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THE EMERGENCE OF DYMANIC COMPLEXITIES IN MODELS OF OPTIMAL GROWTH: THE ROLE OF IMPATIENCE.*

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Abstract

The presence of chaotic behavior in optimal growth models, with finite and infinite planning horizon, is discussed.
A density theorem is given which relates the optimal paths of infinite horizon models to those of the finite horizon case. The theorem proves that chaotic solutions become possible when the discount parameter is small enough.
Besides two simple examples are given. The first considers a standard one-sector neo-classical model where wealth effects are allowed to be included among the arguments of the utility function. The second is a two-sector model with linear technology where the optimizing agents have to maximize utility through the optimal choices of consumption and work effort.

CONTENTS

1. INTRODUCTION

A large part of the literature on optimal economic growth studies the role played by the social rate of impatience in determining stability conditions for the optimal steady state solutions.

A well known result assures that in multisector models of economic growth the optimal paths converge to a unique steady state when the future utilities are not discounted, i.e. when the social rate of discount δ is equal to one and the rate of impatience is then equal to zero.

Some famous examples due to Kurz [14], Sutherland [23], Weitzman (see Samuelson [20]) show that global stability is not assured, in general, when δ is smaller than one. That is to say that in the absence of specific hypothesis we do not have any precise knowledge on the dynamic behavior of the optimal accumulation rules.

However some interesting conditions for stability have been given using assumptions on the curvature of the utility function when δ is in a small neighbourhood of one, as for example in Brock and Scheinkman [3], and in McKenzie [17].

The aim of this paper is that of studying the behaviours of the optimal paths arising from a general multisectorial model of optimal growth when the rate of impatience is very high, that is δ is in a small neighbourhood of zero: we will show that in this case the optimal paths can be of "almost any type".

From this result it follows that also the so called "chaotic paths" are admissible as solutions of optimal problems: we will give two examples of this fact.

In Section 2 the main assumptions of the model are described and we show how a dynamical system can be associated to every problem of mathematical optimization. In Section 3 we study some mathematical properties of the dynamic processes associated to concave optimal models and we conclude the analysis stating an important genericity result.

The two examples of chaotic dynamics given in Section 4 are general enough to prove that these types of phenomena can easily raise in optimal growth problems, even if the very simple one- or two-sector economies are considered.

We conclude the paper discussing some possible relations between our results and the well famous Turnpike Theorems.

2. ACCUMULATION PATHS THAT ARE DYNAMICAL SYSTEMS.

In this Section we introduce the general framework which underlies our study of the problem of economic growth. As it is easily seen the pure mathematical model is general enough to couple with many others economic optimization problems.

We are concerned with the quasi-stationary model in the discrete time version. We use the reduced form for the objective function, where the evaluation of each period depends on the initial and terminal states (usually a vector of capital stocks) instead of events within the period (tipically the consumption flows). When the model is written in this form it allows for very general types of dependence of the social utility function on the states of the economy: the vector of states can be thought of as a general description of the economy and the usual identification of it with the per-capita capital stocks is a simple matter of tradition and simplicity.

This greater generality is not a minor point in our approach: it enables us to think of the forms that the objective function assumes as belonging to a very broad class. Two particular cases are discussed below in Section 4. The economic model we have in mind is essentially McKenzie's [16] or [18,par.7] with only some minor modifications.

As we said we do not assume technical progress, nor the existence of exhaustible natural resources, so that the technology remains the same in every period. The utility function is thought as well as invariant with respect to time and independent from the past events: this guarantees that the only way in which time affects our world is through the discount factor, in this way $u_t(\bullet) = u(\circ)\delta^t$, for all t = 1,2,3,... is the istantaneous utility function.

As usual the population is assumed growing at a given exogenous rate , the participation ratio is constant and full employment will be achieved in every period. We can express the per-capita quantity of the capital stocks using the n-component vector $k_t \varepsilon R_+^n$.

The technology is represented by a non-empty valued and continuous correspondence $F(\circ)$, associating with each initial state vector $k_{t-1}\ \varepsilon R_+^n$ a set $F(k_{t-1})$ of possible final states among which the society selects the next-period's initial state vector according to an opportune maximization rule.

The following basic assumptions are made:

(A.1) The feasible transformation set $D = \{(k_{t-1}, k_t), k_t \varepsilon F(k_{t-1})\ \}$ is a non-empty, convex and compact subset of $K \times K$, where K is a compact subset of the positive orthant of the n-dimensional euclidean space. This implies that $F:K \rightarrow K$ is a continuous set-valued correspondence.

(A.2) The utility function $u: K \times K \rightarrow R$ is a strictly concave and continuous function.

(A.3) From $(x,y,)\varepsilon D$ it follows $(x',y')\varepsilon D$ for all $0 \leqslant y' \leqslant y$ and

$x' \geqslant x$.

(A.4) There exists a k_o with $(k_o,k)\varepsilon D$ and with $k>k_o$.

Some brief comments are useful.

The first hypothesis brings two kinds of restriction: convexity of D implies the concavity of the production correspondence F which is a relevant limitation to the generality of the model, see on this point McKenzie [18,par.2]. Most, but not all, of our arguments can be extended to the non-convex case: namely those based on compactness of D and on continuity of $u(\cdot)$. The compactness of D is the second relevant limitation in (A.1) but, in our opinion, the mildest one.

Economically speaking we have only to think at the existence of a primary factor which is essential to the production but which is available only in a limited amount, say labor for instance; then if the technology is such that it is impossible to produce something with nothing the boundedness of D is a logical consequence. Infact we encounter this assumption very often in the current literature, see for example McKenzie [16] and [18], Gale [11], and Sutherland [23]. In some places it is stated in a more indirect form, that is:

- There exists positive values C and $\gamma<1$ such that $||k||>C$ and $(k,k')\varepsilon D$ implies $||k'||<\gamma||k||$, (McKenzie [16]).

From this hypothesis it is easy to derive a compact transformation set on which the optimal problem is defined (see again McKenzie [16, p.357]).

Finally: (A.3) is the well known "free disposal" assumption and (A.4) simply states that an expansible capital stock does exist: they are both stated for the sake of completeness and will not play a mayor role in the subsequent discussion.

Let us call a (feasible) capital accumulation path with initial state k_0 any sequence $\{(k^*_{t-1},k^*_t)\}$ for $t = 1,\ldots,T$ of pairs (k^*_{t-1},k^*_t) in D such that $k^*_0 = k_0$. Now the classical problem of the optimal growth theory can be stated as: find an accumulation path $\{(k^*_{t-1},k^*_t)\}$ for $t =1,\ldots,T$ such that it solves:

$$\text{Max: } \sum_{t=1}^{T} u(k_{t-1},k_t)\delta^{t-1}$$

$$\text{s.t. } (k_{t-1},k_t)\varepsilon D$$

$$k_T \geqslant 0 \varepsilon K \text{ if } T<\infty$$

$$k_0 \text{ given}$$

To underline its dependence on the initial conditions, the time horizon and the discount parameter we will call this problem $P(k_0, \delta, T)$.

Because we are interested in the infinite horizon case and we are going to use the finite horizon problem as a simple explanatory device only, the final constraint has been relaxed to the non-negativity of k_T for $T< \infty$ (i.e. no positive bequest are required). Under (A.1) - (A.4) problem $P(k_0, \delta, T)$ has one and only one solution for every k_0 in K .

Let us note that, at this stage of the analysis most of the current literature tackles the problem of the existence of prices supporting the optimal path in a competitive economy. Our purpose is quite different and we do not need the properties of the supporting price sequence: this follows from the fact that we do not use any duality argument in our reasoning, we apply instead the dynamical system theory. Nevertheless the interested reader can easily see that, with the weak requirement of the existence of an interior solution, a vector of prices supporting the optimal solution of $P(k_o, \delta, T)$ can be derived, see McKenzie [17, par.3]

This permits the interpretation of all the following results as possible outcomes of the dynamic evolution of an economy which is in competitive equilibrium. This last fact has some possible implications for the equilibrium business cycle theory, but they will not been explicitely considered here.

Now we can move to the core of our argument showing how one can associate to each Problem $P(k_o, \delta, T)$ a dynamical system over K.

We begin by considering the case $T = +\infty$ where no terminal condition is required. Let $\{k_o^*, k_1^*, \ldots\}$ be the unique solution of $P(k_o, \delta, \infty)$. Define as τ_δ: K→K the map sending k_o^* to k_1^* . By the Bellman's Principle it follows that: $k_1^* = \tau_\delta(k_o^*)$, $k_2^* = \tau_\delta(k_1^*)$, ... , $k_t^* = \tau_\delta(k_{t-1}^*)$.

In other words: the values of the capital stocks which belong to the optimal solution of $P(., \delta, \infty)$ are generated by the one parameter family of discrete dynamical systems: $k_t^* = \tau_\delta(k_{t-1}^*)$, $k_o^* = k_o$, with $\delta \varepsilon (0,1)$ as the discount parameter.

In a strict sense it is not possible to consider $P(k_o, \delta, T)$ as a dynamical system when T is finite, both for the presence of a terminal condition and because the system will end up after a finite number of steps. In view of the future use of $P(k_o, \delta, T)$ with $T< \infty$, the following approach, already adopted in Montrucchio [19], can be used here.

We assume that the system is controlled from $t = 0$ to $t = + \infty$ by an agent taking a sequence of optimal decisions, each one lasting for a finite number of periods, say T . This technique is frequently encountered in the economic literature on optimal short-run programming, see Benhabib and Day [1], Day and Kennedy [6] and particularly Intrilligator [12] where a general argument is given; the technique can be used also in the analysis of some overlapping generations models.

The above case can arise when the technology and the preferences are not varying over time, but for some institutional or psychological reason the agent has the power of deciding his action only for a finite sub-period of the economy's life.

Then the T-miopically optimal sequence $\{k_o^*, k_1^*, \ldots\}$ of capital stocks is the solution of the following programming problem:

$$\text{Max: } \sum_{t=hT+1}^{(h+1)T} u(k_{t-1},k_t)\delta^{t-1}$$

$$\text{s.t } k_t \varepsilon F(k_{t-1}) \quad (1)$$

$$k_o \varepsilon K \text{ given,}$$

$$h = 0,1,2,...$$

Using this approach it is easily seen that for any unique solution $\{k_o^*, k_1^*, ...\}$ of $P(k_o, \delta, T)$, $(T< \infty)$, we can define a map θ: $K \to K$ associating k_T^* to k_o^*, that is θ is the initial-state to terminal-state map. The dynamical system is then derived considering the h = 0,1,2,..... iterations of the same mapping and it is defined as:

$$k_h^* = \theta(k_{h-1}^*), \ k_o \varepsilon K \quad (2)$$

$$h = 0, 1, 2, ...$$

A brief consideration will suggest that θ is in this case the Poincare map of the dynamical system associated to (1), in this way a fixed point of θ corresponds to a T-periodic cycle for the accumulation path which solves (1).

3. A MATHEMATICAL DIGRESSION

This Section is entirely devoted to a mathematical analysis of the properties of the optimal dynamic processes θ and τ_δ, the theoretical implications of which will be examined in Section 4 and 5.

In order to underlie the dependence of θ on δ and T we will write $\theta(\delta.T)$ for the general case and simply θ when $T = 1$, as the Poincare map does not depend on δ in this case.

Let us start with the extreme situation where $T = 1$ and $P(k_o, \delta, T)$ simply reduces to:

$$\begin{aligned} &\text{Max: } u(k_o, k_1) \\ &\text{s.t. } k_1 \varepsilon F(k_o) \\ &k_o \varepsilon K, \text{ given.} \end{aligned} \tag{3}$$

Because we are interested only in the mathematical properties of the problem we will not seek any economic interpretation of it. The solution of (3) gives raise to the short-run dynamic system $k_t^* = \theta(k_{t-1}^*)$ where θ is found by solving:

$$\theta(k_o) = \text{Argmax}\{u(k_o, k_1), \text{ s.t. } k_1 \varepsilon F(k_o)\} \tag{4}$$

We are now in position to give our first theorem. This could be named "The Inverse Problem Theorem": it asserts that under our hypothesis the short-run dynamics can be of any type, provided that $\theta(k^*) \varepsilon F(k^*)$.

THEOREM 3.1 Let $\theta: K \to K$ be given, where $K = \bar{X}$ is a compact and convex subset of R^n and X is open. Let θ be a C^2-function on X, continuously extendable with its derivatives on K. Then there exists a strictly concave and continuous function $u: K \times K \to R$ such that:

$$u\{k_o, \theta(k_o)\} = \underset{k_1 \varepsilon K}{\text{Max}}\ u(k_o, k_1)$$

Proof.

Let us define $u(k_o,k_1)$ as:

$$u(k_o,k_1) = a - M[1/2||k_1||^2 - \langle k_1, \theta(k_o)\rangle + (L/2)||k_o||^2],$$

where we take $a \varepsilon R$ and $M,L>0$. In Montrucchio [22] it is proved that u is a strictly concave function on $K \times K$ when L is positively high enough. Q.E.D.

Theorem 3.1 is a major tool for obtaining the genericity results we claim for: it asserts that for T = 1 every type of dynamics can be a solution of a concave-maximization problem. This fact implies that even very "well conformed" utility functions can produce "undesirable" optimal paths. That is to say that assumptions (A.1)-(A.4) are too much generic to guarantee the kind of dynamics we are usually looking for. It becomes then straightforward to look for supplementary acceptable conditions on $u(\cdot)$ and $F(\cdot)$, such that θ results in a well conformed dynamics.

Let us move to examining the case T>1. In Section 2 we showed that optimal paths of infinite programmings $P(k_o, \delta, \infty)$ are generated by the policies τ_δ . Similarly, in the case of finite horizon, the dynamic behavior will be summarized in the Poincare mapping $\theta(\delta,T):K\to K$. Note that, generally speaking, both τ_δ and $\theta(\delta,T)$ are analitically intractable.

Let us indicate with $W_{\delta,T}(k_o)$ and $W_\delta(k_o)$ the usual value functions, respectively for the finite and the infinite horizon case, that is:

$$W_{\delta,T}(k_o) = \text{Max} \sum_{t=1}^{T} u(k_{t-1},k_t)\delta^{t-1} \qquad (5)$$
$$\text{s.t. } k_t \varepsilon F(k_{t-1})$$

and:

$$W_\delta(k_o) = \text{Max} \sum_{t=1}^{\infty} u(k_{t-1},k_t)\delta^{t-1} \qquad (6)$$
$$\text{s.t. } k_t \varepsilon F(k_{t-1})$$

Then $W_{\delta,T}(k_o)$ and $W_\delta(k_o)$ turn out to be strictly concave for every value $0<\delta<1$, moreover they satisfy the Bellman's equations:

$$W_{\delta,T}(k_o) = \text{Max}\{u(k_o,k_1) + \delta W_{\delta,T-1}(k_1)\}, \qquad (7)$$
$$\text{s.t. } k_1 \varepsilon F(k_0)$$

$$W_{\delta,0} = 0, \text{ and} \qquad (8)$$

$$W_\delta(k_o) = \text{Max } \{u(k_o,k_1) + \delta W_\delta(k_1)\} \qquad (9)$$
$$\text{s.t. } k_1 \varepsilon F(k_o)$$

It also a well known result that $W_{\delta,T}$ can be obtained by iteration of a map acting on the space of continuous functions. Formally: let $C^0(K)$ be the space of all the real-valued continuous functions on K, endowed with the uniform topology.

Consider the operator U_δ: $C^0(K) \to C^0(K)$ defined as:

$$(U_\delta f)(x) = \text{Max}\{u(x,y) + \delta f(y) \text{ s.t. } y \in F(x)\} \tag{10}$$

It has been proved, see Denardo [8] and Flynn [10] for a recent exposition, that U_δ turns out to be a contraction with modulus δ, i.e.: $||U_\delta(f) - U_\delta(g)|| \leq \delta||f - g||$. Moreover: the value function W_δ is the unique fixed point of U_δ, i.e.: $U_\delta(W_\delta) = W_\delta$, and the $W_{\delta,T}$ are obtained by iterations of U_δ starting at 0, that is: $W_{\delta,T} = U_\delta^{(T+1)}(0)$, where $U_\delta^{(T+1)} = U_\delta \circ U_\delta^T$.

We need three preliminary Lemmas:

<u>LEMMA</u> <u>3.1</u> The maps $\delta \mapsto W_\delta$ and $\delta \mapsto W_{\delta,T}$ are continuous from [0,1) into $C^0(K)$.

<u>Proof</u>. It follows easily considering the above discussion and the estimates: $|W_{\delta^1,T} - W_{\delta^2,T}| < |\delta^1 - \delta^2|M/(1-\delta^1)(1-\delta^2)$, where we have set $M = \sup|u(k,k')|$. Q.E.D.

Define now as $\Xi^0(K \times K)$ the space of all the continuous and strictly concave functionals over $K \times K$, endowed with the uniform topology, (C^0-topology), and define also as $C^0(K;K)$ the space of all the continuous maps from K to K with the C^0-topology. Define then as $C_F^0(K;K)$ the closed subspace of $C^0(K;K)$ which contains all the maps $f(\cdot)$ such that $f(k) \in F(k)$ for every $k \in K$. Here F indicates the production correspondence defined in Section 2. Now we can prove:

LEMMA 3.2 Let θ be the map defined in (4), then the map $J: u \to \theta$ between $\Xi^0(K \times K)$ and $C^0_F(K;K)$ is continuous.

Proof. It follows easily from the Maximum Lemma (see Berge [2]). Infact the map $(u,k_o,k_1) \mapsto u(k_o,k_1)$ from the product space $\Xi^0(K \times K) \times K \times K$ in R is continuous. For the Maximum Principle the map $(u,k_o) \mapsto \text{Argmax}\{u(k_o,k_1),\ k_1 \in F(k_o)\}$ of $\Xi^0(K \times K) \times K$ in K is continuous. Hence the compactness of K implies that the mapping from u to θ of the space $\Xi^0(K \times K)$ in $C^0_F(K;K)$ is continuous.
Q.E.D.

Finally we can state:

THEOREM 3.2 Under Assumptions (A.1)-(A.4) the following properties are true

a) τ_δ and $\theta(\delta,T)$ are continuous maps from K to K
b) The maps $\delta \mapsto \tau_\delta$ and $\delta \mapsto \theta(\delta,T)$ are continuous from the interval $[0,1)$ into $C^0(K;K)$.

Moreover the following two limit-relations hold:

- $C^0\text{-}\lim_{\delta \to 0^+} \theta(\delta,T) = \theta^T$

- $C^0\text{-}\lim_{\delta \to 0^+} \tau_\delta = \theta$

Proof. We first prove the Theorem for the case in which it is $T = \infty$, that is for the function τ_δ .

Remember that τ_δ has been obtained as:

$$\tau_\delta(k) = \text{Argmax } \{u(k,k') + \delta W_\delta(k'), \text{ s.t. } k' \varepsilon F(k)\}.$$

Then point a) follows immediately from the Maximum Lemma, see Berge [2].

With regard to part b), let us recall from Lemma 3.1 that the map $\delta \mapsto W_\delta$ from $[0,1)$ into $C^0(K)$ is continuous as well as the map $\delta \mapsto \{u(k,k') + \delta W_\delta(k')\}$ from $[0,1)$ into $\Xi^0(K \times K)$.

Then Lemma 3.2 implies the continuity of the composition of the latter with the map J defined in Lemma 3.2 itself. But this composition of maps gives raise to the map $\delta \mapsto \tau_\delta(k)$ considered in the part b) of our Theorem.

The C^0-lim is easily understood considering that:

$$\tau_0(k) = \text{Argmax } \{u(k,k')\} = \theta(k)$$

by definition.

The case $T < \infty$, i.e. the map $\theta(\delta,T)$, is completely analogous to the former if the following recursive relations are considered.

For the Bellman's Principle one has:

$$\theta(\delta,T) = \theta(\delta,T-r) \circ \theta^{(r)}(\delta,T)$$

after having defined $\theta^{(r)}$ as:

$$\theta^{(r)}(\delta,T)\colon k_0 \mapsto k_r^*, \text{ for } 1<r<T,$$

where $\{k_0^*, k_1^*, \ldots.\}$ is the optimal solution of $P(k_0, \delta, T)$.

From this fact we deduce that the map $\theta(\delta,T)$ can be factorized as:

$$\theta(\delta,T) = \theta \circ \theta^{(1)}(\delta,2) \circ \theta^{(1)}(\delta,3) \circ \ldots. \circ \theta^{(1)}(\delta,T),$$

and then the result $\theta(0,T) = \theta^T$ follows.

Because we also have:

$$\theta^{(1)}(\delta,T) = \text{Argmax } \{u(k,k') + \delta W_{\delta,T-1}(k'), \text{ s.t. } k' \varepsilon F(k)\}$$

it is now immediate to see that we can repeat for $\theta^{(1)}(\delta,T)$ the same argument we used above for τ_δ .
Finally, because $\theta(\delta,T)$ is the composition of a finite number of maps $\theta^{(1)}(\delta,\cdot)$ the propositions a) and b) follow also for the finite horizon case. Q.E.D.

Some explanations on the implications of Theorem 3.2 seem to be useful at this point:

1) The optimal paths of capital accumulation are generated by the dynamic system $k_t^* = \tau_\delta(k_{t-1}^*)$ where the continuous map τ_δ is the one that maximizes the concave functional $\{u(k,k') + \delta W_\delta(k')\}$. For δ varying on $[0,1)$ we have a one parameter family of dynamical systems. An analogous argument is true for the Poincare mapping $\theta(\delta,T)$.

Moreover the Lemma 3.2 tells us that from concave functionals close to each other we derive short-run choice functions close to each other (according to the C^0-topology). Because the optimal paths of the finite horizon problems are derived by composition of short-run choice functions the result will extend to them.

2) Theorem 3.2 states that not only τ_δ and $\theta(\delta,T)$ are continuous on K, but also that some relevant limit relations hold between them. The latter are very important because they imply that the capital accumulation paths of models with infinite horizon become as close as we like (in the C^0-topology) to accumulation paths of myopic (T=1) models when the discount parameter δ becomes close enough to zero.

3) Finally the existence of steady state values of the capital stocks k for $P(k^{o},\delta,\infty)$ is easily assured by the continuity of τ_δ and the compactness of K via the Brower fixed point Theorem. An analogous argument can be conducted on $\theta(\delta,T)$ to prove the existence of a fixed point for the map $\theta(\delta,T)$: $K\to K$. Now, remembering that $\theta(\delta,T)$ has been defined as the Poincare map of the dynamic system underlying (1), it turns out that in correspondence to a fixed point of $\theta(\delta,T)$ there exists a T-periodic cycle which is an optimal solution to (1) when this maximization problem has to be replicated over time.

These three facts, when associated with the result of Theorem 3.1, will suggest that also the dynamic motions coming from the models with infinite horizon can be of almost every type if the discounting of the agents manifest a high degree of impatience, that is if δ is very small.

The next proposition we will prove provides a formal version of the above intuition.

As before let $C^0(K;K)$ be the space of all the continuous functions from K to K and let $C^0_F(K;K)$ be the closed subspace consisting of all the functions $f(\cdot)$ such that $f(k)\in F(k)$ for each $k\in K$, both spaces are endowed with the C^0-topology. Then we state:

<u>THEOREM</u> <u>3</u>.<u>3</u> Let $K = \bar{X}$, where X is open in R^n, then under the hypothesis (A.1)-(A.4) and the additional one:

(A.5) $\text{int}F(k) \neq \emptyset$ for every $k\in K$

the set of optimal accumulation paths τ_δ: $K\to K$ which are solutions of $P(k_o,\delta,\infty)$ with K and $F(\cdot)$ fixed and with $u(\cdot,\cdot)$ ranging on $\Xi^0(K\times K)$ and δ on $(0,1)$, is dense in the space $C^0_F(K;K)$.

Proof. Let us begin by recalling that any function $f\varepsilon C^0_F(K;K)$ can be approximated by functions with range contained in the interior of the set $F(k)$. In fact, being $k\mapsto \text{int}F(k)$ lower semi-continuous, Michael's Theorem (see for example Florenzano [10,p.44]) implies that there exists a continuous selector $\mu(k)$ such that $\mu(k)\varepsilon \text{int}F(k)$ for any $k\varepsilon K$.

Thus the family of functions: $f_\alpha(k) = \mu(k) + \alpha[\ f(k) - \mu(k)]$ has the property:

$$C^0\text{-}\lim f_\alpha = f \text{ as } \alpha\to 1^- \text{ and } f_\alpha(k)\varepsilon \text{int}F(k)$$

for every $k\varepsilon K$ and each $\alpha<1$.

As a second step we use the fact that the space $C^2(\bar{X};K)$ is dense in $C^0_F(K;K)$. This relation follows from the point above and from the standard result of density of $C^2(\bar{X};K)$ in $C^0(K;R^n)$.

Now: consider any function $f\varepsilon C^0_F(K;K)$ and any neighbourhood of this function in $C^0_F(K;K)$. For the density argument we just recalled there will be a $\theta\varepsilon C^2_F(\bar{X};K)$ which lies in such a neighbourhood Theorem 3.1 assures that there will exist a concave function $u(k,k')$ such that:

$$\theta(k) = \text{Argmax } \{\ u(k,k'),\ \text{s.t. } k'\varepsilon F(k)\ \}.$$

Thus, from Theorem 3.2 , it follows that the maps τ_δ associated with $u(k,k')$ belong to that neighbourhood for δ in an appropriate right-neighbourhood of zero. This fact concludes the proof of the Theorem. Q.E.D.

The density Theorem just proved is the central result of this paper and, in our opinion, it allows a better understanding of the mechanisms which originate dynamical complexities for the optimal paths of capital accumulation.

As the range of behaviors of maps $f: K \to K$ belonging to $C^0_F(K;K)$ is very wide, then any such behavior is a candidate for maps of the type τ_δ too.

From the last argument we have to conclude that chaotic behaviors are, at least, "logically possible" .This conclusion is a direct implication of the very well known fact that most of the strange attractors and chaotic movements are persistent properties of maps (i.e. such properties are preserved under C^0-perturbations). From a purely mathematical point of view our statement is reinforced by some recent results, (see Butler and Pianigiani [4] and Siegberg [22]), according to which in the space of all the continuous functions from a real and compact interval on itself there exists an open and dense set of functions that are chaotic.

At this point we are lead to conjecture that erratic solutions for optimal programming problems can be found more or less easily and that their existence cannot be labelled as exceptional even in the simple one-dimensional growth model.

In the multidimensional case some analogous results on the degree of ergodicity suggest that erratic behaviors become more and more probable as the dimensionality of the model increases.

Unfortunately an analytic treatment of high dimension models seems, up to now, a prohibitive task.

4. TWO EXAMPLES OF OPTIMAL CHAOTIC PATHS OF CAPITAL ACCUMULATION.

Our last step is to provide a first concrete confirmation of the conjecture with which we ended the last Section. We will do it by giving a couple of simple examples of optimal accumulation problems that, under very general assumptions, are associated to chaotic maps τ_δ for low values of the discount parameter. The two examples we propose are, in some ways, "ad hoc" ones: we mean that we have chosen those specifications for the utility functions and/or for the production functions in order to get some desired results.

Nevertheless these functions satisfy all the standard hypothesis of the theory of optimal growth and they are not "exceptional" at all among those currently used in economic theory. This implies that we do not see any reason, on a pure a-priori ground, to refuse them as irrational or "irrealistic". On the contrary we can actually think of many "concrete" situations in which production or utility functions of these kinds can be usefully adopted.

Both examples produce a one dimensional map τ_δ: $K \to K$ as the final description of the optimal dynamics, with K being an interval on the real line.

This choice has been done for technical reasons: as a matter of fact maps τ_δ: $R^n \to R^n$ with $n \geqslant 2$ are, in most of the cases, analytically intractable and there is no definite method to work with nonlinear maps of dimension two or three or more when they are written in explicit form.

On the contrary the case $n = 1$ has been extensively studied in the last few years and, in particular, there is now a great deal of available knowledges on the characteristics of maps of the interval on itself that show erratic behaviors. A particularly good and extensive survey of these results is given in Collet and Eckmann [5]. They studied there a large set of "strange" behaviors for a parameterized family of dynamical systems. These dynamical systems are produced by the iteration of maps of the unimodal type.

A mapping f of the interval $[a,b]$ into itself is defined unimodal if:

U1) f is continuous;

U2) there exists a point c, $a<c<b$, such that $f'(c) = 0$;

U3) f is strictly increasing on $[a,c]$ and strictly decreasing on $[c,b]$.

The named authors consider a one parameter family of unimodal maps f_μ with $\mu \varepsilon [0,1]$ and, corresponding to various values of μ, they find that f_μ exhibits various kinds of strange behaviors: that is to say "chaos" in the Li and Yorke [15] sense, sensitivity to initial conditions, entropy, etc. .

From our point of view the upshot of their analysis can be summarized as follows. Once a parameterized unimodal map f_μ is given then we can almost surely calculate values of μ such that the dynamical system $x_{t+1} = f_\mu(x_t)$ exhibits some form of strange behaviors.

This simple but important result will be extensively used in the present Section. Infact we will simply concentrate on finding conditions under which the optimal policies τ_δ are unimodal and dependent on some parameters. Then the existence of chaotic regions for these parameters follows from the Collet and Eckmann's results and we will avoid to afford the explicit cumbersome computations. The careful reader is obviously referred to [5].

(4.1) Wealth effects in a one-sector neoclassical model.

It has recently been showed, see Dechert [7], that in the standard one-sector optimal growth model with a neoclassical production function and with people maximizing the present value of a discounted stream of future consumption flows over an infinite horizon, the optimal sequences of the capital stock are monotonic and chaos is therefore, impossible.

We will study now a very simple model where all but one of these assumptions are retained: namely we assume that net wealth, as it is measured by the existing capital stock $k_t \varepsilon K \subset R_+$, enters as an argument of the utility function.

For our purposes we need an analytic form of the utility function and we will adopt the standard Cobb-Douglas form.

The optimal problem is then stated as:

$$\text{Max: } \sum_{t=0}^{\infty} c_t^{\alpha} k_t^{1-\alpha} \delta^{t-1} \tag{11}$$

$$\text{s.t. } 0 \leq c_t \leq f(k_{t-1}) - k_t$$

$$k_t \varepsilon K \text{ and } k_o \text{ given in K.}$$

where K is an interval on R_+. For the moment we will not choose any specific form for the production function f. As in [7] we simply allow for the case in which $f(\cdot)$ have regions where the marginal productivity of capital is negative and, obviously, we assume strict concavity of $f(\cdot)$ over K.

It is very simple to calculate the map $\theta(k_o)$ we have defined in (4) above; for the problem at hand it results to be:

$$k_1 = f(k_o)(1-\alpha) = \theta(k_o,\alpha) \tag{12}$$

The mapping $\theta(\cdot,\alpha)$: $K \to K$ is the desired one-parameter unimodal map: infact there are many possible specifications of an aggregate concave production function such that (12) turns out to be unimodal. The ranging of α over the interval (0,1) will provide the degree of freedom sufficient to obtain those erratic dynamics we claimed to exist. For example the very simple specification $f(k_t) = k_t^{\beta} - k_t^{\gamma}$, satisfies all the three assumptions U1)- U3) above when $0<\beta<1$ and $\gamma>1$.

The authors have carried explicit calculations on this map and it shows chaotic behaviors in the sense of Li and Yorke [15] for a relevant set of values of the parameters.

Recalling the limit relations we proved to hold between θ and τ_δ, (see Theorem 3.2 point b) above), it is straightforward to conclude that there exists a neighbourhood $U_\varepsilon(0)$ of the origin such that, for all the $\delta \varepsilon U_\varepsilon(0)$ the map τ_δ which solves (11) is chaotic.

This very simple analysis suggests that the introduction of wealth effects in the standard neoclassical model of optimal accumulation can destroy the tipical monotonic behaviors which are usually accepted in the literature. It is immediate to conjecture that other natural modifications of the standard assumptions (e.g. those regarding the convexity of the technology set) can lead to non-standard behaviors for appropriate choices of the parameters.

We think that some brief observations can help the reader to understand this point and, in some sense, to discover the technical "trick" which underlies our result as opposed to Dechert's one.

The latter can be intuitively understood in a very simple way. An optimal accumulation path $\{k_0^*, k_1^*, \ldots\}$ is monotonic if and only if the policy function τ_δ is monotonic. Now the first derivative of τ_δ for the one-dimensional case is easily calculated using its definition:

$$\tau_\delta(k) = \text{Argmax}\{u(k,k') + \delta W_\delta(k'), \text{ s.t. } k' \varepsilon F(k)\} \qquad (13)$$

From the above:

$$\tau'_\delta(k) = - u_{12}(k,k')[u_{22}(k,k')+\delta W''_\delta(k')]^{-1} \qquad (14)$$

follows easily using the first order conditions; the notation is self evident.Now the concavity of u and W_δ implies that for every k along the optimal path:

$$\text{sign } \tau'_\delta(k) = \text{sign } u_{12}(k,k') \qquad (15)$$

Equation (15) explains both Dechert's and our result. Infact given the standard specification of the neoclassical one-sector optimal growth model, where $u(k_t,k_{t+1}) = u[f(k_t)-k_{t+1}]$ it is possible to see that, along any optimal path the sign of u_{12} is positive and not changing.

This happens even if the production function is allowed to exhibit zones of negative marginal productivity: the optimal path will never take up values in those zones.

When the technology or the tastes are such that the sign of u_{12} is negative then the optimal trajectory is an oscillating one: in this case there is the possibility of finding optimal n-cycles by studying the existence of fixed points for the n-th iterate of the policy function τ_δ .

What is more important,for us, is that the specification (11) introduces the possibility of switching for the sign of u_{12} also along optimal paths. This is the main building block of our result because, obviously, a unimodal map has to be non-monotonic: the presence of the parameters gives the residual degree of freedom to obtain chaos.

Notice that this argument makes sense for one-dimensional policy functions only. Nevertheless it is a powerful argument: whenever an optimal dynamics can be described by a one-dimensional policy function exotic phenomena are possible only if the specification of the model allows τ_δ to be non-monotonic. We have seen that this amounts to u_{12} changing sign over K K: this can be obviously satisfied by many different economic assumptions.

We have chosen the wealth effect both for its simplicity and because it has some "historical precedents" in the classical growth theory. We like infact to recall the work of Kurz [13] where it was shown that, in the continuous time case, the introduction of the capital stock as an argument of the utility function generates a (finite) molteplicity of steady states, some of which are unstable. Due to the greater sensibility of the discrete-time formulation we have shown that an analogous change of hypothesis is able to produce a result which appears even more disturbing but, in some sense, also more fascinating.

(4.2) Labor-leisure choice in an underdeveloped economy.

Our second example of possible chaotic optimal programs is a little more elaborated than (4.1) because it studies a two-sector economy. We consider a very simple world with a technology of the Gale's [11] type: there exist two sectors in the economy, one produces a consumption good using only the stock of capital whereas the second sector produces the capital good using both capital and labor.

The technology can be expressed in a linear form, the supply of labor is exogenously given and growing at a constant rate.

Let A be the input and B the output matrices, L is the vector of labor inputs, γ is the rate of depreciation of capital:

$$A = \begin{bmatrix} 0 & 0 \\ a_{21} & a_{22} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & 0 \\ (1-\gamma)a_{21} & b_{22} \end{bmatrix}$$

$$L = [0,1]'$$

Indicate with $v = [v^1, v^2]'$ the levels of activity in the two sectors. The constraints will be:

$$Lv_t \leqslant \lambda^{t-1} l_o \qquad (16)$$

and

$$Av_t \leqslant Bv_{t-1} \qquad (17)$$

where v_t is the level of activity at time t, l_o is the amount of available labor in the first period, λ is the rate of growth of the supply of labor.

We will indicate with $u_t = u(v_t/\lambda^{t-1})$ the istantaneous per-capita utility function. It is very well known, see [11], that the model can be reduced to one with constant supply of labor. We will do it here very briefly, the cautios reader is referred to [11] for more details.

Define $A' = \lambda A$, $B' = B$, $L' = L$ and $u_t^1 = u(v_t)$.

Assuming that the system is productive, that is $\lambda a_{22} < b_{22}$, we can formulate the optimal problem as follows:

$$\text{Max: } \sum_{t=0}^{\infty} u(v_t)\delta^{t-1} \tag{18}$$

$$\text{s.t. } \lambda a_{21}v_t^1 + \lambda a_{22}v_t^2 \leqslant (1-\gamma^1)\lambda a_{21}v_{t-1}^1 + (1+\sigma)\lambda a_{22}v_{t-1}^2$$

$$0 \leqslant v_t^1, \quad 0 \leqslant v_t^2 \leqslant l_o/l = \bar{l}$$

where $\gamma^1 > 0$ and $\sigma > 0$ are proportionality factors between A' and B'. The presence of the vector v_t as the argument of u is easily understood: people obtain utility from consuming more of the consumption good, but they experience a reduction in their utility from the time they spend working.

The space V of all the couples (v^1, v^2) is not compact because v^1 is unbounded: we get the compactness of V simply assuming a finite upper bound $\bar{v}^1$ for the level of consumption. This can be done ex-post, that is after that problem (18) has been solved: because to choose an infinite level of activity for the consumption sector implies the use of an infinite amount of labor, which is excluded by our assumptions.

Another difference with the standard programming problem we have been considering in Section 3 is due to the fact that the utility function does not have the initial states and the terminal states as arguments: also in this case the problem is easily solved considering that, if u is strictly concave, the sequence of v_t which is an optimal solution to (18) is still unique. Thus a dynamical system can be associated again to (18) and all the Theorems of Section 3 still hold.

To study the problem let us re-define our variables as:

$$\lambda a_{21} v_t^1 = \nu_t^1, \ \lambda a_{22} v_t^2 = \nu_t^2 \qquad (19)$$

and (18) becomes the following $P(\nu_o, \delta, \infty)$:

$$\text{Max:} \sum_{t=0}^{\infty} u(\nu_t)\delta^{t-1} \qquad P(\nu_o, \delta, \infty)$$

$$\text{s.t. } \nu_t^1 + \nu_t^2 \leqslant (1-\gamma^1)\nu_{t-1}^1 + (1+\sigma)\nu_{t-1}^2$$

$$\nu_t^1 \geqslant 0 \text{ and } 0 \leqslant \nu_t^2 \leqslant \bar{\bar{1}}$$

Now indicate with $\nu_{t-1} \mapsto F(\nu_{t-1})$ the production correspondence constraining problem $P(\nu_o, \delta, \infty)$ a geometrical illustration of which is given in Figure 1. below.

The solution to $P(\nu_o, \delta, \infty)$ will be a map $\tau_\delta : \nu_{t-1}^* \to \nu_t^*$, from which, using (19) the time path of the optimal levels of activity is immediately derived.

Because all the conditions of Theorem 3.2 are met we know that: $\lim \tau_\delta = \theta$ for $\delta \to 0^+$, where θ satisfies the condition: $\theta(\nu_{t-1}^*) \varepsilon F(\nu_{t-1}^*)$. The dynamical system induced by iteration of θ is a bidimensional one but we can reduce it to a one-dimensional dynamic system using the following considerations.

It is easily seen that all the points ν_t on the same segment with slope $-(1-\gamma^1)/(1+\sigma)$ are mapped into the same $F(\nu_t)$: it follows that the induced dynamic becomes monodimensional after only one iteration.

A closer idea of this reduction of dimensionality comes from the consideration of the monotonicity of u in ν^1: this simply amounts to assuming that the agents are locally non-satiable in the consumption good. Then we conclude that the agent will always choose ν^*_{t+1} on the north-east border of $F(\nu^*_t)$ in every period t.

In this case it is easy to express the one dimensional dynamic taking as the state variable the projection ζ on the ν^1 axis, in the direction -1, of the point ν^*_t .

We will indicate with $[\nu^*_1(\zeta), \nu^*_2(\zeta)]$ the maximum of u on the intersection between the line $\nu^1 + \nu^2 = \zeta$ and the feasible set $F(\nu)$. With some simple calculations we can derive the one dimensional dynamic θ as:

$$\zeta_t = (\sigma+\gamma^1)\nu^*_2(\zeta_{t-1}) + (1-\gamma^1)\zeta_{t-1} \tag{20}$$

We can concentrate our analysis on the first addendum because the second one, $(1-\gamma^1)\zeta_{t-1}$, is linear. Let us note that for $\gamma^1 \to 1$ (that is to say for a very high rate of capital depreciation) the second element of (20) disappears and the map θ reduces to $(1+\sigma)\nu^*_2(\zeta_{t-1})$.

Obviously the characteristics of θ as a dynamical system will depend on the form of the function $\nu^*_2(\zeta_{t-1})$. The latter depends on $u(\nu_t)$ so that we have to choose an analytic specification for the utility function. We have chosen the following:

$$u = (\nu^1)^{\alpha}(\nu^2 + \varepsilon\nu^1)^{\beta}(\bar{\bar{1}} - \nu^2)^{\kappa} \qquad (21)$$

which is concave (if $\alpha + \beta + \kappa \leqslant 1$) and it is also monotone in the consumption good ν^1. The particular features of (21) come from its non-monotonic behavior with respect to ν^2, which is an index of the amount of time spent at work. To be more precise we show in Figure 2.a, 2.b, and 2.c three sections of the graph of u with respect to ν^2 corresponding to three different values of ν^1, specifically for $\nu^1 = 0$, for $0<\nu^1\leqslant\beta/\varepsilon\kappa$ and for $\nu^1 > \beta/\varepsilon\kappa$.

The reader can see that the behavior of u with respect to ν^2 is non standard as long as the activity level of the consumption good industry is smaller or equal of a certain critical amount. Infact when ν^1 is in that small right interval of zero we assume the agent receives a positive utility from working up to a certain number of hours: after that point leisure time becomes desired as it is usually assumed. Note that the critical level of ν^1 can be made arbitrarily small by increasing κ and reducing $\alpha + \beta$, without influencing the qualitative results of our analysis.

Many different examples can be used to give an economic rationale to this behaviors: we think , first of all, at the tipical situation of an underdeveloped country with a large amount of unemployed labor, small opportunities of consumption and a very small stock of capital. It seems quite intuitive to think that in such a situation the optimal programmer will not evaluate leisure time as a source of positive utility for the members of the society.

This is, in our opinion, quite natural for two reasons: first the citizens do not posses anything to consume during their free time (case $\nu^1 = 0$) or, at most, a very small amount of consumption good is available; then surviving is not assured. This argument roughly correspond to the claim of a complementarity between leisure and consumption for very low level of the latter.

Also: with no consumption available the citizens will find highly valuable the time spent at work, because this produce the capital stock necessary in the production of the next period consumption good.

We like to stress that the non standard behavior of u is limited to a small neighbourhood of zero: infact the graph of the utility function respect to ν^2 assumes the ordinary downward sloping form when a certain positive amount of consumption good becomes available to the society.

Finally (21) can be interpreted in terms of a preference ordering over consumption and leisure such that when the former falls short of a certain minimum amount the ordering of the latter looses the monotonic relation with its quantities. With similar arguments the utility function (21) is extendable to the study of a household intertemporal choice between consumption and leisure. The presumption that a starving and unemployed person will greatly appreciate his first few hours of works sounds quite realistic, indeed.

Turning back to the mathematics of the model let us note that our claim on the existence of chaotic behaviors in situations of this type follows from the fact that the function $\nu_2^*(\zeta)$ is a unimodal map for $\varepsilon < \beta/(\alpha+\beta)$. With some simple calculations we see that the implicit function defining $\nu_2^*(\zeta)$ has the following expression:

$$\beta(1-\varepsilon)[\varepsilon\zeta+(1-\varepsilon)\nu_2^*]^{-1} = \alpha[\zeta-\nu_2^*]^{-1} + \kappa[\bar{l}-\nu_2^*]^{-1} \qquad (22)$$

The graph of (22) has the form depicted in Figure 3. for values of the parameters such that: $\varepsilon < \beta/(\alpha+\beta)$. The point indicated as $\bar{\zeta}$ can be calculated and it is equal to:

$$\bar{\zeta} = [\beta - \varepsilon(\alpha + \beta)]\bar{\bar{l}}(\kappa\varepsilon)^{-1}$$

These arguments are enough to conclude that the dynamic induced iterating θ (and the τ_δ for low values of δ) can be very rich and even chaotic, depending on the relative values of the parameters.

Also in this case we will avoid the explicit calculations referring again the reader to the techniques illustrated in [5] for unimodal maps. We like only to stress that also in this case complicated phenomena arise for the non invertible character of $\nu_2^*(\zeta)$ and for the presence of the parameterizing factor $(\sigma+\gamma^1)$ in front of it, (see equation (20)).

The two parameters σ and γ^1 can be used to modulate the curvature and the slope of the map θ and it can be seen that erratic phenomena will become more and more accentuated the larger σ and γ^1 are.

We do not intend to discuss further the economic implications of our analysis. As a matter of fact our purpose here is merely that of suggesting a method to discover unexpected results in standard and, apparently, well conformed economic models. It is not our intention to suggest any possible "realistic" application of these results. Nevertheless the reader would find amazing to compare the typical temporal path of a chaotic map with many business cycle data.

5. CONCLUSIONS

In this paper we have shown that optimal trajectories coming from economic problems of optimal intertemporal programming exhibit a large variety of behaviors depending on the values of the discount parameter δ. In particular when δ is in a small neighbourhood of zero they may reach erratic behaviours. This fact has several implications, for example the extreme sensitivity on initial data which leads to impredictability, exactly as in random systems.

On the other hand it is opportune to remember that such a variety of phenomena comes to be (at least partially) destroyed as δ increases and approaches the value one. Perhaps Figure 4. can give a useful metaphoric picture of these evolutions: the image has been elaborated for a specific map coming from a problem $P(k_0, \delta, \infty)$ with:

$$u(k_{t-1},k_t) = M - ak_{t-1}^2 k_t + ak_{t-1}k_t - 1/2k_t^2 - L/2k_{t-1}^2 ,$$
$$\text{with } (k_{t-1},k_t) \varepsilon D = [0,1] \times [0,1]$$

where the value of the parameter a is close to 4 and $L>a^2$. We refer to Montrucchio [19] for details. It represents the tipical bifurcation diagram of a one-parameter map on the intervall (the so-called Feigenbaum scenario, see [5]).

The disappearing of the strange attractors as δ increase is very strongly related, in the contest of economic growth, to the well famous Turnpike Theorems. Shortly speaking we can say that, under suitable assumptions, the Turnpike results guarantee that all optimal paths shrink and converge to a unique stationary solution. This fact can be related to ours by immagining the values one and zero as the "Alice's mirror image" of one other: when the parameter δ moves from one to zero the well organized and regular world of the Turnpike Theorems becomes its contrary: chaotic movements and impredictability of the social choices appear.

Because δ is, fundamentally, an inverse measure of people's degree of impatience we see that in the absence of myopia (i.e. $T = +\infty$), and of uncertainty regarding future events chaos is caused by the impatience in planning. Of course if myopia is present (i.e. the case $T < \infty$) the dynamics of the capital stocks become more and more complicated and erratic behaviors can appear also for "non small" δ. In our opinion the finite horizon case needs further investigation and we conjecture that very interesting phenomena can arise under various acceptable conditions. Unfortunately the analytical treatment is not simple and we will tackle the problem in a following work.

A final brief observation is useful: our two examples suggest that when people increase the degree of "rationality" of their choices, that is they take care of many state-variables in evaluating their utility, chaotic paths become more and more probable. On the other side if they adopt "naive" decision rules the optimal trajectories have a simpler and more regular set of attractors. We think that this other dycotomy could also be useful for future reseraches. It suggests a non-monotonic relation between the degree of "rationality" (as it is usually defined) and "predictability of economic behaviors".

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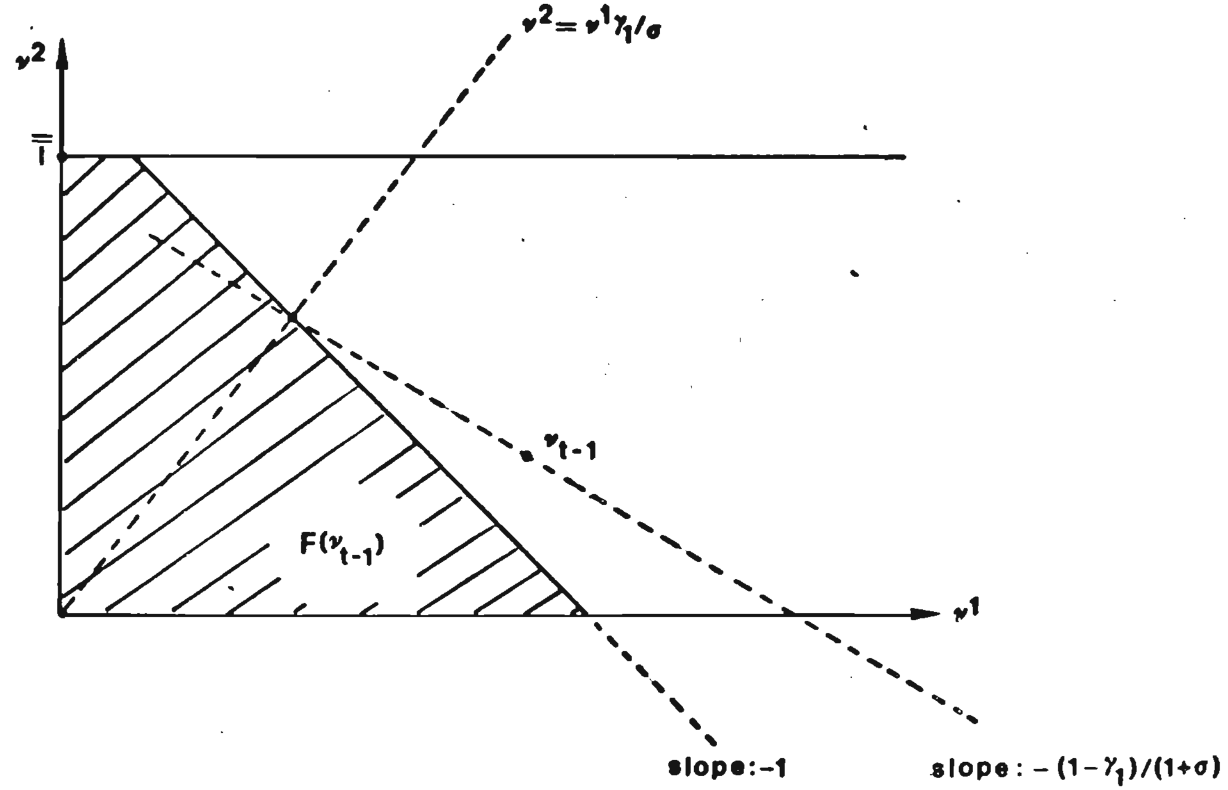
ν^2
$\bar{\bar{l}}$
$\nu^2 = \nu^1\gamma_1/\sigma$
ν_{t-1}
$F(\nu_{t-1})$
ν^1
slope:-1
slope: $-(1-\gamma_1)/(1+\sigma)$

Figure 1

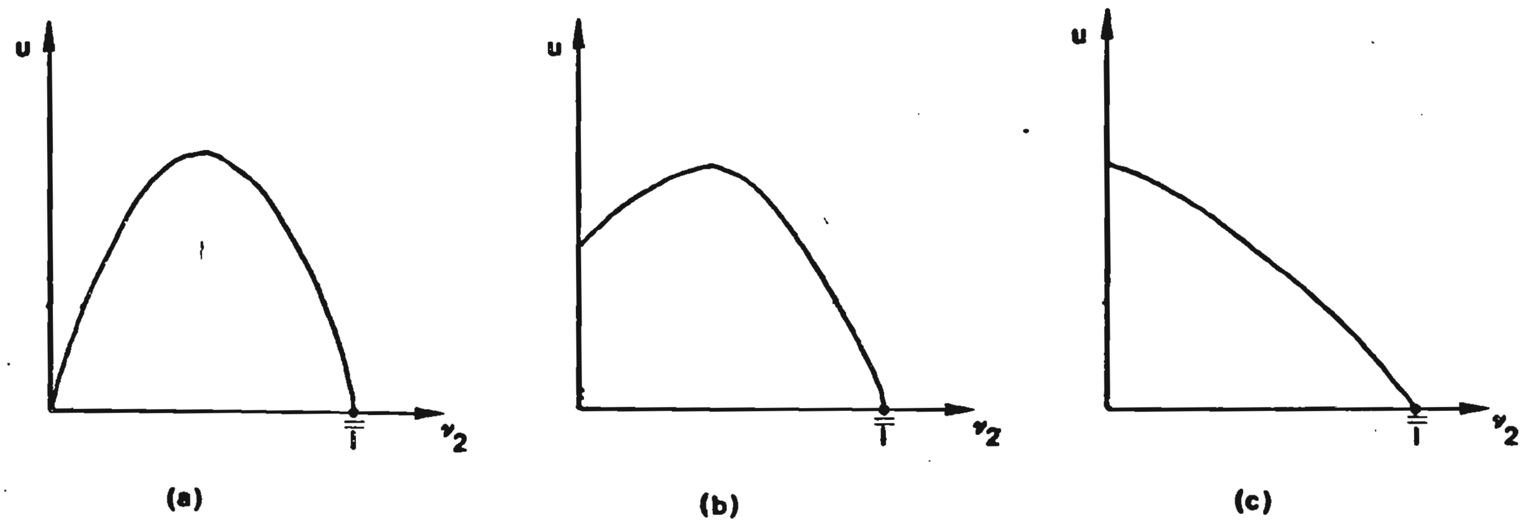

Figure 2

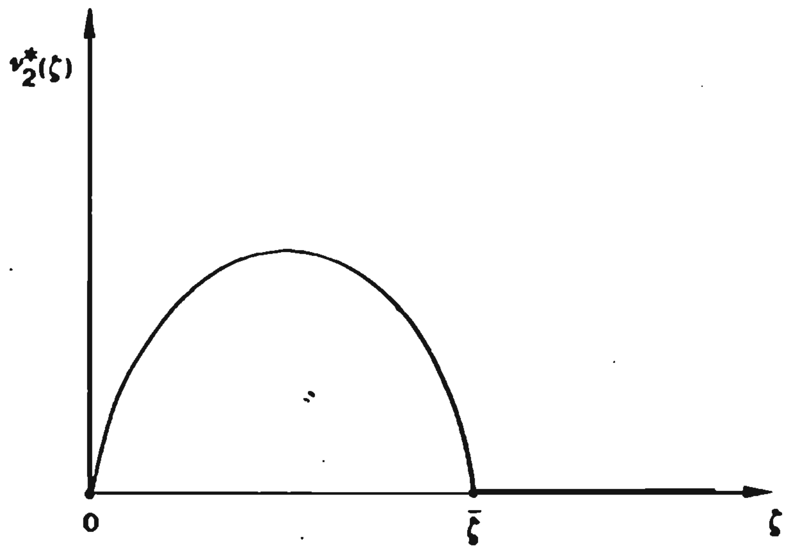

Figure 3

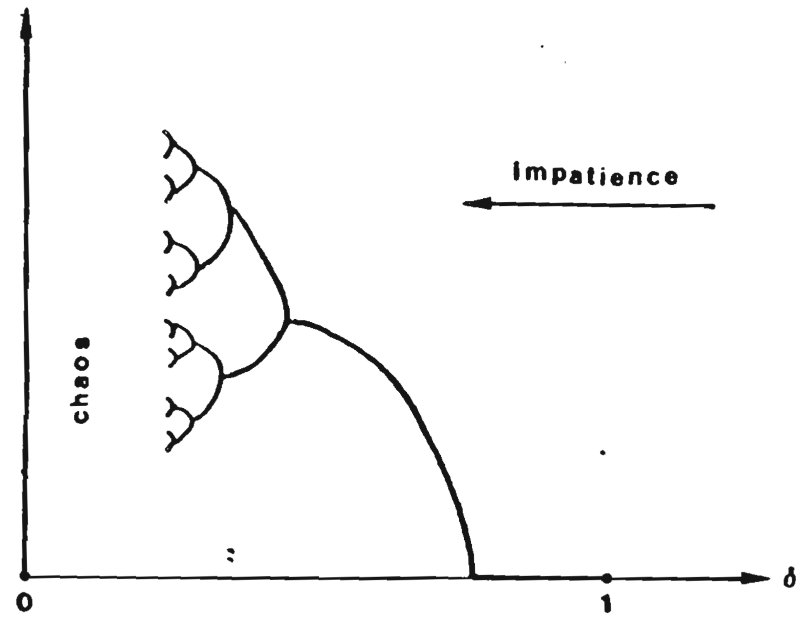

Figure 4

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