awards vector; (b) in two steps, by first assigning to each claimant his "minimal right", namely the difference between the endowment and the sum of the claims of the other claimants, or 0 if this difference is negative, and in a second step, after having revised down all claims by the awards of the first step, applying the rule to divide what remains of the endowment. (This property is the dual of *claims truncation invariance*, so its analysis can be easily derived from our previous analysis of that second property. The family $\mathcal{G}^N$ is not closed under duality however.)

Let $N \in \mathcal{N}$ be given with $|N| = 2$. If a rule satisfies *equal treatment of equals* and *minimal rights first*, then for each $c \in \mathbb{R}^N_+$, its path of awards for c contains the segment [seg$[(c_i - \frac{\min c_k}{2}, c_j - \frac{\min c_k}{2}), c]$. When $0 < c_1, c_2$ and $c_1 \neq c_2$, this segment is non-degenerate and this inclusion never holds for a rule in $\mathcal{G}^N$.

• **Composition down** (Moulin, 2000) says that if the endowment decreases, we should be able to obtain the new awards vector in either one of the following two ways: (a) directly, that is, ignoring the initial awards vector; or (b) using the awards vector initially chosen as claims vector.

The analysis of this property is the most delicate: indeed, the family of rules in $\mathcal{G}^N$ for $N \in \mathcal{N}$ with $|N| = 2$ satisfying it is quite complex, particularly so if *claims continuity* is not imposed. Because *claims continuity* is a very natural property, the result we present below involves both *claims continuity* and *composition down*. In the appendix, we give a series of examples

[2] This is only a necessary condition. For necessary and sufficient conditions, see Thomson (2006).

indicating the various ways in which dropping *claims continuity* enlarges the family of admissible rules.

Let $N \in \mathcal{N}$ be given with $|N| = 2$. An important ingredient in the characterization of the family of rules in $\mathcal{G}^N$ satisfying *claims continuity* and *composition down* is the following characterization of the entire family of rules satisfying *composition down* (Lemma 1). (This result holds for populations of any size, but we only need it for the two-claimant case.)

Consider a family of

(a) continuous and weakly monotone curves in $\mathbb{R}^N_+$ emanating from the origin and such that,

(b) given any point in $\mathbb{R}^N_+$, there is a curve in the family passing through it;

(c) following any one of these curves up from the origin, if we encounter a point at which the curve splits into branches, these branches never meet again. (Otherwise, we would have a "cycle".)

A family of curves satisfying (a)-(c) constitute a **weakly monotone** (from (a)) **space-filling** (from (b)) **tree** (from (c)).

Lemma 1 (Thomson, 2006) *Let $N \in \mathcal{N}$.*

(a) A rule on $\mathcal{C}^N$ satisfies composition down *if and only if there is a weakly monotone space-filling tree in $\mathbb{R}^N_+$ such that, for each $c \in \mathbb{R}^N_+$, the path of awards of the rule for c is obtained by identifying a branch emanating from the origin and passing through c, and taking the part of it that lies in* $\text{box}[0, c]$.

(b) If the rule is claims continuous, *all branches of the tree with which it is associated are unbounded.*

Figure 3a shows a few branches of such a tree and a claims vector c that belongs to more than one branch. There is more than one branch because of a split above c (at c'), but all branches passing through c coincide in $\text{box}[0, c]$. Given any such branch, the path for c is the part of it that lies in the box.

Figures 3b and 4 illustrate what is required for a rule in $\mathcal{G}^N$ for $N \in \mathcal{N}$ with $|N| = 2$ to satisfy *claims continuity* and *composition down*.

Consider a continuous curve C^1 below the 45° line that is

(i) strictly visible from the point at infinity in the direction $(1, 0)$ (strictness here means that the curve contains no non-degenerate horizontal segment),

(ii) visible "from the other side" from the origin, ("from the other side" means that the intersection of ($*$) the union of the half-lines of the form $\{(x_1+t, x_2)\colon x \in C^1, t \geq 0\}$ for $x' \in C^1$ and ($**$) the union of the segments of the form $\mathrm{seg}[0, x']$ for $x' \in C^1$, is C^1;

(iii) has an endpoint on the horizontal axis, or is asymptotic to this axis, and

(iv) if C^1 contains a point of positive ordinate, and $\sup\{\frac{\alpha_2}{\alpha_1}\colon \alpha \in C^1\}$ is reached—let α^1 denote the point of lowest ordinate at which this is so—then C^1 contains $\{\lambda\alpha^1\colon \lambda \geq 1\}$. Otherwise, C^1 is asymptotic to the ray of slope $\sup\{\frac{\alpha_2}{\alpha_1}\colon \alpha \in C^1\}$.

Let $\boldsymbol{\mathcal{C}^1}$ be the class of curves C^1 defined in this manner. Let $\boldsymbol{\mathcal{C}^2}$ be the class of curves defined in a parallel manner to the way we defined $\mathcal{C}^1$, by exchanging the roles played by the two coordinates.

Our next result is that a rule satisfying *composition down* is entirely specified once a pair $C^1 \in \mathcal{C}^1$ and $C^2 \in \mathcal{C}^2$ is given. In the statement of the theorem, it is explained how to derive from C^1 and C^2 the tree with which the rule is associated.

Theorem 1 *Let $N \in \mathcal{N}$ be given with $|N| = 2$. A rule in $\mathcal{G}^N$ satisfies claims continuity and composition down if and only if there are $C^1 \in \mathcal{C}^1$ and $C^2 \in \mathcal{C}^2$ such that the rule is obtained in the manner described in Lemma 1 from the weakly monotone space-filling tree constructed as follows from C^1 and C^2:*

($*$) *For each $x \in C^1$, $\mathrm{seg}[0, x] \cup \{(x_1+t, x_2)|t \geq 0\}$ is a branch.*

For each $x \in C^2$, $\mathrm{seg}[0, x] \cup \{(x_1, x_2+t)|t \geq 0\}$ is a branch.

($**$) *For each α that is not proportional to a point of either C^1 and C^2, the ray in the direction α is a branch.*

The curve C^1 may consist of a single point on the horizontal axis (the abuse of language here seems unavoidable). Then, below the 45° line, the tree with which the rule is associated is that of the proportional rule.

If both C^1 and C^2 reach the 45° line, let us denote by α^1 the point of lowest ordinate at which C^1 does so, by α^2 the point of lowest ordinate at which C^2 does so, and by d the point of highest ordinate among α^1 and α^2. Then, two branches of the tree are associated with each point u on the 45° line such that $u_2 \geq d_2$. (In Figure 4a, $\alpha^1 = \alpha^2 = d$, so that the half-line $\{(d_1+t, d_2+t)|t \geq 0\}$ is contained in both C^1 and C^2. To each u in this half-line are associated two branches of the tree.)

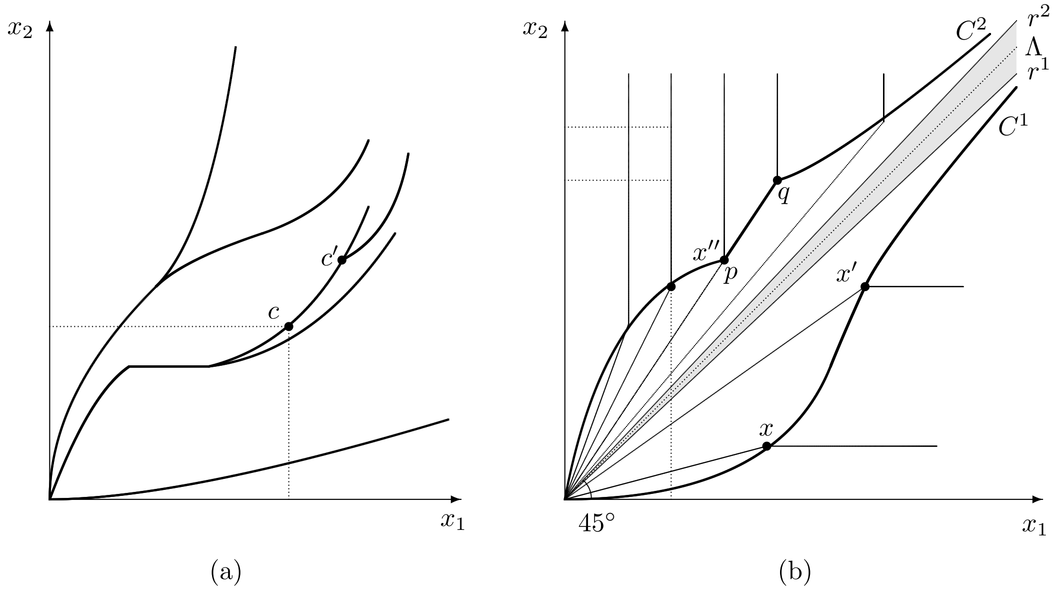

Figure 3: Claims continuity and composition down. (a) Illustrating Lemma 1: this panel shows a typical tree from which the paths of awards of a rule satisfying *claims continuity* and *composition down* are generated. (b) Illustrating Theorem 1. For a rule in $\mathcal{G}^N$ satisfying *claims continuity* and *composition down*, there are two curves C^1 and C^2, C^1 being strictly visible from the point at infinity in the direction $(1, 0)$ and visible from the origin (in fact, in the example illustrated here, it is strictly visible), and C^2 being strictly visible from the point at infinity in the direction $(0, 1)$, and visible from the origin (here, C^2 is only weakly visible from the origin because it contains a non-degenerate segment (seg$[p, q]$) that is lined up with the origin). From C^1 and C^2, a tree is constructed with which the rule is associated.

The 45° line has to be a branch because of *equal treatment of equals*. It is indeed included as a branch if either C^1 or C^2 reach it. Otherwise, its inclusion follows from $(**)$.

The axes have to be branches because we do not require claims to be positive. Thus, for a claims vector of the form $(c_1, 0)$ say, the path is seg$[(0, 0), (c_0, 0)]$. The horizontal axis is indeed in the tree if C^1 reaches it. Otherwise, its inclusion follows from $(**)$. The vertical axis is in the tree if C^2 contains $(0, 1)$. Otherwise, its inclusion follows from $(**)$.

Figure 3b shows that C^1 may reach the horizontal axis at the origin, Figure 4a, that it may reach it at a point of positive abscissa; and Figure 4b, that it may be asymptotic to it.

Figure 4b shows that each of C^1 and C^2 can have two asymptotic directions. If that is the case for C^1, one of them is necessarily the direction $(1, 0)$. If it is the case for C^2, one of them is necessarily the direction $(0, 1)$.

Without the visibility properties (i) and (ii) imposed on C^1 and C^2, the "treeness" of the family of curves generated by following the instructions of Theorem 1 would be violated, which in turn would lead to a violation of *composition down*. Indeed it is easy to see that in the two-claimant case, *com-*

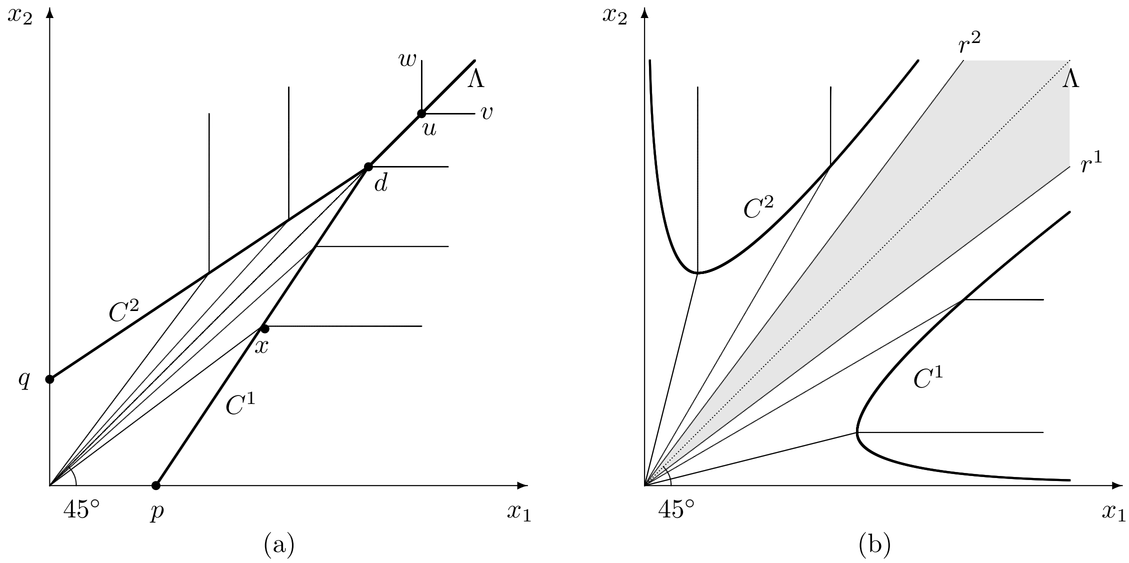

Figure 4: Claims continuity and composition down. Two other rules in $\mathcal{G}^N$ for $N \equiv \{1,2\}$ satisfying *claims continuity* and *composition down*. Both are *anonymous*. (a) The intersection with the 45° line of each of the curves C^1 and C^2 from which the tree to which the rule represented here is associated is the half-line $\{x \in \mathbb{R}^N_+\colon$ there is $t \in \mathbb{R}_+$ such that $x = d + (t,t)\}$. (b) The cones spanned by C^1 and C^2 here do not cover the whole of $\mathbb{R}^N_+$. The missing part (shaded) is filled with rays.

position down implies *claims monotonicity* (Thomson, 2006). Example 1 illustrates a violation of the latter property and because of this logical relation, it illustrates a violation of the former as well. To see this, note that the paths of awards for c and c' defined there cross at $\tilde{c} \equiv (1,2)$: $S(c,2) = S(c',2) = \tilde{c}$. Thus, if S satisfied *composition down*, we would have that for each $E \leq 2$, $S(c,E) = S(S(c,2),E) = S(\tilde{c},E) = S(S(c',2),E) = S(c',E)$: for endowments no greater than 2, the paths of S for c and c' would coincide, but they don't.

For a rule in $\mathcal{G}^N$ to be *anonymous* in addition to satisfying *claims continuity* and *composition down*, C^1 and C^2 should be symmetric of each other with respect to the 45° line. For a rule in $\mathcal{G}^2$ to be *anonymous*, the same pair of symmetric curves should be used in $\mathbb{R}^N_+$ for each $N \in \mathcal{N}$ with $|N| = 2$.

Proof: (of Theorem 1) Let S be a rule satisfying the two axioms of the theorem. If there is no $c \in \mathbb{R}^N_+$ such that the path of S for c has a kink,

S is the proportional rule. Suppose otherwise: there is $c \in \mathbb{R}^N_+$ with $c_1 \neq c_2$ such that the path of S for c has a kink—let us call it x. Obviously $c_2 > 0$. Without loss of generality, suppose $0 < c_1 < c_2$.

Step 1: For each $c'_2 \geq x_2$, the path for (c_1, c'_2) has a kink and this kink is x. This is clearly true for each $c'_2 \geq 0$ such that $x_2 \leq c'_2 \leq c_2$; indeed, the paths of awards for any such claims vector (c_1, c'_2) has to be a subset of the tree and we would have a cycle otherwise. Also, if there were an upper bound to $\{c'_2 \in \mathbb{R}_+ :$ the path for (c_1, c'_2) has at kink at $x\}$, the tree associated with S would have a bounded branch, in contradiction with Lemma 1b.

Step 2: For each $c'_1 > c_1$ and each c'_2 such that $\frac{c'_2}{c'_1} > \frac{c_2}{c_1}$, the path for c' has a kink—let us call it x'—such that $\frac{x'_2}{x'_1} \leq \frac{x_2}{x_1}$. Otherwise, once again, we would have a cycle. The same conclusion applies to each $c'_1 < c_1$. Thus, the locus of kinks is a weakly visible curve C^2 from the origin. It has an asymptotic direction r^2 that is at least as steep as the vector $(1, 1)$.

Each ray r whose slope is intermediate between 1 and the slope of r^2 has to be a branch of the network of curves associated with S, for this network to be a tree.

A curve C^1 can be constructed in the same manner below the 45° line, with symmetric properties. □

• **Composition up** says that if the endowment increases, we should be able to obtain the new awards vector in either one of the following two ways: (a) directly, that is, ignoring the initial awards vector; (b) first assigning the initial awards, then adding to them the awards vector that results by applying the rule to the problem of dividing the increment, claims having been revised down by the initial awards. The conclusions concerning this property are obtained by duality from the ones we reached above for *composition down*. (*Composition up* is the dual of *composition down*.)

The only rules in $\mathcal{G}^N$ for $N \in \mathcal{N}$ with $|N| = 2$ to satisfy the property are obtained by duality from the ones identified for *composition down*.

• **Lorenz comparisons.** Next, we ask when rules in $\mathcal{G}^N$ can be Lorenz ranked. Informally speaking, rule S Lorenz dominates rule S' if for each problem, its distribution of awards is more in favor of agents who receive the least (under *order preservation of awards*, these are the agents with the smallest claims). Formally, $\boldsymbol{S}$ **Lorenz dominates** $\boldsymbol{S'}$ if for each problem $(c, E) \in \mathcal{C}^N$ and, letting $x \equiv S(c, E)$ and $x' \equiv S'(c, E)$ and calling $\tilde{x}$ and $\tilde{x}'$

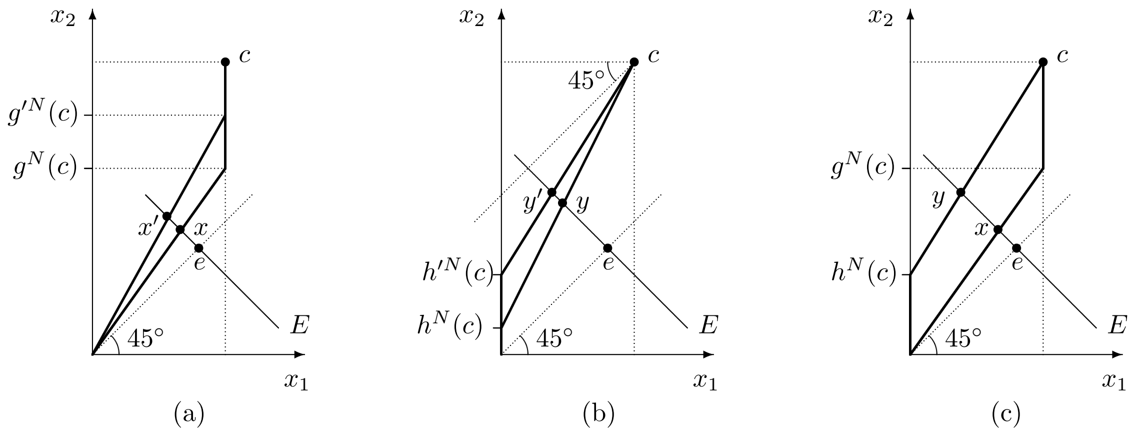

Figure 5: Lorenz domination between members of $\mathcal{G}^N$ and $\mathcal{H}^N$. Here, $N \equiv \{1, 2\}$. (a) Since $g^N(c) < g'^N(c)$, the rule in $\mathcal{G}^N$ associated with g^N Lorenz dominates the rule in $\mathcal{G}^N$ associated with g'^N for all problems with claims vector c. (b) Since $h'^N(c) < h^N(c)$, the rule in $\mathcal{H}^N$ associated with h'^N Lorenz dominates the rule associated in $\mathcal{H}^N$ associated with h^N for all problems with claims vector c. (c) Each rule in $\mathcal{G}^N$ Lorenz dominates each rule in $\mathcal{H}^N$. This is true no matter what c is.

the vectors obtained from x and x' by rewriting their coordinates in increasing order, we have $\tilde{x}_1 \geq \tilde{x}'_1$, $\tilde{x}_1 + \tilde{x}_2 \geq \tilde{x}'_1 + \tilde{x}'_2$, and so on.

Given $N \in \mathcal{N}$ with $|N| = 2$, S Lorenz dominates S' simply if for each $c \in \mathbb{R}^N_+$, the path of awards of S for c is "everywhere at least as close to the 45° line" as the path of awards of S' for c. Let S and S' be rules in $\mathcal{G}^N$, associated with the functions g^N and g'^N respectively. In Figure 5a, $x \equiv S(c, E)$ is closer to the point of equal coordinates on the budget line of equation $t_1 + t_2 = E$ (the point e) than $x' \equiv S'(c, E)$.

1. Given $N \in \mathcal{N}$ with $|N| = 2$ and any two rules S and S' in $\mathcal{G}^N$, it is always true that for each particular $c \in \mathbb{R}^N_+$, either for each $E \in [0, \sum c_i]$, $S(c, E)$ Lorenz dominates $S'(c, E)$, or for each $E \in [0, \sum c_i]$, $S'(c, E)$ Lorenz dominates $S(c, E)$. As c varies, the domination could be reversed. However, S Lorenz dominates S' if and only if, denoting by g^N and $g^{N'}$ the two functions with which S and S' are associated, $g^N \leq g'^N$ (Figure 5a).

2. By a similar argument, given $N \in \mathcal{N}$ with $|N| = 2$ and any two rules S and S' in $\mathcal{H}^N$, S Lorenz dominates S' if and only if, denoting by h^N and $h^{N'}$ the two functions with which S and S' are associated, $h^N \leq h'^N$ (Figure 5b).

3. It is also easily seen by inspection that each rule in $\mathcal{G}^2$ Lorenz dominates each rule in $\mathcal{H}^2$ (Figure 5c).

4 Extending the compromise from two claimants to arbitrarily many claimants

The rules defined in the previous section are two-claimant rules and the question arises what to do for more claimants. It is not obvious at all how to generalize to that case the simple idea that motivated their introduction. So we proceed axiomatically, invoking a principle that has played a fundamental role in addressing this type of issue in a great variety of literatures. Starting from our two-claimant definition, we require the extension to general populations to pass the following test: for each problem and each subpopulation of the claimants it involves, consider the problem involving this subpopulation in which the endowment is the sum of the amounts that have been awarded to them: this is the "reduced problem relative to the subpopulation and the awards vector chosen for the initial problem"; (alternatively, the reduced problem is the problem that results after some claimants have left with their awards and the situation is reassessed at that point); we require that in this reduced problem, the rule should assign to each claimant the same amount as it did in the initial problem.

Consistency: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $N' \subset N$, if $x \equiv S(c, E)$, then $x_{N'} = S(c_{N'}, \sum_{N'} x_i)$.

The following is an important family of *consistent* rules (Young, 1987). Let $\Lambda \equiv [\underline{\lambda}, \bar{\lambda}]$ be a subset of the extended reals, and $f \colon \mathbb{R}_+ \times \Lambda \to \mathbb{R}_+$ be a continuous function such that for each $c_0 \geq 0$, $f(c_0, \underline{\lambda}) = 0$, $f(c_0, \bar{\lambda}) = c_0$, and $f(c_0, \cdot)$ is nowhere decreasing. Let $\boldsymbol{\mathcal{F}}$ be the class of all such functions. Then, the **parametric rule $\boldsymbol{S}$ associated with $\boldsymbol{f}$** is the rule that selects, for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, the awards vector $(f(c_i, \lambda))_{i \in N}$, where $\lambda \in \Lambda$ solves the equation $\sum_N f(c_i, \lambda) = E$.

Let $\boldsymbol{\Gamma}$ be the class of all functions $G \colon \mathbb{R}_+ \to \mathbb{R}_{++}$ that are nowhere decreasing and such that the function $c_0 \in \mathbb{R}_{++} \to \frac{G(c_0)}{c_0}$ is nowhere increasing (these properties imply that G is continuous). Figure 6 illustrates the definition. We will show that if a rule coincides, for each $N \in \mathcal{N}$ such that $|N| = 2$, with a member of $\mathcal{G}^N$, and is *consistent*, then there is $G \in \Gamma$ such that for each two-claimant population $N \in \mathcal{N}$ and each $c \in \mathbb{R}^N_+$, $g^N(c) = (\max c_k) \frac{G(\min c_k)}{G(\max c_k)}$. Here is the description of the rule for general populations. It is a parametric rule, so we define it by giving a representation of it:

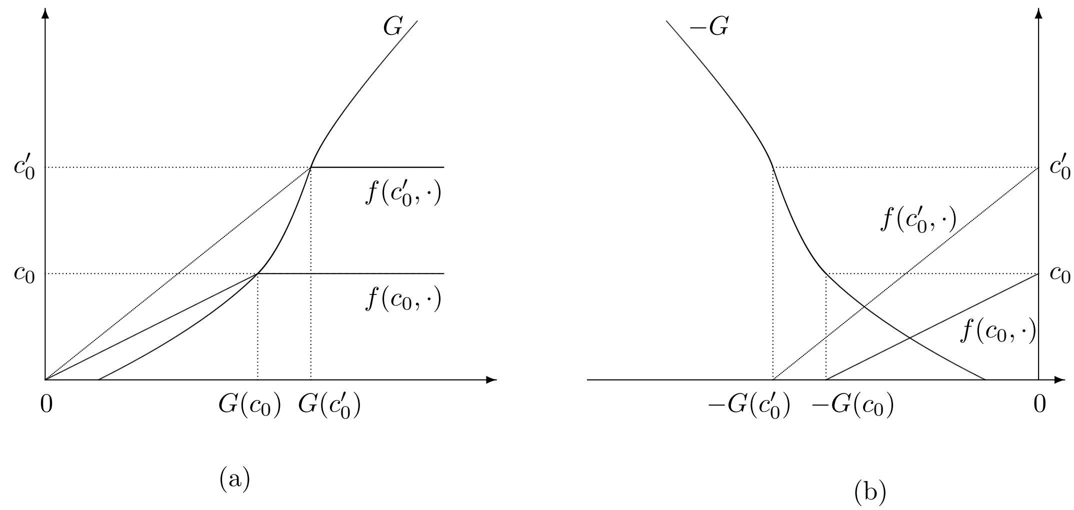

Figure 6: Parametric representations of members of $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{T}}$. The argument of G is measured vertically. (a) Given $G \in \Gamma$, the rule $S^G \in \mathcal{R}$ associated with G admits the representation $f\colon \mathbb{R}_+ \times [0,\infty[\to \mathbb{R}_+$ such that for each $c_0 \in \mathbb{R}_+$, the schedule for c_0 consists of seg$[(0,0),(G(c_0),c_0)]$ and continues with a horizontal half-line. (b) Given $G \in \Gamma$, the rule T^G associated with G admits the representation $f\colon \mathbb{R}_+ \times]-\infty,0] \to \mathbb{R}_+$ such that for each $c_0 \in \mathbb{R}_+$, the schedule for c_0 follows the horizontal axis until $(-G(c_0),0)$ and concludes with seg$[(-G(c_0),0),(0,c_0)]$.

Rule $\boldsymbol{S^G}$ associated with $\boldsymbol{G \in \Gamma}$: Let $\Lambda \equiv [0,\infty[$. Let $f\colon \mathbb{R}_+ \times \Lambda \to \mathbb{R}_+$ be such that for each $c_0 \in \mathbb{R}_+$, the graph of $f(c_0,\cdot)$ is the union of seg$[(0,0),(G(c_0),c_0)]$ and the horizontal half-line $\{(t,c_0)\colon t \geq G(c_0)\}$. Then, S^G is the parametric rule admitting the representation f.

Let $\boldsymbol{\mathcal{S}} \equiv \{S^G\colon G \in \Gamma\}$. The following are two important members of the family. If there is $a > 0$ such that for each $c_0 \in \mathbb{R}_+$, $G(c_0) = a$, then $S^G = P$, and if there is $a > 0$ such that for each $c_0 \in \mathbb{R}_+$, $G(c_0) = ac_0$, then $S^G = CEA$.

Our next theorem fully describes the *consistent* extensions of those two-claimant rules in $\mathcal{G}^2$ that do have such extensions.

Theorem 2 *A rule coincides, for each two-claimant population $N \in \mathcal{N}$, with a member of $\mathcal{G}^N$, and is consistent, if and only if it is a member of $\mathcal{S}$.*

For the proof, we need one additional concept and one lemma. A rule is **conversely consistent** if for each claimant set, each problem that these claimants may face, and each awards vector for this problem, if this vector is such that for each two-claimant subpopulation, its restriction to the subpopulation would be chosen by the rule for the associated reduced problem, then it should be chosen for the initial problem. The formal statement is as follows: For each $N \in \mathcal{N}$, each $(c,E) \in \mathcal{C}^N$, and each award vector x of (c,E),

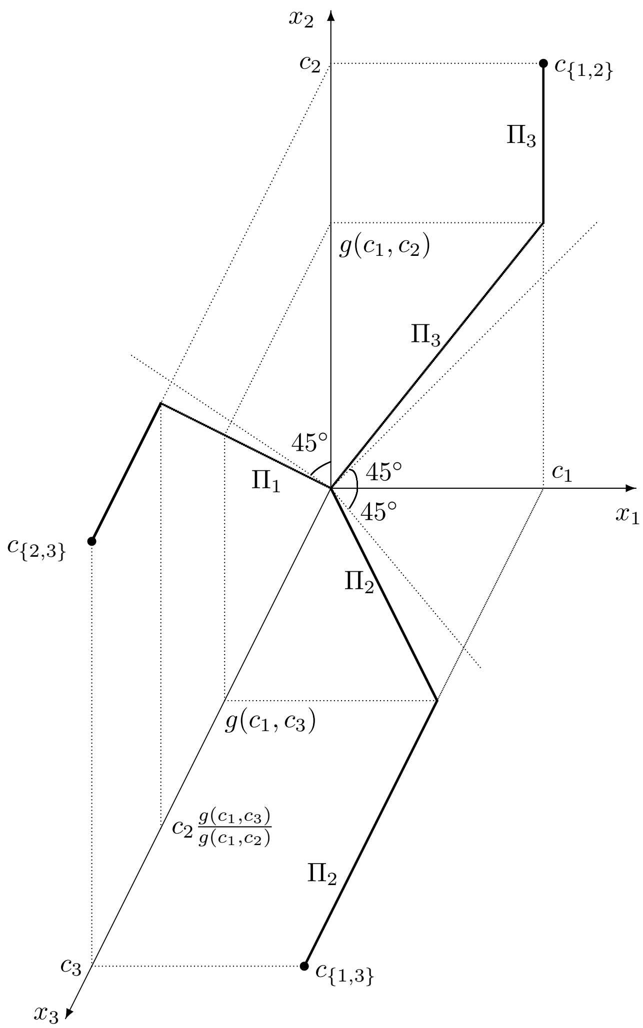

Figure 7: Illustrating Step 2 of the proof of Theorem 2. From the first segments in the paths for $c_{\{1,2\}}$ and $c_{\{1,3\}}$, Π_3 and Π_2, we deduce the first segment in the path for $c_{\{2,3\}}$, Π_1. Because this segment should lie between the 45° line in $\mathbb{R}^{\{2,3\}}$ and $\text{seg}[(0,0),(c_2,c_3)] \subset \mathbb{R}^{\{2,3\}}$, we obtain two monotonicity properties of g (from which we deduce two monotonicity properties of a function γ associated with g).

if for each $N' \subset N$ with $|N| = 2$, $x_{N'} = S(c_{N'}, \sum_{N'} x_i)$, then $x \equiv S(c, E)$. The **Elevator Lemma** (Thomson, 2007, 2012d) asserts that for each pair of rules S and S', if S is *consistent*, S' is *conversely consistent*, and they coincide in the two-claimant case, then in fact they coincide in general.

Proof: First, it is clear that all rules in $\mathcal{S}$ satisfy the requirements of the theorem. Conversely, let S be a rule satisfying these requirements. We show that there is $G \in \Gamma$ such that $S = S^G$.

Step 1: *S* is anonymous. This is because (i) for each $N \in \mathcal{N}$ with $|N| = 2$, the N-component of S satisfies *equal treatment of equals*, and (ii) *equal treatment of equals* in the two-claimant case and *consistency* imply *anonymity* (Chambers and Thomson, 2002). Thus, there is a single function $g\colon \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ such that for each $N \in \mathcal{N}$ with $|N| = 2$, and each $c \in \mathbb{R}^N_+$—to fix notation, let $N \equiv \{i, j\}$ and suppose that $0 < c_i \leq c_j$—the path of S for c is bro.seg$[0, (c_i, g(c)), c]$. Obviously, $c_i \leq g(c) \leq c_j$.

Step 2: Defining a function γ over a subinterval of $\mathbb{R}_+$ from which *G* will be derived, and establishing its monotonicity properties. Let $N \equiv \{1, 2, 3\}$ and $c_1 > 0$. Let $c_2, c_3 \in \mathbb{R}_+$ be such that $0 < c_1 \leq c_2 < c_3$ and $c \equiv (c_1, c_2, c_3)$. It follows directly from the definition of *consistency* that the path of S for c, when projected onto each two-dimensional subspace pertaining to a two-claimant subset N' of N is a subset of its path for the projection $c_{N'}$ of c onto this subspace. Moreover, if S is *endowment monotonic*, and therefore *endowment continuous*, then in fact, the projection of its path for c is its path for $c_{N'}$. The result applies here because the two-claimant components of S are *endowment monotonic*, and if a *consistent* rule is *endowment monotonic* in the two-claimant case, it satisfies this property in general. (Using the terminology of Hokari and Thomson, 2008, this property is "lifted" by *consistency*.)

The path of S for (c_1, c_2) contains seg$[(0, 0), (c_1, g(c_1, c_2))]$ and its path for (c_1, c_3) contains seg$[(0, 0), (c_1, g(c_1, c_3))]$. By *consistency*, the projections of its path for c onto $\mathbb{R}^{\{1,2\}}$ and $\mathbb{R}^{\{1,3\}}$ contain these segments. Thus, its path for c contains seg$[(0, 0, 0), (c_1, g(c_1, c_2), g(c_1, c_3))]$. By *consistency* again, and projecting onto $\mathbb{R}^{\{2,3\}}$, its path for (c_2, c_3) contains $\sigma \equiv$ seg$[(0, 0), (g(c_1, c_2), g(c_1, c_3))]$. Since $c_2 < c_3$, this is possible only if g satisfies the following two properties:

(i) $g(c_1, c_2) \leq g(c_1, c_3)$. This is necessary and sufficient for σ to be at least as steep as the 45° line in $\mathbb{R}^{\{2,3\}}$.

(ii) $\frac{g(c_1,c_2)}{c_2} \geq \frac{g(c_1,c_3)}{c_3}$. Indeed, the extension of σ to the line of equation

$x_2 = c_2$ meets this line at a point whose third coordinate is $c_2 \frac{g(c_1,c_3)}{g(c_1,c_2)}$. It should be no greater than c_3. This is necessary and sufficient for this extension to be at most as steep as the path of the proportional rule for (c_2, c_3).

Let $\gamma\colon [c_1, \infty[\to \mathbb{R}_+$ be defined by $\gamma(a) \equiv g(c_1, a)$. Note that $\gamma(c_1) = c_1$. Properties (i) and (ii) can be rewritten as monotonicity properties γ satisfies.

Step 3: Identifying the function $\boldsymbol{G \in \Gamma}$ with which $\boldsymbol{S}$ is associated. Let $\{c_1^k\}$ be a decreasing sequence in $\mathbb{R}_{++}$ such that $c_1^k \to 0$ as $k \to \infty$. For each $k \in \mathbb{N}$, let γ^k be constructed in the manner in which γ was constructed. For each pair $k, k' \in \mathbb{N}$ such that $k < k'$, we now assert that over the common part of their domains of definition, namely $[c_1^k, \infty[$, the functions γ^k and $\gamma^{k'}$ are proportional. Indeed, by Step 2, for each pair $c_2, c_3 > c_1^k$, the initial segment of the path of S for (c_2, c_3) lies in the direction of $(\gamma^k(c_2), \gamma^k(c_3))$, and since $c_1^k > c_1^{k'}$, by Step 2 again, it also lies in the direction of $(\gamma^{k'}(c_2), \gamma^{k'}(c_3))$. Thus, $\frac{\gamma^k(c_2)}{\gamma^k(c_3)} = \frac{\gamma^{k'}(c_2)}{\gamma^{k'}(c_3)}$, so that $\gamma^{k'}(c_3) = \frac{\gamma^{k'}(c_2)}{\gamma^k(c_2)}\gamma^k(c_3)$. This is true for each $c_2 \geq c_1^k$. Since $\gamma^{k'}(c_1^{k'}) = c_1^{k'}$ and $\frac{\gamma^{k'}(a)}{a}$ is a nowhere-increasing function of a, it follows that $\gamma^{k'}(c_1^k) \leq \gamma^k(c_1^k)$, so that for each $a \in \mathbb{R}_{++}$, the sequence $\{\gamma^k(a)\}$ is decreasing. Let $\tilde{G}(a) \equiv \lim_{k\to\infty} \gamma^k(a)$. It is easy to see that the function $\tilde{G}\colon \mathbb{R}_+ \to \mathbb{R}_+$ inherits the two monotonicity properties of γ established in Step 2. If there is $a > 0$ so that $\tilde{G}(a) = 0$, then by the monotonicity properties, $\tilde{G} = 0$. Then, let $b > 0$ and $S = S^G$ for the function $G \in \Gamma$ that assigns to each a the value b. Otherwise, $\tilde{G} > 0$. Then, $\tilde{G} \in \mathcal{G}$ and $S = S^{\tilde{G}}$.

Step 4: Concluding: Steps 1 and 2 together imply that on the domain of two-claimant problems in which claims are unequal, $S = S^G$. Since S satisfies *equal treatment of equals*, the equality $S = S^G$ holds, trivially, for two-claimant problems in which claims are equal. By hypothesis, S is *consistent* and since S^G is *consistent* and *endowment monotonic*, it is *conversely consistent* (Chun, 1999). Thus, by the Elevator Lemma, $S = S^G$ for any number of claimants.

□

By duality, we also obtain a characterization of the family of rules that coincide for each $N \in \mathcal{N}$ with $|N| = 2$, with a member of $\mathcal{H}^N$, and are *consistent*. These rules are also indexed by functions $G \in \Gamma$ and they have parametric representations, as illustrated in Figure 6b. Given $G \in \Gamma$, for each $N \in \mathcal{N}$ with $|N| = 2$, and each $c \in \mathbb{R}^N_{++}$, $h^N(c) = (\max c_k)[1 - \frac{G(\max c_k)}{G(\min c_k)}]$.

Rule $\boldsymbol{T^G}$ associated with $\boldsymbol{G \in \Gamma}$: Let $\Lambda \equiv]-\infty, 0]$. Let $f\colon \Lambda \times \mathbb{R}_+$ be such that for each $c_0 \in \mathbb{R}_+$, the graph of $f(c_0, \cdot)$ is the union of the horizontal half-line $\{(t, 0))\colon t \leq -G(c_0)\}$, and $\text{seg}[(-G(c_0), 0), (0, c_0)]$. Then, T^G is the parametric rule admitting the representation f.

Let $\boldsymbol{\mathcal{T}} \equiv \{T^G\colon G \in \Gamma\}$. The following are two important members of the family. If there is $a > 0$ such that for each $c_0 \in \mathbb{R}_+$, $G(c_0) \equiv a$, then $T^G = P$, and if for each $c_0 \in \mathbb{R}_+$, $G(c_0) \equiv c_0$, then $T^G = CEL$.

Theorem 3 *A rule coincides, for each two-claimant population $N \in \mathcal{N}$, with a member of $\mathcal{H}^N$, and is consistent, if and only if it is a member of $\mathcal{T}$.*

5 Other properties of the members of the families $\mathcal{S}$ and $\mathcal{T}$.

Having identified which members of the family $\mathcal{G}^2$ have *consistent* extensions, and characterized these extensions, we now look for those of the resulting rules that satisfy the properties we introduced in Section 3. Two proof techniques are available to establish positive results here. Often both apply.

(a) One technique takes advantage of the fact that the rules in $\mathcal{S}$ are parametric rules; thanks to the parametrization of a rule, we can often determine that a particular property is met, or identify what it takes for the property to be met. (Here, as our starting point is not the entire parametric family but a subfamily, the answers are more easily attainable than they are when the search is within the larger family; indeed, for some properties, it is still unknown which subfamily of the parametric family satisfy them.)

(b) When a two-claimant rule in $\mathcal{G}^2$ satisfies the property, the other technique is to exploit the "lifting" results of Hokari and Thomson (2008): a property is **lifted by consistency** from the two-claimant case to arbitrarily many claimants if whenever a rule satisfies the property in the two-claimant case and the rule is *consistent*, then it satisfies the property in general. It is **lifted by consistency with the assistance of some other property (properties)** if this implication holds as soon as the rule satisfies this other property (these other properties).

• That all rules in $\mathcal{S}$ satisfy *anonymity* (and therefore *equal treatment of equals*), *endowment monotonicity*, and *continuity* can be seen immediately from their representations. For *equal treatment of equals* and *endowment* *monotonicity*, it also follows from the fact that these properties are lifted by *consistency*. For *anonymity* and *continuity*, it follows from the fact that each of these properties is lifted with the assistance of *endowment monotonicity*, which all rules in $\mathcal{G}^2$ satisfy.

• Because the constrained equal awards rule is the only two-claimant rule in our family to satisfy the $\frac{1}{|N|}$-*lower bound*, or *conditional full compensation*, or *claims truncation invariance*, it follows from the Elevator Lemma (Section 4) that it is the only rule in $\mathcal{S}$ to have any of these properties.

• For a parametric rule of representation $f \in \mathcal{F}$ to satisfy *order preservation in awards*, its schedules should be ordered as claims are ($c'_0 > c_0$ implies that $f(c'_0, .) \geq f(c_0, .)$). This is obviously the case for the rules in $\mathcal{S}$ (Figure 6).

For *order preservation in losses*, let $N \in \mathcal{N}$, $(c, E) \in \mathcal{C}^N$, and $i, j \in N$ with $c_i < c_j$. Let $\lambda \in \mathbb{R}_+$ be such that $\sum f(c_k, \lambda) = E$. Let $x \equiv S(c, E)$. If $\lambda < G(c_i)$, let $x'_j \in \mathbb{R}_+$ be such that $\frac{x'_j}{\lambda} = \frac{G(c_i)}{c_j}$. We have that $x'_i \geq f(c_j, \lambda)$ and $c_j - x'_j \geq c_i - x_i$. Thus, $c_j - x_j \geq c_j - x'_j \geq c_i - x_i$, as required by *order preservation in losses*. If $\lambda \geq G(c_i)$, then $x_i = c_i$: thus, $c_i - x_i = 0 \leq c_j - x_j$, as required by *order preservation in losses*. Thus, this property is met by all rules in $\mathcal{S}$.

• All rules in $\mathcal{S}$ are *claims monotonic* because a parametric representation f of each member of the family is such that the schedules $f(c_0, .)_{c_0 \in \mathbb{R}_+}$ are ordered as claims are. (Not all parametric rules satisfy the property.) To see this, let $G \in \Gamma$ and $S \in \mathcal{S}$ be the rule associated with G. Because for each $c_0 \in \mathbb{R}_+$, $f(c_0, .)$ is strictly monotone until it reaches the value c_0 (for $\lambda = G(c_0)$), it follows that for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, the function $\sum_N f(c_i, .)$ is strictly monotone until it reaches the value $\sum_N c_i$ (for $\lambda = G(\max c_i)$). Thus, for each $E < \sum c_i$, the solution to the equation $\sum_N f(c_i, \lambda) = E$ giving the awards vector is unique. Now, let $c, c' \in \mathbb{R}^N_+$ and $i \in N$ be such that $c'_i > c_i$ and $c'_{-i} = c_{-i}$. Let $E \leq \sum c_i$. When agent i's claim increases from c_i to c'_i, in the calculation of the awards vector, the function $f(c_i, .)$ is replaced by the function $f(c'_i, .)$. The graph of the latter is everywhere on or above that of the former. Thus, the graph of $f(c'_i, .) + \sum_{N \setminus \{i\}} f(c_j, .)$ is everywhere on or above that of $\sum_N f(c_j, .)$. Thus, the solution in λ to the equation $\sum_N f(c_j, \lambda) = E$ is at least as large as the solution in λ to the equation $f(c'_i, \lambda) + \sum_{N \setminus \{i\}} f(c_j, \lambda) = E$. This implies that each agent in $N \setminus \{i\}$ receives at most as much as initially, and since awards add up to E, that agent i receives at least as much as initially.

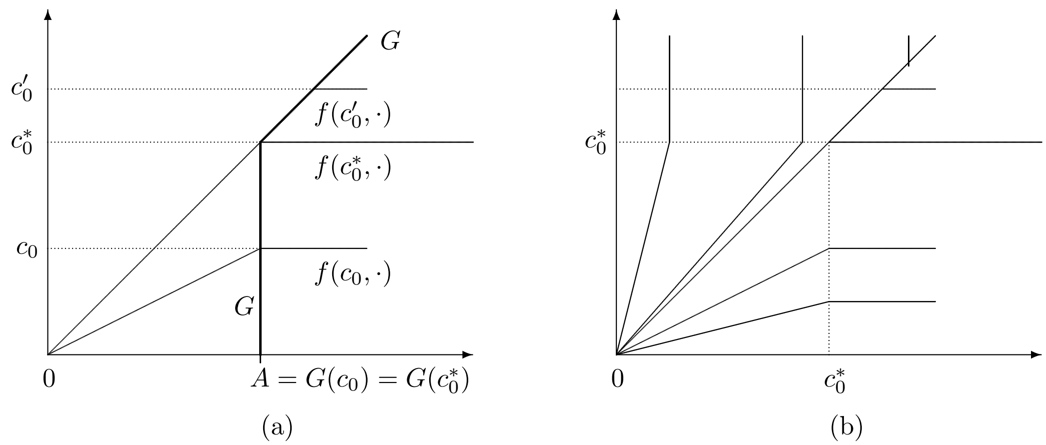

Figure 8: Identifying the rules in $\mathcal{S}$ satisfying composition down. (a) A parametric representation of a rule in $\mathcal{S}$ satisfying *composition down*. (b) The tree associated with such a rule for the two-claimant case.

If a rule in $\mathcal{G}^2$ is *claims monotonic* and it has a *consistent* extension, this extension is *claims monotonic* because all rules in $\mathcal{G}^2$ are *endowment monotonic* and *claims monotonicity* is lifted with the assistance of *endowment monotonicity* (Hokari and Thomson, 2008).

• To identify which rules in $\mathcal{S}$ are *homogeneous*, we use the fact that they are parametric. Let $S \in \mathcal{S}$ and $G \in \Gamma$ be the function with which it is associated. To simplify notation, suppose $N \equiv \{1, 2\}$. Let $c \in \mathbb{R}^N_+$ with $0 < c_1 < c_2$. The coordinates of the kink in the path of S for c are c_1 and $c_2 \frac{G(c_1)}{G(c_2)}$. The slope of the segment from the origin to this kink is $\frac{c_2}{c_1} \frac{G(c_1)}{G(c_2)}$. Let $\alpha > 0$. By *homogeneity*, the slope of the segment to the kink in the path for αc should be the same. After canceling out terms, we obtain $\frac{G(c_1)}{G(c_2)} = \frac{G(\alpha c_1)}{G(\alpha c_2)}$. It follows from Aczél (1987) that there are $t \in \mathbb{R}$ and $a > 0$ such that $G(c_1) = a c_1^t$. We obtain as particular cases the constrained equal awards and proportional rules. Then, *homogeneity* holds for each population, not just for two-claimant populations.

• No rule in $\mathcal{G}^2$ satisfies *minimal rights first*. Thus, no rule in $\mathcal{S}$ does.

• Next, we identify the rules in $\mathcal{S}$ that satisfy *composition down*.

We will show that the only rules in $\mathcal{S}$ to satisfy *composition down* constitute a one-dimensional family $\boldsymbol{\mathcal{S}}^*$ of rules defined as follows: Let $c_0^* \in \bar{\mathbb{R}}_+$ and $G \in \Gamma$ be defined by setting $G(c_0) \equiv A$ for each $c_0 \leq c_0^*$ and $G(c_0) \equiv c_0$ otherwise (Figure 8a). For $c_0^* = 0$, we obtain the constrained equal awards rule and for $c_0^* = \infty$, we obtain the proportional rule. These are the only two rules in the family that are *homogeneous*. Figure 8b illustrates the definition for $N \in \mathcal{N}$ with $|N| = 2$, say $N \equiv \{1,2\}$: the tree from which the paths are generated consist of

(a) rays in the square $\{c \in \mathbb{R}_+^N\colon$ for each $i \in N$, $c_i \leq c_0^*\}$,

(b) horizontal half-lines in the region outside of the square and below the 45° line,

(c) vertical half-lines in the region outside of the square and above the 45° line.

Theorem 4 *A rule in $\mathcal{S}$ satisfies composition down if and only if it belongs to the family $\mathcal{S}^*$.*

Proof: let $S \in \mathcal{S}$ be given and $G \in \Gamma$ be the function with which it is associated.

Step 1: If there are $c_0, c_0' \in \mathbb{R}_+$ with $c_0 < c_0'$ such that $G(c_0) = G(c_0')$, then for each $c_0'' < c_0'$, we have $G(c_0'') = G(c_0')$. Suppose by contradiction that there is $c_0'' < c_0$ for which $G(c_0'') \neq G(c_0)$. Since G is nowhere decreasing, $c_0'' < c_0$ and $G(c_0'') < G(c_0)$. Let $N \equiv \{1,2\}$, $c_1 \equiv c_0''$ and $c_2 \equiv c_0$. Then the path for c has a kink x^* of abscissa c_0''. Since S is *claims continuous*, as can be seen directly from the definition (but all parametric rules are), if follows from Theorem 1 that the tree with which the component of S relative to N is associated contains $\{x \in \mathbb{R}^N\colon x = x^* + t(0,1)\}$. Now, the path for (c_0'', c_0') meets this branch at a point of ordinate greater than x_2^*. Thus the paths for S are not generated by a tree, in contradiction with Lemma 1.

It follows from Step 1 that either $c_0^* = \infty$—then G is constant and $S = P$—or there is $c_0^* \geq 0$ such that G is constant over $[0, c_0^*]$ and strictly increasing over $[c_0^*, \infty[$. Step 2 pertains to this second case.

Step 2: For each $c_0 > c_0^*$, we have $\frac{c_0}{G(c_0)} = \frac{c_0^*}{G(c_0^*)}$. Suppose by contradiction that there is $c_0 > c_0^*$ such that the equality fails. Because G is such that $\tilde{c}_0' \in \mathbb{R}_+ \to \frac{G(\tilde{c}_0')}{\tilde{c}_0}$ is nowhere increasing, then $\frac{c_0}{G(c_0)} > \frac{c_0^*}{G(c_0^*)}$. Let $N \equiv \{1,2\}$, $c_1 \equiv c_0^*$ and $c_2 \equiv c_0$. Because $G(c_0) > G(c_0^*)$, the path for c begins with a segment that is less steep than $\text{seg}[(0,0), c]$. Let $c_0' < c_0$ be the award to agent 2 when agent 1 reaches full compensation. By the monotonicity of G and the definition of c_0^*, $G(c_0') > G(c_0^*)$. Thus, for the claims vector (c_0^*, c_0'), agent 1 reaches full compensation before agent 2 does. Thus, the paths for

(c_0^*, c_0) and (c_0^*, c_0') meets at the origin and at (c_0^*, c_0'), and at no other point: they are not generated by a tree, in contradiction with Lemma 1.

Combining Steps 1 and 2, we conclude that there is $c_0^* \in \bar{\mathbb{R}}_+$ and $A \geq 0$ so that the graph of G consists of seg$[(A, 0), (A, c_0^*)]$ together with all vectors of the form $t(A, c_0^*)$ for $t > 1$.

$\square$

The family of rules satisfying *composition down* and *consistency* has recently been characterized (Chambers and Moreno-Ternero, 2014). Our family $\mathcal{S}^*$ is a subfamily.

• It follows by duality that the only rules in $\mathcal{S}$ to satisfy *composition up* are the proportional and constrained equal losses rules.

• Next are several variable-population properties.

Population monotonicity says that upon the arrival of new claimants, each initial claimant should be awarded at most as much as initially. *Replication invariance* says that when a problem is replicated, all clones of each initial claimant should be awarded the amount he was awarded initially. *Converse consistency* has already been introduced. All parametric rules satisfies all of these properties. Thus, all rules in $\mathcal{S}$ satisfy them.

• We conclude with a discussion of Lorenz comparisons. The lifting of an order $\prec$ on the space of rules (it could be the Lorenz order or some other order) can be defined in the manner in which we defined the lifting of a property: an **order $\prec$ is lifted by consistency** (Thomson, 2012) if, whenever two rules S and S' are such that in the two-claimant case, $S \prec S'$, and both are *consistent*, then for arbitrarily many claimants, $S \prec S'$. The notion of **assisted lifting** for orders is defined in the obvious way. The next lemma identifies very mild conditions under which the Lorenz order is lifted.

Lemma 2 (Thomson, 2012) *Let S and S' be two rules satisfying* order preservation *of awards in the two-claimant case,* endowment monotonicity *in the two-claimant case, and* bilateral consistency. *If S Lorenz dominates S' in the two-claimant case, then S Lorenz dominates S' in general.*

This lemma is useful for us because (i) the two auxiliary properties it involves are met by all rules in $\mathcal{G}^2$, and (ii) Lorenz domination is very easily checked in the two-claimant case, as we have seen (Section 3).

Theorem 5 *(a) Let S and S' be two rules in $\mathcal{G}^2$ associated with functions $(g^N)_{N\in\mathcal{N},|N|=2}$ and $(g'^N)_{N\in\mathcal{N},|N|=2}$. If, for each $N \in \mathcal{N}$ with $|N| = 2$, $g^N \leq g'^N$, and both rules have consistent extensions—let us call these extensions $\tilde{S}$ and $\tilde{S}'$—then $\tilde{S}$ Lorenz dominates $\tilde{S}'$.*

(b) Also, each rule in $\mathcal{S}$ Lorenz dominates each rule in $\mathcal{T}$ (Figure 5c).

Proof: Both (a) and (b) follow from the fact that all of the rules under discussion are *endowment monotonic* and from Lemma 2. $\square$

6 Concluding comment

We have proposed a new way of compromising between principles that are central in the adjudication of conflicting claims,the principles of proportionality and the principle of equality (of awards or of losses). In applications, rules should be easy to understand and we believe that our proposal meet this criterion. Flexibility is also desirable, and we have seen that, by appropriately selecting from our family, we could achieve many desirable properties.

Our proofs in Sections 4 and 5 bring together several concepts and techniques that have been important in the literature on claims problems, and on multiple occasions, taking advantage of these advances has given us very simple ways of reaching our conclusions. They are the following: the observation that *consistency* can (essentially) be expressed as a projection property of paths of awards, (the observation that can be exploited very generally in settling the issue of existence *consistent* extensions and constructing these extensions when they exist; Thomson, 2007); certain logical relations between properties of rules; the Elevator Lemma; the concepts of lifting and assisted lifting of properties and orders by means of *consistency* (Hokari and Thomson, 2008); and the notion of a parametric rule (Young, 1987). We expect the future literature on the adjudication of conflicting claims to be greatly helped by relying on these concepts and techniques.

Appendix

As mentioned above, the family of rules in $\mathcal{G}^N$ that satisfy *composition down* but not *claims monotonicity* is complex. We only give examples to illustrate the various ways in which the family of rules satisfying these axioms (Theorem 1) is enlarged by dropping this second property. Each of the examples is defined by delimiting subregions in awards space and specifying what to do in these regions. The complexity of the family comes from the choices we have in specifying the boundary between these regions and, for points that belong to the boundaries of two adjacent regions, in deciding whether they should be thought of as belonging to one region or as belonging to the other. These choices cannot be made independently however.

Let L^1 be the region of awards space above the 45° line. Figure 9a shows a rule for which L^1 is partitioned into two cones, R^1 (shaded) and R^2. The boundary ray shared by these cones is denoted r. A typical branch of the tree in R^1 consists of a segment in Λ emanating from the origin followed by a vertical segment to r, excluding the point at which r is reached. A typical branch of the tree in R^2 consists of a segment in r followed by an unbounded vertical segment.

Figure 9b shows a rule for which L^1 is partitioned into three cones, R^1, R^2 (shaded) , and R^3. The boundary ray shared by R^1 and R^2 is denoted r^1. The boundary ray shared by R^2 and R^3 is denoted r^2. A typical branch of the tree in R^1 consists of a segment in Λ emanating from the origin followed by a vertical segment to r, excluding the point at which r^1 is reached. A typical branch of the tree in R^2 consists of a ray from the origin. A typical branch of the tree in R^3 consists of a segment in r^2 followed by an unbounded vertical segment.

Figure 9c shows a rule for which L^1 is partitioned into two cones, R^1 (shaded) and R^2. The boundary ray shared by these cones is denoted r. In R^1, there is a downward sloping continuous curve D^1 (the segment seg$[b, a]$) that is visible from below from the origin. A typical branch of the tree in R^1 consists of a segment to D^1 emanating from the origin followed by a vertical segment to r, excluding the point at which r is reached. In R^2, there is a downward sloping continuous curve D^2 (the segment seg$[d, b]$) that is visible from below from the origin. A typical branch of the tree in C^2 consists of a segment to D^2 emanating from the origin followed by an unbounded vertical segment.

Figure 9d shows a rule for which L^1 is partitioned into two regions, R^1

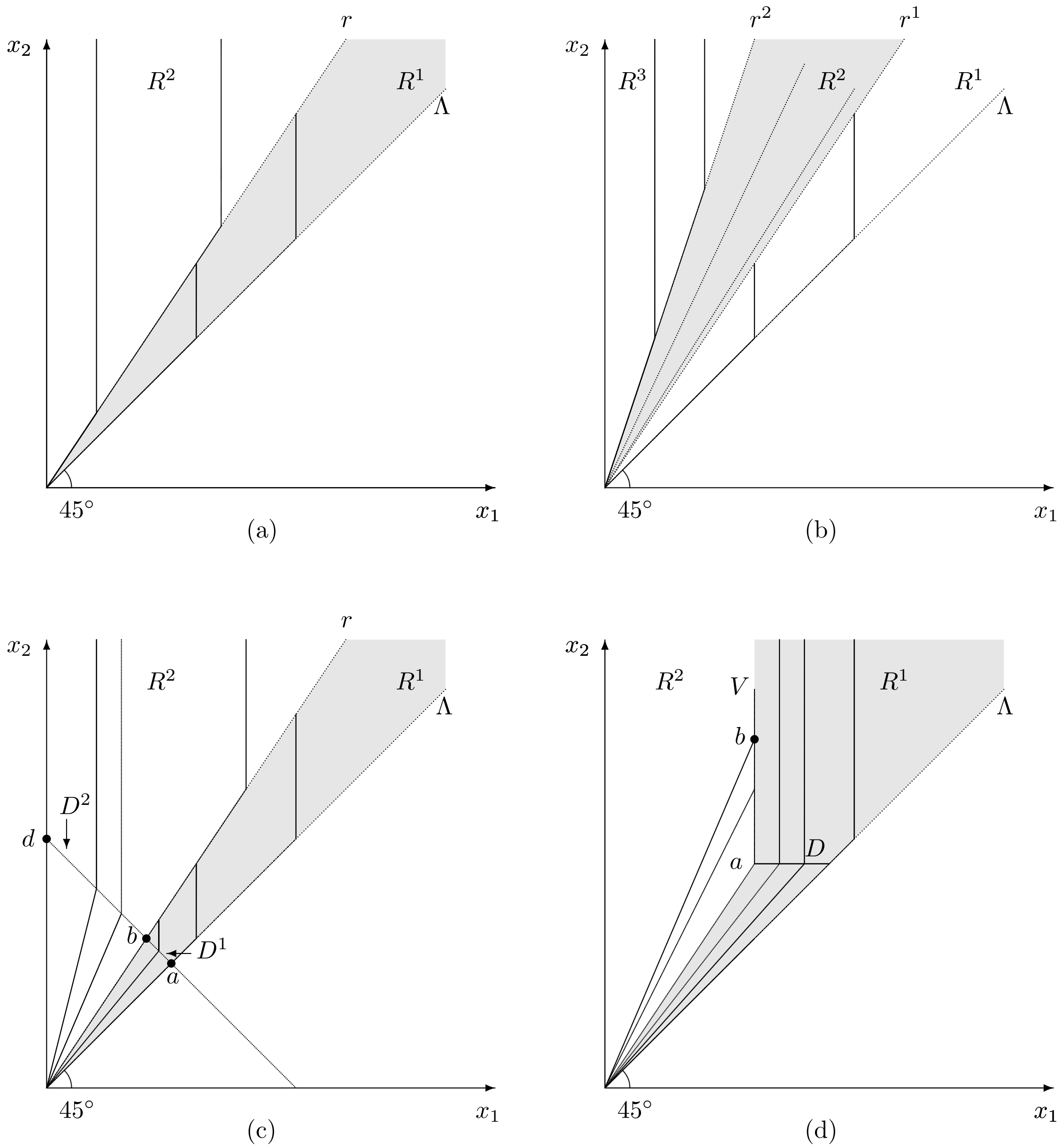

Figure 9: Composition down. Illustrating different possibilities if *claims continuity* is not imposed. (a) A simple partitioning of L^1 into two cones. All branches of the tree consist of two segments. The loci of kinks are rays. (b) Here, we have added a cone in which branches are rays. (c) Here, the loci of kinks are not rays, but visible curves from the origin, D^1 and D^2. (d) Here, some of the branches are bounded segments, $\text{seg}[(0,0), b[$, for example.

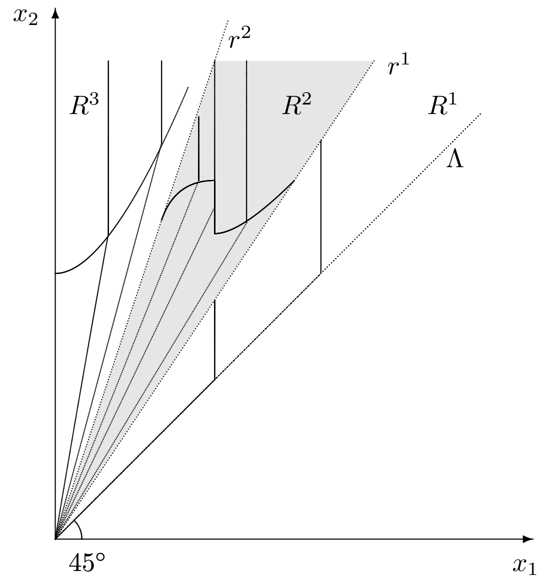

Figure 10: Composition down. Illustrating another possibility.

(shaded) and R^2, defined as follows. There is a point $a \in L^1$ such that R^1 consists of the union of the cone with boundary rays Λ and the ray through a as well as all points in L^1 whose abscissa is at least a_1. There is a downward slope curve D from a to Λ. Region R^2 is the complement. The unbounded vertical half-line V with lowest point is a is part of the boundary of R^2. A typical branch of the tree in R^1 consists of a segment to D^1 followed by an unbounded vertical segment. A typical branch of the tree in R^2 consists of a segment to V.

There is also a point b on the vertical segment V such that all branches in R^2 whose limit point is in seg$[a, b]$ do not contain their upper limit point, and all branches in R^2 whose limit point is in the vertical half-line with lowest endpoint is b do contain their upper limit point. Then, seg$[0, a] \cup$ seg$[a, b]$ is a branch of the tree.

Figure 10 shows a rule that exhibits all of these features described in Figures 9a-d. It illustrates the general definition.

7 Acknowledgements

I thank Yoichi Kasajima for his useful comments.

8 References

Aczél. J., A short course on functional equations, D. Reidel Publishing Company, Dordrecht (1987).

R. Aumann, and M. Maschler, Game theoretic analysis of a bankruptcy problem from the Talmud, J. Econ Theory 36 (1985) 195-213.

C. Chambers, and W. Thomson, Group order preservation and the proportional rule for bankruptcy problems, Math Social Sciences 44 (2002) 235-252.

Y. Chun, , Equivalence of axioms for bankruptcy problems, Int J. Game Theory 28 (1999) 511-520.

I. Curiel, I., M. Maschler and S. H. Tijs, Bankruptcy games, Zeitschrift für Op Research 31 (1987) A143-A159.

N. Dagan, N., R. Serrano, and O. Volij, A non-cooperative view of consistent bankruptcy rules, Games and Econ Behavior 18 (1997) 55-72.

N. Dagan, N. and O. Volij, The bankruptcy problem: a cooperative bargaining approach, Math Social Sciences 26 (1993), 287-297.

J.M. Giménez-Gómez, and J. Peris, A proportional approach to claims problems with a guaranteed minimum, European J. Operational Research 232 (2014) 109-116.

C. Herrero, and A. Villar, Sustainability in bankruptcy problems, TOP 10 (2002) 261-273.

T. Hokari, and W. Thomson, On properties of division rules lifted by bilateral consistency, J. Math Econ 44 (2008) 211-231.

J. Moreno-Ternero, and A. Villar, The Talmud rule and the securement of agents' awards, Math Social Sciences 47 (2004) 245-257.

J. Moreno-Ternero, and A. Villar, The TAL-family of rules for bankruptcy problems, Soc Choice Welfare 27 (2006a) 231-249.

J. Moreno-Ternero, and A. Villar, On the relative equitability of a family of taxation rules, J. Public Econ Theory 8 (2006b) 283-291.

H. Moulin, Priority rules and other asymmetric rationing methods, Econometrica 68 (2000) 643-684.

B. O'Neill, A problem of rights arbitration from the Talmud, Math Social Sciences 2 (1982) 345-371.

W. Thomson, Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey, Math Social Sciences 45 (2003) 249-297.

W. Thomson, W., *How to Divide When there Isn't Enough*, mimeo, 2006.

W. Thomson, On the existence of consistent rules to adjudicate conflicting claims: a geometric approach, Rev Econ Design 11 (2007) 225-251.

W. Thomson, Two families of rules for the adjudication of conflicting claims, Soc Choice Welfare 31 (2008) 667-692.

W. Thomson, Lorenz rankings of rules for the adjudication of conflicting claims, Economic Theory 50 (2012a) 547-569.

W. Thomson, On the axiomatics of resource allocation: Interpreting the consistency principle, Econ and Phil 28 (2012b) 385-421.

W. Thomson, Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: an update, mimeo, 2012c.

W. Thomson, Consistent allocation rules, mimeo, 2012d.

P. Young, On dividing an amount according to individual claims or liabilities, Math Op Research 12 (1987) 398-414.