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# BARGAINING SOLUTIONS AND STABILITY OF GROUPS

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## *Abstract*

We propose to evaluate solutions to abstract problems of fair division on the basis of the extent to which they differentially affect the fortunes of two individuals initially present, when the number of agents increases while their opportunities do not. We introduce the concept of the relative guarantee structure of a solution to quantify this possibility. The Kalai–Smorodinsky and Egalitarian solutions offer maximal guarantees in the class of anonymous solutions, and, in particular, they strictly dominate the Nash solution. In that class, the Kalai–Smorodinsky solution is the only weakly Pareto-optimal solution to offer maximal guarantees and to satisfy scale invariance while the Egalitarian solution is the only weakly Pareto-optimal solution to offer maximal guarantees and to satisfy independence of irrelevant alternatives.



Key words: relative guarantee structure; Kalai–Smorodinsky solution; Egalitarian solution; Nash solution.

# *Bargaining Solutions and Stability of Groups*

1. ***Introduction.*** We consider abstract problems of fair division in circumstances in which the number of agents may increase while their opportunities do not. We propose to evaluate solutions to such problems on the basis of the extent to which they differentially affect the fortunes of two individuals initially present. We suggest that the stability of society is more likely to be preserved by solutions for which such changes remain small. We define the notion of the ***relative guarantee structure*** of a solution as a way of quantifying these changes and we use this measure to rank solutions.

We show that the Kalai-Smorodinsky (1975) and Egalitarian (Kalai, 1977) solutions are best among all solutions satisfying the minimal requirement of Anonymity, while the Nash (1950) solution is strictly inferior (we derive explicit formulas for these three solutions). Moreover, the requirement on a solution that it offers maximal guarantees can be used to characterize the Kalai-Smorodinsky and Egalitarian solutions. In addition to this requirement, these characterizations involve two alternative sets of standard conditions, differing only in that Scale Invariance is used for the first result while Independence of Irrelevant Alternatives (Nash, 1950) is used for the latter.

2. ***Preliminaries.*** Our analysis is placed within the context of Nash's bargaining theory, as generalized by Thomson (1983a) to accomodate variations in the number of agents. There is an infinite set of potential agents, I = {1,2,...}. $\mathcal{P}$ is the set of all finite subsets of I, with generic elements P,

Q,.... Given $P \in \mathcal{P}$, $\mathbb{R}^P_+$ is the utility space pertaining to the group P. $\Sigma^P$ is the class of ***problems*** that P may face: each $S \in \Sigma^P$ is a convex, compact, and comprehensive ($\forall x,y \in \mathbb{R}^P_+$, if $y \leqq x$ and $x \in S$, then $y \in S$) subset of $\mathbb{R}^P_+$ containing at least one $x > 0$.[1,2] A ***solution*** F is a mapping defined on $\Sigma \equiv \bigcup_{\mathcal{P}} \Sigma^P$, associating with each $P \in \mathcal{P}$ and each $S \in \Sigma^P$, a unique point of S, F(S). F(S) is interpreted as the predicted, or recommended, depending upon the context, compromise for S. For the ***Nash*** (1950) solution, ***N(S)*** is the maximizer of $\prod_P x_i$ for $x \in S$; for the ***Kalai-Smorodinsky*** (1975) solution, ***K(S)*** is the maximal point of S on the segment connecting the origin to a(S), where for each $i \in P$, $a_i(S) \equiv \max\{x_i | x \in S\}$; for the ***Egalitarian*** (Kalai, 1977) solution, ***E(S)*** is the maximal point of S with equal coordinates.

The following properties of solutions will be useful.

***Weak Pareto-Optimality (WPO)***: For all $P \in \mathcal{P}$, for all $S \in \Sigma^P$, for all $x \in \mathbb{R}^P_+$, if $x > F(S)$, then $x \notin S$.

***Anonymity (AN)***: For all P, $P' \in \mathcal{P}$ with $|P| = |P'|$, for all $S \in \Sigma^P$, $S' \in \Sigma^{P'}$, for all one-to-one functions $\gamma: P \to P'$, if $S' = \{x' \in \mathbb{R}^{P'} | \exists\, x \in S \text{ with } x'_{\gamma(i)} = x_i \;\; \forall\, i \in P\}$, then $F_{\gamma(i)}(S') = F_i(S)$ for all $i \in P$.

$\lambda: \mathbb{R}^P \to \mathbb{R}^P$ is an ***independent person by person, positive linear transformation*** if there exists $\alpha \in \mathbb{R}^P_{++}$ with $\lambda_i(x) = \alpha_i x_i$ $\forall i \in P$. Let $\Lambda^P$ be the class of these transformations.

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[1]Vector inequalities: $x > y$, $x \geq y$, $x \geqq y$.

[2]The usual specification of a bargaining problem involves a distinguished element of S, usually called the disagreement point and denoted d. Here, utilities are assumed to be normalized so that d = 0. This allows us to eliminate all references to d in the notation, with no essential loss of substance.

***Scale Invariance* (*S.INV*):** For all $P \in \mathcal{P}$, for all $S \in \Sigma^P$, for all $\lambda \in \Lambda^P$, $F(\lambda(S)) = \lambda(F(S))$.

***Independence of Irrelevant Alternatives* (*IIA*):** For all $P \in \mathcal{P}$, for all $S, S' \in \Sigma^P$, if $S' \subset S$ and $F(S) \in S'$, then $F(S') = F(S)$.

***WPO*** says that it should not be feasible to make all agents better off than they are at the compromise; ***AN*** says that no other information is available than that contained in the mathematical description of the problem so that if two agents enter symmetrically in that description, they should be treated symmetrically by the solution; ***S.INV*** says that the theory should be unaffected by linear transformations of the utility scales; ***IIA*** says that if an alternative is thought to be the best compromise for a particular problem, it should still be thought best for any subproblem that still contains it.

The Nash solution satisfies all four properties, the Kalai-Smorodinsky solution satisfies all but ***IIA*** and the Egalitarian solution satisfies all but ***S.INV***.

Other notation: $e_P$ is the vector of all ones in $\mathbb{R}^P$. Given $S, S' \in \mathbb{R}^P$, $cch\{S,S'\}$ is the smallest convex and comprehensive set containing $S$ and $S'$.

**3. *The problem*.** Although our work is formulated in the abstract framework of bargaining theory, the concrete problem of resource allocation constitutes an important motivation for it. Consider the classical problem of fair division: there is a vector of resources available to a group of consumers who have equal claims on it. How should these goods be allocated? And how should they be reallocated if the number of claimants happens to increase, while the resources at their disposal remain fixed? Given a solution to the problem of fair division, let $z_i$ and $z_i'$ be the consumptions assigned to agent i, one of the agents initially present, before and after the arrival of the new agents. The ratio $u_i(z_i')/u_i(z_i)$ of agent i's final to initial utility is a measure of how he is affected by this event. A small ratio indicates that he greatly suffers. But, the ratio could be greater than 1, indicating that he has actually gained, in spite of the fact that the number of claimants on the fixed resources has increased. In order to evaluate the solution, we propose to determine how any two members of the initial group fare relative to each other. If one of them is very negatively affected while the other is hardly affected, or perhaps even gains, one might expect resentment on the part of the disadvantaged agent and his opposition to the solution. On the other hand, a solution for which agents are similarly affected is more likely to preserve the stability of society.

This is the issue with which we will be concerned here, but as mentioned earlier, we will operate in the abstract framework of bargaining theory.

Let P be the initial group and $S \subset \mathbb{R}^P$ be the image in utility space of the set of feasible allocations of the initial economy. The group P enlarges to Q and $T \subset \mathbb{R}^Q$ is the new feasible set. Since resources remain fixed, $S \equiv T_P$ ($T_P$ is the projection of T onto $\mathbb{R}^P$). Given two individuals i and j in P, the change in their relative fortunes is given by the ratio

$$\frac{F_i(T)/F_i(S)}{F_j(T)/F_j(S)},$$

if this ratio is well defined on the extended real line. In order to evaluate the extent to which F is likely to preserve the stability of the group, we determine the smallest value of the ratio as S and T vary.

$$\text{Let } \epsilon_F(i,j,P,Q) \equiv \begin{cases} \inf\{r \mid S \in \Sigma^P,\ T \in \Sigma^Q,\ T_P = S, \\ \quad r = \dfrac{F_i(T)/F_i(S)}{F_j(T)/F_j(S)} \text{ is well defined}\} \text{ if this infimum exists,}^3 \\ 0 \text{ otherwise.} \end{cases}$$

If F satisfies **AN**, $\epsilon_F(i,j,P,Q)$ depends on the cardinalities of P and Q\P only, denoted m and n respectively, and it can be written as $\epsilon_F^{mn}$. Then, necessarily, $\epsilon_F^{mn} \leqq 1$ for all $(m,n) \in (\mathbb{N}\backslash 1)\times\mathbb{N}$. Let $\epsilon_F \equiv \{\epsilon_F^{mn} \mid (m,n) \in (\mathbb{N}\backslash 1)\times\mathbb{N}\}$ be the ***relative guarantee structure of F***. $\epsilon_F^{mn}$ measures the maximal change in the relative fortunes of two agents initially part of a group of cardinality m upon the arrival of n additional agents. Seen positively, a high value of $\epsilon_F^{mn}$ indicates that each original agent is guaranteed to not be very differentially affected from any other agent as new agents come in.

In the next section we compare solutions on the basis of their relative guarantee structures.

**4.** ***The results.*** Our aim here is to compare solutions on the basis of their relative guarantee structures. We will in fact be able to rank the major solutions. First, we consider the Kalai-Smorodinsky and Egalitarian solutions.

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[3]Most solutions, and in particular the three that we examine in detail here, are such that for all $P \in \mathcal{P}$ and for all $S \in \Sigma^P$, $F(S) > 0$. Therefore, the ratio under study happens to be well defined for all pairs {S,T} satisfying $S = T_P$.

*Theorem* 1. $\epsilon_K^{mn} = \epsilon_E^{mn} = 1$ for all $(m,n) \in (\mathbb{N}\backslash 1)\times\mathbb{N}$.

*Proof*. Given any i, j, P, Q as in the definition of $\epsilon_K^{mn}$, note that $K_i(S)/K_j(S) = a_i(S)/a_j(S)$. Also, $K_i(T)/K_j(T) = a_i(T)/a_j(T)$. Since $S = T_P$ implies $a_i(S) = a_i(T)$ and $a_j(S) = a_j(T)$, we have $\epsilon_K^{mn} = 1$.

We omit the straightforward proof that $\epsilon_E^{mn} = 1$.

**Q.E.D.**

Together with our earlier observation that if F is anonymous, then $\epsilon_F^{mn} \leqq 1$ for all $(m,n) \in (\mathbb{N}\backslash 1)\times\mathbb{N}$, Theorem 1 says that the Kalai-Smorodinsky and Egalitarian solutions are best among all anonymous solutions. Moreover, it is worth noting that for both of these solutions the ratio under study is equal to 1 for all pairs {S,T} satisfying $S = T_P$. This means that these two solutions dominate all anonymous solutions for *all* such pairs. Of course, if T happens to be a symmetric problem (invariant under all exchanges of agents), the ratio $\dfrac{F_i(T)/F_i(S)}{F_j(T)/F_j(S)}$ is equal to 1 for any weakly Pareto-optimal and anonymous F. However, if T is not a symmetric problem the ratio will usually be strictly smaller than 1. This is in particular the case for the Nash solution, to which we now turn.

*Theorem* 2. $\epsilon_N^{mn} = \dfrac{B^2-2-\sqrt{(B^2-2)^2-4}}{2}$ where $B = \sqrt{m(m+n)} - m+2$ for all $(m,n) \in (\mathbb{N}\backslash 1)\times\mathbb{N}$.

*Proof*. Let $P \equiv \{1,\ldots,m\}$ with $m \geqq 2$ and $Q \equiv \{1,\ldots,m+n\}$ with $n \geqq 1$. We have to solve the following problem:

P1: Find $\inf\left\{\frac{N_1(T)/N_1(S)}{N_2(T)/N_2(S)} \middle| S \in \Sigma^P,\ T \in \Sigma^Q,\ S = T_P\right\}$.

Since N satisfies ***S.INV***, we can assume that $N(T) = e_Q$ and since N satisfies ***IIA***, that $S = cch\{x, e_P\}$ with $x = N(S)$. Then, $N(T) = e_Q$ if and only if x lies below the hyperplane in $\mathbb{R}^Q$ supporting at $e_Q$ the set $\{y \in \mathbb{R}^Q_+ | \prod_Q y_i \geqq 1\}$. The equation of this hyperplane is $\sum_Q y_i = m+n$. Also, $x = N(cch\{x, e_P\})$ if and only if $e_P$ lies below the hyperplane in $\mathbb{R}^P$ supporting at x the set $\{x' \in \mathbb{R}^P_+ | \prod_P x'_i \geqq \prod_P x_i\}$. The equation of this hyperplane is $\sum_P (x'_i/x_i) = m$. Then P1 reduces to

***P2***. Find $\inf\{x_2/x_1 | \sum_P x_i \leqq m+n,\ \sum_P (1/x_i) \leqq m\}$.

This problem takes place entirely in $\mathbb{R}^P$.

First, we note that we can impose $\sum_P x_i = m+n$. Indeed, starting from x with $\sum_P x_i < m+n$ and $\sum_P (1/x_i) \leqq m$, let $\bar{x} \equiv (x_1+\epsilon, x_2, \ldots, x_m)$ where $\epsilon > 0$ is small enough so that $\sum_P \bar{x}_i \leqq m+n$. Then, $\sum_P (1/\bar{x}_i) = 1/\bar{x}_1 + \sum_{P\backslash 1} (1/\bar{x}_i) = 1/(x_1+\epsilon) + \sum_{P\backslash 1} (1/x_i) \leqq \sum_P (1/x_i) \leqq m$. Since $\bar{x}_2/\bar{x}_1 < x_2/x_1$, we are done.

Next, we note that we can impose $\sum_P (1/x_i) = m$. Indeed, given x with $\sum_P x_i = m+n$ and $\sum_P (1/x_i) < m$, let $\bar{x} \equiv (x_1, x_2-\epsilon, \ldots, x_m)$, where $\epsilon > 0$ is small enough so that $\sum_P (1/\bar{x}_i) \leqq m$. Since $\sum_P \bar{x}_i \leqq \sum_P x_i = m+n$, and $\bar{x}_2/\bar{x}_1 < x_2/x_1$ we are done.

Finally, we claim that we can set $x_i = x_j$ for all $i, j \in P' \equiv P\backslash\{1,2\}$. (of course, P' is empty if m = 2). Indeed, supposing $m > 2$, let $x \in \mathbb{R}^P_+$ with $\sum_P x_i = m+n$ and $\sum_P (1/x_i) = m$ be given. Let $\bar{x}$ be such that $\bar{x}_1 = x_1$, $\bar{x}_2 = x_2$ and $\bar{x}_i = (\sum_{P'} x_j)/(m-2)$ for all $i \in P'$. Note that $\sum_P \bar{x}_i = x_1 + x_2 + (m-2)(\sum_{P'} x_j)/(m-2)$

$= \sum_P x_i = m+n$. Also, $\sum_P (1/\bar{x}_i) = 1/x_1 + 1/x_2 + (m-2)^2/\sum_{P'} x_j \leq \sum_P (1/x_i) = m$ since $\frac{1}{\sum_{P'} x_j/(m-2)} \leq \frac{1}{m-2} \sum_{P'} (1/x_j)$ by convexity of the function $h(t) = 1/t$. Finally, since $\bar{x}_2/\bar{x}_1 = x_2/x_1$, we are done.

After eliminating from the two constraints the common value of $x_j$ for $j \in P'$, we are then led to solving the following problem:

*P3*. Find $\inf\{x_2/x_1 \,|\, 1/x_1 + 1/x_2 + (m-2)^2/(m+n-x_1-x_2) = m\}$.

Forming the Lagrangian

$$L(x_1, x_2, \lambda) = \frac{x_2}{x_1} + \lambda\left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{(m-2)^2}{m+n-x_1-x_2} - m\right],$$

we obtain after differentiation

$$\frac{\partial L(x_1,x_2,\lambda)}{\partial x_1} = \frac{-x_2}{x_1^2} + \lambda\left[\frac{-1}{x_1^2} + \frac{(m-2)^2}{(m+n-x_1-x_2)^2}\right] = 0,$$

$$\frac{\partial L(x_1,x_2,\lambda)}{\partial x_2} = \frac{1}{x_1} + \lambda\left[\frac{-1}{x_2^2} + \frac{(m-2)^2}{(m+n-x_1-x_2)^2}\right] = 0.$$

Since $x > 0$, the coefficients of $\lambda$ are non-zero. Then, eliminating $\lambda$ between these two equations gives

$$\frac{x_2}{x_1} = \frac{\dfrac{-1}{x_1^2} + A}{\dfrac{1}{x_2^2} - A} \qquad \text{where } A = \left[\frac{m-2}{m+n-x_1-x_2}\right]^2,$$

and after cross multiplication,

$$\frac{1}{x_2} - x_2 A = \frac{-1}{x_1} + x_1 A.$$

Setting $S \equiv x_1 + x_2$ and $P \equiv x_1x_2$, this equality becomes $\frac{S}{P} = SA$, and since $S \neq 0$, $\frac{1}{P} = \left[\frac{m-2}{m+n-S}\right]^2$. Inserting this expression into the equality constraint written as $\frac{S}{P} + \frac{(m-2)^2}{m+n-S} = m$ yields $\frac{S}{P} + \frac{m+n-S}{P} = m$, i.e. $P = \frac{m+n}{m}$. Using this expression for P in $\frac{m+n-S}{m-2} = \sqrt{P}$ we obtain $S = m+n - (m-2)\sqrt{\frac{m+n}{m}}$.

The expression we are looking for is the smallest root of the following equation in t:

$$t^2 - \frac{S^2-2P}{P} t + 1 = 0,$$

which after replacing S and P by their values as functions of m and n, becomes

$$t^2 - (B^2-2)t + 1 \text{ where } B = \sqrt{m(m+n)} - m+2.$$

The smallest root is $\frac{B^2-2-\sqrt{(B^2-2)^2-4}}{2}$, the desired expression.

The derivation of the second order condition is relegated to Appendix 1.

**Q.E.D.**

It can be shown that for each fixed m, $\epsilon_N^{mn}$ is a decreasing and convex function of n and that $\epsilon_N^{mn} \to 0$ as $n \to \infty$. Also, for each fixed n, $\epsilon_N^{mn}$ is a decreasing and convex function of m and $\epsilon_N^{mn} \to 8 / [n^2+8n+8+(n+4)\sqrt{n^2+8n}]$ as $m \to \infty$ (the calculations can be found in appendix).

An intuitive reason why the Nash solution does not offer very good relative guarantees is that, as opposed to the Kalai-Smorodinsky and Egalitarian solutions, which keep the agents' utilities tied together, it responds to "stretchings" of the feasible set in somewhat unpredictable ways. It is also for that reason that it violates Population Monotonicity (which says that as a result of an increase in the number of agents, unaccompanied by an expansion of opportunities, all agents initially present weakly lose; see Thomson (1983a,b)), a property which is satisfied by both the Kalai-Smorodinsky and Egalitarian solutions. Not surprisingly, the minimum of the ratio appearing in the definition of $\epsilon_N^{mn}$ is attained precisely for a pair {S,T} for which the Nash solution violates the property. Indeed, using the expressions for $x_1 + x_2$ and $x_1x_2$ obtained in the proof of Theorem 2, it can be shown that $x_2 = N_2(S) < 1$ while $N_2(T) = 1$ (see Appendix 1 for the calculations). Agent 2 has gained in spite of the fact that the group of claimants has enlarged from P to Q.

Theorems 1 and 2 together imply that the Kalai-Smorodinsky and Egalitarian solutions are both strictly superior to the Nash solution from the viewpoint of relative guarantees. We have already noted that these two solutions offer maximal relative guarantees among all anonymous solutions. We show next that it is possible to characterize them with the help of two alternative sets of standard conditions together with the requirement of maximal relative guarantees. The proofs of these results, which are based on constructions that are somewhat similar to those used in the characterizations appearing in Thomson (1983a,b), can be found in Appendix 2.

***Theorem* 3.** The Kalai-Smorodinsky solution is the only solution satisfying ***WPO***, ***AN***, and ***S.INV*** that offers maximal relative guarantees.

***Theorem* 4.** The Egalitarian solution is the only solution satisfying ***WPO***, ***AN*** and ***IIA*** that offers maximal relative guarantees.

## 5. *Concluding comments*

This study should be a useful complement to an earlier contribution by Thomson and Lensberg (1983), where solutions were evaluated on the basis of the extent to which an agent could see his own situation deteriorate, upon the arrival of new agents unaccompanied by an expansion of their opportunities. A notion of absolute guarantees was introduced there and used to rank solutions. It was found that the Kalai-Smorodinsky solution performed strictly better than the Nash and Egalitarian solutions, and that no solution satisfying Weak Pareto-Optimality and Anonymity could do better.

Two main differences should be noted between the two studies. First, according to the criterion used there to rank solutions, the Egalitarian solution performed extremely badly (it offers no absolute guarantees at all), while for the criterion under consideration here this solution is just as good as the Kalai-Smorodinsky solution. The second difference is that while we are able here to characterize the Kalai-Smorodinsky solution by using the condition that the solution offers maximal *relative* guarantees, the condition that the solution offers maximal *absolute* guarantees, together with the same list of complementary axioms, was not sufficient there to yield a characterization. Although the Kalai-Smorodinsky solution was more easily distinguishable from its main competitors, isolating it in the class of solutions satisfying Weak Pareto-Optimality and Anonymity was actually more difficult.

The Thomson-Lensberg study and the current study offer new viewpoints from which to evaluate solutions to the bargaining problem. The differences between the conclusions of the two papers should reinforce what is perhaps the main lesson to be drawn from the developments of the axiomatic theory of bargaining that took place over the last ten years: no unique solution has emerged as *the* best solution to the bagaining problem, but a few solutions have kept reappearing as major actors in study after study. These solutions are the Nash, Kalai-Smorodinsky and Egalitarian solutions. In the present study, the spotlight has been on the latter two.

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## *Appendix 1*

The purpose of this appendix is to check the second-order conditions for the optimization problem of Theorem 2.

Using the notation $A = \left[\frac{m-2}{m+n-x_1-x_2}\right]^2$, we obtain the following bordered Hessian

$$|\bar{H}| = \begin{vmatrix} 0 & -\frac{1}{x_1^2} + A & -\frac{1}{x_2^2} + A \\ -\frac{1}{x_1^2} + A & \frac{2x_2}{x_1^3} + \lambda\left[\frac{2}{x_1^3} + \frac{2A}{m+n-x_1-x_2}\right] & -\frac{1}{x_1^2} + \frac{2A\lambda}{m+n-x_1-x_2} \\ -\frac{1}{x_2^2} + A & -\frac{1}{x_1^2} + \frac{2A\lambda}{m+n-x_1-x_2} & \frac{2\lambda}{x_2^3} + \frac{2A\lambda}{m+n-x_1-x_2} \end{vmatrix}$$

With $C \equiv \frac{-1}{x_1^2} + A$, the first-order conditions become $-\frac{1}{x_2^2} + A = \frac{x_1}{x_2}\left(\frac{1}{x_1^2} - A\right)$ $= -\frac{x_1}{x_2}C$. Let $D \equiv \frac{2A\lambda}{m+n-x_1-x_2}$.

Therefore, we have

$$|\bar{H}| = \begin{vmatrix} 0 & C & -\frac{x_1}{x_2}C \\ C & \frac{2x_2}{x_1^3} + \frac{2\lambda}{x_1^3} + D & -\frac{1}{x_1^2} + D \\ -\frac{x_1}{x_2}C & -\frac{1}{x_1^2} + D & \frac{2\lambda}{x_2^3} + D \end{vmatrix}$$

$$= -C^2 \begin{vmatrix} 1 & -\frac{1}{x_1^2} + D \\ -\frac{x_1}{x_2} & \frac{2\lambda}{x_2^3} + D \end{vmatrix} - C^2 \begin{vmatrix} \frac{x_1}{x_2} & \frac{2x_2}{x_1^3} + \frac{2\lambda}{x_1^3} + D \\ -\frac{x_1^2}{x_2^2} & -\frac{1}{x_1^2} + D \end{vmatrix}$$

$$= -C^2\left[\frac{2\lambda}{x_2^3} + D + \frac{x_1}{x_2}\left(-\frac{1}{x_1^2} + D\right) + \frac{x_1}{x_2}\left(-\frac{1}{x_1^2} + D\right) + \frac{x_1^2}{x_2^2}\left[\frac{2x_2}{x_1^3} + \frac{2\lambda}{x_1^3} + D\right]\right]$$

$$= -C^2\left[\frac{2\lambda}{x_2^3} - \frac{2}{x_1x_2} + \frac{2}{x_1x_2} + \frac{2\lambda}{x_1x_2^2} + D\left[\frac{x_1^2}{x_2^2} + \frac{2x_1}{x_2} + 1\right]\right]$$

$$= -C^2\left[\frac{2\lambda}{x_2^2}\left[\frac{1}{x_2} + \frac{1}{x_1}\right] + D\left[\frac{x_1}{x_2} + 1\right]^2\right]$$

From the second first-order condition, we have

$$\lambda = \frac{\frac{1}{x_1}}{\frac{1}{x_2^2} - A} = \frac{x_2^2}{x_1(1-Ax_2^2)}.$$

From the first-order conditions, $x_1 + x_2 = S = m + n - (m-2)\sqrt{\frac{m+n}{m}}$. Therefore, $A = \frac{m}{m+n}$, so that $0 < A < 1$. Also, from the first-order conditions, $0 < x_2 < 1$. Indeed, $x_1$ and $x_2$ are obtained as solutions to the following equation

$$t^2 - (x_1+x_2)t + x_1x_2 = 0$$

$$t^2 - St + P = 0$$

$$t = \frac{S \pm \sqrt{S^2-4P}}{2},$$

where $S = m+n-(m-2)\sqrt{\frac{m+n}{m}}$ and $P = \frac{m+n}{m}$. Since $x_2 < x_1$, we have

$$x_1 = \frac{S+\sqrt{S^2-4P}}{2}, \quad x_2 = \frac{S-\sqrt{S^2-4P}}{2}.$$

Now, substituting S and P with their expressions as functions of m and n, we obtain $x_2 < 1$.

Altogether, we have $0 < Ax_2^2 < 1$, so that $\lambda > 0$. Since $A > 0$ and $m + n - x_1 - x_2 > 0$, all the terms appearing in the bracket multiplying $-C^2$ are positive. Therefore, $|\bar{H}| < 0$.

***Remark.*** It is of interest to note that the minimum of $\frac{N_i(S)/N_i(T)}{N_j(S)/N_j(T)}$ is obtained at a point where the axiom of population monotonicity is violated.

Next we study the behavior of $\epsilon_N^{mn}$ as a function of m and n. To simplify notation, we will from here on write $\epsilon$ instead of $\epsilon_N^{mn}$.

***Lemma.*** $\epsilon_N^{mn}$ is a decreasing and convex function of n, for each fixed m.

***Proof.*** First, we compute $\frac{\partial B}{\partial n}$.

$$\frac{\partial B}{\partial n} = \frac{1}{2}\,[m(m+n)]^{-\frac{1}{2}}\,m = \frac{1}{2}\,\left(\frac{m}{m+n}\right)^{\frac{1}{2}} > 0.$$

$$\frac{\partial \epsilon}{\partial n} = \frac{1}{2}\left[1 - \frac{1}{2}[(B^2-2)^2-4]^{-\frac{1}{2}}2(B^2-2)\right]2B\frac{\partial B}{\partial n}$$

$$= B\,\frac{\partial B}{\partial n}\,\{1-[(B^2-2)^2-4]^{-\frac{1}{2}}\,(B^2-2)\}.$$

The expression in braces is negative since

$$1 < \frac{B^2-2}{\sqrt{(B^2-2)^2-4}} \qquad \text{and} \qquad B > 2.$$

Therefore $\frac{\partial \epsilon}{\partial n} < 0$.

Next, we compute the second-order partial derivative.

$$\frac{\partial^2 B}{\partial n^2} = -\,\frac{1}{4}\,[m(m+n)]^{-\frac{3}{2}}\,m^2 < 0.$$

$$\frac{\partial^2\epsilon}{\partial n^2} = (\frac{\partial B}{\partial n})^2 \{1 - [(B^2-2)^2-4]^{-\frac{1}{2}} (B^2-2)\}$$

$$+ B \frac{\partial B^2}{\partial^2 n} \{1 - [(B^2-2)^2-4]^{-\frac{1}{2}} (B^2-2)\}$$

$$+ 2B^2 (\frac{\partial B}{\partial n})^2 [(B^4-4B^2)^{-\frac{3}{2}}(B^2-2)^2 + (B^4-4B^2)^{-\frac{1}{2}} ].$$

Since the expression on the second line is positive, it is enough to show that

$$1 - (B^4-4B^2)^{-\frac{1}{2}}(B^2-2) + 2B^2(B^4-4B^2)^{-\frac{3}{2}} (B^2-2)^2 + 2B^2(B^4-4B^2)^{-\frac{1}{2}}$$

is positive.

Indeed,

$$-(B^4-4B^2)^{-\frac{3}{2}} \{(B^2-2)(B^4-4B^2) - 2B^2(B^2-2)^2 - 2B^2(B^4-4B^2)\}$$

$$= -[B^2(B^2-4)]^{-\frac{3}{2}} B^4(-3B^2+10) > 0,$$

since $B > 2$, $B^2(B^2-4) > 0$ and $-3B^2+10 < 0$. Therefore, $\frac{\partial^2\epsilon}{\partial n^2} > 0$. ***Q.E.D.***

***Lemma:*** $\epsilon_N^{mn}$ is a decreasing and convex function of m, for each fixed n.

***Proof.*** $\frac{\partial B}{\partial m} = \frac{1}{2} [m(m+n)]^{-\frac{1}{2}} (2m+n) - 1$

$$= \frac{1}{2} \frac{2m+n}{\sqrt{m(m+n)}} - 1 > 0.$$

$$\frac{\partial \epsilon}{\partial m} = B \frac{\partial B}{\partial m} \{1 - [(B^2-2)^2-4]^{-\frac{1}{2}} (B^2-2)\},$$

therefore, $\frac{\partial \epsilon}{\partial m} < 0.$

The second-order partial derivative is

$$\frac{\partial^2 B}{\partial m^2} = - \frac{1}{4} [m(m+n)]^{-\frac{3}{2}} (2m+n)^2 + \frac{1}{2} [m(m+n)]^{-\frac{1}{2}} 2$$

$$= - \frac{1}{4} n^2 [m(m+n)]^{-\frac{3}{2}} < 0.$$

Therefore, by a calculation similar to that of $\frac{\partial^2 \epsilon}{\partial n^2}$, we conclude that $\frac{\partial^2 \epsilon}{\partial m^2} > 0.$

$\epsilon$ is a convex function of m. ***Q.E.D.***

Next we study the asymptotic behavior of $\epsilon_N^{mn}$ as $m \to \infty$ and as $n \to \infty$.

$$B = \sqrt{m(m+n)} - m + 2$$

$$= \frac{m(n+4) - 4}{\sqrt{m(m+n)} + m - 2}.$$

As $m \to \infty$, $B \to \frac{n+4}{2}$ and $B^2-4 \to \frac{n^2+8n}{4}$.

Since we have

$$\epsilon = \frac{(B^2-2)^2 - (B^2-2)^2 + 4}{2(B^2 - 2 + \sqrt{(B^2-2)^2-4})}$$

$$= \frac{2}{B^2 - 2 + \sqrt{B^2(B^2-4)}} ,$$

as $m \to \infty$,

$$\epsilon \to \frac{2}{\frac{n^2+8n+8}{4} + \sqrt{\left(\frac{n^2+8n+16}{4}\right)\left(\frac{n^2+8n}{4}\right)}} = \frac{8}{n^2+8n+8+\sqrt{(n^2+8n)(n^2+8n+16)}} .$$

Also, as $n \to \infty$, $\epsilon = \dfrac{2}{(\sqrt{m(m+n)}-m+2)^2-2+\sqrt{[(\sqrt{m(m+n)}-m+2)^2-2]^2-4}} \to 0.$

## *Appendix 2*

Here, we characterize the Kalai-Smorodinsky and Egalitarian solutions.

First, we formally introduce the requirement on a solution that it offers maximal relative guarantees.

***Maximal Relative Guarantee Structure* (*MRGS*).** For all $P,Q \in \mathcal{P}$ with $P \subset Q$, for all $i,j \in P$, $\epsilon_F(i,j,P,Q) = 1$.

Now, we characterize the Kalai-Smorodinsky solution with the help of ***MRGS***. The proof is similar to, but more direct than, Thomson's (1983a) characterization of that solution based on population monotonicity.

***Proposition* 1.** The Kalai-Smorodinsky solution satisfies ***WPO***, ***AN***, ***S.INV*** and ***MRGS***.

***Proof*.** Straightforward.

***Proposition* 2.** If a solution F satisfies ***WPO***, ***AN***, ***S.INV*** and ***MRGS***, then it is the Kalai-Smorodinsky solution.

***Proof*.** First, we show that $F = K$ on $\Sigma^P$ for $|P| = 2$. To fix the ideas, we let $P \equiv \{1,2\}$ and we observe that by ***S.INV*** it is enough to prove that $F(S) = K(S)$ for all $S \in \Sigma^P$ with $a(S) = e_P$. We introduce a third agent whom, without loss of generality, we take to be agent 3, and we construct a problem $T \in \Sigma^Q$, where $Q \equiv \{1,2,3\}$, such that

$$S^1 \equiv \{(y_2,y_3) \in \mathbb{R}^{\{2,3\}} \mid \exists\ (x_1,x_2) \in S \text{ with } y_2 = x_1 \text{ and } y_3 = x_2\},$$

$$S^2 \equiv \{(y_3,y_1) \in \mathbb{R}^{\{3,1\}} \mid \exists\ (x_1,x_2) \in S \text{ with } y_3 = x_1 \text{ and } y_1 = x_2\},$$

and

$$T \equiv \text{cch}\{S, S^1, S^2\}.$$

By **WPO** and **AN**, we have $F(T) = e_Q$.

Since by **WPO** there exists $i \in P$ such that $F_i(S) > 0$, then by **MRGS** applied to $\{1,2\}$, we find

$$\frac{F_1(T)/F_1(S)}{F_2(T)/F_2(S)} = 1 \implies F_1(S) = F_2(S).$$

By **WPO** and the fact that K satisfies **WPO**, we obtain $F(S) = K(S)$.

Given $Q \in \mathcal{P}$ with $|Q| > 2$, and $T \in \Sigma^Q$, we show that $F(T) = K(S)$ by considering $P \subset Q$ with $|P| = 2$ and noting thart $F(S) = K(S) = \lambda a(S)$ for some $\lambda > 0$ by the first step. Therefore, by **MRGS**, $F_P(T) = \mu a(S)$ for some $\mu > 0$. Since $a_P(T) = a(S)$, we obtain, by repeated application of this argument, $F(T) = \mu a(T)$ and by **WPO**, $F(T) = K(T)$.

**Q.E.D.**

Finally, we characterize the Egalitarian solution. The proof bears some similarity to Thomson's (1983b) characterization of the solution based on population monotonicity. It essentially involves showing that if $S \in \Sigma^P$, then S is the intersection with $\mathbb{R}^P$ of a problem in $\mathbb{R}^Q$, for some $Q \supset P$, whose solution outcome can be shown to have equal coordinates. An application of **MRGS** then implies that the solution outcome of S also has to have equal coordinates.

***Proposition* 3.** The Egalitarian solution satisfies **WPO**, **AN**, ***IIA*** and **MRGS**.

***Proof*.** Straightforward.

***Proposition* 4.** If a solution F satisfies **WPO**, **AN**, ***IIA*** and **MRGS**, then it is the Egalitarian solution.

***Proof***. Let $P \in \mathcal{P}$ and $S \in \Sigma^P$ be given. Without loss of generality, assume that $E(S) = e_P$. Let $m \equiv \max\{\sum_P x_i \mid x \in S\}$. Note that $m \geqq |P|$. Let $Q \in \mathcal{P}$ with $P \subset Q$ and $|Q| \equiv q$ be the smallest integer with $q > n$, and $T \in \Sigma^Q$ be defined by $T \equiv \{x \in \mathbb{R}^Q \mid \sum_Q x_i \leqq q\}$. Finally, let $T' \equiv \text{cch}\{S, e_Q\}$. By **WPO** and **AN**, $F(T) = e_Q$. Since $T' \subset T$ and $F(T) \in T'$, by ***IIA***, $F(T') = F(T)$. It can be checked that $T'_P = S$. By **WPO**, there exists $i \in P$ such that $F_i(S) > 0$. Therefore, for all $j \in P$, $j \neq i$, by **MRGS**, we find that $\dfrac{F_i(T)/F_i(S)}{F_j(T)/F_j(S)} = \dfrac{E_i(T)/F_i(S)}{E_j(T)/F_j(S)} = \dfrac{F_j(S)}{F_i(S)} = 1$.

Therefore, $F_i(S) = F_j(S)$ for all $i, j \in P$. Finally, by **WPO**, and the fact that E satisfies **WPO**, we obtain $F(S) = E(S)$.

***Q.E.D.***

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