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ABSTRACT

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We analyze bargaining problems with known feasible sets and uncertain disagreement points. We impose on solutions the requirement of ***disagreement point concavity,*** which says that uncertainty affects all agents in the same way. This requirement is dual to one considered by Myerson [1981] and others for situations where the disagreement point is known but the feasible set is uncertain. Together with weak Pareto optimality, individual rationality and continuity, it characterizes a new family of solutions, the ***directional solutions:*** let δ be a function associating with each feasible set a non-negative direction. Then, the ***directional solution relative to*** δ is obtained by choosing as solution outcome of each problem the maximal feasible point such that the vector of utility gains from the disagreement point is in the direction determined by applying δ to the feasible set. These solutions generalize the weighted egalitarian solutions. This subfamily, as well as the egalitarian solution, which it contains, can be obtained by imposing various additional axioms, expressing that the solution should respond appropriately to certain changes in the shape of the feasible set.

1. Introduction

Most axiomatic studies of the bargaining problem, as formulated by Nash [1950], have been concerned with the responsiveness of solutions to variations in the feasible set. Recently, however, several papers have appeared which focus on the disagreement point (Livne [1986a,b], Peters[1986b] and Thomson [1987]). The present paper is also devoted to an analysis of the role played by the disagreement point in bargaining.

Specifically, we study situations in which the feasible set is known but the disagreement point is not. To illustrate, suppose bargaining takes place today, without the precise location of the disagreement point being known, this uncertainty being resolved tomorrow. One option open to the bargainers is to wait until tomorrow and to solve then whatever problem has come up. The other option is to use the expected value of the disagreement point as new disagreement point and to solve the resulting problem today. We impose on solutions the requirement that agents always agree on solving the problem today. This requirement of *disagreement point concavity* is dual to one considered by Myerson [1981] (variants of which are studied by Perles and Maschler [1981] and Peters [1986a]) for situations where the disagreement point is known but the feasible set is not.

We show that disagreement point concavity, when used in conjunction with the standard conditions of weak Pareto optimality, individual rationality and continuity, suffices to characterize a new family of solutions, which we call *directional solutions*. They are defined as follows. Let δ be a function associating with each feasible set a non-negative direction. Then, the *directional solution relative to* δ is defined by choosing as solution outcome of each problem the maximal feasible point such that the vector of utility gains from the disagreement point is in the direction determined by applying δ to the feasible set. These solutions generalize the well-known weighted egalitarian solutions. We also show how this subfamily, as well as the egalitarian solution, which it contains, can be obtained by imposing various additional axioms, expressing that the solution should respond appropriately to certain changes in the shape of the feasible set.

The paper is organized as follows. Section 2 contains some preliminaries and introduces the basic axioms. Section 3 states our main axiom of disagreement point concavity, and characterizes the family of directional solutions. Section 4 characterizes the family of weighted egalitarian solutions, and its distinguished member, the egalitarian solution. Finally, section 5 investigates in greater detail the role played by each axiom in our main result, as well as the importance of our choice of domain.

2. Preliminaries

An n-person bargaining problem, or simply a *problem,* is a pair (S, d), where S is a subset of $\Re^n$ and d is a point in S, such that

(1) S is convex and closed,

(2) S lies below some hyperplane with a positive normal, i.e., there exist $p \in \Re^n_{++}$ [1] and $c \in \Re$, such that for all $x \in S, p \cdot x \leqq c$,

(3) S is *comprehensive,* i.e., for all $x \in S$ and for all $y \in \Re^n$, if $y \leq x$, [2] then $y \in S$,

(4) there exists $x \in S$ with $x > d$.

S is the *feasible set.* Each point x of S is a *feasible alternative.* The coordinates

[1] $\Re^n_{++} \equiv \{x \in \Re^n | x_i > 0 \text{ for all } i\}$.

[2] Vector inequalities: given $x, y \in \Re^n, x \geqq y, x \geq y, x > y$.

of x are the von Neumann-Morgenstern utility levels attained by the n agents through the choice of some joint action. d is the *disagreement point* (or status quo). The intended interpretation of (S,d) is as follows: the agents can achieve any point of S if they unanimously agree on it. If they do not agree on any point, they end up at d. Let Σ^n be the class of all n-person problems and Γ^n be the class of all feasible sets satisfying (1), (2) and (3).

A *solution* is a function $F: \Sigma^n \rightarrow \Re^n$ such that for all $(S,d) \in \Sigma^n, F(S,d) \in S$. $F(S,d)$, the value taken by the solution F when applied to the problem (S,d), is called the *solution outcome of* (S,d).

We are interested in solutions satisfying the following axioms.

Weak Pareto Optimality (WPO). For all $(S,d) \in \Sigma^n$ and for all $x \in \Re^n$, if $x > F(S,d)$, then $x \notin S$.

Let $WPO(S) \equiv \{x \in S |$ for all $x' \in \Re^n, x' > x$ implies $x' \notin S\}$ be the set of *weakly Pareto-optimal points of S*. Similarly, let $PO(S) \equiv \{x \in S |$ for all $x' \in \Re^n, x' \geq x$ implies $x' \notin S\}$ be the set of *Pareto-optimal points of S*. If $WPO(S) = PO(S)$, then S is *strictly comprehensive*.

Individual Rationality (IR). For all $(S,d) \in \Sigma^n$, $F(S,d) \geqq d$.

Let $IR(S,d) \equiv \{x \in S | x \geqq d\}$ be the set of *individually rational points of (S,d)*.

In the formulation of the next axiom, convergence of a sequence of sets is evaluated in the Hausdorff topology.

Continuity (CONT). For all sequences $\{(S^k, d^k)\} \subset \Sigma^n$ and for all $(S,d) \in \Sigma^n$, if $S^k \rightarrow S$ and $d^k = d$ for all k, then $F(S^k, d^k) \rightarrow F(S,d)$.

WPO requires that there be no feasible alternative at which all agents are better off than at the solution outcome. IR requires that no agent be worse off at the solution outcome than at the disagreement point. CONT requires that a small change in the feasible set cause only a small change in the solution outcome. Note that we require continuity with respect to the feasible set, not with respect to the disagreement point.

The following notation and terminology will be used frequently. Given $x_1, \ldots, x_k \in \Re^n$, $con\{x_1, \ldots, x_k\}$ is the *convex hull* of these points (the smallest convex set containing them,) $comp\{x_1, \ldots, x_k\}$ is their *comprehensive hull* (the smallest comprehensive set containing them,) and $concomp\{x_1, \ldots, x_k\}$ is their *convex and comprehensive hull* (the smallest convex and comprehensive set containing them.) Given $A \subset \Re^n$, $Int(A)$ is the relative interior of A. Δ^{n-1} is the (n-1)-dimensional simplex. Finally, given $x \in \Re^n$ and $\delta \in \Delta^{n-1}$, $\ell(x, \delta)$ is the line passing through x in the direction δ.

3. The Axiom of Disagreement Point Concavity. The Main Characterization

First, we introduce our main axiom.

Disagreement Point Concavity (D.CAV). For all $(S^1, d^1), (S^2, d^2) \in \Sigma^n$ and for all $\alpha \in [0, 1]$, if $S^1 = S^2 \equiv S$, then

$$F(S, \alpha d^1 + (1-\alpha)d^2) \geqq \alpha F(S, d^1) + (1-\alpha)F(S, d^2).$$

Note that $(S, \alpha d^1 + (1-\alpha)d^2)$ is a well-defined element of Σ^n.

This axiom, which should be compared to the axiom of concavity used by Myerson [1981], can be motivated on the basis of very similar considerations of timing of social choice. Consider agents *today,* who, *tomorrow,* will face one of two equally likely problems (S, d^1) and (S, d^2), having the same feasible set, but different disagreement points. The agents have two options: either they wait until tomorrow for the uncertainty to be lifted and solve then whatever problem has come up, or they consider the problem obtained by taking as disagreement point the average of d^1 and d^2 and solve that problem today. The expected payoff vector associated with the first option is $\frac{F(S,d^1)+F(S,d^2)}{2}$ and that associated with the second option is $F(S, \frac{d^1+d^2}{2})$, since $\frac{d^1+d^2}{2}$ is the corresponding "expected" disagreement point. If either $F(S, \frac{d^1+d^2}{2})$ weakly dominates $\frac{F(S,d^1)+F(S,d^2)}{2}$ or the reverse holds, all agents agree on what to do. A conflict may arise if neither of these inequalities holds. Imposing D.CAV on the solutions prevents such a conflict.

Related axioms were introduced by Myerson [1981] (under the name of concavity), Perles and Maschler [1981] (under the name of super-additivity) and Peters [1986a] (under the name of partial super-additivity). All of these axioms are concerned with uncertainty about the feasible set, while disagreement point concavity is concerned with uncertainty about the disagreement point.

Our main theorem is a characterization of a new family of solutions, [3] which generalize the egalitarian solution. [4]

Definition. Given $\delta: \Gamma^n \to \Delta^{n-1}$, the *directional solution relative to* δ, E^δ, is defined by setting, for each $(S, d) \in \Sigma^n$, $E^\delta(S, d)$ equal to $\ell(d, \delta(S)) \cap WPO(S)$.

Throughout this paper, the following lemmas will be useful. Note that they do not involve CONT.

Lemma 1. *Let F be a solution satisfying WPO, IR and D.CAV. Also let* $(S, d) \in \Sigma^n$

[3] Actually we characterize continuous members of the family.

[4] See formal definition in Section 4.

be such that $F(S,d) \in Int(PO(S))$. Then for all $x \in \ell \cap Int(S)$, where ℓ is the line passing through d and F(S,d), $F(S,x) = F(S,d)$.

Proof. First, note that ℓ is well-defined and that $(S,x) \in \Sigma^n$ for all $x \in \ell \cap Int(S)$. Let $x \in \ell \cap Int(S)$ such that $x \neq d$ be given.

(a) $x \in (d, F(S,d))$. Let $\bar{\lambda} \in (0,1)$ be such that $x = \bar{\lambda} d + (1-\bar{\lambda})F(S,d)$, and $\{\lambda^k\} \subset (0,1)$ be such that $\lambda^k < \bar{\lambda}$ for all k and $\lambda^k \to \bar{\lambda}$. Also, let $x^k \equiv \frac{x-\lambda^k d}{1-\lambda^k}$ for all k. Note that $(S,x^k) \in \Sigma^n$ for all k. By D.CAV, $F(S,x) \geqq \lambda^k F(S,x^k) + (1-\lambda^k)F(S,d)$. As $k \to \infty$, $x^k \to F(S,d)$ and by IR and the fact that $F(S,d) \in PO(S)$, $F(S,x^k) \to F(S,d)$. Therefore, we obtain $F(S,x) \geqq F(S,d)$. Since $F(S,d) \in PO(S)$, $F(S,x) = F(S,d)$.

(b) $d \in (x, F(S,d))$. By a similar argument, we could show that $F(S,d) \geqq F(S,x)$. Since $F(S,d) \in Int(PO(S))$ and by WPO, $F(S,x) \in WPO(S)$, we have $F(S,x) = F(S,d)$. Q.E.D.

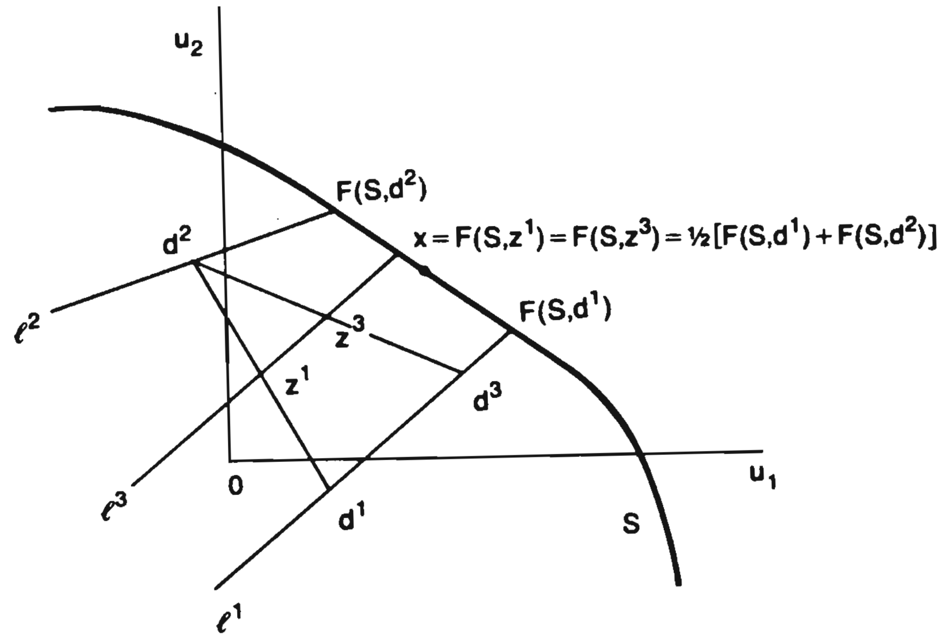
Lemma 2. *Let F be a solution satisfying WPO, IR and D.CAV. Also let $S \in \Gamma^n$ and $d^1, d^2 \in Int(S)$. If $\alpha F(S,d^1) + (1-\alpha)F(S,d^2) \in PO(S)$ for all $\alpha \in [0,1]$ and $F(S,d^1) \in Int(PO(S))$, then the line ℓ^1 passing through d^1 and $F(S,d^1)$ is parallel to the line ℓ^2 passing through d^2 and $F(S,d^2)$.*

Proof. Let S, d^1, d^2, ℓ^1 and ℓ^2 be as in the Lemma. By Lemma 1, for all $y \in \ell^1 \cap Int(S)$, $F(S,y) = F(S,d^1)$. Let $d^3 \in \ell^1 \cap Int(S)$ be such that $d^3 \neq d^1$. Without loss of generality, suppose that $d^3 \in (d^1, F(S,d^1))$. Now let $z^i \equiv (\frac{1}{2})(d^i + d^2)$ for $i = 1,3$. By D.CAV, we have

$$F(S,z^i) \geqq \frac{1}{2}\{F(S,d^i) + F(S,d^2)\} = \frac{1}{2}\{F(S,d^1) + F(S,d^2)\} \equiv x \qquad \text{for } i = 1,3.$$

Since $x \in PO(S)$, we have $F(S,z^i) = x$ for $i = 1,3$. Now let ℓ^3 be the line passing through z^1 and z^3. For all $z \in \ell^3 \cap Int(S)$ such that $z^3 = \lambda z^1 + (1-\lambda)z$ for some

$\lambda \in (0,1)$, by D.CAV and $\lambda < 1$, we have $x \geq F(S,z)$. Since $F(S,d^1) \in Int(PO(S))$ and $F(S,d^2) \in PO(S)$ imply $x \in Int(PO(S))$, we have by WPO, $F(S,z) = x$. By IR, ℓ^3 passes through x. This is possible only if ℓ^1 is parallel to ℓ^2. Q.E.D.



Proof of Lemma 2.
Figure 1.

Now we present our main result.

Theorem 1. *A solution satisfies WPO, IR, CONT and D.CAV if and only if it is a directional solution E^δ with an additional property, that δ be continuous.*

Proof. It is obvious that all E^δ satisfy WPO, IR and D.CAV and, if δ is continuous, CONT. Conversely, let F be a solution satisfying the four axioms. Let $(S,d) \in \Sigma^n$ be a polygonal problem such that $Int(PO(S)) \neq \emptyset$. Let $\{S^i | i \in I\}$, where $I \subseteq \mathbf{N}$, and $S^i \equiv \{x \in \Re^n | \sum p^i_j x_j \leq c^i$ for some $p^i \in \Delta^{n-1}$ and $c^i \in \Re\}$ be a minimal collection such that $S = \cap_{i \in I} S^i$. Let $i \in I$ be such that $p^i > 0$. Then, by WPO and IR, there exists $x \in Int(S)$ such that $F(S,x) \in Int(PO(S) \cap PO(S^i))$. By Lemma 2, for all $d \in Int(S)$, if $F(S,d) \in PO(S) \cap PO(S^i)$, then the line connecting x and $F(S,x)$ is parallel to the line connecting d and $F(S,d)$. Let the common direction be denoted $\delta(S^i)$. By IR, $\delta(S^i) \geq 0$. Also, for all $d \in Int(S)$, if $\ell(d, \delta(S^i)) \cap PO(S^i) \equiv y \in Int(PO(S))$, then $F(S,d) = y$. Indeed, if $y \in Int(PO(S^i) \cap PO(S)) \equiv A^i$, then the conclusion follows from Lemma 2. If $y \notin A^i$, let $y^1 \in A^i$, $d^1 \equiv y^1 - \delta(S^i)$, $d^2 \equiv \frac{1}{2}(d+d^1)$, $y^2 \equiv \frac{1}{2}(y+y^1)$ (see Figure 2). By D.CAV, $F(S,d^2) \geqq \frac{1}{2}(F(S,d^1) + F(S,d)) = \frac{1}{2}(y^1 + F(S,d))$. Since $y^1 \in A^i$, $F(S,d^2) = y^2$. Therefore, $y \geqq F(S,d)$, and since $y \in Int(PO(S))$, and F satisfies WPO, we conclude that $y = F(S,d)$, as desired.

Now let S^i and S^j be such that $PO(S^i) \cap PO(S^j) \cap Int(PO(S)) \neq \emptyset$. Without loss of generality, suppose $i = 1$ and $j = 2$. We claim that $\delta(S^1) = \delta(S^2)$. Let $a \in PO(S^1) \cap PO(S^2) \cap Int(PO(S))$, $y^1 \in A^1$, $d^1 \equiv y^1 - \delta(S^1)$ and $d^2 \equiv a - \delta(S^2)$. By the previous step, $F(S,d^1) = y^1$ and $F(S,d^2) = a$. By Lemma 2 applied to d^1 and d^2, we conclude that the line passing through d^1 and $F(S,d^1)$ is parallel to the line passing through d^2 and $F(S,d^2) = a$. Therefore, $\delta(S^1) = \delta(S^2)$. Repeating the argument, we have $\delta(S^i) = \delta(S^1)$ for all S^i with $p^i > 0$. We denote this common non-negative direction by $\delta(S)$.

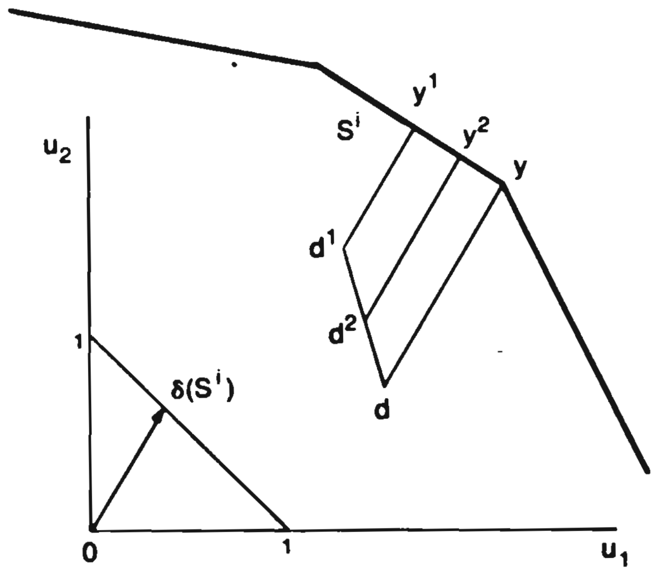
We now claim that $F(S,d) = E^\delta(S,d)$. If $\ell(d,\delta(S)) \cap WPO(S) \in Int(PO(S))$, then, by the previous argument, $F(S,d) = E^\delta(S,d)$. Otherwise, let $\{(S^k,d)\} \subset \Sigma^n$ be a sequence of polygonal problems such that for all k, $d \in Int(S^k)$, and

$PO(comp\{IR(S^k, d)\}) \cap PO(S^k) \subset Int(PO(S^k))$, and such that $S^k \to S$. By the previous argument, $F(S^k, d) = E^\delta(S^k, d)$ for all k, and by CONT, $F(S, d) = E^\delta(S, d)$.

Since an arbitrary feasible set $S \in \Gamma^n$ can be approximated by a sequence of polygonal elements of Γ^n, we conclude, by applying CONT again, that $F(S, d) = E^\delta(S, d)$ for all $(S, d) \in \Sigma^n$.

Finally, we note that CONT implies the continuity of δ in the Hausdorff topology.

Q.E.D.



Proof of Theorem 1.
Figure 2.

4. Further Characterizations

In this section, we impose additional axioms to characterize an important subfamily of the family of directional solutions, namely, the family of weighted egalitarian solutions, as well as its distinguished member, the egalitarian solution (see, for example, Kalai [1977]).

Definitions. Given $\alpha \in \Delta^{n-1}$, the *weighted egalitarian solution with weights* α, E^α, is defined by setting, for all $(S,d) \in \Sigma^n$, $E^\alpha(S,d)$ equal to $\ell(d,\alpha) \cap WPO(S)$. The *egalitarian solution*, E, is the member of this family obtained by choosing $\alpha_i = \alpha_j$ for all i,j.

The following axiom, introduced by Nash [1950], has been widely used to characterize various solutions. It says that if an alternative has been judged superior to all others in some feasible set, then it should also be judged superior to all alternatives in any subset (to which it belongs) provided the disagreement point is kept constant.

Independence of Irrelevant Alternatives (IIA). For all $(S^1,d^1),(S^2,d^2) \in \Sigma^n$, if $S^2 \subseteq S^1$, $d^1 = d^2$ and $F(S^1,d^1) \in S^2$, then $F(S^2,d^2) = F(S^1,d^1)$.

Theorem 2. *A solution satisfies WPO, IR, CONT, D.CAV and IIA if and only if it is a weighted egalitarian solution.*

Proof. Obviously, all E^α satisfy the five axioms. Conversely, let F be a solution satisfying the five axioms. By Theorem 1, the first four axioms imply that F is a directional solution E^δ relative to some continuous δ. Let $S^1, S^2 \in \Gamma^n$ be given. Given $c \in \Re$, let $S^c \equiv \{x \in \Re^n | \sum x_i \leqq c\}$. Then there exist c^i, for $i = 1,2$, such that for all $c \leqq c^i$, $S^c \cap S^i \neq \emptyset$. Let $\bar{c} \equiv min\{c^1, c^2\}$.

(a) If, for $i = 1,2$, there exists x^i such that $x^i \in Int(PO(S^i)) \cap PO(S^{\bar{c}})$, then let $T^i \equiv S^{\bar{c}} \cap S^i$ and $d^i \equiv x^i - \delta(S^i)$ for $i = 1,2$. By definition of F, $F(S^i, d^i) = x^i$ for $i = 1,2$. By IIA applied to the pairs $\{(S^i, d^i), (T^i, d^i)\}$, for $i = 1,2$, $F(T^i, d^i) = F(S^i, d^i)$. Therefore, $\delta(T^i) = \delta(S^i)$ for $i = 1,2$. We could show similarly that $\delta(T^i) = \delta(S^{\bar{c}})$ for $i = 1,2$. Altogether, we have $\delta(S^1) = \delta(S^2)$.

(b) Otherwise, let $\{S^{ik}\}$, for $i = 1,2$, be a sequence of problems such that (a) $S^{ik} \to S^i$ and (b) $Int(PO(S^{ik})) \cap PO(S^{\bar{c}}) \neq \emptyset$ for all k. Since, by (i), $\delta(S^{ik}) = \delta(S^{\bar{c}})$ for all i and k, the continuity of δ implies that $\delta(S^i) = \delta(S^{\bar{c}})$ for all i. Q.E.D.

An interesting variant of Theorem 2 can be obtained by modifying IR and IIA as follows.

Strong Individual Rationality (SIR). For all $(S,d) \in \Sigma^n$, $F(S,d) > d$.

Weak Independence of Irrelevant Alternatives (WIIA). For all $(S^1, d^1), (S^2, d^2) \in \Sigma^n$, if $S^2 = comp\{F(S^1, d^1)\}$ and $d^1 = d^2$, then $F(S^2, d^2) = F(S^1, d^1)$.

SIR, introduced by Roth [1977], requires that all agents be better off at the solution outcome than at the disagreement point. WIIA, introduced by Thomson and Myerson [1980], is a considerable weakening of IIA, since it applies to (at most) one subproblem of the original problem. Both SIR and WIIA are satisfied by all well-known solutions.

Theorem 3. *A solution satisfies WPO, SIR, CONT, D.CAV and WIIA if and only if it is a weighted egalitarian solution with positive weights.*

Proof. First, note that all the E^α, for $\alpha > 0$, satisfy the five axioms. Conversely, let F satisfying the five axioms be given. By Theorem 1, $F = E^\delta$ for some continuous δ. By SIR, $\delta(S) > 0$ for all $S \in \Gamma^n$. Let $S^1, S^2 \in \Gamma^n$ be such that $WPO(S^1) \cap WPO(S^2) \neq \emptyset$.

Let $x \in WPO(S^1) \cap WPO(S^2)$, $T \equiv comp\{x\}$ and $d \equiv x - \delta(S^1)$. Since $\delta(S^1) > 0$, $x > x - \delta(S^1)$, so that $(S^1, d), (T, d) \in \Sigma^n$. By definition of F, $F(S^1, d) = x$. By WIIA applied to the pair $\{(S^1, d), (T, d)\}$, $F(T, d) = x$. Therefore, $\delta(T) = \delta(S^1)$. Similarly, we could show that $\delta(T) = \delta(S^2)$. Altogether, we have $\delta(S^1) = \delta(S^2)$.

Now let S^1 and S^2 be arbitrary elements of Γ^n. Let $T \in \Gamma^n$ be such that $WPO(T) \cap WPO(S^1) \neq \emptyset$ and $WPO(T) \cap WPO(S^2) \neq \emptyset$. By applying the above argument twice, $\delta(T) = \delta(S^1)$ and $\delta(T) = \delta(S^2)$. Therefore, $\delta(S^1) = \delta(S^2)$ for all $S^1, S^2 \in \Gamma^n$. Q.E.D.

The egalitarian solution is the only weighted egalitarian solution satisfying the following axiom.

Symmetry (SY). For all $(S, d) \in \Sigma^n$ and for all permutations $\pi: \{1, ..., n\} \to \{1, ..., n\}$, if $S = \pi(S)$ and $d = \pi(d)$, then $F_i(S, d) = F_j(S, d)$ for all $i, j = 1, ..., n$.

This says that if the only information available on the conflict situation is contained in the mathematical description of (S, d), and (S, d) is a symmetric problem, then there is no ground for favoring one agent at the expense of another.

Corollary 1. *A solution satisfies (i) WPO, IR, CONT, D.CAV, IIA and SY, or (ii) WPO, SIR, CONT, D.CAV, WIIA and SY if and only if it is the egalitarian solution.*

These two statements are direct consequences of Theorems 2 and 3, respectively.

Next, we investigate which invariance properties are satisfied by the E^δ. A *positive affine transformation* is a function $\lambda: \Re^n \to \Re^n$ given by $a \in \Re^n_{++}$ and $b \in \Re^n$, such that for all $x \in \Re^n$, $\lambda(x) \equiv (a_1 x_1 + b_1, \ldots, a_n x_n + b_n)$.

Scale Invariance. For all $(S,d) \in \Sigma^n$ and for all positive affine transformations $\lambda: \Re^n \to \Re^n$, $F(\lambda(S), \lambda(d)) = \lambda(F(S,d))$.

Theorem 4. *A directional solution E^δ satisfies scale invariance if and only if for some i, $\delta(S) = e_i$ for all $S \in \Gamma^n$, where e_i is the i^{th} unit vector of Δ^{n-1}.*

Note that in that case E^δ satisfies the stronger property of *ordinal invariance,* that is, E^δ is invariant under all monotone increasing transformations of the utility functions. Finally, we note that E^δ satisfies *homogeneity,* that is, E^δ is invariant under positive affine transformations with equal multiplicative coefficients ($a_i = a_j$ for all i,j), if and only if $\delta(\lambda S) = \delta(S)$ for all $S \in \Gamma^n$ and for all $\lambda \in \Re_{++}$.

5. Variants

In Theorem 1, we identified all solutions satisfying WPO, IR, CONT and D.CAV. Here we investigate in greater detail the role played by every one of these axioms, by exhibiting additional solutions that would be made admissible by removing it, and evaluating the extent to which it could be strengthened or substituted for. We also consider alternative domains of problems.

(i) *Weak Pareto Optimality.*

First, we ask whether in Theorem 1 the optimality axiom can be strengthened to the following, while the other axioms are retained.

Pareto Optimality (PO). For all $(S,d) \in \Sigma^n$ and for all $x \in \Re^n$, if $x \geq F(S,d)$, then $x \notin S$.

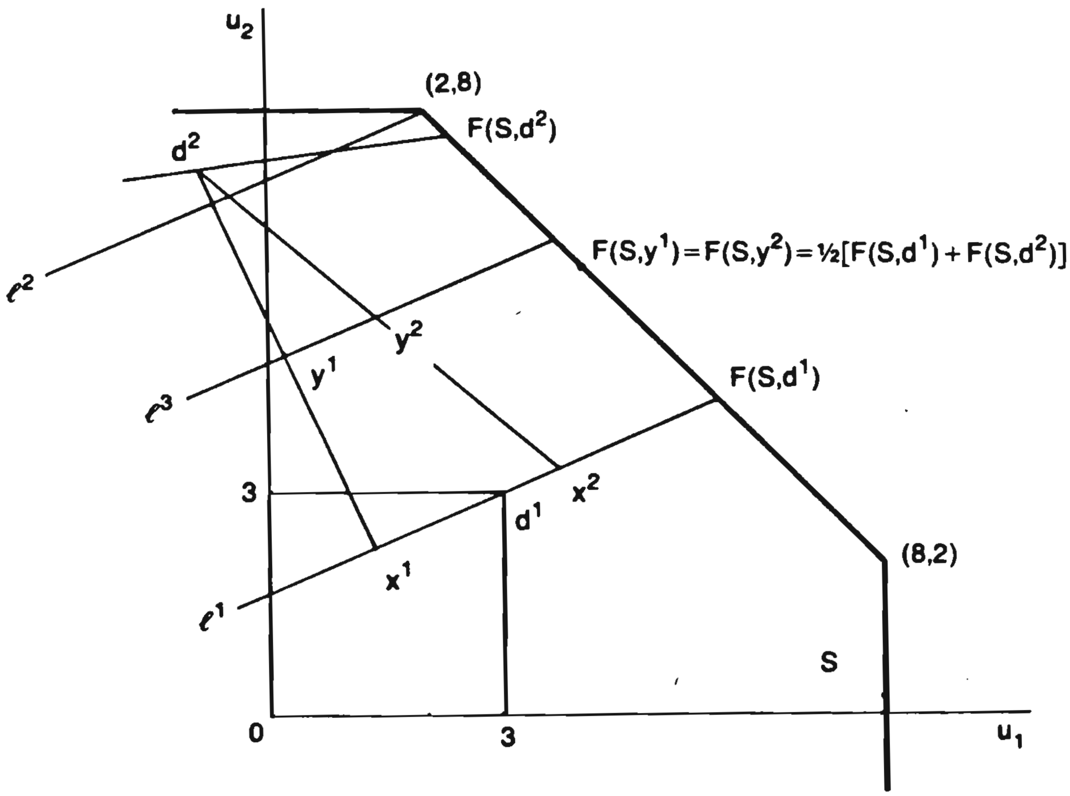
PO says that F should exhaust all gains from cooperation.

The answer is negative, even if CONT is abondoned, as shown in the following Theorem.

Theorem 5. *There is no solution satisfying PO, IR and D.CAV.*

Proof. See Figure 3. Let $S \equiv concomp\{(8,2),(2,8)\}$ and $d^1 \equiv (3,3)$. By PO and IR, $F(S,d^1) \in Int(PO(S))$ and $F(S,d^1) \geq (3,3)$. Let ℓ^1 be the line connecting d^1 and $F(S,d^1)$. By Lemma 1, $F(S,d) = F(S,d^1)$ for all $d \in \ell^1 \cap Int(S)$. If ℓ^1 is not a horizontal line, then let ℓ^2 be the line passing through (2,8) and parallel to ℓ^1 and d^2 be a point in $Int(S)$ and strictly above ℓ^2. If ℓ^1 is a horizontal line, then let ℓ^2 be the horizontal line passing through (8,2) and d^2 be a point in Int(S) and strictly below ℓ^2. By PO, $F(S,d^2) \in PO(S)$. Note that ℓ^1 is not parallel to the (well-defined) line passing through d^2 and $F(S,d^2)$. This is in contradiction with Lemma 2.

This argument can easily be generalized to an arbitrary $n \geq 3$. Q.E.D.



Proof of Theorem 5.
Figure 3.

If WPO is dropped from the list of axioms of Theorem 1, a large family of solutions become admissible. First, for all $(S, d) \in \Sigma^n$, let F be defined by $F(S, d) \equiv d$. This "disagreement solution" satisfies IR, CONT and D.CAV, but not WPO. More generally, let $k: \Re_+ \to \Re_+$ be a concave function such that $k(t) \leq t$ for all $t \in \Re_+$. Also, let $\delta: \Sigma^n \to \Delta^{n-1}$ be a continuous function. Then, given $(S, d) \in \Sigma^n$, let $F(S, d) \equiv d + k(\|u\|)u/\|u\|$ where $u \equiv E^{\delta(S)}(S, d) - d$, and $\|u\|$ is the Euclidean norm of u. Any such F also satisfies IR, CONT and D.CAV.

Note that a member of this family satisfies IIA if $\delta(S)$ is constant for all $S \in \Gamma^n$ and there exists $\bar{t}$ such that $k(t) = t$ for all $t < \bar{t}$ and $k(t) = \bar{t}$ for all $t \geq \bar{t}$. These *truncated egalitarian solutions* were introduced and characterized by Thomson [1985].

(ii) *Individual Rationality.*

First, we show that IR and IIA can be replaced in Theorem 2 and Corollary 1 by the following axiom.

Independence of Non-Individually Rational Alternatives (INIR). For all $(S, d) \in \Sigma^n$, $F(S, d) = F(comp\{IR(S, d)\}, d)$.

This axiom, introduced by Peters [1986a], says that the non-individually rational alternatives are irrelevant to the determination of the solution outcome. It is a natural condition since agents are guaranteed their utilities at d. It is also a very weak condition which is satisfied by all the major solutions.

Theorem 6. *A solution satisfies WPO, CONT, D.CAV and INIR if and only if it is a weighted egalitarian solution.*

Proof. Since it is obvious that all E^α satisfy the four axioms, we only show the converse statement. Let F be a solution satisfying the four axioms. First, it is clear that WPO, INIR and CONT together imply IR. By Theorem 1, it follows that F is a directional solution E^δ relative to some continuous δ. Now let $S \in \Gamma^n$ be such that $Int(PO(S)) \neq \emptyset$. Let $x \in Int(PO(S))$. We claim that $\delta(S) = \delta(comp\{x\})$. Let $\ell \equiv \{x - \lambda\delta(S) | \lambda > 0\}$, let $\{d^k\}$ be a sequence of points in ℓ such that $d^k \to x$ and finally let $S^k \equiv comp\{IR(S, d^k)\}$. By INIR, we have, for all k, $F(S, d^k) = F(S^k, d^k) = x$, which implies that $\delta(S^k) = \delta(S)$. Since $S^k \to comp\{x\}$, the continuity of δ implies that $\delta(comp\{x\}) = \delta(S)$.

Next, let S be an arbitrary problem in Γ^n and $x \in WPO(S)$. By CONT, we obtain $\delta(comp\{x\}) = \delta(S)$.

Now we apply the argument used in the proof of Theorem 3 and obtain the desired conclusion. Q.E.D.

Corollary 2. *A solution satisfies WPO, CONT, D.CAV, INIR and SY if and only if it is the egalitarian solution.*

If IR is dropped from the list of Theorem 1, again many additional solutions become admissible. In fact, any solution that is independent of the disagreement point satisfies D.CAV trivially, but of course violates IR. The other requirements can easily be satisfied.

(iii) *Continuity.* Given $(S, d) \in \Sigma^n$, set $F(S, d) \equiv x$ if $S \equiv comp\{x\}$ for some $x \in \Re^n$. Otherwise, set $F(S, d) \equiv E(S, d)$. F satisfies WPO, IR and D.CAV, but not CONT.

(iv) *Disagreement Point Concavity.* The Nash and Kalai-Smorodinsky solutions [5] sat-

[5] Given $(S, d) \in \Sigma^n$, the Nash [1950] solution outcome of (S, d), $N(S, d)$, is the unique maximizer of the product $\Pi_{i \in N}(x_i - d_i)$ over $IR(S, d)$, and the Kalai-Smorodinsky [1975] solution outcome of (S, d), $K(S, d)$, is the maximal point of S on the segment connecting d and $a(S, d)$, where, for each i, $a_i(S, d) \equiv max\{x_i | x \in IR(S, d)\}$.

isfy WPO, IR and CONT, but not D.CAV. Many other examples could be constructed.

(v) *domain*

(a) Let $\tilde{\Sigma}^n$ be the domain of problems satisfying all properties defining Σ^n except perhaps for convexity. On this domain, all E^δ satisfy the following weakening of D.CAV, which is still an appealing condition since it guarantees that all agents *agree* whether to wait or not.

Disagreement Point Uniformity. For all $(S^1, d^1), (S^2, d^2) \in \tilde{\Sigma}^n$ and for all $\alpha \in [0,1]$, if $S^1 = S^2 \equiv S$ and $(S, \alpha d^1 + (1-\alpha)d^2) \in \tilde{\Sigma}^n$, then either

$$F(S, \alpha d^1 + (1-\alpha)d^2) \geqq \alpha F(S, d^1) + (1-\alpha)F(S, d^2)$$

or

$$F(S, \alpha d^1 + (1-\alpha)d^2) \leqq \alpha F(S, d^1) + (1-\alpha)F(S, d^2).$$

(b) Let $\Sigma^n_B \subset \Sigma^n$ be the set of all members of Σ^n that are bounded from above, and Γ^n_B be the corresponding domain of feasible sets. On Σ^n_B, which is a commonly considered domain, the four axioms of Theorem 1 still characterize the E^δ.

In addition, the following generalization of the directional solutions can be characterized by weakening the individually rationality axiom. Let $B^{n-1} \equiv$

$\{x \in \Re^n \mid \sum |x_i| = 1 \text{ and } -x \notin \Re^n_+\}$ and let $\bar{\ell}(d, \delta(S))$ be the half-line originating at d in the direction $\delta(S)$.

Definition. Given $\delta : \Gamma^n_B \to B^{n-1}$, the *generalized directional solution relative to* δ is defined by setting, for each $(S,d) \in \Sigma^n_B$, $E^\delta(S,d)$ equal to $\bar{\ell}(d, \delta(S)) \cap WPO(S)$.

Note that for $(S,d) \in \Sigma_B^n$, the intersection $\bar{\ell}(d,\delta(S)) \cap WPO(S)$ always exists (and is a singleton), so that these solutions are indeed well-defined. On Σ^n, they would not necessarily be well-defined.

The characterization uses the following axiom, which considerably weakens IR.

Boundary (BOUND). For all sequences $\{(S^k,d^k)\} \subset \Sigma_B^n$ and for all $(S,d) \in \Sigma_B^n$, if $S^k = S$ for all k and $d^k \to x \in PO(S)$, then $F(S^k,d^k) \to x$.

Theorem 7. *A solution on Σ_B^n satisfies WPO, CONT, D.CAV and BOUND if and only if it is a generalized directional solution.*

The proof of Theorem 7 is similar to that of Theorem 1, and we omit it.

Next, consider the following appealing conditions:

Cutting (CUT). For all $(S^1,d^1),(S^2,d^2) \in \Sigma_B^n$, and for all i, if $S^2 \subseteq S^1$, $d^1 = d^2$ and $x \in S^1 \backslash S^2$ implies $x_i \leq F_i(S^1,d^1)$, then $F_j(S^2,d^2) \leq F_j(S^1,d^1)$ for all $j \neq i$.

Translation Invariance (T.INV). For all $(S^1,d^1),(S^2,d^2) \in \Sigma_B^n$, and for all $t \in \Re^n$, if $S^2 = S^1 + t$ ($\equiv \{x^2 \in \Re^n | \exists x^1 \in S^1$ with $x^2 = x^1 + t\}$) and $d^2 = d^1 + t$, then $F(S^2,d^2) = F(S^1,d^1) + t$.

CUT, which is a variant of a condition introduced by Thomson and Myerson [1980] under the same name, requires that eliminating alternatives unfavorable to agent i causes all other agents to (weakly) lose. T.INV requires the invariance of the solution under translations. These are very weak conditions satisfied by all well-known solutions (except that CUT is not satisfied by the Perles-Maschler [1981] solution. [6])

[6] The definition of the Perles-Maschler [1981] solution is somewhat complicated, so we will not

Theorem 8. *A solution on Σ_B^n satisfies WPO, SIR, CONT, D.CAV, CUT and T.INV if and only if it is a weighted egalitarian solution.*

Proof. Obviously, all E^α satisfy all six axioms. Conversely, let F be a solution on Σ_B^n satisfying the six axioms. It was noted earlier that, even on Σ_B^n, the first four axioms imply that $F = E^\delta$ for some δ. Let $S^1, S^2 \in \Gamma_B^n$ be given. First, we assume that S^1 and S^2 are the comprehensive hulls of compact sets. Let $a_i(S^k) \equiv max_{x \in S^k} x_i$ and $b^i(S^k) \in PO(S)$ be such that $b^i_i(S^k) = a_i(S^k)$ for all i and k. Let $t^i, r^i \in \Re^n$, for all i, be defined by $t^i \equiv b^i(S^1) - a(S^2)$ and $r^i \equiv b^i(S^2) - a(S^1)$. By T.INV, $\delta(S^2 + t^i) = \delta(S^2)$ and $\delta(S^1 + r^i) = \delta(S^1)$ for all i.

Let $d \equiv b^1(S^2) + t^1 - \delta(S^1)$. Since $\delta(S^1) > 0$ by SIR, $(S^1, d) \in \Sigma_B^n$ and $F(S^1, d) = b^1(S^2) + t^1$. Also, $(S^2 + t^1) \subseteq S^1$ and if $x \in S^1 \backslash (S^2 + t^1)$, then $x_1 \leq F_1(S^1, d)$. Therefore, by applying CUT to the pair $\{(S^2 + t^1, d), (S^1, d)\}$, we have $F_j(S^2 + t^1, d) \leq F_j(S^1, d)$, for all $j \neq 1$. This implies $\delta_1(S^2) = \delta_1(S^2 + t^1) \geq \delta_1(S^1)$. By a similar argument applied to the pair $\{S^1 + r^1, S^2\}$, we conclude that $\delta_1(S^1) \geq \delta_1(S^2)$. Therefore, $\delta_1(S^1) = \delta_1(S^2)$.

We repeat the argument for all i, and obtain $\delta_i(S^1) = \delta_i(S^2)$ for all i. Altogether, we have $\delta(S^1) = \delta(S^2)$, as desired.

If S^1 and S^2 are not the comprehensive hulls of compact sets, we apply CONT.

Q.E.D.

---

give it in full. For our purposes, it suffices to have it for the following problems: given $(S, d) \in \Sigma^2$ such that $S \equiv comp\{x, y\}$ for some $x, y \in \Re^2$, the Perles-Maschler solution outcome of (S, d), $PM(S, d)$, is the midpoint of $PO(S) \cap IR(S, d)$; also, given a symmetric problem $(S, d) \in \Sigma^2$, $PM(S, d)$ is the maximal feasible point of equal coordinates. To prove the statement in parenthesis, let $S^1 \equiv comp\{(1, \frac{2}{3}), (0, 1)\}$ and $S^2 \equiv comp\{(1, 0), (\frac{3}{4}, \frac{3}{4}), (0, 1)\}$, and $d^1 = d^2 = 0$. Then $PM(S^1, d^1) = (\frac{1}{2}, \frac{5}{6})$ and $PM(S^2, d^2) = (\frac{3}{4}, \frac{3}{4})$, which is in contradiction to CUT since agent 2 loses although (S^2, d^2) is obtained from (S^1, d^1) by "cutting" to the South-East of $F(S^1, d^1)$. Perles and Maschler [1981] also contains an example showing that their solution violates CUT.

It is well-known that the Nash, Kalai-Smorodinsky and Perles-Maschler solutions satisfy PO and IR. Therefore, Theorem 5 implies that these solutions do not satisfy D.CAV. This can be seen directly in the example that follows.

The main purpose of the example however is to show that this negative result still holds if we restrict the domain to the class of economic problems of fair division, *even in the one-commodity and two-agents case.* [7] Such a problem is a triple (u, Ω, ω) with $u = (u_1, u_2)$, where for each i, $u_i: \Re_+ \rightarrow \Re$ is agent i's *utility function*, $\Omega \in \Re_+$ is the *aggregate endowment* and $\omega_i \in \Re_+$ such that $\omega_1 + \omega_2 \leq \Omega$ is the amount of the commodity each agent receives when the conflict occurs. A *feasible allocation* for (u, Ω, ω) is a vector $x = (x_1, x_2) \in \Re^2_+$ such that $x_1 + x_2 \leq \Omega$. Let $A(\Omega)$ be the set of feasible allocations of (u, Ω, ω). Given (u, Ω, ω), let

$$S(u, \Omega) \equiv \{\bar{u} \in \Re^2 | \text{ there exists } x \in A(\Omega) \text{ with } u_i(x_i) = \bar{u}_i \text{ for all } i\} \qquad \textit{and}$$

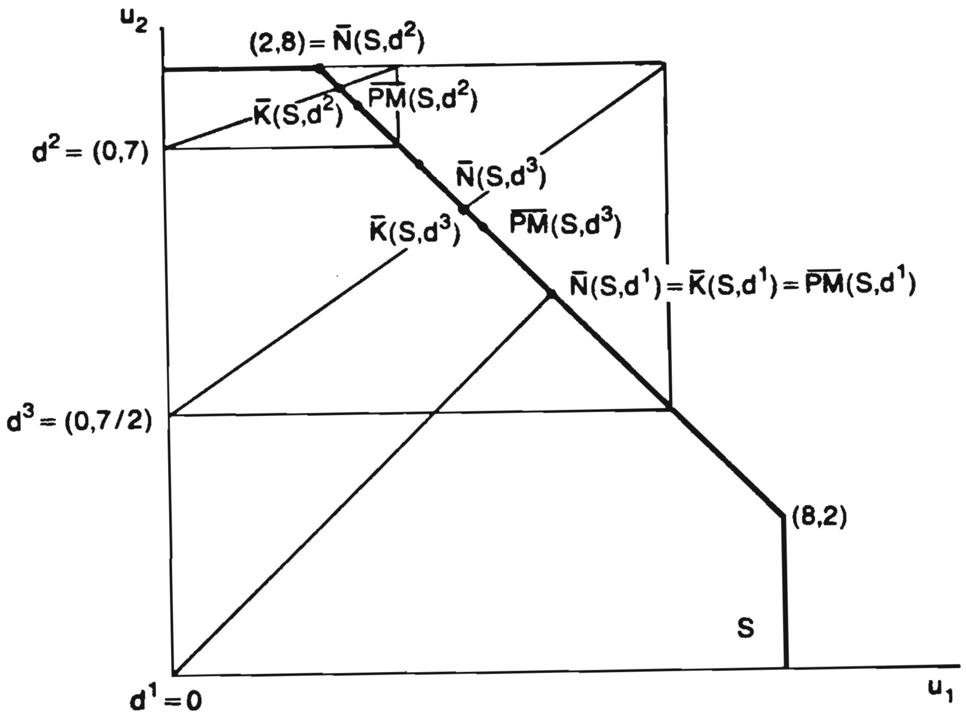
$$d(u, \omega) \equiv (u_1(\omega_1), u_2(\omega_2)).$$

Given a bargaining solution $F: \Sigma^2 \rightarrow \Re^2$, let $\bar{F}$ be the solution to the problem of fair division defined by associating with each (u, Ω, ω) such that $(S(u, \Omega), d(u, \omega)) \in \Sigma^2$, the allocation(s) $x \in A(\Omega)$ such that $u(x) = F(S(u, \Omega), d(u, \omega))$.

Now let $u_1 = u_2 = u: \Re \rightarrow \Re$ be defined by $u(x) \equiv min\{x, 8\}$ and $\Omega \equiv 10$. Then we have $S(u, \Omega) = con\{(0,0), (8,0), (8,2), (2,8), (0,8)\}$. Also, let $\omega^1 \equiv (0,0)$, and therefore, $d^1 \equiv d(u, \omega^1) = (0,0)$. We obtain $\bar{N}(S, d^1) = \bar{K}(S, d^1) = \bar{PM}(S, d^1) = (5,5)$. Next, by setting $\omega^2 \equiv (0,7)$, we have $d^2 \equiv d(u, \omega^2) = (0,7)$, which implies $\bar{N}(S, d^2) = (2,8)$, $\bar{K}(S, d^2) = (\frac{9}{4}, \frac{31}{4})$ and $\bar{PM}(S, d^2) = (\frac{5}{2}, \frac{15}{2})$. Now, let $\omega_3 \equiv (0, \frac{7}{2})$. Note that $\omega^3 = (\frac{1}{2})(\omega^1 + \omega^2)$. We have $d^3 \equiv d(u, \omega^3) = (0, \frac{7}{2})$, which

[7] Chun and Thomson [1984] discuss the usefulness of bargaining theory in solving such problems.

implies that $\bar{N}(S, d^3) = (\frac{13}{4}, \frac{27}{4})$, $\bar{K}(S, d^3) = (\frac{169}{44}, \frac{271}{44})$ and $\bar{PM}(S, d^3) = (\frac{17}{4}, \frac{23}{4})$. Since $(\frac{1}{2})\{\bar{N}(S, d^1) + \bar{N}(S, d^2)\} \not\leq \bar{N}(S, d^3)$, $(\frac{1}{2})\{\bar{K}(S, d^1) + \bar{K}(S, d^2)\} \not\leq \bar{K}(S, d^3)$ and $(\frac{1}{2})\{\bar{PM}(S, d^1) + \bar{PM}(S, d^2)\} \not\leq \bar{PM}(S, d^3)$, the desired conclusion follows.



The Nash, Kalai-Smorodinsky and Perles-Maschler solutions do not satisfy D.CAV for economic problems of fair division with one commodity and two agents.
Figure 4.

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