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in object assignment problems

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Abstract:

We consider the problem of reallocating objects among a group of agents, each possibly endowed with several objects and possibly consuming several. We search for allocation rules that are immune to manipulation through merging and splitting endowments. Unfortunately, on several natural domains of preferences, we show that no rule is efficient, meets the endowments lower bound, and s robust to either one of these strategic moves.

Keywords: object-reallocation problems; merging-proofness; splitting-proofness.

1 Introduction

We consider a group of agents, each endowed with a set of indivisible goods, called "objects", and equipped with a preference relation over subsets of the union of their endowments. An "allocation rule" associates with each economy of this type a redistribution among them of their endowments. We are concerned about two kinds of opportunities to manipulate a rule to their advantage that agents may have and we investigate the existence of rules that are immune to the manipulations.

First, imagine two agents merging their endowments; one of them shows up with his augmented endowment, and the other leaves; the rule is applied;

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We thanks Shin Sato for his comments. Filename: BuChenThomson.tex

the agent with the augmented endowment may now be assigned a bundle that can be partitioned into two bundles, one for himself and one for the agent who left, that make both of them at least as well off as they would have been otherwise, and at least one of them better off.

Alternatively, an agent may split his endowment into two parts. He then invites some outsider, someone who has no endowment, and gives him control over one part of his endowment. In the enlarged economy, the two of them may be assigned bundles whose union the agent who invited the outsider prefers to what he would have been assigned otherwise.

We are interested in selections from the correspondence that selects for each economy its efficient allocations that meet the endowments lower bound (each agent finds his component of the chosen allocation at least as desirable as his endowment). We inquire about the existence of such selections that are immune to one or the other of these behaviors. Our main results are negative: even if preferences are restricted in natural ways, there is no such selection.

Vulnerability of rules to this type of behaviors is examined in the context of classical economies by Thomson (2014), who reports negative results for rules that are central to the literature. The current note suggests that the problem may be quite pervasive.

2 Model and axioms

The model pertains to the problem of reassigning among a group of agents the resources that they privately own. Here, the resources are indivisible; we call them "objects".

Since the form of manipulation that we consider involve variations in the set of "active" agents, we need to cast our analysis in a variable-population framework. There is an infinite set of "potential agents", $\mathbb{N}$. To define an economy, we draw a finite subset of them from this infinite population. Let $\mathcal{N}$ be the class of finite subsets of $\mathbb{N}$, with generic elements N and N'. Endowments consist of objects, denoted $a, b, \ldots$. For each $i \in \mathbb{N}$, let ω_i be agent i' s endowment. Preferences are defined over sets of objects. An **economy with agent set $\boldsymbol{N}$** is obtained by specifying, for each agent, an endowment and a preference relation defined over the subsets of the union of everyone's endowment. Formally, it is a pair (R, ω), where for each $i \in N$, ω_i is agent i's endowment, and R_i designates agent i's preference relation.

This relation defined over subsets of $\bigcup \omega_i$, is complete, transitive, and anti-symmetric. Let P_i be the strict preference associated with R_i and I_i the indifference relation. Let $\boldsymbol{\mathcal{E}^N}$ be the set of all economies with agent set N. An **allocation** for $e \equiv (R, \omega) \in \mathcal{E}^N$ is a list of bundles $x \equiv (x_i)_{i \in N}$ such that $\bigcup x_i = \bigcup \omega_i$. Let $\boldsymbol{X(e)}$ denote the set of allocations of e. A **solution** is a correspondence defined over $\bigcup \mathcal{E}^N$ that associates with each $N \in \mathcal{N}$ and each $e \equiv (R, \omega) \in \mathcal{E}^N$ a non-empty subset of $X(e)$. A **rule** is a single-valued solution. We designate the intersection of two correspondences φ and φ' by $\varphi \cap \varphi'$.

A preference relation is **size monotone** if given two sets of objects, the one with more items is at least as desirable as the one with fewer items. It is **additive** if numbers can be assigned to objects in such a way that two sets can be compared by comparing the sums of the numbers assigned to the objects in each set.[1]

Two basic solutions are defined next. We will be interested in rules that select from their intersection. An allocation is "efficient" if there is no other allocation that each agent finds at least as desirable and at least one agent prefers. Our first solution is the solution that associates with each economy its set of efficient allocations. The notation we choose for it is in reference to Pareto:

Efficiency solution, $\boldsymbol{P}$: For each $N \in \mathcal{N}$, each $(R, \omega) \in \mathcal{E}^N$, $x \in P(R, \omega)$ if and only if there is no $x' \in X(e)$ such that for each $i \in N$, $x'_i \; R_i \; x_i$ and for at least one $i \in N$, $x'_i \; P_i \; \omega_i$.

Next is the solution that associates with each economy its set of allocations such that each agent finds his component of it at least as desirable as his endowment:

Endowments lower bound solution, $\boldsymbol{B}$: For each $N \in \mathcal{N}$ and each $(R, \omega) \in \mathcal{E}^N$, $x \in B(R, \omega)$ if and only if for each $i \in N$, $x_i \; R_i \; \omega_i$.

Next, we turn to strategic requirements. We formulate two kinds of manipulation in which agents may engage, and require immunity of rules to the manipulation. Let φ be a rule.

[1] Note that we do not require an agent's preference to be strict. Nevertheless, since the preferences in our proofs are strict, our impossibility results (Propositions 1-3) hold for strict domains.

• Consider a pair of agents; imagine that one of them entrusts his endowment to the other and withdraws; the rule is applied to the resulting economy; at the chosen allocation, the second agent's assignment may be a bundle that can be partitioned between the two of them in such a way that each of them is at least as well off as he would have been if they had not merged their endowments, and at least one of them is better off. Again, we formulate a requirement of immunity to this sort of behavior:

Endowments-merging–proofness: For each $N \in \mathcal{N}$, each $(R,\omega) \in \mathcal{E}^N$, each $\{i,j\} \subset N$, and letting $x \equiv \varphi(R,\omega)$, $\omega'_i \equiv \omega_i \cup \omega_j$, and $x' \equiv \varphi(R_{N\setminus\{j\}}, \omega'_i, \omega_{N\setminus\{i,j\}})$, it is not the case that there is a bundle $(\hat{x}_i, \hat{x}_j)$ such that $\hat{x}_i \cup \hat{x}_j = x'_i$, and for each $k \in \{i,j\}$, $\hat{x}_k \; R_k \; x_k$, and for at least one $k \in \{i,j\}$, $\hat{x}_k \; P_k \; x_k$.

A weaker requirement is that the rule should not be such that both agents end up better off. We refer to it as **weak endowments-merging–proofness**.

• Symmetrically, an agent may transfer some of his endowment to some agent who was not initially present and whose endowment is the empty set; the rule is applied, and the union of what the two of them are assigned is a bundle that the first agent prefers to his initial assignment. We require immunity to this sort of behavior:[2]

Endowments-splitting–proofness: For each $N \in \mathcal{N}$, each $(R,\omega) \in \mathcal{E}^N$, each $i \in N$, each $j \notin N$, each $R'_j \in \mathcal{R}$, and each pair (ω'_i, ω'_j) of bundles such that $\omega'_i \cup \omega'_j = \omega_i$, and letting letting $x \equiv \varphi(R,\omega)$, and $x' \equiv \varphi(R, R'_j, \omega'_i, \omega_{N\setminus\{i\}}, \omega'_j)$, $x_i \; R_i \; (x'_i \cup x'_j)$.

Here, a weaker requirement is that the rule should not be such that agent i can partition his assignment into two bundles, one for himself and one for the person he invited in, so that both of them end up better off.

[2]Alternatively, we could allow the second agent to have his own endowment and require that the union of their assignments be such that this agent be able to retrieve his endowment, the first agent ending up better off with the remaining resources. Finally, we could declare the manipulation successful if there is a partitioning of the union of their assignments that make both of them at least as well off as they would have been without the maneuver and at least one of them better off.

Related strategic moves have been defined in other models. In the context of the adjudication of conflicting claims, it is their claims that agents can merge or split (O'Neill, 1982). In the context of queueing, agents can split or merge jobs (Moulin, 2007). In each case, one may require a rule to not be vulnerable to these actions.

Counterparts of these properties for classical economies are studied by Thomson (2014).

3 The results

We establish two impossibility results. In the proofs, we slightly abuse notation and omit brackets around sets of objects. For instance, instead of $\{a, b\}$, we write ab. We present preferences in table form. We place endowments in boxes.

Proposition 1. *On the size-monotone domain, no selection from the Pareto–and–endowments-lower-bound solution is weakly endowments-merging-proof.*

Proof. Let $\varphi \in B \cap P$.

Let $N \equiv \{1, 2, 3, 4\}$, $\omega_1 \equiv a$, $\omega_2 \equiv b$, $\omega_3 \equiv cd$, and $\omega_4 \equiv ef$. Preferences $(R_i)_{i \in N}$ are described in the following table.

R_1	R_2	R_3	R_4
$\forall S$ s.t. $\lvert S\rvert \geq 4$ } $\vdots$	$\forall S$ s.t. $\lvert S\rvert \geq 3$ } $\vdots$	$\forall S$ s.t. $\lvert S\rvert \geq 4$ } $\vdots$	$\forall S$ s.t. $\lvert S\rvert \geq 5$ } $\vdots$
bdf	df	adf	$abdf$
aef	ab	bcd	$cdef$
$\forall S$ s.t. $3 \geq \lvert S\rvert \geq 2$ and $S \neq bdf, aef$ } $\vdots$	$\forall S$ s.t. $\lvert S\rvert = 2$ and $S \neq df, ab$ } $\vdots$	$\forall S$ s.t. $\lvert S\rvert \geq 3$ and $S \neq adf, bcd$ } $\vdots$	$\forall S$ s.t. $4 \geq \lvert S\rvert \geq 3$ and $S \neq abdf, cdef$ } $\vdots$
d	f	ad	bf
c	e	ac	be
$\boxed{a}$	$\boxed{b}$	$\boxed{cd}$	$\boxed{ef}$
$\forall S$ s.t. $\lvert S\rvert = 1$ and $S \neq d, c, a$ } $\vdots$	$\forall S$ s.t. $\lvert S\rvert = 1$ and $S \neq f, e, b$ } $\vdots$	$\forall S$ s.t. $\lvert S\rvert \leq 2$ and $S \neq ad, ac, cd$ } $\vdots$	$\forall S$ s.t. $\lvert S\rvert \leq 2$ and $S \neq bf, be, ef$ } $\vdots$

Let $e \equiv (R, \omega) \in \mathcal{E}^N$ be the economy so defined. Let $w \in X(e)$ be defined by $w_1 \equiv c$, $w_2 \equiv f$, $w_3 \equiv ad$, and $w_4 \equiv be$. Let $x \in X(e)$ be defined by $x_1 \equiv c$, $x_2 \equiv e$, $x_3 \equiv ad$, and $x_4 \equiv bf$. Let $y \in X(e)$ be defined by $y_1 \equiv d$, $y_2 \equiv f$, $y_3 \equiv ac$, and $y_4 \equiv be$. Let $z \in X(e)$ be defined by $z_1 \equiv d$, $z_2 \equiv e$, $z_3 \equiv ac$, and $z_4 \equiv bf$.

We omit the proof that $B(e) \cap P(e) = \{w, x, y, z\}$. Since $\varphi \in BP$, there are four cases to consider.

Case 1: $\varphi(e) = w$.

Agents 1 and 4 merge their endowments and agent 4 withdraws. Let $e' \equiv (R_{-4}, (aef, b, cd)) \in \mathcal{E}^{N\setminus\{4\}}$ be the resulting economy. Let $w' \in X(e')$ be defined by $w'_1 \equiv bdf$, $w'_2 \equiv e$, and $w'_3 \equiv ac$. We omit the proof that $B(e') \cap P(e') = \{w'\}$. Thus, $\varphi(e') = w'$. Let $\hat{w}_1 \equiv d$ and $\hat{w}_4 \equiv bf$. Then, $\hat{w}_1 \cup \hat{w}_4 = w'_1$. Since $\hat{w}_1 \; P_1 \; w_1$ and $\hat{w}_4 \; P_4 \; w_4$, *weak endowments-merging-proofness* is violated.

Case 2: $\varphi(e) = x$.

Agents 1 and 2 merge their endowments and agent 1 withdraws. Let $e' \equiv (R_{-1}, (ab, cd, ef)) \in \mathcal{E}^{N\setminus\{1\}}$ be the resulting economy. Let $x' \in X(e')$ be defined by $x'_2 \equiv df$, $x'_3 \equiv ac$, and $x'_4 \equiv be$. We omit the proof that $B(e') \cap P(e') = \{x'\}$. Thus, $\varphi(e') = x'$. Let $\hat{x}_1 \equiv d$ and $\hat{x}_2 \equiv f$. Then, $\hat{x}_1 \cup \hat{x}_2 = x'_2$. Since $\hat{x}_1 \; P_1 \; x_1$ and $\hat{x}_2 \; P_2 \; x_2$, *weak endowments-merging-proofness* is violated.

Case 3: $\varphi(e) = y$.

Agents 3 and 4 merge their endowments and agent 3 withdraws. Let $e' \equiv (R_{-3}, (a, b, cdef)) \in \mathcal{E}^{N\setminus\{3\}}$ be the resulting economy. Let $y' \in X(e')$ be defined by $y'_1 \equiv c$, $y'_2 \equiv e$, and $y'_4 \equiv abdf$. We omit the proof that $B(e') \cap P(e') = \{y'\}$. Thus, $\varphi(e') = y'$. Let $\hat{y}_3 \equiv ad$ and $\hat{y}_4 \equiv bf$. Then, $\hat{y}_3 \cup \hat{y}_4 = y'_4$. Since $\hat{y}_3 \; P_3 \; y_3$ and $\hat{y}_4 \; P_4 \; y_4$, *weak endowments-merging-proofness* is violated.

Case 4: $\varphi(e) = z$.

Agents 2 and 3 merge their endowments and agent 2 withdraws. Let $e' \equiv (R_{-3}, (a, bcd, ef)) \in \mathcal{E}^{N\setminus\{2\}}$ be the resulting economy. Let $z' \in X(e')$ be defined by $z'_1 \equiv c$, $z'_3 \equiv adf$, and $z'_4 \equiv be$. We omit the proof that $B(e') \cap P(e') = \{z'\}$. Thus, $\varphi(e') = z'$. Let $\hat{z}_2 \equiv f$ and $\hat{z}_3 \equiv ad$. Then, $\hat{z}_2 \cup \hat{z}_3 = z'_3$. Since $\hat{z}_2 \; P_2 \; z_2$ and $\hat{z}_3 \; P_3 \; z_3$, *weak endowments-merging-proofness* is violated. □

The next proposition pertains to a narrower domain, but the violation that we establish is for *endowments-merging–proofness* itself, not for its weak version.

Proposition 2. *On the intersection of the size-monotone and additive domains, no selection from the Pareto–and–endowments-lower-bound solution is endowments-merging-proof.*

Proof. Let $\varphi \in B \cap P$.

Let $N \equiv \{1, 2, 3\}$, $\omega_1 \equiv a$, $\omega_2 \equiv b$, and $\omega_3 \equiv c$. Preferences $(R_i)_{i \in N}$ are described in the following table.

R_1	R_2	R_3
abc	abc	abc
ac	ac	ab
bc	ab	ac
ab	bc	bc
c	a	a
$\boxed{a}$	c	b
b	$\boxed{b}$	$\boxed{c}$

Let $e \equiv (R, \omega) \in \mathcal{E}^N$ be the economy so defined. Let $x \in X(e)$ be defined by $x_1 \equiv c$, $x_2 \equiv b$, and $x_3 \equiv a$. Let $y \in X(e)$ be defined by $y_1 \equiv c$, $y_2 \equiv a$, and $y_3 \equiv b$.

We omit the proof that $B(e) \cap P(e) = \{x, y\}$. Since $\varphi \in BP$, there are two cases to consider.

Case 1: $\varphi(e) = x$.

Agents 1 and 2 merge their endowments and agent 1 withdraws. Let $e' \equiv (R_{-1}, (ab, c))$ be the resulting economy. Let $x' \in X(e')$ be defined by $x'_2 \equiv ac$, and $x'_3 \equiv b$. We omit the proof that $B(e') \cap P(e') = \{x'\}$. Thus, $\varphi(e') = x'$. Let $\hat{x}_1 \equiv c$ and $\hat{x}_2 \equiv a$. Then, $\hat{x}_1 \cup \hat{x}_2 = x'_2$. Since $\hat{x}_1 = x_1$ and $\hat{x}_2 \; P_2 \; x_2$, *endowments-merging-proofness* is violated.

Case 2: $\varphi(e) = y$.

Agents 1 and 3 merge their endowments and agent 3 withdraws. Let $e' \equiv (R_{-3}, (ac, b))$ be the resulting economy. Let $y' \in X(e')$ be defined by $y'_1 \equiv ac$, and $y'_2 \equiv b$. We omit the proof that $B(e') \cap P(e') = \{y'\}$. Thus, $\varphi(e') = y'$. Let $\hat{y}_1 \equiv c$, and $\hat{y}_3 \equiv a$. Then, $\hat{y}_1 \cup \hat{y}_3 = y'_1$. Since $\hat{y}_1 = y_1$ and $\hat{y}_3 \; P_3 \; y_3$, *endowments-merging-proofness* is violated. $\square$

Next, we turn to our second property and for it too, we establish a negative result.

Proposition 3. *On the size-monotone domain, no selection from the Pareto–and–endowments-lower-bound solution is endowments-splitting-proof.*

Proof. Let $\varphi \in B \bigcap P$.

Let $N \equiv \{2, 3\}$, $\omega_2 \equiv ab$, and $\omega_3 \equiv cd$. Preferences $(R_i)_{i \in N}$ are as described in the following table.

R_2	R_3
$\forall S$ s.t. $\lvert S \rvert \geq 3$ } $\vdots$	$\forall S$ s.t. $\lvert S \rvert \geq 2$ } $\vdots$
bc	bd
ac	ad
$\boxed{ab}$	$\boxed{cd}$
$\forall S$ s.t. $\lvert S \rvert = 2$ and $S \neq bc, ac, ab$ } $\vdots$	$\forall S$ s.t. $\lvert S \rvert = 2$ and $S \neq bd, ad, cd$ } $\vdots$
b	d
c	b
a	c
$\forall S$ s.t. $\lvert S \rvert = 1$ and $S \neq b, c, a$ } $\vdots$	$\forall S$ s.t. $\lvert S \rvert = 1$ and $S \neq d, b, c$ } $\vdots$

Let $e \equiv (R, \omega) \in \mathcal{E}^N$ be the economy so defined. Let $x \in X(e)$ be defined by $x_2 \equiv ac$ and $x_2 \equiv bd$. Let $y \in X(e)$ be defined by $y_1 \equiv bc$ and $y_2 \equiv ad$.

We omit the proof that $B(e) \cap P(e) = \{x, y\}$. Since $\varphi \in BP$, there are two cases to consider.

Case 1: $\varphi(e) = x$.

Agent 2 invites agent 1, with preference $R_1 = R_2$ and no endowment, and provides him with b as endowment. Let $e' \equiv ((R_1, R), (b, a, cd))$. Let $x' \in X(e')$ be defined by $x'_1 \equiv b$, $x'_2 \equiv c$, and $x'_3 \equiv ad$. We omit the proof that $B(e') \cap P(e') = \{x'\}$. Thus, $\varphi(e') = x'$. Since $bc = x'_1 \cup x'_2 \; P_2 \; x_2 = ac$, *endowments-splitting-proofness* is violated.

Case 2: $\varphi(R, \omega) = y$.

Agent 3 invites agent 4, with preference $R_4 = R_3$ and no endowment, and provides him with d as endowment. Let $e' \equiv ((R, R_3), (ab, c, d)) \in \mathcal{E}^{N'}$. Let $y' \in X(e')$ be defined by $y'_2 \equiv ac$, $y'_3 \equiv b$, and $y'_4 \equiv d$. We omit the proof that $B(e') \cap P(e') = \{y'\}$. Thus, $\varphi(e') = y'$. Since $bd = y'_3 \cup y'_4 \; P_3 \; y_3 = ad$, *endowments-splitting-proofness* is violated. □

We observe that in the proof of Proposition 3, the preferences of the new agent are the same as that of the agent who invited him. So, we need not invoke "exotic" preferences to prove it.

To establish the independence of the axioms in our propositions, consider the rule that, for each problem, assigns to each agent his endowment. This rule is obviously *endowments-merging–proof* as well as *endowments-splitting–proof*, but it is not *efficient*. Other rules can be defined with these properties that do better from the *efficiency* viewpoint: for many economies, they achieve a Pareto improvement over the endowment profile. Any selection from the Pareto solution is *efficient* and violates both strategic requirements.

4 Related literature

Other manipulation opportunities in economies of the kind considered here have been studied in earlier literature. An agent may benefit by withholding some of his endowment or by destroying some of it (Atlamaz and Klaus, 2007). An agent may benefit by artificially augmenting it by borrowing from the outside, or from one of his fellow traders (Atlamaz and Thomson, 2006). An agent may benefit by grouping the objects he owns into bundles (Klaus, Dimitrov and Haake, 2006). Most results concerning immunity of rules to these behaviors are negative. It is unfortunate that we had to add to the list. Note that these other types of manipulation do not involve changes in the set of active agents. Such changes are the focus of the present note. In the context of claims problems, the property of immunity to the merging or splitting of claims studied in the literature cited in the introduction is akin to the properties we analyzed here. In that context, it can be met however. (The proportional rule passes these tests.)

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