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William Thomson

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# For claims problems, another compromise between the proportional and constrained equal awards rules

William Thomson
Department of Economics
University of Rochester
Rochester, NY 14627
wth2@mail.rochester.edu*

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**Abstract**

For the problem of adjudicating conflicting claims, we propose to compromise in the two-claimant case between the proportional and constrained equal awards rules by taking, for each problem, a weighted average of the awards vectors these two rules recommend. We allow the weights to depend on the claims vector, thereby generating a large family of rules. We identify the members of the family that satisfy particular properties. We then ask whether the rules can be extended to populations of arbitrary sizes by imposing "consistency": the recommendation made for each problem should be "in agreement" with the recommendation made for each reduced problem that results when some claimants have received their awards and left. We show that only the proportional and constrained equal awards rules qualify. We also study a dual compromise between the proportional and constrained equal losses rules.

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# 1 Introduction

We consider the problem of allocating a social endowment of a single infinitely divisible resource among a group of people with incompatible claims on it. A "rule" specifies for each such "claims problem" a division of the amount available among the claimants (O'Neill, 1982). How to distribute the liquidation value of a bankrupt firm among its creditors is the primary application of the model, but taxation is also covered: there, the issue is to determine how much of the cost of a public project each of its users should contribute when users differ in their incomes. Alternatively, it could be on the basis of the benefits they derive from the project, or on a combination of incomes and benefits, that contributions could be based. In what follows, we use language that is appropriate for the claims problem interpretation of the model, and refer to the amount assigned to each claimant as his "award".

Two focal rules in the literature (for surveys, see Thomson, 2003, 2006, 2015a) are the "proportional rule", for which awards are proportional to claims, and the "constrained equal awards rule", for which awards are made as equal as possible subject to no one receiving more than his claim. Neither fully captures the intuition that many people have about how claims problems should be solved however. Indeed, and although the proportional rule is almost universally used, the feeling is common that some steps should be taken in the direction of equality, especially when the amount to divide is small in relation to claims. On the other hand, the insistence on equality (subject to no one receiving more than his claim) embodied in the constrained equal awards rule appears too rigid to many. Thus, some compromise should be found between proportionality and (constrained) equality. Several have been defined in the literature, and some have come out of axiomatic analysis.

For the two-claimant case, we propose here to simply average, for each problem, the recommendations made by the proportional and constrained equal awards rules. It is a common research strategy in the axiomatics of resource allocation to start with two agents, and we adopt it here: our intuition is stronger in that case, as conceptual and technical issues having to do with the treatment of groups do not arise. As for averaging, it is a meaningful operation in the context of claims problems because the set of vectors from which it is natural to choose for each problem is a convex set.

To add flexibility though, we allow the weights placed on these two recommendations to depend on the claims vector, thereby obtaining a rich family of rules.[1] It would be unreasonable to let the weights vary in some arbitrary way however, and our first task is to identify the restrictions that the weight function should satisfy for the resulting rule to enjoy particular properties. We review most of the properties that have been important in the literature. Some are in fact met by all members of our family. Some are met provided some restrictions are placed on the weight function; we identify these restrictions and describe the resulting rules. Some are satisfied by no member of the family. We also show that members of our family can easily be compared in the Lorenz order.

The next question is what to do for more claimants. How should our definition be extended to such situations? Our strategy here is to invoke "consistency", a versatile principle that has successfully guided the theory of resource allocation in a wide variety of contexts. (For a survey of the literature, see Thomson, 2012c). Informally, here, consistency says that the choice made for each population and each problem that this population could face should be "confirmed" in the "reduced problem" each subpopulation faces when the members of the complementary subpopulation have received their awards and left the scene: the rule should assign to each remaining claimant the amount it did in the initial problem. Multiple interpretations can be given to the consistency of a rule. An important one is robustness under partial implementation. Possible fairness underpinnings are evaluated in Thomson (2012b, 2012c).

Of course, not all two-claimant rules have what can be called a "consistent extension" to arbitrary populations. The two-claimant proportional and constrained equal awards rules are members of our family, and when the proportional formula is applied in general, that is, to problems involving populations of arbitrary sizes, consistency is satisfied, as is well-known and easily checked. The same is true if the constrained equal awards formula is applied in general. But let us consider the next simplest case, when within our family, the weights assigned to the proportional and constrained equal awards rules are both positive but independent of the identity of the two claimants involved and of their claims. It is equally easy to see that using the same weights for any number of claimants does not deliver a consistent rule. So, let us allow the weights to depend on the identity of the claimants who are present and on their claims. Some new opportunities may arise then.

[1] In the context of a different model, compromises of this type have been studied by Moulin (1987) and Chun (1988).

Moreover, in the search for consistency, there is no reason why one should insist that an average of the proportional and constrained equal awards formulas be used for problems involving more than two claimants. So, which of our two-claimant rules have consistent extensions and what are these extensions? This is the question we ask next. Disappointingly, the answer is that in fact, within our family, only the proportional rule and the constrained equal awards rule have such extensions, and for consistency, one should apply the proportional formula for each population or the constrained equal awards formula for each population.

We also propose a way of compromising between the proportional rule and the "constrained equal losses rule", the rule that divides the endowment so as to make the losses incurred by all claimants, that is, the differences between their claims and their awards, as equal as possible subject to no one receiving a negative amount. This case can be handled by standard "duality" arguments, exploiting our earlier conclusions. Here, we find that, within this second family, only the two rules that serve as points of departure, the two-claimant proportional rule and the two-claimant constrained equal losses rule, have consistent extensions. These extensions are obtained by applying the proportional formula for each population or by applying the constrained equal losses formula for each population.

**Related literature.** In an earlier paper, we proposed to compromise between the two-claimant proportional and constrained equal awards rules by means of a different type of operation, namely by averaging, for each problem, parallel to the axis along which the larger claim is measured (Thomson, 2015b). (The only rules that are in common to the family defined here and that family are the proportional and constrained equal awards rules.) We found a non-trivial subfamily of the resulting rules that had consistent extensions.

Another possibility is to choose, for each claims vector, the path of awards to consist of the union of a segment contained in the 45° line and a segment to the claims vector (Thomson, 2007). Giménez-Gómez and Peris (2014)'s proposal is along the same lines. This formulation amounts to averaging, for each problem, the recommendations made by the proportional and constrained equal awards rules parallel to the 45° line. (Again, the only rules that are in common to our family and that family are the proportional and constrained equal awards rules.)

One can also ask about compromising in the two-claimant case between the constrained equal awards and constrained equal losses rules, and here too, different types of averaging have been considered. It turns out that averaging in the manner suggested here is incompatible with consistency unless either all the weight is always placed on the constrained equal awards rule or all the weight is always placed on the constrained equal losses rule (Thomson, 2007). On the other hand, averaging parallel to the 45° rule, or for each problem, parallel to the axis along which the larger claim is measured, lead to two subfamilies of families that had been introduced as generalizations of the Talmud rule, the ICI and CIC families (Thomson, 2008). These subfamilies are the TAL family and its dual (Moreno-Ternero and Villar, 2006a,b).

Other ways of compromising between rules have been defined that are not based on averaging operations. Following the axiomatic approach from the outset, Moulin (2000) obtains a family of two-claimant rules that can be understood as providing a compromise between the proportional, constrained equal awards, and constrained equal losses rules. Each member of the family requires that awards space be partitioned into cones and that within each cone, either the proportional rule or a "compressed" version of the constrained equal awards or constrained equal losses rules be used. These rules do not necessarily assign equal amounts to claimants with equal claims, which is desirable in some situations. The only rules that are common to our family and Moulin's family are the proportional and constrained equal awards rules. Extending Moulin's rules to more than two claimants in a consistent way however requires that the set of potential claimants be partitioned into ordered priority classes, and that within each class, one of the following be used, the proportional rule, a weighted constrained equal awards rule, or a weighted constrained equal losses rule. So, once again, and apart from the extra freedom gained by dropping the insistence on treating equals equally, the same rules emerge as the only viable candidates as the ones that come out of our analysis here.

Recent contributions to the literature on claims problems concern the impact of uncertainty in the data of the problem (Xue, 2015; Ertemel and Kumar, 2015), the possibility that these data be integers (Chen, 2015), and experimental testing (Cappelen, Luttens, Sorensen, and Tungodden, 2015).

## 2 The model and our two-claimant proposal

There is an infinite set of potential "claimants", indexed by the natural numbers, $\mathbb{N}$. Let $\mathcal{N}$ be the family of all finite subsets of $\mathbb{N}$ and $\mathcal{N}^2$ the subfamily of two-claimant subsets. A **claims problem with claimant set $\boldsymbol{N} \in \boldsymbol{\mathcal{N}}$** is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum c_i \geq E$. An **awards vector of $\boldsymbol{(c, E)}$** is a vector $x \in \mathbb{R}^N_+$ such that $\sum x_i = E$ and $x \leqq c$. The **deficit** in $(c, E)$ is the difference $\sum c_i - E$. Let $\boldsymbol{\mathcal{C}^N}$ be the class of all problems with claimant set $N$. A **rule on $\boldsymbol{\mathcal{C}^N}$** is a function that associates with each $(c, E) \in \mathcal{C}^N$ a unique awards vector of $(c, E)$. The **path of awards of $\boldsymbol{S}$ for $\boldsymbol{c} \in \mathbb{R}^{\boldsymbol{N}}_+$** is the locus of the awards vector $S$ selects for $(c, E)$ as $E$ varies from 0 to $\sum c_i$. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. For the **proportional rule, $\boldsymbol{P}$**, for each $i \in N$, claimant $i$'s award is $\lambda c_i$, $\lambda$ being chosen, as in the next two definitions, so that awards add up to $E$; for the **constrained equal awards rule, $\boldsymbol{CEA}$**, claimant $i$'s award is $\min\{c_i, \lambda\}$; for the **constrained equal losses rule, $\boldsymbol{CEL}$**, it is $\max\{c_i - \lambda, 0\}$. (Historical references are in O'Neill, 1982.)

Given $a \in \mathbb{R}^N_+$, **box$[\boldsymbol{0, a}]$** is the set $\{x \in \mathbb{R}^N_+ \colon 0 \leqq x \leqq a\}$. Given $a, b, c \in \mathbb{R}^N$, **seg$[\boldsymbol{a, b}]$** is the segment connecting $a$ and $b$, and to exclude $b$ for example, we use the notation **seg$[\boldsymbol{a, b}[$**. Also, **bro.seg$[\boldsymbol{a, b, c}]$** is the broken segment $\text{seg}[a, b] \cup \text{seg}[b, c]$. The simplex in $\mathbb{R}^N$ is denoted $\boldsymbol{\Delta^N}$.

We start with the two-claimant case. As a compromise between the proportional and constrained equal awards rules, we propose a weighted average, but we let the weights depend on the claims vector.[2] Let $N \in \mathcal{N}^2$. Let $i, j$ be the two members of $N$ and $\Lambda^N$ be the class of functions $\lambda^N \colon \mathbb{R}^N_+ \to [0, 1]$. Given $\lambda^N \in \Lambda^N$, and $(c, E) \in \mathcal{C}^N$, let $S^{\lambda^N}(c, E)$ select as awards vector for $(c, E)$ the average of $P(c, E)$ and $CEA(c, E)$ with weights $\lambda^N(c)$ and $1 - \lambda^N(c)$, namely

$$S^{\lambda^N}(c, E) \equiv \lambda^N(c)P(c, E) + (1 - \lambda^N(c))CEA(c, E).$$

Let $\boldsymbol{\mathcal{S}^N}$ be the family of rules so defined. The definition is illustrated in Figure 1a for $i = 1$ and $j = 2$. If for each $c \in \mathbb{R}^N_+$, $\lambda^N(c) = 1$, then $S = P$, and if for each $c \in \mathbb{R}^N_+$, $\lambda^N(c) = 0$, then $S = CEA$. To extend the definition to the domain $\bigcup_{N \in \mathcal{N}^2} \mathcal{C}^N$, we specify for each $N \in \mathcal{N}^2$ a function $\lambda^N \in \Lambda^N$.

Our superscript to $\lambda^N$ indicates that the identity of the two claimants may be taken into account in specifying the weights. This adds to the flexi-

[2] A standard convexity operator would use constant weights across problems.

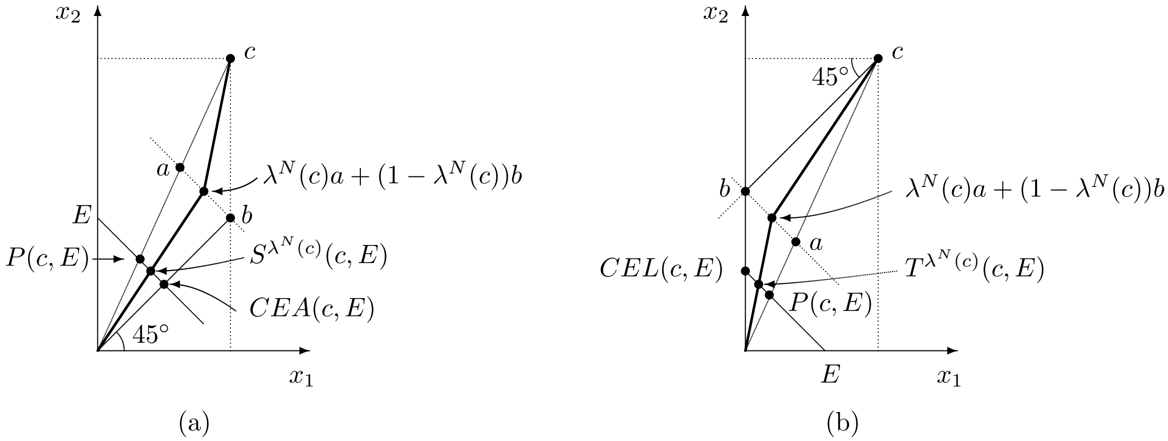


**Figure 1: Compromising between the proportional and constrained equal awards (or constrained equal losses) rules.** Here, $N \equiv \{1,2\}$ and $c \in \mathbb{R}^N_+$ is such that $c_1 < c_2$. (a) For the member of family $\mathcal{S}^N$ associated with $\lambda^N \in \Lambda^N$, the path of awards for $c$ is bro.seg$[(0,0), \lambda^N(c)a + (1-\lambda^N(c))b, c]$, where $a \equiv P(c, 2c_1)$ and $b \equiv CEA(c, 2c_1)$. (b) For the member of family $\mathcal{T}^N$ associated with $\lambda^N \in \Lambda^N$, the path for $c$ is bro.seg$[(0,0), \lambda^N(c)a + (1-\lambda^N(c))b, c]$, where $a \equiv P(c, c_2 - c_1)$ and $b \equiv CEL(c, c_2 - c_1)$.

bility of our definition. Whether a claimant is a corporation or a household for example can be taken into consideration. Altogether, we start with a collection of rules indexed by two-claimant populations. Let $\mathcal{S}^2$ be the family of lists $(S^N)_{N \in \mathcal{N}^2}$, where for each $N \in \mathcal{N}^2$, $S^N \in \mathcal{S}^N$.

We also offer a symmetric way of compromising between the proportional and constrained equal losses rules. Again, let $N \in \mathcal{N}^2$. Let $i$ and $j$ be the two members of $N$. For each $(c, E) \in \mathcal{C}^N$, let

$$T^{\lambda^N}(c,E) \equiv \lambda^N(c)P(c,E) + (1-\lambda^N(c))CEL(c,E).$$

Let $\boldsymbol{\mathcal{T}^N}$ be the family of rules so defined. If, for each $c \in \mathbb{R}^N_+$, $\lambda^N(c) = 1$, then $T^\lambda = P$; also, if for each $c \in \mathbb{R}^N_+$, $\lambda^N(c) = 0$, then $T^\lambda = CEL$. Let $\mathcal{T}^2$ be the family of lists $(T^N)_{N \in \mathcal{N}^2}$, where for each $N \in \mathcal{N}^2$, $T^N \in \mathcal{T}^N$.

This second definition can be derived from our first definition through duality (Aumann and Maschler, 1985). **Two problems are dual** if they have the same claims vector but the endowment in one is equal to the deficit in the other. The dual of a rule $S$ is the rule that, for each problem, divides the endowment in the manner in which $S$ divides the deficit in the dual problem:

**Dual of rule $\boldsymbol{S}$, $\boldsymbol{S^d}$:** For each $(c, E) \in \mathcal{C}^N$,

$$S^d(c,E) \equiv c - S(c, \sum c_i - E).$$

It is easy to see that, for each $\lambda^N \in \Lambda^N$, $S^{\lambda^N}$ and $T^{\lambda^N}$ are dual.

The concept of duality also applies to properties of rules: two **properties are dual** if whenever a rule satisfies one of them, the dual rule satisfies the other.

# 3 Properties

Although the rules that we proposed have geometrically simple paths of awards, the family they constitute is rather large because our definition includes no restriction on the weights placed on the proportional and constrained equal awards rules. The question of how to specify the weights can be answered in two ways. A first answer is that it is up to the user of the theory to choose them so as to express his or her relative preference for one or the other of the proportionality and equality principles. The other is to proceed axiomatically and to identify the restrictions implied by properties of rules that are found desirable. This is the approach we follow in this section. Given a property, we ask which of the rules in $\mathcal{S}^2$ satisfy it. For some properties, they all do; for others, none does; for some, it depends on the weights; for each property in that last category, we identify the subset of rules in $\mathcal{S}^2$ that do satisfy it. (Some properties apply non-trivially only when the number of claimants is greater than 2; then, there is nothing to say about $\mathcal{S}^2$ concerning them.)[3] We only state as formal theorems our conclusions pertaining to two properties for which the class of admissible rules is particularly complex.

For each property except for *anonymity*, defined soon, a rule in $\mathcal{S}^2$ satisfies the property if and only if for each $N \in \mathcal{N}^2$, the component of the rule pertaining to population $N$ satisfies it. Thus, it is enough to understand the issue for some $N \in \mathcal{N}^2$. For simplicity of notation, we then choose $N \equiv \{1, 2\}$.

• **Equal treatment of equals** says that two agents with equal claims should

[3]The terminology concerning properties of rules is not uniform in the literature. In each of the following pairs of terms or expressions, the first one is the one we use here whereas the second one is used in the paper where the concept is proposed: the "$\frac{1}{2}$-truncated claims lower bound" versus "securement", "order preservation under endowment variation" versus "super-modularity", "homogeneity" versus "scale invariance", "minimal rights first" versus "composition from minimal rights", "composition down" versus "path independence", and "composition up" versus "composition".

be assigned equal awards. It follows directly from their definition that all rules in $\mathcal{S}^N$ satisfy the property.

• **Anonymity** says that an exchange of the names of two agents in a problem should be accompanied by an exchange of their awards. Let $N \in \mathcal{N}^2$ and $\lambda^N \in \Lambda^N$. Obviously, $S^{\lambda^N}$ is *anonymous* if and only if $\lambda^N$ itself is invariant under exchanges of the names of the members of $N$.

For rules in $\mathcal{S}^2$, *anonymity* also means invariance with respect to the replacement of agents by others. A rule in $\mathcal{S}^2$—let $(\lambda^N)_{N \in \mathcal{N}^2}$ be the list of functions with which it is associated—is *anonymous* if this list satisfies the following requirement. Let $N, N' \in \mathcal{N}^2$ and $r\colon N \to N'$ be a bijection. Let $c \in \mathbb{R}^N_+$ and $c' \in \mathbb{R}^{N'}_+$ be such that, abusing notation slightly, $r(c) = c'$. Then, it should be the case that $\lambda^{N'}(c') = r(\lambda^N(c))$.

• **Order preservation** (Aumann and Maschler, 1985) says that awards should be ordered as claims are (*order preservation in awards*), and that so should losses (*order preservation in losses*). We return to $N \equiv \{1, 2\}$, and do so until Section 4. For simplicity, let $c \in \mathbb{R}^N_+$ be such that $c_1 < c_2$. A rule *preserves order* if its path of awards for $c$ lies on or above the 45° line and on or below the line of slope 1 passing through $c$. It follows directly from their definition that all rules in $\mathcal{S}^N$ *preserve order*.

• **The $\frac{1}{|N|}$-truncated claims lower bound** (Moreno-Ternero and Villar, 2004) says that each claimant should be assigned at least the fraction $\frac{1}{|N|}$ of his claim truncated at the endowment. The property implies that for each $c \in \mathbb{R}^N_+$, the path of awards for $c$ should include a segment of slope 1 emanating from the origin. The constrained equal awards rule is the only rule in $\mathcal{S}^N$ to have that feature, and it passes this test.

• **Conditional full compensation** (Herrero and Villar, 2002) says that if an agent's claim is such that by substituting it for the claim of each agent whose claim is greater, there is now enough to compensate everyone, the claimant should be fully compensated. Let $c \in \mathbb{R}^N_+$. The property implies that the path of awards for $c$ should contain the segment from the greatest point of equal coordinates that is dominated by $c$ to $c$. The constrained equal awards rule is the only rule in $\mathcal{S}^N$ to pass this test.

• **Endowment monotonicity** says that if the endowment increases, each claimant's award should be at least as large as it was initially: paths of awards should be monotone curves. It follows directly from their definition that all rules in $\mathcal{S}^2$ are *endowment monotone*.

• **Endowment continuity, claims continuity**, and **full continuity** say that (a) for each claims vector, small changes in the endowment should not lead to large changes in the chosen awards vector; (b) for each endowment, small changes in the claims vector should not lead to large changes in the chosen awards vector; and (c) small changes in the data of a problem should not lead to large changes in the chosen awards vector.

It follows directly from their definition that all rules in $\mathcal{S}^N$ are *endowment continuous*.

Let $\lambda^N \in \Lambda^N$. Then, $S^{\lambda^N}$ is *claims continuous*, or *fully continuous*, if and only if $\lambda^N$ is continuous.

• **Order preservation under endowment variation** (Dagan, Serrano, and Volij, 1997) says that as the endowment increases, changes in awards should be ordered as claims are. Let $c \in \mathbb{R}^N_+$ be such that $c_1 < c_2$. This means that the slope of the path of awards for $c$ (when defined) should be at least 1. All rules in $\mathcal{S}^N$ trivially satisfy the property.

• **Claims monotonicity** says that if an agent's claim increases, his award should be at least as large as it was initially. Let $c, c' \in \mathbb{R}^N_+$ be such that $c_2 = c'_2$ and $c_1 < c'_1$. Then, the path of awards for $c'$ should lie to the southeast of the path for $c$, a parallel statement holding when agent 1's claim is held fixed and agent 2's claim increases.

The following example shows that a rule in $\mathcal{S}^N$ is not necessarily *claims monotonic*:

**Example 1** *Let $c \equiv (1, 4)$ and $c' \equiv (2, 4)$. Let $\lambda^N \in \Lambda^N$ be such that $\lambda^N(c) = 0$ (thus, the path of awards of $S^{\lambda^N}$ for $c$ is that of the constrained equal awards rule), and $\lambda^N(c') = 1$ (thus, its path of awards for $c'$ is that of the proportional rule). Let $E \equiv 2$. Then, $S^{\lambda^N}(c, E) = (1, 1)$ and $S^{\lambda^N}(c', E) = (\frac{2}{3}, \frac{4}{3})$: as agent 1's claim increases from 1 to 2, his award decreases from $1$ to $\frac{2}{3}$. (The definition of $S^{\lambda^N}$ can easily be completed so that $S^{\lambda^N}$ is continuous.)*

Let us see what it takes for a rule in $\mathcal{S}^N$ to be *claims monotonic*. Let $c_2 \in \mathbb{R}_+$. Let $C(c_2)$ denote the locus of the point of intersection of the line of slope $-1$ passing through $(c_1, c_1)$ with seg$[(0,0), (c_1, c_2)]$, as $c_1$ varies in $[0, c_2]$ (Figure 2a). Let $C(c_1)$ be defined in a parallel way.

**Theorem 1** *Let $N \equiv \{1, 2\}$. (a) A rule $S$ in $\mathcal{S}^N$ is claims monotonic if*

*(i) there is a function $f\colon \mathbb{R}^2_+ \to \mathbb{R}_+$ such that for each $c_2 \in \mathbb{R}_+$, the locus $K(c_2)$ of the point $k(c_1, c_2) \equiv (c_1, c_1) + f(c_1, c_2)(-1, 1)$ as $c_1$ varies in $[0, c_2]$*

*lies on or below $C(c_2)$ and on or above the 45° line, and is such that for each pair $c_1, c'_1$ with $c_1 < c'_1$, the slope of* $\text{seg}[(0,0), k(c_1, c_2)]$ *is at least as large as the slope of* $\text{seg}[(0,0), k(c'_1, c_2)]$ *(Figure 2b), and*

*(ii) for each pair $c_2, c'_2 \in \mathbb{R}_+$ with $0 < c_2 < c'_2$, and for each $c_1 \leq c_2$, $f(c_1, c'_2) \geq f(c_1, c_2)$ (Figure 3a).*

*Now, given $c \in \mathbb{R}^N_+$ with $c_1 \leq c_2$, the path of $S$ for $c$ is* $\text{bro.seg}[(0,0), k(c_1, c_2), c]$.

*(i') and (ii') are two statements parallel to (i) and (ii), obtained by exchanging the roles of the two coordinates.*

*Given $c \in \mathbb{R}^N_+$ with $c_1 \geq c_2$, the path of $S$ for $c$ is defined in a symmetric way.*

*(b) If the rule is* claims continuous*, for each $c_2 \in \mathbb{R}_+$, the function $f(., c_2)$ is continuous.*

For an intuitive description of the result, it will help to introduce one more concept. We say that a curve is **visible from below from the origin** if for an observer standing at the origin, and thinking of the curve as opaque, no part of it would hide any other part of it; also, the segment from the origin to each point in the curve lies on or below the curve. The limit case, when the curve contains non-degenerate segments lined up with the origin, is allowed. For a curve that is **strictly visible from below from the origin**, there is no such segment. The slope requirement on $k$ simply means that for each $c_2 \in \mathbb{R}_+$, the locus $K(c_2)$ is visible from below from the origin.

**Proof:** (a) Statement (i) is necessary and sufficient for the path for $(c'_1, c_2)$ to indeed lie to the southeast of the path for $(c_1, c_2)$ if $c'_1 > c_1$. Statement (ii) guarantees that for each $c_1 \in \mathbb{R}_+$ and each pair $c_2, c'_2 \in \mathbb{R}_+$ with $c_1 \leq c_2 < c'_2$, agent 2's award when his claim is $c'_2$ is at least as large as his award when his claim is $c_2$.

(b) We omit the straightforward proof. □

For the proportional rule, for each $c_2 \geq 0$, $K(c_2) = C(c_2)$ (visibility is strict) and for the constrained equal awards rule, $K(c_2) = \text{seg}[(0,0), (c_2, c_2)]$ (visibility is nowhere strict). In Figure 2b, $\text{seg}[p, q]$ illustrates the fact that $K(c_2)$ need not be strictly visible with a second example. For an example violating *claims continuity*, consider the function $f$ such that the graph of the resulting locus $K(c_2)$ consists of the subset of $C(c_2)$ that has as endpoints the origin and some arbitrary point $a$, excluding $a$, and $\text{seg}[(\frac{a_1+a_2}{2}, \frac{a_1+a_2}{2}), (c_2, c_2)]$.

• **Homogeneity** says that for each problem and each $\alpha > 0$, multiplying the data defining the problem by $\alpha$ results in a new problem whose chosen

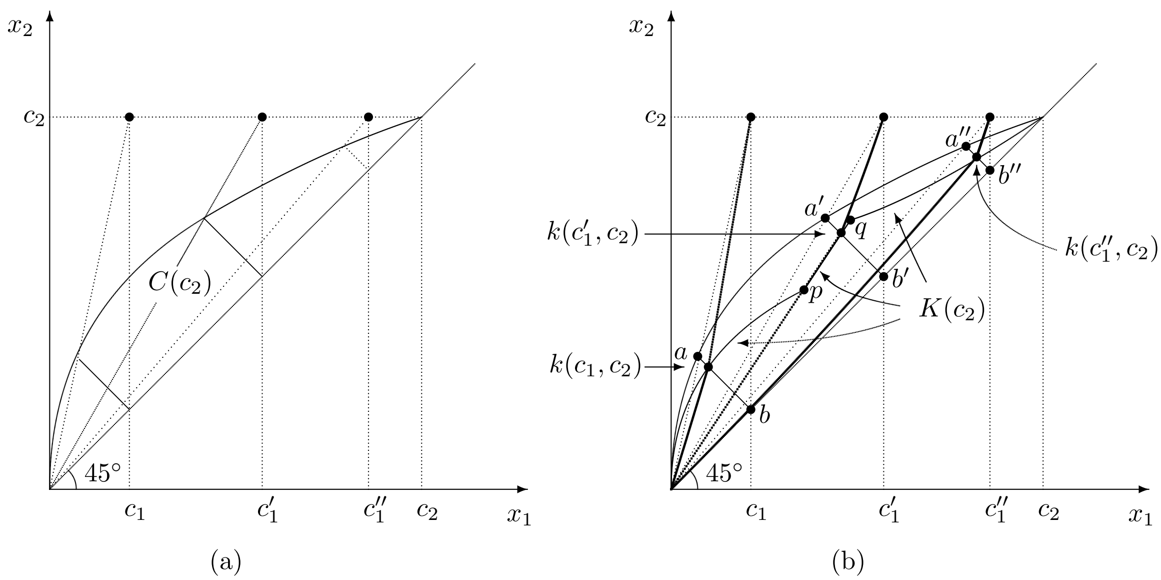


**Figure 2: Identifying the two-claimant rules that are claims monotonic**. Here, $N \equiv \{1,2\}$. Fix $c_2 > 0$. (a) The curve $C(c_2)$ is an upper boundary for the locus $K(c_2)$, a set that contains all the kinks of the paths of awards of $S$ for $(c_1, c_2)$ as $c_1$ varies in $[0, c_2]$. A lower boundary is seg$[(0,0),(c_2,c_2)]$. The locus $K(c_2)$ should be visible from below from the origin. (b) This panel shows a typical locus $K(c_2)$. It has three parts, a strictly concave part, from the origin to $p$, a segment lined up with the origin, seg$[p,q]$, and a slightly convex section, from $q$ to $(c_2, c_2)$. It also shows the paths of awards for three claims vectors, $(c_1, c_2)$, $(c_1', c_2)$, and $(c_1'', c_2)$.

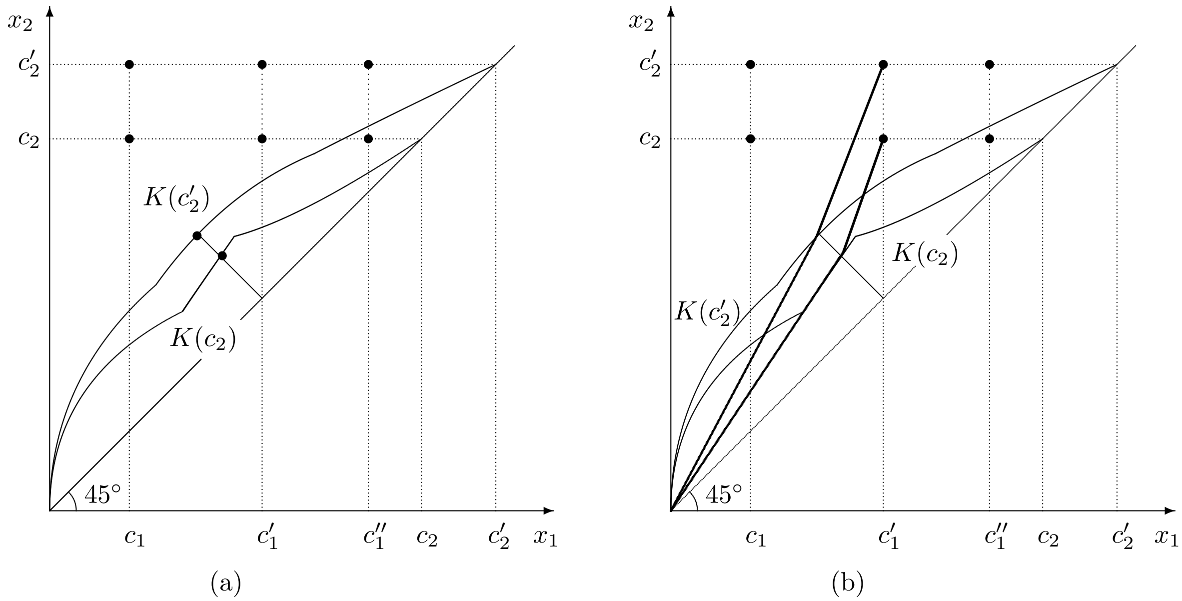


**Figure 3: Identifying the two-claimant rules that are claims monotonic** (Part 2). Here, $N \equiv \{1, 2\}$. (a) Given $c_2, c_2'$ with $c_2 < c_2'$, this panel shows possible shapes for $K(c_2)$ and $K(c_2')$. For *claims monotonicity* to hold when agent 2's claim increases from $c_2$ to $c_2'$, $K(c_2')$ should lie to the northwest of $K(c_2)$, as it does here. (b) This panel shows the paths of awards for $(c_1', c_2)$ and $(c_1', c_2')$.

awards vector should be obtained by multiplying the chosen awards vector of the initial problem by $\alpha$.

Let $\lambda^N \in \Lambda^N$. For $S^{\lambda^N}$ to be *homogeneous*, it is necessary and sufficient that for each $\alpha > 0$, its path for $\alpha c$ be obtained by subjecting its path for $c$ to a scale expansion of factor $\alpha$. This holds if and only if $\lambda^N(\alpha c) = \lambda^N(c)$.

• **Claims truncation invariance** (Dagan and Volij, 1993) says that for each problem, replacing each claim that is greater than the endowment by the endowment should not affect the chosen awards vector.

*Equal treatment of equals*, which each rule $S \in \mathcal{S}^N$ satisfies (see above), and *claims truncation invariance* together imply that for each $c \in \mathbb{R}^N_+$,

(†) the path of awards of $S$ for $c$ contains seg$[0, (\frac{\min_i c_i}{2}, \frac{\min_i c_i}{2})]$.[4]

Indeed, for an endowment that is no greater than the smaller claim, both truncated claims are equal, and by *equal treatment of equals*, awards are equal then. When $c_1, c_2 > 0$ and $c_1 \neq c_2$, the constrained equal awards rule is the only rule in $\mathcal{S}^N$ satisfying (†). Moreover, this rule is *claims truncation invariant.* Thus, it is the only rule in $\mathcal{S}^N$ that is *claims truncation invariant.*

[4]This is only a necessary condition. For necessary and sufficient conditions, see Thomson (2006).

• **Minimal rights first** (Curiel, Maschler, and Tijs, 1987) says that we should be able to solve each problem in either one of the following two ways: (a) directly; (b) in two steps, by first assigning to each claimant his "minimal right", namely the difference between the endowment and the sum of the claims of the other claimants, or 0 if this difference is negative, and then, after having revised all claims down by the awards of the first step, applying the rule to divide what remains of the endowment. This property is the dual of *claims truncation invariance*, so its analysis can be easily derived from our previous analysis of that property.

The issue can also be addressed directly. If a rule satisfies *equal treatment of equals* and *minimal rights first*, then for each $c \in \mathbb{R}^N_+$, its path of awards for $c$ contains the segment [seg$[(c_i - \frac{\min_k c_k}{2}, c_j - \frac{\min_k c_k}{2}), c]$. When $0 < c_1, c_2$ and $c_1 \neq c_2$, this segment is non-degenerate and this inclusion never holds for a rule in $\mathcal{S}^N$.

• **Composition down** (Moulin, 2000) says that if the endowment decreases, we should be able to obtain the new awards vector in either one of the following two ways: (a) directly, that is, ignoring the initial awards vector; or (b) using the initial awards vector as claims vector.

An ingredient in the characterization of the subfamily of $\mathcal{S}^N$ of rules satisfying *composition down* is the following characterization of the entire family of rules satisfying *composition down* (Lemma 1).

Consider a network of

(a) continuous and weakly monotone curves in $\mathbb{R}^N_+$ emanating from the origin and such that,

(b) given any point in $\mathbb{R}^N_+$, there is a curve in the family passing through it;

(c) following any one of these curves up from the origin, if we encounter a point at which the curve splits into branches, these branches never meet again.

A family of curves satisfying these three conditions constitute a **weakly monotone** (from (a)), **space-filling** (from (b)) **tree** (from c).

We state the following lemma for an arbitrary population $N \in \mathcal{N}$ but we will need it only in the two-claimant case.

**Lemma 1** (Thomson, 2006) *Let $N \in \mathcal{N}$.*

*(a) A rule on $\mathcal{C}^N$ satisfies* composition down *if and only if there is a weakly monotone space-filling tree in $\mathbb{R}^N_+$ such that, for each $c \in \mathbb{R}^N_+$, the*

*path of awards of the rule for c is obtained by identifying a branch emanating from the origin and passing through c, and taking the part of it that lies in* box$[0, c]$.

*(b) If the rule is* claims continuous, *all branches of the tree with which it is associated are unbounded.*

We return to $N \equiv \{1, 2\}$. Figure 4a shows a few branches of such a tree and a claims vector $c \in \mathbb{R}^N_+$ that belongs to more than one branch. There is more than one branch because of a split above $c$ (at $c'$), but all branches passing through $c$ coincide in box$[0, c]$. Given any such branch, the path for $c$ is the part of it that lies in the box.

Figures 4b illustrates what is required for a rule in $\mathcal{S}^N$ to satisfy *composition down.*

Consider a continuous, weakly monotone, and unbounded curve $C^1$ below the 45° line that is visible from below from the origin and let $\boldsymbol{\mathcal{C}^1}$ be the class of all such curves. Let $\boldsymbol{\mathcal{C}^2}$ be the class of curves defined in a parallel manner to the way we defined $\mathcal{C}^1$, by exchanging the roles played by the two coordinates.

Our next result is that a rule satisfying *composition down* is entirely specified once a pair $C^1 \in \mathcal{C}^1$, $C^2 \in \mathcal{C}^2$ is given, together with, for each $i \in N$, a partition $D^i = \{D^{iP}, D^{iCEA}\}$ of $C^i$ satisfying certain properties. The statement of the theorem includes instructions on how to derive from these objects the tree with which the rule is associated. Informally, each curve $C^i$ serves as a boundary between a region of awards space in which branches of the tree are those of the proportional rule and a region in which branches are those of the constrained equal awards rule. Thus, for each given claims vector, there is actually no compromising between the recommendations made by the proportional and constrained equal awards rules. It is only across claims vectors that compromising occurs. The partition $\{D^{iP}, D^{iCEA}\}$ specifies, for each point of $C^i$, whether it should be considered as a point of the first region or as a point of the second region. For a curve $C^i$ that is strictly visible from the origin, there are no restrictions on its partition; it can be chosen in some arbitrary manner.

**Theorem 2** *Let $N \equiv \{1, 2\}$. (a) A rule in $\mathcal{S}^N$ satisfies* composition down *if and only if there are $C^1 \in \mathcal{C}^1$ and $C^2 \in \mathcal{C}^2$ and for each $i \in N$, a partition $D^i = \{D^{iP}, D^{iCEA}\}$ of $C^i$ satisfying the following requirements:*

*Let $x \in C^1$. If $x$ does not belong to a non-degenerate segment in $C^1$ lined up with the origin or parallel to the first axis, $x$ can be assigned to either $D^{1P}$*

*or $D^{1CEA}$. If $x$ belongs to a non-degenerate segment in $C^1$ that is lined up with the origin, and it is assigned to $D^{1P}$, then so should all other points of the segment that are closer to the origin. If $x$ belongs to a non-degenerate segment contained in $C^1$ that is parallel to the first axis, and it is assigned to $D^{1CEA}$, then so should all other points of the segment whose first coordinate is smaller.*

*A symmetric statement holds for the curve $C^2$.*

*Now, the rule is obtained in the manner described in Lemma 1 from the weakly monotone space-filling tree whose branches are the following:*

*(i) For each $a \geq 0$, (ia) a branch consisting of $\text{seg}[(0,0),(a,a)]$ together with the half-line $\{x \in \mathbb{R}^2_+ : \text{ for some } t \geq 0, x = (a,a) + t(1,0)\}$, if this half-line does not intersect $C^1$; otherwise, when the intersection is non-empty—it is a (perhaps degenerate) segment $s$—and in $s$, there is either a point $\alpha$ of maximal first coordinate that belongs to $D^{1CEA}$ or a point $\beta$ of minimal first coordinate that belongs to $D^{1P}$; then (ib) in the first case, a branch consisting of $\text{seg}[(0,0),(a,a)]$ and $\text{seg}[(a,a),\alpha]$ and in the second case, a branch consisting of $\text{seg}[(0,0),(a,a)]$ and $\text{seg}[(a,a),\beta[$.*

*(ii) For each $r \in \Delta^N$ such that the ray in direction $r$ is on or below $C^1$, (iia) this ray is a branch if it only intersects $C^1$ at the origin; otherwise, the ray intersects $C^1$ at more than one point—the intersection is a (perhaps degenerate) segment $s$—and in $s$, there is either a point $\alpha$ of maximal first coordinate that belongs to $D^{1P}$ or a point $\beta$ of minimal first coordinate that belongs to $D^{1CEA}$; then (iib) in the first case, $\text{seg}[(0,0),\alpha]$ is a branch, and in the second case, $\text{seg}[(0,0),\beta[$ is a branch.*

*(iii) All branches obtained from $C^2$ by exchanging the roles played by the two coordinates should be included.*

*(b) If in addition, the rule is* claims continuous, *then in each of the regions into which the non-negative quadrant is partitioned by the 45° line, it coincides with either the proportional rule or with the constrained equal awards rule.*

Without the visibility properties (i) and (ii) imposed on $C^1$ and $C^2$, the "treeness" of the family of curves generated by following the instructions of Theorem 2 would be violated, which in turn would lead to a violation of *composition down*. It is easy to see that in the two-claimant case, *composition down* implies *claims monotonicity* (Thomson, 2006). Example 1 illustrates a violation of the latter property and because of this logical relation, it illustrates a violation of the former as well. To see this, note that the paths of

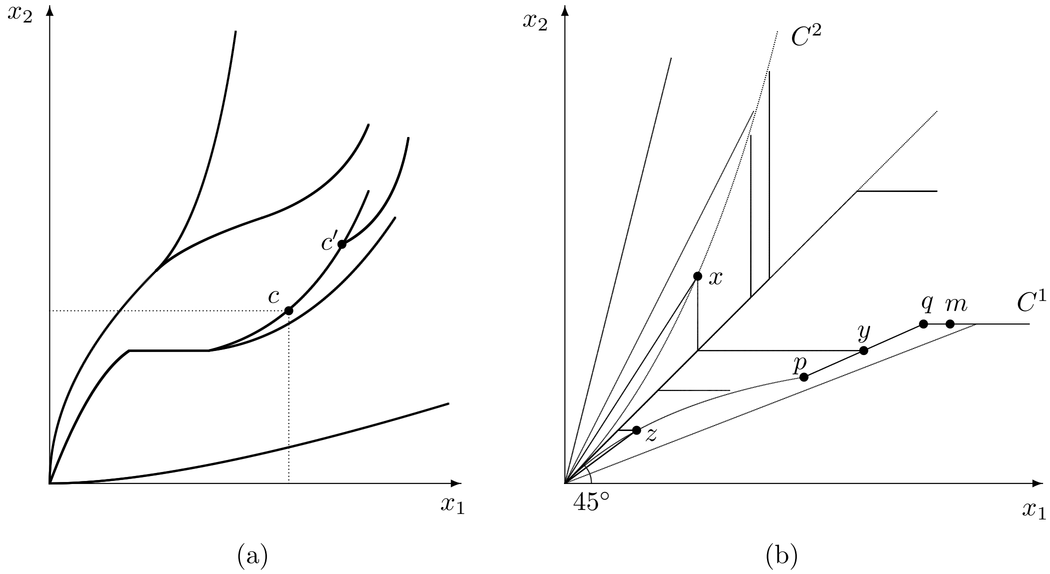


**Figure 4: Composition down.** (a) Illustrating Lemma 1: this panel shows a typical tree from which the paths of awards of a rule satisfying *claims continuity* and *composition down* are generated. (b) Illustrating Theorem 2. For a rule in $\mathcal{S}^N$ satisfying *composition down*, there are two curves $C^1 \in \mathcal{C}^1$ and $C^2 \in \mathcal{C}^2$ from which a tree is constructed with which the rule is associated. Here, $C^1$ is only weakly visible from the origin because it contains a non-degenerate segment, seg$[p, q]$, that is lined up with the origin; on the other hand, $C^2$ is strictly visible.

awards for $c$ and $c'$ defined there cross at $\tilde{c} \equiv (1, 2)$: $S(c, 2) = S(c', 2) = \tilde{c}$. Thus, if $S$ satisfied *composition down*, we would have that for each $E \leq 2$, $S(c, E) = S(S(c, 2), E) = S(\tilde{c}, E) = S(S(c', 2), E) = S(c', E)$: for endowments no greater than 2, the paths of $S$ for $c$ and $c'$ should coincide, but they do not.

Again, because for two claimants a rule satisfying *composition down* also satisfies *claims monotonicity*, from the curves $C^1$ and $C^2$ and their partitions $D^1$ and $D^2$ associated with the rule as explained in Theorem 2, one should be able to recover the loci $K(c_1)$ for $c_1 > 0$ and $K(c_2)$ for $c_2 > 0$ associated with it as explained in Theorem 1. It is easy to see that for each $c_2 > 0$, $K(c_2)$ is just like the example described at the end of the paragraph following the proof of Theorem 1. It consists of a subset of the locus $K(c_2) = C(c_2)$ of the proportional rule, from the origin to some point $a$, together with a subset of the locus $K(c_2) = \text{seg}[(0, 0), (c_2, c_2)]$ of the constrained equal awards rule, from a point $b$ whose coordinates add up to those of $a$, to $(c_2, c_2)$ itself, $a$ being included in the first component of this union if and only if $b$ is not included in the second component of the union. For $c'_2 > c_2$, two points $a'$ and $b'$ can be identified that play a role analogous to that played by the points $a$ and $b$, but the coordinates of $a'$ add up to at least the coordinates of $a$, as they should for *claims monotonicity* with respect to agent 2's claim to hold. The loci $K(c_1)$ for $c_1 > 0$ are determined in a symmetric way.

For a rule in $\mathcal{S}^N$ to be *anonymous* in addition to satisfying *composition down*, $C^1$ and $C^2$ should be symmetric of each other with respect to the 45° line and so should their partitions $D^1$ and $D^2$. For a rule in $\mathcal{S}^2$ to satisfy these two properties, the same pair of symmetric curves and symmetric partitions of these curves should be used in $\mathbb{R}^N_+$ for each $N \in \mathcal{N}^2$.

The statement of Theorem 2 is tedious, but the example of Figure 4b should convey the intuition. The curve $C^2$ is strictly visible from the origin, and no vertical line intersects it at more than one point, so there are no restrictions on how it is partitioned. On the other hand, the curve $C^1$ contains a non-degenerate segment lined up with the origin, seg$[p, q]$, and a half-line parallel to the first axis, $L \equiv \{x \in \mathbb{R}^2_+ : \text{ for some } t \in \mathbb{R}_+, x = q + t(1, 0)\}$. As a result, if the path for some $y \in$ seg$[p, q]$ is chosen to be the path of $P$, that is, if $y \in D^{1P}$, so should the path for each claims vector between $p$ and $y$. Similarly, if the path for some $m \in L$ is chosen to be the path of $CEA$, that is, if $m \in D^{1CEA}$, so should the path for each claims vector between $q$ and $m$.

**Proof:** (of Theorem 2) Let $S$ be a rule satisfying *composition down*.

**Step 1:** There is no $c \in \mathbb{R}^N_+$ such that the path of $S$ for $c$ has a kink unless it is the path of $CEA$.

Suppose otherwise: there is $c \in \mathbb{R}^N_+$ such that the path of $S$ for $c$ has a kink—let us call it $a$. Since $S$ satisfies *equal treatment of equals*, $c_1 \neq c_2$. Without loss of generality, suppose $0 < c_1 < c_2$. Then seg$[a, c]$ is non degenerate. Let $x$ be a point of its relative interior. Note that $x < c$. By *composition down*, the path for $x$ is bro.seg$[(0, 0), a, x]$. This path has a kink, namely $a$, whose coordinates add to twice the smallest coordinate of $c$, namely $c_1$. By the definition of the family $\mathcal{S}^N$, if the path for $x$ has a kink, the sum of its coordinates is twice the smallest coordinate of $x$, namely $x_1 < a_1$. This contradiction concludes the proof of Step 1.

**Step 2:** Below the 45° line, award space is partitioned into two regions bounded by a curve $C^1 \in \mathcal{C}^1$. For each $c$ above $C^1$, the path of awards of $S$ is that of the constrained equal awards rule. For each $c$ below $C^1$, the path is that of the proportional rule.

To see this, let $c$ below the 45° line be such that the path for $c$ is that of $CEA$. Then, for each $c'$ in the cone spanned by the path for $c$ such that $c'_2 \geq c_2$, the path for $c'$ is that of $CEA$. Indeed, if it were that of $P$, the paths for $c$ and $c'$ would have a non-trivial intersection (a cycle would be created). Now, for each $c_2 > 0$, consider the set $V(c_2)$ of values of $c_1$ for which the path for $(c_1, c_2)$ is that of $CEA$ and for each $c_2$, let $v(c_2)$ be the supremum of $V(c_2)$. It is easy to check that the graph of $v$ is a curve in $\mathcal{C}^1$.

A curve $C^2$ can be constructed in the same manner above the 45° line, with symmetric properties.

(b) If for some $c_2 > 0$, $0 < v(c_2) < \infty$, the tree associated with $S$ has a finite branch, and therefore, by Lemma 1, $S$ is not *claims continuous*. This implies that to obtain this property, the rule should coincide above the 45° line with either the proportional rule, or with the constrained equal awards rule. The same statement holds below the 45° line. These choices can be made independently above and below the 45° line, but if *anonymity* is imposed as well, they have to match; we end up with either the proportional rule or the constrained equal awards rule. □

• **Composition up** (Young, 1988) says that if the endowment increases, we should be able to obtain the new awards vector in either one of the following two ways: (a) directly, that is, ignoring the initial awards vector; (b) first assigning the initial awards, revising claims down by these awards, then adding to them the awards vector that results by applying the rule to the problem in which claims are these revised amounts, and the amount to divide is the increment in the endowment. The conclusions concerning this property are obtained by duality from the ones we just reached for *composition down*. (*Composition up* is the dual of *composition down*.)

The only rules in $\mathcal{S}^N$ for $N \in \mathcal{N}^2$ to satisfy the property are obtained by duality from the ones identified for *composition down*.

• **Lorenz comparisons** (Hougaard and Thorlund-Peterson, 2001; Moreno-Ternero and Villar, 2006b; Bosmans and Lauwers, 2011; Thomson, 2012a) Next, we ask when rules in $\mathcal{S}^N$ can be Lorenz ranked. Informally speaking, rule $S$ Lorenz dominates rule $S'$ if for each problem, its distribution of awards is more in favor of agents who receive the least (under *order preservation of awards*, these are the agents with the smallest claims). Formally, ***S* Lorenz dominates *S′*** if for each problem $(c, E) \in \mathcal{C}^N$ and, letting $x \equiv S(c, E)$ and $x' \equiv S'(c, E)$ and calling $\tilde{x}$ and $\tilde{x}'$ the vectors obtained from $x$ and $x'$ by rewriting their coordinates in increasing order, we have $\tilde{x}_1 \geq \tilde{x}'_1$, $\tilde{x}_1 + \tilde{x}_2 \geq \tilde{x}'_1 + \tilde{x}'_2$, and so on.

Let $N \in \mathcal{N}^2$. Given two rules $S$ and $S'$ defined on $\mathcal{C}^N$, $S$ Lorenz dominates $S'$ simply if for each $c \in \mathbb{R}^N_+$, the path of awards of $S$ for $c$ is everywhere at least as close to the 45° line as the path of awards of $S'$ for $c$. Let $\lambda^N$ and

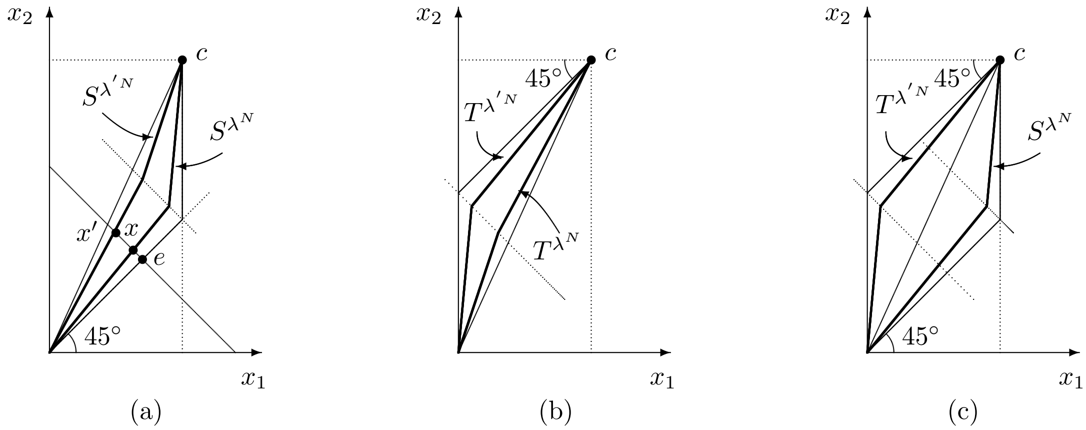


**Figure 5: Lorenz domination between members of $\mathcal{S}^N$ and $\mathcal{T}^N$.** Here, $N \equiv \{1, 2\}$. (a) If $\lambda^N(c) < \lambda'^N(c)$, the rule $S^{\lambda^N}$ Lorenz dominates the rule $S^{\lambda'^N}$ for all problems with claims vector $c$. (b) If $\lambda'^N(c) < \lambda^N(c)$, the rule $T^{\lambda^N}$ Lorenz dominates the rule $T^{\lambda'^N}$ for all problems with claims vector $c$. (c) Each rule in $\mathcal{S}^N$ Lorenz dominates each rule in $\mathcal{T}^N$. This is true no matter what $c$ is.

$\lambda'^N \in \Lambda^N$. In Figure 5a, $x \equiv S^{\lambda^N}(c, E)$ is closer to the point of equal coordinates on the line of equation $t_1 + t_2 = E$ (the point $e$) than $x' \equiv S^{\lambda'^N}(c, E)$.

1. For each $\lambda^N$ and $\lambda'^N \in \Lambda^N$, it is always true that for each particular $c \in \mathbb{R}^N_+$, either for each $E \in [0, \sum c_i]$, $S^{\lambda^N}(c, E)$ Lorenz dominates $S^{\lambda'^N}(c, E)$, or for each $E \in [0, \sum c_i]$, $S^{\lambda'^N}(c, E)$ Lorenz dominates $S^{\lambda^N}(c, E)$. As $c$ varies, the domination could be reversed. However, given $\lambda^N, \lambda^{N'} \in \Lambda^N$, $S^{\lambda^N}$ Lorenz dominates $S^{\lambda'^N}$ if and only if $\lambda^N \leq \lambda'^N$ (Figure 5a).

2. By a similar argument, $T^{\lambda^N}$ Lorenz dominates $T^{\lambda'^N}$ if and only if $\lambda^N \geq \lambda'^N$ (Figure 5b).

3. It is also easily seen by inspection that each rule in $\mathcal{S}^N$ Lorenz dominates each rule in $\mathcal{T}^N$ (Figure 5c).

# 4 Extending the definition from two claimants to arbitrarily many claimants

The rules defined in the previous section are two-claimant rules and the question arises as to how to extend them to more claimants. Thus, we are now looking for rules defined on $\mathcal{C} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{C}^N$. We proceed axiomatically and invoke a principle that has played a fundamental role in addressing this type of issues in a great variety of literatures. Starting from our two-claimant definition, we require its extension to general populations to pass the following test: for each problem and each subpopulation of the claimants it involves, consider the problem faced by this subpopulation in which the endowment is the sum of the amounts that have been awarded to them (equivalently, the difference between the endowment in the initial problem and the sum of the awards to the members of the complementary subpopulation). We refer to it as the "reduced problem relative to the subpopulation and the awards vector chosen for the initial problem". We require that in this reduced problem, the rule assign to each claimant the same amount as it did in the initial problem:[5]

**Consistency:** For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $N' \subset N$, if $x \equiv S(c, E)$, then $x_{N'} = S(c_{N'}, \sum_{N'} x_i)$.

Our next theorem says that the only rules in $\mathcal{S}^2$ that have *consistent* extensions to $\mathcal{C}$ are the proportional and constrained equal awards rules. These extensions are the variable-population versions of these rules.

**Theorem 3** *The only rules on $\mathcal{C}$ that coincide, for each two-claimant population $N \in \mathcal{N}$, with a member of $\mathcal{S}^N$, and are* consistent, *are the proportional rule and the constrained equal awards rule.*

The proof exploits the fact that *consistency* of a rule implies that for each population of claimants $N \in \mathcal{N}$ and each claims vector $c \in \mathbb{R}^N_+$ for that population, the projection $\Pi_{N'}$ of its path of awards $\Pi$ onto the subspace $\mathbb{R}^{N'}$ pertaining to a subpopulation $N' \subset N$ is a subset of its path of awards $\Pi'$ for the projection $c_{N'}$ of $c$ onto $\mathbb{R}^{N'}$. Moreover, if the rule is *endowment continuous*, $\Pi_{N'}$ and $\Pi'$ actually coincide. The question that has to be addressed is the converse. Suppose that the choice has been made of how to solve two-claimant problems, so that paths of awards are known in the two-claimant case. Now, given a population $N$ of size greater than 2, and a claims vector $c \in \mathbb{R}^N_+$, is there a path for $c$ with the property that for each two-claimant subpopulation $N'$ of $N$, its projection onto the two-dimensional coordinate subspace $\mathbb{R}^{N'}$ coincides with the path chosen for the projection $c_{N'}$ of $c$ onto that subspace?

A rule is **conversely consistent** if for each claimant set, each problem that these claimants may face, and each awards vector for this problem, if this

[5]The central result on *consistency* for claims problems is due to Young (1987).

vector is such that for each two-claimant subpopulation, its restriction to the subpopulation is the choice the rule would make for the associated reduced problem, then the rule chooses it for the initial problem: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each award vector $x$ of $(c, E)$, if for each $N' \subset N$ with $|N| = 2$, $x_{N'} = S(c_{N'}, \sum_{N'} x_i)$, then $x \equiv S(c, E)$. The **Elevator Lemma** (Thomson, 2007, 2012c) asserts that for each pair of rules on $\mathcal{C}$, if one is *consistent*, the other *conversely consistent*, and they coincide in the two-claimant case, then in fact they coincide in general.

**Proof:** We already know that the proportional and constrained equal awards rules satisfy the requirements of the theorem. Conversely, let $S$ be a rule on $\mathcal{C}$ satisfying these requirements. We show that either $S = P$ or $S = CEA$.

**Step 1: $\boldsymbol{S}$ is anonymous**. This is because (i) on $\mathcal{C}^2$, $S$ satisfies *equal treatment of equals*, and (ii) *equal treatment of equals* in the two-claimant case and *consistency* imply *anonymity* (Chambers and Thomson, 2002).

Thus, there is a single function $\lambda\colon \mathbb{R}^2_+ \to [0, 1]$ such that for each $N \in \mathcal{N}^2$ and each $c \in \mathbb{R}^N_+$—to fix notation, let $N \equiv \{i, j\}$ and suppose that $0 < c_i \leq c_j$—the path of $S$ for $c$ is bro.seg$[0, \lambda(c)P(c, 2c_i) + (1 - \lambda(c))(1 - CEA(c, 2c_i)), c]$.

If a rule is *endowment monotonic* in the two-claimant case and *consistent*, then it is *endowment monotonic* in general (Hokari and Thomson, 2008)[6], and therefore *endowment continuous* in general, since *endowment monotonicity* implies *endowment continuity*.

Thus, the projection implication of *consistency* described earlier can be invoked. In particular, the projection of the path of the rule for a claims vector involving three claimants onto the subspace pertaining to a two-claimant subpopulation coincides with the path for the projection of that claims vector onto that subspace.

**Step 2: For each $\boldsymbol{c} \in \mathbb{R}^2_+$ of unequal coordinates, $\boldsymbol{\lambda(c) \in \{0, 1\}}$.** We argue by contradiction. Without loss of generality, we can take $N \equiv \{1, 2\}$ and $c \in \mathbb{R}^N_+$ such that $c_1 < c_2$. Because $0 < \lambda(c) < 1$, the path for $c$ has a kink—let us call it $x$—and $x_1 < c_1$. Let us call the path $\Pi_3$. We introduce a third claimant, claimant 3, set $c_3 \equiv \frac{x_1}{2}$, and consider the claims vector $(c_1, c_2, c_3) \in \mathbb{R}^{\{1,2,3\}}_+$. Let $\Pi_2$ be the path of $S$ for $(c_1, c_3)$. We show next that the path of $S$ for $(c_1, c_2, c_3)$, $\Pi$, can be recovered from $\Pi_3$ and $\Pi_2$, which have

[6] Using these authors' terminology, *endowment monotonicity* is "lifted" by *consistency*.

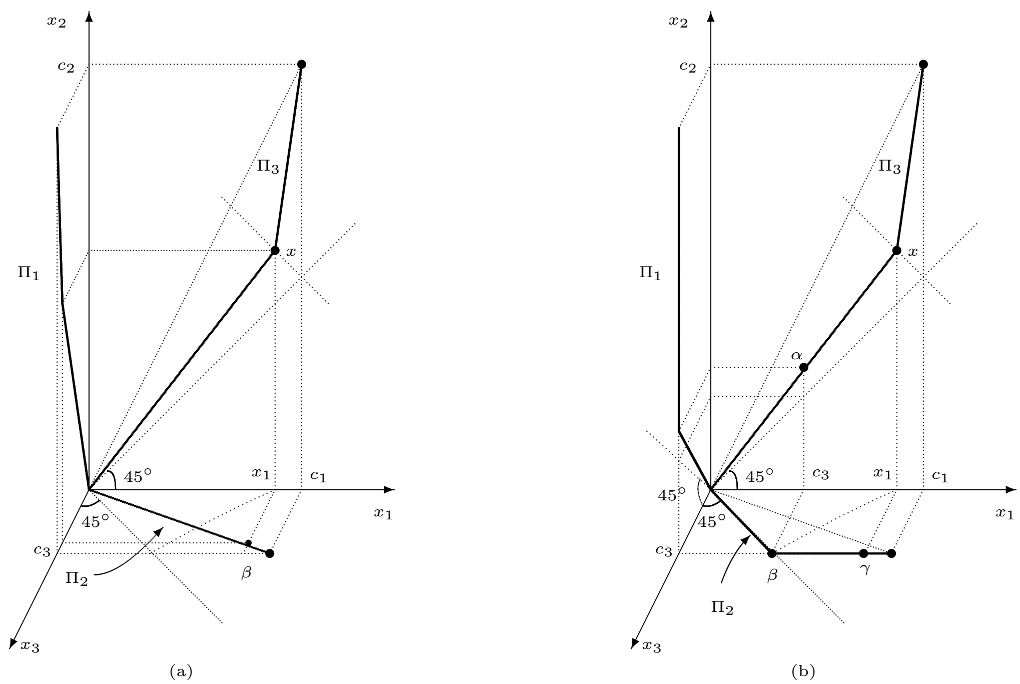


**Figure 6: Illustrating Cases 1 and 2 of the proof of Theorem 3.**

to be its projections unto $\mathbb{R}^{\{1,2\}}$ and $\mathbb{R}^{\{1,3\}}$. We then deduce the path for $(c_2, c_3)$, $\Pi_1$, by projecting $\Pi$ onto $\mathbb{R}^{\{2,3\}}$ and we show that $\Pi_1$ violates what we know the paths of a rule in $\mathcal{S}^2$ look like. We distinguish three cases.

**Case 1: $\boldsymbol{\lambda(c_1, c_3) = 1}$** (Figure 6a). This means that $\Pi_2$ is the path of $P$ for $(c_1, c_3)$, namely seg$[(0,0),(c_1,c_3)]$. Let $\beta$ the point of $\Pi_2$ of first coordinate $x_1$.

The path $\Pi$ consists of two segments, (i) a segment whose projection onto $\mathbb{R}^{\{1,2\}}$ is seg$[(0,0),x]$ and whose projection onto $\mathbb{R}^{\{1,3\}}$ is seg$[(0,0),\beta]$, and (ii) a segment whose projection onto $\mathbb{R}^{\{1,2\}}$ is seg$[x,(c_1,c_2)]$ and whose projection onto $\mathbb{R}^{\{1,3\}}$ is seg$[\beta,(c_1,c_3)]$,

The path $\Pi_1$ is the projection of $\Pi$ onto $\mathbb{R}^{\{2,3\}}$. It too consists of two segments. Since $x_1 + x_2 = 2c_1$, then $x_2 = 2c_1 - x_1$. Also, $\beta_3 = \frac{c_3}{c_1}x_1 = 2c_3$. The coordinates of the kink in $\Pi_1$ are $x_2$ and $\beta_3$. By definition of the rules in $\mathcal{S}^2$, they should add up to twice the smallest coordinate of $(c_2, c_3)$ which, since $c_3 < c_1 < c_2$, is $c_3$. However $2c_1 - x_1 + 2c_3 = 2c_3$ implies $x_1 = 2c_1$, which contradicts the fact that $x_1 < c_1$.

**Case 2: $\boldsymbol{\lambda(c_1, c_3) = 0}$** (Figure 6b). This means that $\Pi_2$ is the path of $CEA$ for $(c_1, c_3)$, namely, since $c_3 < c_1$, bro.seg$[(0,0),\beta,(c_1,c_3)]$, where $\beta \equiv (c_3, c_3)$. Let $\alpha$ be the point of $\Pi_3$ of first coordinate $c_3$ and $\gamma$ the point of seg$[\beta,(c_1,c_3)]$ of first coordinate $x_1$.

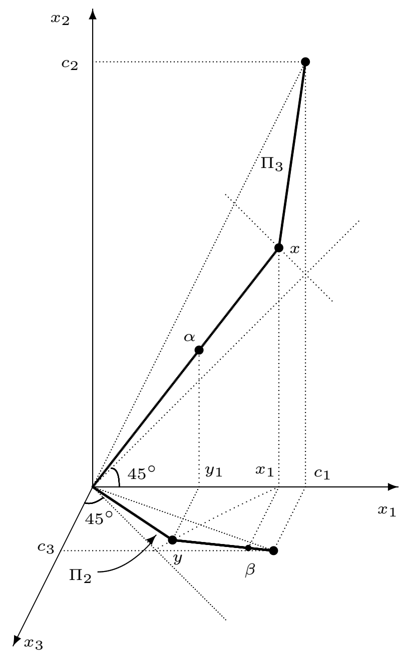


**Figure 7: Illustrating Case 3 of the proof of Theorem 3.**

Thus, using the fact that $c_3 < x_1$, we deduce that $\Pi$ consists of two segments, (i) a segment whose projection onto $\mathbb{R}^{\{1,2\}}$ is seg$[(0,0),\alpha]$, and whose projection onto $\mathbb{R}^{\{1,3\}}$ is seg$[(0,0),\beta]$, (ii) a segment whose projection onto $\mathbb{R}^{\{1,2\}}$ is seg$[\alpha, x]$, and whose projection onto $\mathbb{R}^{\{1,3\}}$ is seg$[\beta,\gamma]$, and (iii) a segment whose projection onto $\mathbb{R}^{\{1,2\}}$ is seg$[x,(c_1,c_2)]$, and whose projection onto $\mathbb{R}^{\{1,3\}}$ is seg$[\gamma,(c_1,c_3)]$.

The path $\Pi_1$ is the projection of $\Pi$ onto $\mathbb{R}^{\{2,3\}}$. It too consists of two segments. They are seg$[(0,0),(\alpha_2,c_3)]$, a segment whose slope is not equal to 1, since $\alpha_2 \neq c_3$, and the segment seg$[(\alpha_2,c_3),(c_2,c_3)]$, which is parallel to the second axis. This second segment is the union of the projections of the two segments described in (ii) and (iii) above. Given the definition of the rules in $\mathcal{S}^2$, the inclusion of a segment parallel to the second axis in the path for $(c_2,c_3)$ is possible only for $CEA$, but then the path should begin with a segment of slope 1, which we have just shown in not the case, a contradiction.

**Case 3: $\boldsymbol{0 < \lambda(c_1,c_3) < 1}$** (Figure 7). Then, $\Pi_2$ still has a kink—let us call it $y$. By the choice of $c_3$, $y_1 < x_1$. Let $\alpha$ be the point of $\Pi_3$ of first coordinate $y_1$ and $\beta$ the point of $\Pi_2$ of first coordinate $x_1$.

The path $\Pi$ consists of three segments, (i) a segment whose projection onto $\mathbb{R}^{\{1,2\}}$ is seg$[(0,0),\alpha]$ and whose projection onto $\mathbb{R}^{\{1,3\}}$ is seg$[(0,0),y]$, (ii) a segment whose projection onto $\mathbb{R}^{\{1,2\}}$ is seg$[\alpha,x]$ and whose projection onto $\mathbb{R}^{\{1,3\}}$ is seg$[y, \beta]$, and (iii) a segment whose projection onto $\mathbb{R}^{\{1,2\}}$ is seg$[x, (c_1, c_2)]$ and whose projection onto $\mathbb{R}^{\{1,3\}}$ is seg$[\beta, (c_1, c_3)]$.

The path $\Pi_1$ is the projection of $\Pi$ onto $\mathbb{R}^{\{2,3\}}$. It too consists of three segments. This contradicts the fact that the paths of rules in $\mathcal{S}^2$ never have more than two segments.

**Step 3: For each $\boldsymbol{c} \in \mathbb{R}^2_+$ of unequal coordinates, $\boldsymbol{\lambda(c) = 1}$ or for each $\boldsymbol{c} \in \mathbb{R}^2$, $\boldsymbol{\lambda(c) = 0}$.** Suppose by contradiction that there are $c_0, c'_0, c''_0, c'''_0 \in \mathbb{R}_+$ with $c_0 \neq c'_0$ and $c''_0 \neq c'''_0$ such that $\lambda(c_0, c'_0) = 1$ and $\lambda(c''_0, c'''_0) = 0$. If two of $c_0, c'_0, c''_0, c'''_0$ are equal, say $c'_0 = c'''_0$, let $c = (c_0, c'_0, c''_0) \in \mathbb{R}^{\{1,2,3\}}$. Here, we deduce, as in the analysis of Case 2 so we omit the details, that the path for $(c_1, c_3)$ consists of a segment whose slope is not equal to 1 and a segment parallel to one of the axes. This is incompatible with the way the paths of rules in $\mathcal{S}^2$ are defined. If $c_0, c'_0, c''_0, c'''_0$ are all distinct, let $c \equiv (c_0, c'_0) \in \mathbb{R}^{\{1,2\}}_+$ and $\bar{c} \equiv (c''_0, c'''_0) \in \mathbb{R}^{\{3,4\}}_+$. Let $(c_0, c'_0, c''_0) \in \mathbb{R}^{\{1,2,3\}}_+$. Because $\lambda(c) = 1$, then (i) $\lambda(c'_0, c''_0) = 1$ as well. Let $(c'_0, c''_0, c'''_0) \in \mathbb{R}^{\{2,3,4\}}_+$. Because $\lambda(c''_0, c'''_0) = 0$, then (ii) $\lambda(c'_0, c''_0) = 0$ as well. (i) and (ii) are in contradiction.

**Step 4: Concluding.** For each $N \in \mathcal{N}^2$ and each claims vector $c \in \mathbb{R}^N_+$ of equal coordinates, the paths of awards of $P$ and $CEA$ coincide with seg$[(0,0), c]$, which is also the path of $S$ for $c$. Then, using Step 3, we conclude that on $\mathcal{S}^2$, either $S = P$ or $S = CEA$. Since $S$ is *consistent* and both $P$ and $CEA$ are *conversely consistent*, if follows by the Elevator Lemma that for any number of claimants either $S = P$ or $S = CEA$. $\square$

By duality, we also obtain a characterization of the family of rules that coincide for each $N \in \mathcal{N}^2$, with a member of $\mathcal{T}^N$, and are *consistent*.

**Theorem 4** *The only rules on $\mathcal{C}$ that coincide, for each two-claimant population $N \in \mathcal{N}$, with a member of $\mathcal{T}^N$, and are consistent, are the proportional rule and the constrained equal losses rule.*

# 5 Concluding comments

We have proposed a simple way of compromising between two principles that are focal in the theory and practice of resource allocation, when fairness is a concern, the principle of proportionality and the principle of equality. As we documented in the introduction, in the context of the adjudication of conflicting claims, notions of proportionality and of equality, of awards or of losses, can be expressed in multiple ways. Additional families of rules that can be thought of as further generalizing the proportional, constrained equal awards, and constrained equal losses rules have come out of recent axiomatic work (Ju, Miyagawa, and Sakai, 2007; Harless, 2015; Stovall, 2014; Flores-Szwagrzak, 2015; Chambers and Moreno-Ternero, 2015). Consistency has played an important role in most of these developments. The present study should contribute to a further understanding of this property in the context of this model, and in particular the extent to which it allows departing from either the proportional or constrained equal awards, or constrained equal losses rules. We have seen that here, the answer is essentially negative, even if the averaging is allowed to depend on the claims vector.

We simply advocated the rules studied here on the basis of the intuitive appeal of their definition. An open question is whether they can be given interesting axiomatic justifications.

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