Two Aspects of Axiomatic Theory of Bargaining

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Replication Invariance of Bargaining Solutions

by

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1. Introduction. This note is concerned with the behavior of bargaining solutions under replications of bargaining problems. Kalai (1975) showed that each member of the one parameter family of non-symmetric generalizations of the two-person Nash solution (1950) is equivalent to the Nash solution under appropriate replications (or limits of such replications). He also noted the invariance of the Kalai-Smorodinsky (1975) solution under replications.

We argue here that these results crucially depend on the manner in which the replications are performed. We propose another replication method which seems to us just as natural and we formulate a corresponding notion of invariance. Now it is the Nash solution that is invariant, while each member of a one parameter subfamily of the family of non-symmetric generalizations of the two-person Kalai-Smorodinsky solution is equivalent to the Kalai-Smorodinsky solution under appropriate replications (or limits of such replications).

The two replication methods are polar opposites. Kalai's method models maximal compatibility of interests between the original agents and the newcomers while the method described here models minimal compatibility. Both methods have natural interpretations and applications in economic contexts. Kalai's method seems particularly appropriate in public good economies and ours in private good economies.

The parallel results obtained in the two studies have relevance to situations in which there is some flexibility in the number of agents actually involved in negotiations. They tell us when groups of similar agents benefit from being represented by one of them and conversely when an agent gains from recruiting supporters with claims similar to his.

2. Definitions, results. Given $n \in \mathbb{N}$, the set of positive integers, let B^n be the class of n-person bargaining problems, i.e., convex, compact and comprehensive[1] subsets of $\mathbb{R}^n_+$ containing at least one positive vector.[2] An n-person solution associates to every $S \in B^n$ a unique point of S, interpreted as the recommended compromise for S. For the n-person Nash solution, N(S) is the maximizer of Πx_i for $x \in S$. For the n-person Kalai-Smorodinsky solution, K(S) is the maximal point of S on the segment $[0,a(S)]$ where $a_i(S) \equiv \max\{x_i | x \in S\}$ for each i. These solutions have non-symmetric generalizations:[3] Given $p = (p_1,\ldots,p_n) \in \mathbb{R}^n_{++}$, $N^p(S)$ is the maximizer of $\Pi x_i^{p_i}$ for x S and $K^p(S)$ is the maximal point of S on the segment $[0,a^p(S)]$ where $a^p_i(S) \equiv p_i a_i(S)$ for each i.[4] With a slight abuse of notation, we will also write N^p as $N^{p_1,\ldots,p_n}$ and K^p as $K^{p_1,\ldots,p_n}$.

Next, we define our notion of replication. Given a two-person problem $S \in B^2$, and given $m,n \in \mathbb{N}$, the (m,n) replica of S is the problem $S^{m,n} \in B^{m+n}$ involving m agents of type 1, indexed by $i \in I_m \equiv \{1,\ldots,m\}$ and n agents of type 2, indexed by $j \in J_n \equiv \{m+1,\ldots,m+n\}$. It is constructed as follows: given $(i,j) \in I_m \times J_n$, let $S_{ij} \equiv \{x \in \mathbb{R}^{m+n} | \exists (x_1',x_2') \in S$ with $x_1' = x_i$ and $x_2' = x_j$; $x_k = 0$ for all $k \notin \{i,j\}\}$. Finally, let $S^{m,n} \equiv \mathrm{cch}\{S_{ij} | (i,j) \in I_m x J_n\}$, where $\mathrm{cch}\{A_1,\ldots,A_t\}$, with $A_1,\ldots,A_t \subset \mathbb{R}^{m+n}$, denotes the convex and comprehensive hull of $A_1,\ldots,A_t$, i.e. the smallest convex and comprehensive subset of $\mathbb{R}^{m+n}$ containing $A_1,\ldots,A_t$.

We are interested in comparing the sum of what the agents of a given type get in $S^{m,n}$ to what the agent they are replicating gets in S. This comparison depends of course on what solution is used to solve $S^{m,n}$ and S. If, for a particular solution, the two numbers are always equal, we say that the solution is replication invariant. We show that the Nash solution has this property. Although the Kalai-Smorodinsky does not, it turns out that the first number is equal to what the second would be if S were solved by the two-person non-symmetric Kalai-Smorodinsky solution with weights p_1 and p_2 proportional to m and n. Conversely, any two-person non-symmetric Kalai-Smorodinsky solution can be approximated by applying the Kalai-Smorodinsky solution after replication of the agents in proportions approaching the degree of asymmetry of the solution. We start with the results that concern the Kalai-Smorodinsky solution.

<u>Theorem</u> <u>1</u>: For each $(m,n) \in \mathbb{N} \times \mathbb{N}$, and for each $S \in B^2$, $mK_i(S^{m,n}) = K_1^{m,n}(S)$ for each $i \in I_m$ and $nK_j(S^{m,n}) = K_2^{m,n}(S)$ for each $j \in J_n$.

<u>Proof</u>: Let $(\alpha,\beta) \equiv K^{m,n}(S)$ and $x \in \mathbb{R}^{m+n}$ be defined by $x_i \equiv \alpha/m$ for each $i \in I_m$ and $x_j = \beta/n$ for each $j \in I_n$. We will prove that $x = K(S^{m,n})$.

Let $\alpha',\beta' \in \mathbb{R}$ be such that the line of equation $\alpha'x_1' + \beta'x_2' = \alpha'\alpha + \beta'\beta$ supports S at (α,β). Given each $(i,j) \in I_m \times J_n$, let $x^{ij} \in \mathbb{R}^{m+n}$ be defined by $x_i^{ij} = \alpha$, $x_j^{ij} = \beta$ and $x_k^{ij} = 0$ for any other coordinate k. Clearly $x^{ij} \in S_{ij}$ and the hyperplane $H \subset \mathbb{R}^{m+n}$ of equation $\alpha'\sum_{I_m} x_i' + \beta'\sum_{J_n} x_j' = \alpha'\alpha + \beta'\beta$ supports S_{ij} at x^{ij}. Since $S^{m,n} = \text{cch}\{S_{ij} | (i,j) \in I_m \times J_n\}$, H supports $S^{m,n}$ as well. Note that $x = \frac{1}{mn} \sum_{(i,j)\in I_m \times J_n} x^{ij}$. Therefore x is a point of $S^{m,n}$ undominated by any other point of $S^{m,n}$. We will be done if we can show that $x = \lambda a(S^{m,n})$ for some λ.

Since $(\alpha,\beta) = K^{m,n}(S)$, $(\alpha,\beta) = \mu(ma_1(S), na_2(S))$ for some μ. Also, $a_i(S^{m,n}) = a_1(S)$ for all $i \in I_m$ and $a_j(S^{m,n}) = a_2(S)$ for all $j \in J_n$. The argument concludes by recalling the definition of x. (λ turns out to be equal to μ.)

QED

<u>Corollary</u>: For each $(m,n) \in \mathbb{N} \times \mathbb{N}$, and for each $S \in B^2$, $K^{m,n}(S)$ can be approximated with any degree of accuracy by $(m_k K_i(S^{m_k,n_k}), n_k K_j(S^{m_k,n_k}))$ where $i \in I_{m_k}$, $j \in J_{n_k}$ and $\{(m_k,n_k)\}$ is a sequence in $\mathbb{N} \times \mathbb{N}$ with the property that $m_k/n_k \to m/n$.

The next result is that the Nash solution is replication invariant.

<u>Theorem 2</u>. For all $(m,n) \in \mathbb{N} \times \mathbb{N}$, for all $S \in B^2$, $mN_i(S^{m,n}) = N_1(S)$ for each $i \in I_m$ and $nN_j(S^{m,n}) = N_2(S)$ for each $j \in J_n$.

<u>Proof</u>: Let $x \in \mathbb{R}^{m+n}$ be as in Th.1 with (α,β) designating $N(S)$ instead of $K^{m,n}(S)$. We claim that $x = N(S^{m,n})$. First of all, we have that $x_i = x_{i'}$ for all $i,i' \in I_n$ and $x_j = x_{j'}$ for all $j,j' \in J_n$. It is indeed well-known that the Nash solution is anonymous, i.e. that if two individuals enter symmetrically in the geometric description of a problem, their Nash payoffs are identical. Now (α,β) is obtained by maximizing $x'_1 x'_2$ for $x' = (x'_1,x'_2) \in S$. This implies that the line of equation $x'_1/\alpha + x'_2/\beta = 2$ supports S at (α,β) as well as the curve of equation $x'_1 x'_2 = \alpha\beta$. As in Th.1, x is a point of $S^{m,n}$ undominated by any other point of $S^{m,n}$. Finally, the hyperplane H' of equation $(\sum_{I_m} x'_i)/\alpha + (\sum_{J_n} x'_j)/\beta = 2$ supports $S^{m,n}$ at x but it also supports there the hypersurface of equation $\prod_{k \in I_m \cup J_n} x'_k = (\frac{\alpha}{m})^m (\frac{\beta}{n})^n$. This means that $x = N(S^{m,n})$.

QED

3. <u>Discussion</u>. The difference between our approach and Kalai's is best explained by reproducing the example that appears in his paper (p.131): there are two players who can receive one dollar if they agree on a division of that dollar. Each agent's utility is linear in what he receives so that the problem they face in the utility space is the 1-dimensional simplex S if the utilities are normalized by assigning utility one to receiving the whole dollar and utility zero to receiving nothing. Both the Nash solution and the Kalai-Smorodinsky solution recommend that the two agents get equal utilities, implying that they receive 1/2 dollar each. Now a third player comes in, identical to agent 2 in that he derives exactly the same utility as agent 2 from what agent 2 receives. This implies that the problem faced by the 3-player group is the largest convex and comprehensive set with projections S on the 1-2 plane and the replica S' of S on the 1-3 plane (obtained from S by having agent 3 playing in S' the role played in S by agent 2). The problem is the intersection cyl{S,S'} of the cylinder based on S with generators parallel to the third axis and of the cylinder based on S' with generators parallel to the second axis. This construction amounts to assuming maximal coincidence of interests among agents 2 and 3. Agent 2's consumption is like a public good for the two of them.

The opposite extreme, in replicating agent 2, is to assume that agent 3 derives from any amount that he, agent 3, consumes, a utility equal to the utility that agent 2 would derive from consuming the same amount but that, for each utility level attainable by agent 1, agents 2 and 3's utilities sum to a constant. Such a situation, which implies minimal compatibility of interests between the two identical agents, is modeled by taking the three-person problem to be cch{S,S'} as we have done here. This would be the appropriate formulation for economies with only private goods.[5] It is of course because the utilities of identical agents can be traded off at a one-to-one ratio that it is particularly interesting to add these agents' payoffs and to compare the sum to what one of them would get if he were to "represent" them all. Such considerations are behind the notion of replication invariance that we proposed.

More generally, when economic problems are considered in which some goods are public goods for only subgroups of agents, replications would have to be modeled by combining Kalai's approach with the one followed here.

Kalai's results and ours reveal a sort of symmetry in the responses of two important solutions to two extreme forms of replication, and help us understand the circumstances under which agents would gain from having more agents similar to them around or would be hurt by it. Such evaluation would be of particular relevance when agents have control over whether new agents join them in the negotiations.

Related invariance properties were recently investigated by Moulin (1983). His analysis, placed in the context of choice problems with transferable utility, involves requirements on solutions such as "invariance under merging and splitting" (of agents). He uses them in characterizations of Egalitarian and Utilitarian solutions.

Finally, we note that the results presented here could just as well be proved for replications of n-person (instead of 2-person) problems.

Footnotes

1. $S \subsetneq \mathbb{R}^n_+$ is comprehensive if for all $x, y \in \mathbb{R}^n_+$, $x \in S$ and $x \geq y$ imply that $y \in S$.
2. Bargaining theory is usually concerned with pairs (S,d) of a convex, compact subset S of $\mathbb{R}^n$ and of a point $d \in S$ strictly dominated by at least one point of S. The bargaining problems considered here differ from those in three ways. First, we have chosen $d = 0$ and ignored it altogether in the notation. Second, we have required all points of S to dominate d. These two differences are immaterial to our analysis. Third, we have imposed comprehensiveness of S: this is because without this restriction the Kalai-Smorodinsky solution may select strictly dominated points where $n \geq 3$, while with the restriction, it always selects weakly Pareto optimal points. (It is true however that the Kalai-Smorodinsky solution outcome of an n-person problem obtained by replicating in the manner described later a two-person problem that is not necessarily comprehensive, would be weakly Pareto-optimal.)
3. The non-symmetric generalizations of N first appear in Harsanyi-Selten (1972). They are also discussed in Roth (1979).

   The non-symmetric generalizations of K discussed here constitute a one-parameter subfamily of the family of non-symmetric generalizations of K first described in Peters and Tijs (1982).
4. Note that for all $\lambda > 0$, $N^{\lambda p} = N^p$ and $K^{\lambda p} = K^p$.
5. Incidentally, it is the approach followed by Kaneko and Wooders (1982) in their study of the cores of partitioning games.

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Monotonicity of Bargaining Solutions with Respect to the Disagreement Point

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Monotonicity of Bargaining Solutions with Respect to the Disagreement Point

1. Introduction. An n-person bargaining problem is a pair (S,d) of a subset S of $\mathbb{R}^n$ and of a point $d \in \mathbb{R}^n$. $\mathbb{R}^n$ is the utility space, S is the feasible set, and d is the disagreement point. If the agents unanimously agree on a point x of S, they get x. Otherwise, they get d. Given a class of n-person bargaining problems, a solution defined on the class is a function F associating to every (S,d) in the class a point $F(S,d) \in S$ interpreted as the compromise recommended for (S,d).

We investigate here whether the best known solutions respond appropriately to changes in d, for fixed S. Given some agent i, assume that d_i increases while d_j remains constant for each $j \neq i$. Since d_i represents agent i's fallback position, one would expect that agent i's final payoff increases (or at least does not decrease). We show that the Nash solution behaves in this way on a class of problems commonly considered; so do the Kalai-Smorodinsky and Egalitarian solutions on the subclass obtained by requiring utility to be freely disposable (a condition which is natural since without it these two solutions would not always yield (even weakly) Pareto-optimal outcomes).

A related requirement is that, in the circumstances described in the above paragraph, not only agent i's payoff does not decrease, but also the payoff of none of the other agents increases, so that they all bear the cost of the improvement in agent i's bargaining position. We show that neither the Nash nor the Kalai-Smorodinsky solution behave in this way, even if utility is freely disposable. However, under that assumption, the Egalitarian solution does.

2. Notation. We will consider two classes of problems. $\bar{\Sigma}^n$ is the class of pairs (S,d) where $S \subseteq \mathbb{R}^n$ is convex, compact, and there exists $x \in S$ with $x > d$.[1] Σ^n is the subclass of $\bar{\Sigma}^n$ of comprehensive problems (if $x \in S$ and $d \leqq y \leqq x$, then $y \in S$; this is the form that we will find convenient to give to the condition that "utility is freely disposable"). Given $(S,d) \in \bar{\Sigma}^n$, its Nash (1950) solution outcome $N(S,d)$ is the point where the product Πx_i is maximized for $x \in S$ with $x \geqq d$; its Kalai-Smorodinsky (1975) solution outcome $K(S,d)$ is the maximal point of S on the segment connecting d to $a(S,d)$, where for each i, $a_i(S,d) \equiv \max\{x_i | x \in S;\ x \geqq d\}$; its Egalitarian solution outcome $E(S,d)$ is the maximal point x of S with $x_i - d_i = x_j - d_j$ for all i,j.

Given a solution F, it will be of interest to know whether F satisfies any of the following conditions (in the statements of which (S,d), (S',d') are arbitrary elements, and (S^k,d^k) an arbitrary sequence, in its domain of definition).

Pareto-optimality (po): For all $x \in \mathbb{R}^n$, if $x \geq F(S,d)$, then $x \notin S$.

Weak Pareto-optimality (wpo): For all $x \in \mathbb{R}^n$, if $x > F(S,d)$, then $x \notin S$.

Scale invariance (s.inv): For all positive affine transformation $\lambda: \mathbb{R}^n \to \mathbb{R}^n$, if $(S',d') = (\lambda(S),\lambda(d))$, then $F(S',d') = \lambda(F(S,d))$.

Independence of irrelevant alternatives (iia): If $S' \subset S$, $d = d'$, $F(S,d) \in S'$, then $F(S',d') = F(S,d)$.

Continuity (cont): If $S^k \to S$ (in the Hausdorff topology) and $d^k \to d$, then $F(S^k,d^k) \to F(S,d)$.

N satisfies all these conditions on $\bar{\Sigma}^n$. K satisfies s.inv on $\bar{\Sigma}^n$, wpo and cont on Σ^n. E satisfies iia on $\bar{\Sigma}^n$, wpo and cont on Σ^n.

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[1]Vector inequalites: $x \geqq y$, $x \geq y$, $x > y$.

Finally, $e \equiv (1,\ldots,1)$; $\mathrm{co}\{x_1,\ldots,x_k\}$ is the smallest convex set containing $x_1,\ldots,x_k$.

3. <u>The</u> <u>results.</u> We start by formulating our first condition of monotonicity with respect to the disagreement point.

<u>d-monotonicity</u> (d-mon): For all (S,d), (S',d'), for all i, if $S' = S$, $d_i' > d_i$ and $d_j' = d_j$ for all $j \neq i$, then $F_i(S',d') \geqq F_i(S,d)$.

<u>Theorem</u> <u>1</u>. The Nash solution satisfies d-monotonicity on $\bar{\sum}^n$.

<u>Proof</u>. Because N satisfies s.inv, we can assume that $d = 0$ and $N(S,d) = e$. Without loss of generality, we can also assume that $d' = (\alpha,0,\ldots,0)$ for $\alpha > 0$. Let $x \equiv N(S,d')$. We have to show that $x_1 \geqq 1$. Let $S' \equiv \mathrm{co}\{e, x, d, d'\}$. Since N satisfies iia, then $N(S',d) = e$ and $N(S',d') = x$.

Since $N(S',d) = e$, the hyperplane of support to the set $\{x' \in \mathbb{R}^n | \Pi x_i' = 1\}$ lies above x, i.e.

(i) $\quad \sum x_i \leqq n.$

Since $N(S',d') = x$, the hyperplane of support to the set $\{x' \in R^n | (x_1'-\alpha) \prod_{i\neq 1} x_i' = (x_1-\alpha) \prod_{i\neq 1} x_i\}$ lies above e, i.e.

(ii) $$\frac{1}{x_1-\alpha} + \sum_{i\neq 1} \frac{1}{x_i} \leqq \frac{x_1}{x_1-\alpha} + n-1.$$

Taking x_1 as fixed, we are led to investigating whether $(x_2,\ldots,x_n) \in \mathbb{R}^{n-1}$ exist satisfying the inequalities

(iii) $$\sum_{i\neq 1} x_i \leqq n-x_1 \text{ and}$$

(iv) $$\sum_{i\neq 1} \frac{1}{x_i} \leqq \frac{x_1-1}{x_1-\alpha} + n-1.$$

Since the function $h:\mathbb{R}_{++} \to \mathbb{R}_+$ defined by $h(t) = 1/t$ is convex, if $(x_2,\ldots,x_n)$ solves (iii) and (iv), so does $(x_2',\ldots,x_n')$ defined by

$x_i' \equiv (\sum_{j \neq 1} x_j)/(n-1)$ for all $i \neq 1$. Assuming then that $x_2 = \ldots = x_n \equiv a$, (iii) and (iv) become

(v) $(n-1)a \leqq n - x_1$

(vi) $\frac{n-1}{a} \leqq \frac{x_1-1}{x_1-\alpha} + n-1$

These inequalities can be jointly satisfied by some a only if $\frac{n-x_1}{n-1} \leqq \frac{(n-1)(x_1-\alpha)}{nx_1-1-\alpha(n-1)}$. For this to be true, it is necessary that the quadratic expression in x_1

$$-nx_1^2 + x_1(\alpha(n-1)+2n) - \alpha n + \alpha - n$$

be positive. However, its discriminant is $\alpha^2(n-1)^2$, and its roots 1 and $1 + \alpha\frac{n-1}{n}$. It is positive only if $x_1 \in \left[1, 1+\alpha\frac{n-1}{n}\right]$, i.e. if $x_1 \geqq 1$.

QED

<u>Theorem 2</u>. The Kalai-Smorodinsky solution satisfies d-monotonicity on $\sum^n$.

<u>Proof</u>. Because K satisfies s.inv, we can assume that d=0. Without loss of generality, we can also assume that $d' \equiv (\alpha,0,\ldots,0)$ for $\alpha > 0$. Let $x \equiv K(S,d)$ and $x' \equiv K(S,d')$. Since $\{y \in S | y \geqq d\} \supseteq \{y \in S | y \geqq d'\}$, then $a(S,d') \leqq a(S,d)$. However, $a_1(S,d) = a_1(S,d')$. Therefore we have

(i) $a_1(S,d) = a_1(S,d')$ and $a_i(S,d) \geqq a_i(S,d')$ for all $i \neq 1$.

By definition of K, we also have

(ii) $x = (1-\lambda)a(S,d)$ for some $\lambda \in [0,1[$ and

(iii) $x' = \mu d' + (1-\mu)a(S,d')$ for some $\mu \in [0,1[$.

Suppose now, by way of contradiction, that $x_1' < x_1$. Then, for at least one $i \neq 1$, $x_i' \geqq x_i$. Otherwise, we would have $x' < x$, in violation of the fact that K satisfies wpo on $\sum^n$. To fix the ideas, suppose $x_2' \geqq x_2$. Then from (i), (ii), (iii) the constraints $x_1' < x_1$ and $x_2' \geqq x_2$ yield

$$x_1' = \mu\alpha + (1-\mu)a_1(S,d') < (1-\lambda)a_1(S,d) = x_1$$

$$x_2' = (1-\mu)a_2(S,d') \geqq (1-\lambda)a_2(S,d) = x_2$$

which in view of (i) give $\mu\alpha < 0$, which is impossible.

QED

<u>Remark</u>. This result also holds on $\sum^n$ if $n = 2$ but not if $n > 2$. This is clear in the first case. To prove the negative statement, we consider the following example, illustrated in Figure 1.

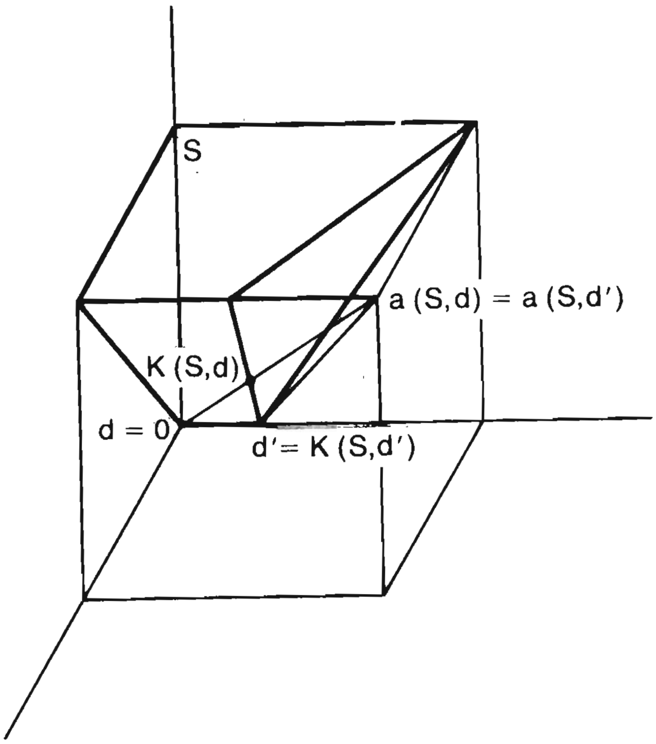

Figure 1

<u>Example</u>. Let $n \equiv 3$. Let $S \equiv \text{co}\{(0,0,0), (1/4,0,0), (1,1,0), (0,1,0), (0,1,1), (1/2,1,1)\}$, $d \equiv (0,0,0)$, $d' \equiv (1/4,0,0)$. Then we have $a(S,d) = a(S,d') = e$, $K(S,d) = (1/3,1/3,1/3)$ and $K(S,d') = (1/4,0,0)$. Agent 1 loses as d changes to d'.

<u>Theorem 3</u>. The Egalitarian solution satisfies d-monotonicity on $\sum^n$.

Proof. Suppose by way of contradiction and without loss of generality, that for some (S,d), $(S,d') \in \sum^n$ with $d = (0,\ldots,0)$ and $d' = (\alpha,0,\ldots,0)$ with $\alpha > 0$, we have that $x_1' < x_1$, where $x \equiv E(S,d)$ and $x' \equiv E(S,d')$. By definition of E, $x_1 = \ldots = x_n$ and $x_1' - \alpha = x_2' = \ldots = x_n'$. Since $x_1' < x_1$, it follows that $x_2' < x_2,\ldots,x_n' < x_n$ so that $x' < x$, in violation of the fact that E satisfies wpo on $\sum^n$.

QED

Remark. This result does not hold on $\sum^n$, even if $n = 2$. Indeed, let $S \equiv co\{(0,0),\ (1,0),\ (2,2)\}$, $d \equiv (0,0)$, $d' \equiv (1/2,0)$. We have that $E(S,d) = (2,2)$ while $E(S,d') = (3/2,1)$. Agent 1 loses as d changes to d'.

We will now strengthen the condition of d-monotonicity by requiring that if d moves in a direction favorable to an agent, all the others be negatively (in the weak sense) affected.

Strong d-monotonicity (st.d-mon). For all (S,d), (S',d'), for all i, if $S' = S$, $d_i' > d_i$ and $d_j' = d_j$ for all $j \neq i$, then $F_j(S',d') \leqq F_j(S,d)$ for all $j \neq i$.

Remark. As stated, this condition is not really stronger than d-mon since no requirement is imposed on $F_i(S',d')$ in relation to $F_i(S,d)$. However, if F satisfies po and st.d-mon, then F satisfies d-mon. Also, if F satisfies wpo, st.d-mon and cont, then F satisfies d-mon. Finally, observe that if n=2, and under these additional conditions, st.d-mon is equivalent to d-mon.

Our results here are mainly negative.

Theorem 4. The Nash solution does not satisfy strong d-monotonicity.

Proof. Let $n \equiv 3$. Let $d \equiv (0,0,0)$ and $d' \equiv (\alpha,0,0)$ for $\alpha > 0$. We will look for $x \in \mathbb{R}^3$ and $\alpha > 0$ such that for $S \equiv \mathrm{co}\{d,d',e,x\}$, we have $e = N(S,d)$, $x = N(S,d')$ and $x_3 > 1$. We argue as in the proof of Theorem 1, that for $e = N(S,d)$ and $x = N(S,d')$ to hold, it is necessary and sufficient that

(i) $\quad x_1 + x_2 + x_3 \leqq 3$

(ii) $\quad \dfrac{1}{x_1-\alpha} + \dfrac{1}{x_2} + \dfrac{1}{x_3} \leqq \dfrac{x_1}{x_1-\alpha} + 2.$

These inequalities are satisfied (at equality) for

$$x_1 = \frac{3}{2},\ x_2 = \frac{2}{5},\ x_3 = \frac{11}{10} \text{ and } \alpha = \frac{71}{62}.$$

Agent 3 gains as d is changed to d'.

QED

Theorem 5. The Kalai-Smorodinsky solution does not satisfy strong d-monotonicity on $\sum^n$.

Proof. The proof is illustrated in Figure 2. Let $n \equiv 3$. Let $d \equiv (0,0,0)$, $d' \equiv (1,0,0)$, and $S \equiv \mathrm{co}\{d, (2,0,0), (2,1,0), (0,2,0), (0,2,2), (0,0,2), (2,0,2), (2,1,2)\}$. We have that $a(S,d) = (2,2,2)$ and $K(S,d) = (4/3,4/3,4/3)$. Also $a(S,d') = (2,3/2,2)$ and $K(S,d') = (7/4,9/8,7/4)$. Agent 3 gains as d changes to d'.

QED

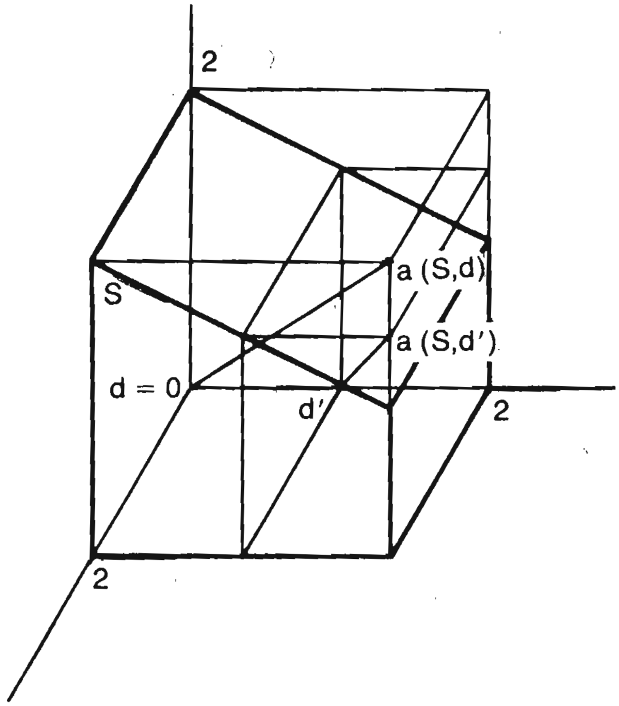

Figure 2

<u>Theorem</u> <u>6</u>. The Egalitarian solution satisfies strong d-monotonicity on $\sum^n$.

<u>Proof</u>. Straightforward.

The above results leave open several questions. One of them is whether a solution on $\sum^n$ can satisfy po, sy, iia and cont without satisfying d-mon. The answer is yes, even if n = 2, as shown by the next example, illustrated in Figure 3.

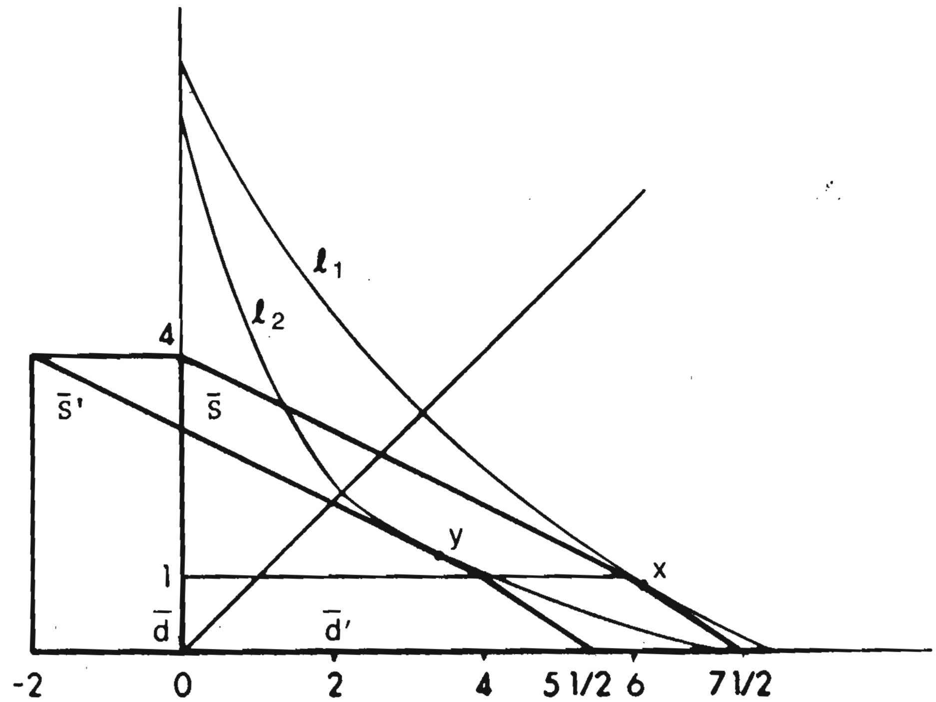

Figure 3

Example. Let $n \equiv 2$. Let $S \equiv co\{(0,0), (7\ 1/2,0), (6,1), (0,4)\}$ and $S' \equiv co\{(-2,0), (5\ 1/2,0), (4,1), (-2,4)\}$. Let $x \in\,](7\ 1/2,0), (6,1)[$, $y \in\,](4,1), (2,2)[$ be given and let $f: \mathbb{R}^2 \to \mathbb{R}$ be a strictly convex, symmetric and increasing function having two level curves ℓ_1 and ℓ_2 respectively tangent to S at x and to S' at y. Then, given any $(S,d) \in \sum^2$, let $F(S,d) \equiv \operatorname{argmax}\{f(x-d) | x \in S\} + d$.

It is easily verified that F satisfies po, sy, iia and cont. To show that F does not satisfy d-mon, let $d \equiv (0,0)$, $d' \equiv (2,0)$. Note that $F(S,d) = x$ and that $F(S,d') = F(S',d) + (2,0) = y + (2,0)$. Since $y_1 + 2 < x_1$, we are done.

The next question is whether a solution on $\sum^n$ can satisfy po, sy, s.inv and cont without satisfying d-mon. Again, the answer is yes, even if n = 2, as shown by the next example, illustrated in Figure 4:

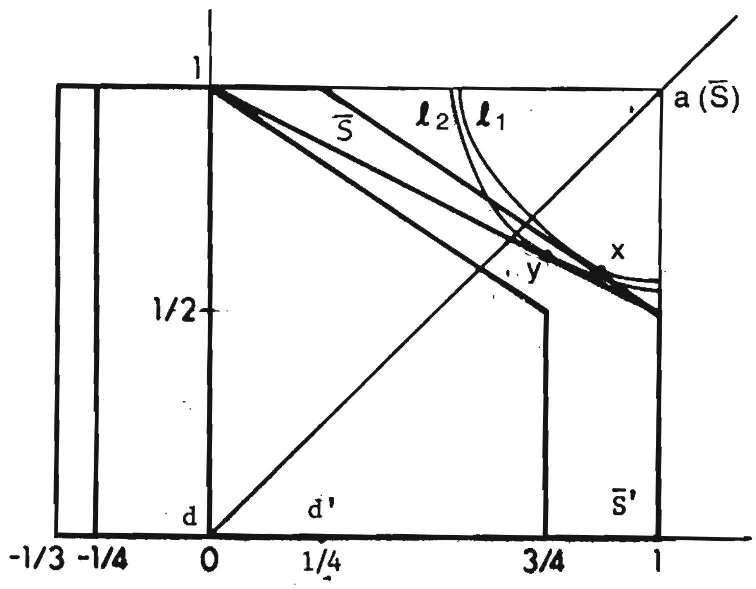

Figure 4

Example. Let $n \equiv 2$. Let $\bar{S} \equiv co\{(0,0), (1,0), (1,1/2), (1/4,1), (0,1)\}$, $\bar{S}' \equiv co\{(1,0), (1,1/2), (0,1), (-1/3,1), (-1/3,0)\}$. Let $x \in [(1,1/2), (1/4,1)]$, $y \in [(1,1/2), (0,1)]$ be given such $x_1 > x_2$, $y_1 > y_2$ and $x_2 <$

y_2, and let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a strictly convex, symmetric and increasing function having two level curves ℓ_1 and ℓ_2 respectively tangent to $\bar{S}$ at x and to $\bar{S}'$ at y. Then, given any $(S,d) \in \sum^2$, let $F(S,d)$ be defined as follows: first, (S,d) is subjected to a positive affine transformation $\lambda\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\lambda(d) = 0$, $\lambda(a(S,d)) = (1,1)$. The maximum x^* of f on $\lambda(S)$ is then determined and finally, $F(S,d)$ is set equal to $\lambda^{-1}(x^*)$.

It is easily verified that F satisfies po, sy, s.inv and cont. To show that F violates d-mon, let $d \equiv (0,0)$, $d' \equiv (1/4,0)$. Note that $F(\bar{S},d) = x$ and that $F(\bar{S}',d) = y$. To compute $F(\bar{S},d')$, we consider the linear transformation λ defined by $\lambda(x') = (\frac{4}{3}x_1' - \frac{1}{4}, x_2')$, and we find that $\lambda(\bar{S},d') = (\bar{S}',d)$. Therefore $F(\bar{S},d') = \frac{3}{4}F(\bar{S}',d) + (1/4,0)$. Since $\frac{3}{4}y_1 + 1/4 < x_1$, we are done.

References

Kalai, E. and M. Smorodinsky, "Other solutions to Nash's bargaining problem," Econometrica 43 (1975), 513-518.

Nash, J. F., "The bargaining problem," Econometrica 18 (1950), 155-162.

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