Simple Methods of Estimation and Inference for Systems Characterized by Deterministic Chaos

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SIMPLE METHODS OF ESTIMATION AND INFERENCE FOR SYSTEMS CHARACTERIZED BY DETERMINISTIC CHAOS

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Abstract

This paper considers deterministic laws of motion that exhibit the ergodic property. The virtues of studying such laws of motion in the space of densities over the state space are shown. A complete characterization of stationary ergodic processes in that space of densities is developed. The asymptotic properties of the kernel estimators of the stationary density and the law of motion in the density space are studied, and shown to hold under weaker conditions than is currently available in the literature. Small sample behavior for the estimators is subjectively shown to be good even when optimal choices of the kernel and smoothing parameters are not exploited.

Keywords: Ergodic Theory, Deterministic Chaos, Kernel Estimation

1. Introduction

Very recently, there has been a surge of interest in economic circles concerning the philosophical and empirical problem of stochastic versus deterministic explanations of some economic processes. The resulting literature suggests that we may be able to see a truly deterministic process as a stochastic one if only we chose to focus on a certain side of our models. The recent literature popularizing the so-called chaotic explanations of economic behavior has in part been inspired by a similar literature in the natural sciences (e.g. see motivations in Brock 1986, Brock and Dechert 1986, Brock, Dechert and Sheinkman, 1987, Brock and Sayers, 1987, & Sheinkman and Le Baron, 1987). In those studies, similarities were drawn between computer algorithms for pseudo-random number generation and deterministic difference equations arising from economic models.

With the familiarity of ARMA models, there was a surprise generated by the result of Sakai and Tokumaru (1980) that most trajectories of the difference equation popularly known as the tent map:

$$\begin{aligned} x_{t+1} &= \frac{x_t}{a}; \quad x_t \in [0, a] \\ &= \frac{(1 - x_t)}{(1 - a)}; \quad x_t \in [a, 1] \end{aligned}$$

generate the same stochastic behavior (in terms of autocorrelation coefficients) as the AR(1) process:

$$y_{t+1} = (2a - 1)y_t + \epsilon_t$$

for i.i.d. ϵ_t. A number of economists started investigating the possibility of generating deterministically chaotic systems from simple economic models. The research program was very successful as the results achieved by Grandmont (1985), Brock and Chamberlain (1984), Boldrin and Deneckere (1987) -to mention only a few- have demonstrated. While the theorists are improving upon their machinery and producing more deterministically chaotic models out of more familiar overlapping generation models (Grandmont, 1985) and growth theoretic ones (Boldrin and Deneckere, 1987), another branch of studies developed. A series of papers from the camp of Brock, Sheinkman, LeBaron, Dechert, Sayers mentioned above started testing for deterministic chaos in regular data sets relying on a dimensionality of correlation test. The theoretical development of that test was again inspired by work in the natural sciences and then mathematically developed in Brock and Dechert (1986).

In this paper, one hopes to deal with econometric issues arising from the question: what kind of statistical inference can be made concerning models in which the variables of interest are characterized by deterministic chaos? Assume that an economist wrote down a dynamic model which gave a deterministic non-linear first order difference equation which he then claims explains the behavior of some economic variable. With the theoretical model generating the deterministic chaos, and the tests of data suggesting that we could not distinguish between some stochastic and deterministic systems, it would be difficult to ignore the possibility that those models may perform well in estimation and forecasting. Two main problems stand out in defiance of econometric investigation, though, the first being the unfamiliarity of the concept of estimation and inference in deterministic models. The second major obstacle is the non-linearity of most of those chaotic laws of motion. By a law of motion, one means an operator $S: X \longrightarrow X$ that defines $x_{t+1} = S(x_t)$. X is referred to as the state of the system. This paper proposes to solve those two problems by introducing an unorthodox tool of investigation. By analyzing the behavior of the deterministic dynamic process in the space of densities over the state space (referred to as the hyper-state space), one can introduce a stochastic element through the initial condition for the law of motion. Also, one will show that the corresponding law of motion in the hyper-state space will be linear despite the non-linearity of the state space law of motion. Moreover, the estimation procedure will be done independently of the functional form and parameter values of the model, and hence the results should be robust to model mis-specification up to the result of chaotic behavior of the state variable. One starts first by outlining what is now a standard procedure in economic theory.

It is natural that one would take the starting point of a maximizing agent as given. Another natural step to take is to follow the vast majority of the modern macro economic literature and make the model *dynamic*. By that one means setting up the maximization problem in such a way that the standard results of dynamic optimization can be used to achieve fast and simple solutions. The standard way to do this (*Sargent,1986 Ch.1*) would be to copy the standard optimal control representation

$$\max_{(y_t)} \sum_{t=0}^{\infty} u_t(x_t, y_t)$$

$$s.t. \;\; x_{t+1} = g_t(x_t, y_t); \;\; \forall t.$$

where *u(.,.)* is the return (utility) in period t, x and y are the state and control (state and choice) variables respectively. It is standard in economic literature to then simplify the model and put

$$u_t(.,.) = \beta^t u(.,.); \;\; g_t(.,.) = g(.,.); \;\; \forall t.$$

The standard result of the dynamic optimization literature * gives a unique solution such that

$$y_t = h(x_t)$$

This result is usually the most interesting to economists and is commonly referred to as the optimal policy choice. For our purposes, the more interesting result is that that relation in turn implies the

* Bellman (1973) together with some simplifying assumptions (Sargent 1987, Chapter 1)

law of motion for x:

$$x_{t+1} = g(x_t, h(x_t)) = S(x_t).$$

Needless to say, some economists find it more direct to use the first order conditions of the original optimization problem to arrive at the law of motion for the state or policy variable. It is well known, however, that solving the first order conditions does not in general give a closed form solution for the policy function. Regardless of the method of achieving the law of motion, this paper starts at the stage where one would normally proceed to do some econometric estimation of the underlying parameters of the model.

The basic idea of the literature on economic chaos is to analyze deterministic laws of motion of the type discussed above where the (highly non-linear) law of motion gives rise to trajectories that look very similar to stochastic ones. Since our treatment of economic chaos in the following sections of the paper deviates significantly from the mainstream literature on the issue, it seems to be wise to devote this last section of the introduction to an overview of that literature. Due to the vastness of the literature at hand, a comprehensive survey would be lengthy, and arguably not of great value for our purposes.

First, one has to outline the definition of chaos and its interest to economists. Appropriately, there is no one definition for chaos. The general agreement among scientists in different fields is to use the term chaos to characterize dynamical systems with unstable equilibria where any conceivable trajectory can be generated by choosing the appropriate initial condition. In economics, the majority of the literature on chaotic systems seems to fall within one of two categories. The first category constitutes the research program of searching for economic models that generate complicated dynamics. This research program resulted in a plethora of economic models of various types which generated chaos. To mention only a few, Grandmont's (1985), generated chaotic business cycles from standard overlapping generations macro-models. Day (1987) generated ergodic chaos (which will be defined and fully analyzed in the following sections) from a Keynesian macro-model. In general, as Brock (1988b, pp. 7-10) indicates in his summary of this research program, relaxing some of the common assumptions in economic models e.g. allowing for high levels of discounting, introducing regions of increasing returns to scale, incomplete markets, disequilibrium dynamics,... etc. will readily result in chaotic behavior. This research program is therefore considered a success, and it suggests very strongly that the possibility of having economic chaos cannot be ignored.

In the following sections, however, little reference will be made to that research program for a number of reasons. The first reason has to do with the above mentioned *likelihood* of occurrence of economic chaos. As we mentioned above, the usual understanding of chaotic behavior in economic models is the existence of at least one initial condition that will result in any trajectory. The obvious weakness of such a definition after it is formalized is that it does not give us a notion of the *likelihood* of occurrence of such chaotic behavior. More formally, what is the probability that the modeled economic system starts at one of the initial conditions that lead to chaotic behavior. With the exception of the work by Day and Shafer (1987), and Day and Lin (1987), it seems that that problem is very rarely addressed in the economic literature. In this paper, we shall define and completely characterize ergodic chaos, which deals precisely with that issue.

The second research program in the field of economic chaos which has been growing very rapidly is the *testing for chaos* in economic data.* The basic idea of those tests as adopted by Brock (1988a and 1988b), Brock and Sayers (1987), Sheinkman and LeBaron (1987), and others was originally used in the natural sciences (Eckman and Ruelle, 1985). The basic concept rests on observing a time series $\{x_t\}$, and examining the quantity

$$C_m(\epsilon,T) = \#\{(t,s), 1 \le t,s \le T | \|x_t^m - x_s^m\| < \epsilon\}/T_m^2$$

where

$$T_m = T - (m-1)$$

and

$$x_t^m = (x_t, ..., x_{t-m+1})$$

This statistic clearly measures the frequency of observing the same pattern approximately repeat in the data, thus giving a measure of correlation in the data. Then, a quantity d referred to as the dimensionality of autocorrelation is defined as

$$d = \lim_{m\uparrow\infty} \lim_{\epsilon\downarrow 0} \lim_{T\uparrow\infty} \frac{ln[C_m(\epsilon,T)]}{ln[\epsilon]}$$

which is clearly approximated for a large m. The slope of $ln[C_m(\epsilon,T)]$ with respect to $ln[\epsilon]$ is then used (Brock 1988a and 1988b, Brock and Sayers, 1986, Sheinkman and LeBaron, 1987) to test for the dimensionality of the model, which vaguely tests for the degree of complexity of the model needed to represent the data. As this paper is being written, more work is being done by those researchers to improve their tests and find their statistical properties. At this point in time, however, it seems that that research program functions to primarily satisfy the curiosity of the researchers regarding the possibility of certain time series being generated from low dimensional deterministic processes. Since that is not one of the objectives of our current research, this research program will not play a major role in shaping the following sections except for the motivation to find estimation techniques that will be robust to the possibility of chaotic data generating processes.

* It is worth noting at this point that since the earlier version of this paper has been written, the authors referred to below have replaced the expression *testing for chaos* by the expression *testing for extra structure.*

For our purposes, since the economic model in question remains generic, one starts by defining the simplest possible economic model that is rich enough to demonstrate all the interesting aspects of our approach. The model is a non-overlapping generations model where each individual lives for only one period and is succeeded by his son the moment he dies. The only product in that economy is mangos, and a mango seed takes one period (generation) to grow into a tree and bear one fruit, after which it dies instantly. A son inherits all the trees that his father planted, chooses the number of mangos to eat and the number of seeds to plant. If a person decides not to plant a seed, then that causes him some pain, either because of a bequest motive or because he would have to eat the seed, which is not a pleasant experience. Notice that the family line can look like an infinitely lived individual with infinite discounting of future utility, but the dynamics of the system still stem from the bequest motive. The model, thus could be written as follows:

$$\max_{L_t} \; u(C_t, L_t)$$

$$s.t. \quad L_t \leq C_t$$

$$and \quad C_{t+1} = g(C_t, L_t)$$

where the first constraint is that the number of seeds (units of labor) thrown cannot exceed the number of mangos eaten. The second constraint specifies the technology of growing trees, and drives the dynamics of the system. To further simplify the system, we specify the Cobb-Douglas utility function $u(C, L) = CL$ which together with the first constraint will obviously give us $C_t = L_t; \; \forall t$. We then specify the technology that drives the system and study the behavior of consumption (our state variable) in the state space as well as the density space. The discontinuities in the technology can easily be explained by a soil corrosion argument.

$$C_{t+1} = 2L_t \;\; mod \;\; 1$$

and, thus, our law of motion is

$$C_{t+1} = 2C_t \;\; mod \;\; 1$$

2. Defining the corresponding law of motion in the hyper-state space

At this point, we shall take the result of the deterministic dynamic optimization problem

$$x_{t+1} = S(x_t)$$

as our starting point. Although the law of motion for the state variable x is completely deterministic, one could study the evolution of the economy over time within a probabilistic framework. For although each infinitely lived individual's behavior of x follows the law of motion S (which is assumed to be identical for all individuals), starting from an initial distribution of individuals in the state space, the evolution of that distribution could have rather interesting characteristics. Another way to look at that treatment is that it introduces a stochastic element through the randomness of the initial condition of our difference equation as suggested by Richard Bellman (1973). We are now ready to define the induced law of motion in the density space. We start by defining a complete normalized measure space $(X, \Im, \mu)$, i.e. $\mu(X) = 1$. A transformation

$S: X \longrightarrow X$ is non-singular if

$$B \in \Im; \ \ \mu(S(B)) = 0, \ \ \Rightarrow \mu(B) = 0.$$

For every non singular law of motion S,

$$\exists! P: D \longrightarrow D$$

defined by:

$$\int_A Pf(x)d\mu(x) = \int_{S^{-1}(A)} f(x)d\mu(x)$$

where D is the space of all densities defined on X, i.e. the space of nonnegative L^1 functions on X whose integral is 1 and P is called the ***Frobenius-Perron operator*** corresponding to S (Lasota and Mackey 1985, pp. 36-37). The following properties of the operator P follow directly from the definition:

a). P is a Markov operator.

b). P is linear.

c). If $S_n = S \circ S \circ \ldots \circ S$, then the corresponding F-P operator is $P_n = P^n$

What property c). gives us is that as long as time is discrete, (or as long as there is a lower limit on the time it takes an individual to make a decision), the sampling period and the decision making period need not be identical (as long as the former is an integer multiple of the latter). Also, all limiting properties of the F-P operators corresponding to the different periods are the same.

The F-P operator gives us precisely the behavior of the evolution of the density of the initial states over time as the individual goes through the process of maximization which is summarized by the law of motion S. One interpretation, thus could be that there are many individuals in the economy, and as each of them changes his state variable by passing through the operator S, the overall distribution over the state space moves according to P. Another interpretation would be that we would initially put some likelihood measure over the state space and knowing the individual's behavior, we could consistently put a measure of probability on where our representative individual would be after n periods of time, or asymptotically. The distinction between these two interpretations is important since it allows for vastly different uses of the F-P operator.

Our mango economy example simplifies things tremendously since we have $S\colon [0,1] \longrightarrow [0,1]$ and we could specify the measure to be Lebesgue measure. Making the extra assumption that starting densities come from the space of continuous densities on [0,1] which is dense in the space of all densities on [0,1], we could simplify the definition of the F-P operator since the Lebesgue and Riemann integrals will coincide, and we can define:

$$Pf(x) = \frac{d}{dx} \int_{S^{-1}([0,x])} f(s).ds$$

3. The definition of equilibrium and chaos

By abandoning the state space in favor of the hyper-state space, we need to be able to define the counterpart to the most important features of the state space. The main four characteristics of the state space that we naturally discuss are initial conditions, laws of motion, equilibria, and chaos. The concept of initial condition is obviously substituted by that of initial density, and the law of motion by the Frobenius-Perron operator discussed in the previous section. In this section, we define the counterparts of equilibrium and chaos in the hyper-state space to complete the framework. A natural notion of equilibrium, regardless of how we interpret the initial density, could be defined as follows:

Weak Equilibrium: is defined by a stationary density $f*$ such that

$$Pf* = f*$$

Strong Equilibrium: is defined by an asymptotically stable with respect to $\|.\|$ density $f*$ such that

$$\lim_{n\uparrow\infty} \|P^n f - f*\| = 0; \;\; \forall f \in D.$$

In a recent paper, Boldrin and Deneckere (1987) consider simple macro systems that exhibit topological chaos. They also note that one of the drawbacks of topologically chaotic behavior is that it might only be valid for a set of initial conditions that is of measure zero. The same argument is used by Day and Shafer (1987), and Day and Lin (1987) to motivate studying ergodic chaos. For our purposes, we rigorously define the notion (Lasota and Mackey, 1985) of ergodic chaos. We say a non-singular transformation

$$S: X \longrightarrow X$$

is chaotic if it is ergodic; i.e.

$$\forall A \in \Im : S^{-1}(A) = A; \;\; \mu(A) = 0, \;\; or \;\; \mu(X \setminus A) = 0.$$

This is clearly a natural concept of chaos since it states that the only invariant sets are trivial sets, and thus, that as time progresses, one could not trace back the initial condition. Now, we are ready to state the following theorem which describes chaotic behavior of X in the density space. We define D to be the space of all non-negative functions in L^1 with unit norms whose support is X . *In statements of convergence $\forall f \in D$, it is understood that we only consider densities with no point mass with respect to μ.*

*Theorem 1:**

Given $S: (X, \Im, \mu) \longrightarrow (X, \Im, \mu)$ μ preserving, with $\mu(X) = 1$:

S is ergodic if and only if the limit

$$\lim_{n \uparrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k f(x) = 1; \quad a.e.$$

$$\forall f \in D$$

○

Proof of Theorem 1:

For the only if part , one starts by showing that there can be at most one stationary density for S ergodic. Assume that f_1 and f_2 were both stationary densities of P, let $g = f_1 - f_2$, then by the linearity of P, $Pg = g$, and since $g = g^+ - g^-$ (the positive and negative parts of g), g^+ and g^- are also stationary functions for P (by proposition 3.1.3 Lasota and Mackey 1985, p.35). Let $A = \{x: g^+(x) > 0\}$ and let $B = \{x: g^-(x) > 0\}$, then clearly, $A \cap B = \phi$ and $\mu(A) > 0$ and $\mu(B) > 0$. Since f_1 and f_2 are densities that are different. It follows that

$$A \subset S^{-1}(A) \subset \ldots \subset S^{-n}(A)$$

* This theorem is stated and proved in Lasota and Mackey(1985). The following proof draws heavily on the proofs in Lasota and Mackey (1985) and Walters (1982).

and

$$B \subset S^{-1}(B) \subset \ldots \subset S^{-n}(B)$$

Letting $A* = \cup_{n=0}^{\infty} S^{-n}(A)$ and $B* = \cup_{n=0}^{\infty} S^{-n}(B)$, we have $A*$ and $B*$ being invariant sets, that are non-trivial, i.e. $S^{-1}(A*) = A*$ and $\mu(A*) \neq 1$ or 0, and the same for B, but that violates the definition of ergodicity by all invariant sets being trivial, hence we have a contradiction and there could be at most one stationary density. Now, Lasota and Mackey (1985), Theorem 5.2.2 , pp. 85-86 proves the existence of the Cesàro limit for $P^n f$ if $\exists$ a unique stationary density, and we have the obvious result (since by above, the only $f \colon Pf = f$ is $f = 1$, see proof of lemma 1 below):

$$\lim_{n\uparrow\infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k 1 = 1$$

It follows that there exists a unique Cesàro limit for $P^n f$. Since S is measure preserving, it is readily seen that $P1 = 1$ and hence, 1 has to be the unique limit, i.e. we have shown that:

$$\frac{1}{n} \sum_{k=0}^{n-1} P^k f(x) \longrightarrow 1$$

which concludes the proof for the only if part .

For the if part, let the Cesàro convergence to 1 be given, and assume that $f*$ is a stationary density, i.e. $P^k f* = f*$,$\forall k$, then $\lim_{n\uparrow\infty} \frac{1}{n} \sum_{k=0}^{n-1} f* = 1$ and thus, $f* = 1$, and hence, 1 is the unique stationary density for P. Now, assume S is not ergodic; then $\exists A$ non-trivial s.t. $S^{-1}(A) = A$, and letting $B = X \setminus A$, i.e. the complement of A with respect to X. we get $S^{-1}(B) = B$. Now, let us write (by the linearity of P)

$$f* = I_A f * + I_B f* = PI_A f * + PI_B f*$$

then we have $I_A f* = PI_A f*$ and $I_B f* = PI_B f*$ where $I_Y(x)$ is the indicator function which takes the value 1 if $x \in Y$, and 0 otherwise. Letting $f_1 = \frac{I_A f*}{\int I_A f*}$ and $f_2 = \frac{I_B f*}{\int I_B f*}$, it follows that f_1 and f_2 are two distinct stationary densities for P which results in a contradiction. Hence S has to be ergodic and the proof of Theorem 1 is finished. ■

To get an intuitive feeling for the origin of that result, notice that by the Birkhoff Ergodic Theorem (Walters, 1982 p.34), it follows that with S μ-preserving:

$$\frac{1}{n} \sum_{i=0}^{n-1} f(S^i(x)) \longrightarrow f* \in D \quad a.e.$$

and if S is ergodic, $f*$ is constant almost everywhere, and hence, with a normalized measure space, and since $f* \in D$, it must be that starting from any $f \in D$, the density converges to 1 a.e. The result of the Birkhoff ergodic theorem obviously does not deal with the F-P operator but rather with its conjugate, but Lasota and Mackey (1985, thm. 5.2.2) show that the same result is true for the Cesàro limit of the F-P operator if S is ergodic. This theorem is very powerful since it states that given a maximization problem that yields a chaotic S mapping, regardless of what our priors are, or regardless of the initial distribution of the agents in the state space, we should expect the distribution of the individuals to be asymptotically uniform (with respect to the measure μ which we shall usually specify to be Lebesgue measure).

At this stage, we are ready to analyze the chaotic behavior of our mango economy. Figure 3a shows the trajectory of consumption. The driving technology , the diadic transformation, is obviously Lebesgue measure preserving. From figure 1, it is clear that the state space has only two steady state equilibria, namely 0 and 1, both of which are unstable. The trajectory of C does not seem to show any short term periodicity. As a matter of fact, one can easily show that the diadic transformation is chaotic. To show this notice that

$$S^{-1}([0,x]) = [0, \frac{x}{2}] \cup [\frac{1}{2}, \frac{1+x}{2}]$$

which gives:

$$Pf(x) = \frac{d}{dx} \sum_{i=0}^{1} \int_{\frac{i}{2}}^{\frac{(i+x)}{2}} f(s).ds$$

$$= \frac{1}{2} \sum_{i=0}^{2-1} f(\frac{i+x}{2})$$

and clearly, by a simple inductive argument:

$$P^n f(x) = \frac{1}{2^n} \sum_{i=0}^{2^n-1} f(\frac{i+x}{2^n}) \longrightarrow \int_0^1 f(s).ds = 1 \quad as \quad n \uparrow \infty$$

uniformly in x $\forall f$ (*again where we only consider f's that do not have a point mass at a steady state*). Hence, the density converges strongly to the uniform density. Strong convergence implies Cesàro convergence, and by Theorem 1, since the diadic transformation is Lebesgue measure preserving, this process is ergodic. The speed of convergence to the uniform density starting from the density $f(x) = 2x$ is illustrated in Figure 3b.

4. A counter example to demonstrate the importance of measure preservation and analysis in the hyper-state space

At this stage of the paper, we could not say anything about the chaotic behavior of transformations that do not preserve the measure we happen to choose. Since a measure will typically be chosen for convenience (e.g. Lebesgue), there is no guarantee that the law of motion arising from the economic model will preserve that measure. Also, we cannot see the reason for analyzing the hyper-state space. One reason that one may wish to analyze the hyper-state space is that one may suspect that by putting probabilistic weights on the state space, one may be able to find interesting behavior in some cases where the state space exhibits a multiplicity of unstable equilibria, and hence may exhibit chaotic behavior. If one does not wish to assume that the process starts on a stable manifold (which in general is of measure zero, and may not exist in many cases) or at a steady state equilibrium, then we may be able to study the likelihood of the process being in any particular region of the state space given an initial distribution. Again, we can interpret that initial density as a probabilistic prior or as an actual distribution of agents when the process starts. The case of the diadic transformation analyzed above clearly satisfied the condition of having multiple unstable equilibria. In that case, however, the trajectory in the state space seemed to be sufficient in detecting the chaotic behavior of the variable. To see how dramatic the difference between that apparent behavior in the state space and actual behavior in the density space can be, let us change our technology in the mango economy to:

$$C_{t+1} = \frac{L_t}{(1-L_t)} \;\; ; L_t \in [0, \frac{1}{2}]$$

$$C_{t+1} = 2L_t - 1 \;\; ; L_t \in (\frac{1}{2}, 1]$$

This example is especially tailored to yield a transformation which looks essentially similar to the diadic transformation in that the two steady states 0 and 1 are unstable. Moreover, looking at figure 4a, it seems that the trajectory does not have a short period. Notice, however, that the transformation here is not lebesgue measure preserving, and thus, the result of Theorem 1 does not apply. Comparing figures 1 and 2, it is clear that the only change we introduced in figure 2 is the slower speed of repulsion from 0. The result of such a minor change, however is remarkably strong, for one can show* that $\forall \epsilon > 0$:

$$\lim_{n\uparrow\infty} \int_{\epsilon}^{1} P^n f(s).d\mu(s) = 0$$

i.e. asymptotically, all the mass is on the stationary point 0. This result is especially strong; in more general frameworks, we get a limiting distribution over the state space. Such limiting distributions can be viewed as a generalization of Jovanovic's (1986) framework with multiple equilibria. In his framework, he considers systems with multiple equilibria, and notes that the identifications of the system's coefficients will not in general be achieved if we do not know which equilibrium the system is in. He then proceeds to include the particular equilibrium as one of the parameters to estimate, putting a measure on the equilibria under consideration. In our framework, we do not require that the system be in any particular equilibrium in the state space. We can derive the weak or strong

* This is the example referred to as the paradox of the weak repeller in Lasota and Mackey (1985).

equilibrium in the hyper-state space, and that will give us a measure on the entire state space which we can in principle use to estimate the parameters of the model. As we shall argue in the section on estimation, however, the parameters of the model are not necessarily of any interest to us since most of the interesting dynamics can be derived from the law of motion in the hyper-state space.

5. Characterizing chaotic behavior in non-measure preserving transformations

Since we shall be interested in empirically analyzing the behavior of the density space , it would be useful to know when a transformation preserves a certain measure using only the density space behavior. The following lemma has been constructed to achieve such a result:

Lemma 1:

A non-singular transformation $S:(X,\Im,\mu)\longrightarrow(X,\Im,\mu)$ *with* $\mu(X)=1$ *is measure- preserving if and only if* 1 *is a stationary density for P, the F-P operator corresponding to S.* ○

Proof of Lemma 1:

For the if part of the lemma, let 1 be a stationary density, then $P1=1$ and $\forall A\in\Im$, since by definition

$$\int_A Pf d\mu = \int_{S^{-1}(A)} f d\mu$$

we have

$$\int_A 1 d\mu = \int_{S^{-1}(A)} 1 d\mu$$

but that simply says that $\mu(A)=\mu(S^{-1}(A))$, i.e. that S is μ preserving.

For the only if part, let S be measure preserving, then

$$\mu(A)=\mu(S^{-1}(A))$$

and hence

$$\int_A 1 d\mu = \int_{S^{-1}(A)} 1 d\mu$$

but by the definition of the F-P operator,

$$\int_A P1 d\mu = \int_{S^{-1}(A)} 1 d\mu$$

Hence, since this has to hold for all sets $A \in Im$, $P1 = 1$ $a.e.[\mu]$, and since we are not considering sets of measure zero, 1 is a stationary density for P. ■

What this lemma together with the result of Theorem 1 state in terms of our economic interpretation of the convergence of densities is that if the uniform density is a weak equilibrium, then the underlying transformation is chaotic if and only if the uniform density is also a strong equilibrium. However, this result still could be considered to be too restrictive since our characterization of chaotic behavior will only apply in cases where the uniform density is stationary, i.e. where the transformations preserve the chosen measure (which will usually be Lebesgue measure). The following theorem has been constructed especially to generalize that result and allow us to define chaos in the density space under certain circumstances where the transformation in question does not preserve our measure. The trick is to try to find a measure which that transformation preserves and then go back to analyze chaotic behavior using the result of Theorem 1.

Theorem 2:

Given the probability space $(X, \Im, \mu)$, let the non-singular $S: X \longrightarrow X$ have the F-P operator P, and let $\exists f > 0$ such that $Pf = f$ which has no point mass; i.e.

$$\forall A_\epsilon : \mu(A_\epsilon) < \epsilon$$

$$\lim_{\epsilon \downarrow 0} \int_{A_\epsilon} f d\mu = 0$$

then S is ergodic if and only if $\forall g$ with no point mass, f is the Cesàro limit of $P^n g$; i.e.

$$\lim_{n \uparrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k g = f$$

○

Proof of Theorem 2:

Given $f* > 0$ is a stationary density for P which has no point mass, we can define the measure ν as follows

$$\nu(A) = \int_A f * d\mu; \ \ \forall A \in \Im$$

Clearly, S preserves ν since $P_\mu f* = f*$:

$$\nu(A) = \int_A f * d\mu = \int_A P_\mu f * d\mu$$

$$= \int_{S^{-1}(A)} f * d\mu = \nu(S^{-1}(A)); \ \forall A \in \Im$$

and hence, by Theorem 1, since S preserves ν, S is ergodic $[\nu]$ (i.e. all S invariant sets are of ν measure 0 or 1) if and only if the uniform density with respect to ν is the Cesàro limit of $P_\nu^n g$ for all g with no point mass. For the if part of Theorem 2, we need to show that if $f*$ is the Cesàro limit of P_μ, then S is ergodic $[\mu]$. But notice that the uniform density $[\nu]$ corresponds to $f*$ $[\mu]$, since:

$$\nu(A) = \int_A 1 d\nu = \int_A f * d\mu; \ \forall A \in \Im$$

and by the linearity of integration, and linearity of P, consider any initial density $j > 0$ with respect to μ, then both measures μ and ν are absolutely continuous with respect to the measure generated by integrating j with respect to μ. Hence, there exists a corresponding density g with respect to ν such that:

$$\int_A \frac{1}{n} \sum_{k=0}^{n-1} P_\mu^k g d\nu = \int_A \frac{1}{n} \sum_{k=0}^{n-1} P_\nu^k j d\mu$$

and taking limits of both sides , and due to the absence of point mass, using bounded convergence, we can take the limits inside the integration, and since

$$\lim_{n \uparrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} P_\nu^k g = 1$$

and using the definition of the measure ν, we get

$$\lim_{n \uparrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} P_\mu^k j = f*$$

Hence, S is ν ergodic if and only if $f*$ is the Cesàro limit of $P_\mu^n j$ for all $j > 0$ with no point mass. But remembering that we were not interested in $[\nu]$ ergodicity but rather in $[\mu]$ ergodicity, notice that by the definition of ν:

$$\nu(A) = \int_A f * d\mu$$

it follows that $f* > 0$ is the Radon-Nikodym derivative of ν with respect to μ, and thus, $\mu \ll \nu$; and by that absolute continuity, if S is ergodic $[\nu]$, all invariant sets are of ν measure zero and hence of μ measure zero or of ν measure 1, and hence their complements are of ν and μ measure zero. Hence, all invariant sets have to also be of μ measure 0 or 1, and S is ergodic $[\mu]$ which completes the proof of the if part of the Theorem.

For the only if part, we follow the same proof strategy as that of Theorem 1. Since the ergodicity $[\mu]$ of S implies that P_μ has at most one stationary density (this claim is proved in the beginning of the proof of Theorem 1), and by f* being stationary, it follows that f* has to be the unique stationary density for P_μ. And by Theorem 5.2.2 pp. 85-86 in Lasota and Mackey (1985), it follows that f* is also the Cesàro limit for P_μ. ∎

At this stage, we have the result that any non-singular process which is in a weak equilibrium is chaotic if and only if the stationary density defined by that weak equilibrium is also the Cesàro limit density for the corresponding F-P operator. Given a model which suggests the chaotic behavior of the state or policy variable, we could characterize the behavior of that variable in the hyper-state space. The economist does not have to even be able to solve for the closed form of the operator S, as long as he could demonstrate its chaotic behavior. Another popular approach that non-structural form econometricians might find appealing is to simply assume that a certain variable can be explained by a deterministically chaotic law of motion and to proceed with analyzing the behavior in the density space.

6. The framework for estimation

In the preceding sections, we derived a Markov process and characterized its law of motion in the hyperstate space and we want to be able to estimate that law of motion. Since we are assuming that the process is chaotic, and assuming the system has been running long enough, or that we were lucky enough, let the hyperstate be currently in the stationary limiting density. This is not a very uncommon practice in econometric investigations since estimation is usually done under the assumption that the model is in equilibrium. This approach allows the state variable to oscillate as long as the density is kept stationary. In other words, the weakness of the notion of equilibrium in the hyper-state space makes the assumption of estimation under equilibrium weak.

We observe a time series $\{x_n, n \geq 1\}$ which is a Markov process on $(X, \Im, \mu)$. By the chaotic assumption, and the above assumption of observing the process in equilibrium, it follows that the process x_n is strictly stationary. For simplicity, we assume that the process takes values in $\mathbf{R}$, with σ-algebra $\mathbf{B}$ which is the σ- algebra generated by the Borel sets of $\mathbf{R}$. The assumption that the process is univariate will be relaxed in later research, but it seems sufficient as a first step for the simplicity of exposition as well as the fact that most of the economic models generating deterministic chaos are univariate. The existing results on estimation in Markov processes that will be used in Theorems 3 and 4 below will require that one also makes the assumption that Doeblin's condition holds , i.e., that $\exists\lambda$ a finite measure on B with λ ($\mathbf{R}$) > 0, and an integer $\nu \geq 1$ and $\epsilon > 0$ such that $\forall \xi \in X$,

$$Tr^{(\nu)}(\xi, A) \leq 1 - \epsilon \quad \forall \lambda(A) \leq \epsilon$$

where Tr^n is the n-period transition probability. It is clear that this assumption will not hold for the deterministic law of motion that we have. The desired results can be easily recovered, however, by taking a sequence of *truly* stochastic Markov processes which converges to our deterministic process. The details of that procedure are outlined below in the proofs of Theorems 3 and 4 (especially the proof of Theorem 4). Given the assumption of chaotic behavior of X, it follows that there exists only one ergodic class with no cyclic subsets. Given these assumptions and some technical conditions that will have to be adapted for our purposes, Prakasa-Rao (1983) and Roussas (1969) prove consistency and asymptotic normality results for the kernel estimator of the stationary density f, and consistency for the Frobenius- Perron operator which is clearly defined by the conditional density $p(x'|x)$ in the continuous state version of the Chapman-Kolmogorov equation (by the stationarity of f):

$$f(x') = \int p(x'|x)f(x)dx$$

It is obvious, then, that forecasting the probability of the system being in some set at some point in time can be performed by estimating the density at that point in time using the Chapman-Kolmogorov equations, and then integrating the density over the set of interest.

7. Asymptotic Results

In the following statements, K is a symmetric, bounded kernel and h_n is the smoothing factor in the kernel estimators. We observe $(x_1, ..., x_{n+1})$ and estimate the densities of interest by:

i. the stationary density:

$$f_n(x) = \frac{1}{nh_n}\sum_{j=1}^{n} K(\frac{(x - X_j)}{h_n})$$

ii. the joint density:

$$j_n(x, x') = \frac{1}{nh_n}\sum_{j=1}^{n} K(\frac{(x - X_j)}{\sqrt{h_n}})K(\frac{(x' - X_{j+1})}{\sqrt{h_n}})$$

and of course,

iii. the transition conditional density:

$$p_n(x'|x) = \frac{j_n(x, x')}{f_n(x)}$$

Recently, Gallant and Nychka (1987), and others have popularized the use of orthogonal series polynomial expansions of the densities for what they called ***seminonparametric*** estimators. One of the necessary conditions for the consistency of most of those seminonparametric estimators, however, is that the estimated densities have to be in L^2. The reason for this restriction is that most of those series expansions (e.g. trigonometric, Hermite, and Legendre) form systems that do not form a basis for L^1, and hence, the respective estimators are not consistent for some densities in L^1.* For that reason, one prefers to use the kernel estimators which are themselves densities and thus form a basis for L^1. Notice, however, that the results of the following two theorems will prove consistency and asymptotic normality of the stationary density, and consistency of the estimate of the transition density. It is well known, however, that the estimator of the stationary density is biased.

We state the following set of assumptions varying in their strength for use in Theorems 3 and 4. Notice that the following set of assumptions are not mutually exclusive, and though restrictive, there is still a large class of kernels and smoothing parameters that can satisfy all the conditions, and thus, achieve the warranted properties proved in the following two theorems.

1.

i. $\lim_{|x|\uparrow\infty} |x|K(x) = 0$

ii. $\int_{-\infty}^{\infty} x^2 K(x)dx < \infty$.

2.

i. $nh_n \longrightarrow \infty$ as $n \uparrow \infty$.

ii. $\exists\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ of integers tending to ∞ such that $\beta_n \gamma_n \alpha_n^{-1} \rightarrow 0$ and $\alpha_n h_n \rightarrow 0$.

3.

i. $f(x)$ is continuous.

ii. f , the stationary density is absolutely continuous, and satisfies:

$$|f(x') - f(x)| \le M(x)|x - x'|$$

iii. S, the deterministic law of motion, has at most a finite number of discontinuities.

*Theorem 3:**

Given assumptions **1.i, 2.i & 3.i,iii,**

i. $\int_{-\infty}^{\infty} |E[f_n(x)] - f(x)|dx \longrightarrow 0$

And given assumptions **1.i,ii, 2.i,ii, 3.i,ii,iii,**

---

* See Theorems 1, p.294, 5, p.312, and 6, p.317 in Devroye and Györfi (1985) for the respective inconsistency results for the trigonometric, Hermite, and Legendre polynomial estimators.

* For a statement and proof of part i., see Basawa and Prakasa-Rao (1980, pp. 302-304), the proof replacing the assumption of Doeblin's condition holding by 3.iii is clear using an argument similar to that in the proof of Theorem 4. The proof of part ii. provided here uses the results in Basawa and Prakasa-Rao (1980, pp.305-306).

ii. $\sqrt{nh_n}[f_n(x) - E(f_n(x))] \longrightarrow N(0, \sigma^2(x)) \quad \forall x$

where $\sigma^2(x) = f(x) \int K^2(z)dz$. ○

Proof of Theorem 3 part ii.:

We have the following general Central Limit Theorem (this is a special case of Basawa and Prakasa-Rao (1980), Theorem 2.5 p.305):

Theorem:

Given $\{x_n\}$a Markov process satisfying Doeblin's condition. Let $u_n \uparrow \infty$ and let $\{L_n\}$ be a sequence of uniformly bounded functions $L: R \to R$ such that

a. $E|L_n(x_1)|^2 = O(u_n n^{-1})$.

Let $v_n(x_i) = L_n(x_i) - EL_n(x_i)$:

b. $E|v_n(x_1)v_n(x_i)| = O(u_n^2 n^{-2})$ uniformly in $1 \leq i \leq n$.

c. $E|v_n(x_1)v_n(x_i)v_n(x_j)| = O(u_n^3 n^{-3})$ uniformly in $1 \leq i \leq j \leq n$.

d. $nu_n^{-1}var(L_n(x_1)) \to \sigma^2 < \infty$.

and suppose that $\exists$ sequences of integers $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ tending to ∞ such that

$$\beta_n \gamma_n \alpha_n^{-1} \to 0$$

and $\alpha_n u_n n^{-1} \to 0$.

Then

$$\sqrt{u_n^{-1}} \sum_{j=1}^{n} [L_n(x_j) - EL_n(x_j)] \to N(0, \sigma^2).$$

Now, following Basawa and Prakasa-Rao (1980), Theorem 2.6, p. 306, let $u_n = nh_n$, and let

$$L_n(z) = K(\frac{x - z}{h_n})$$

Then, it is clear that the assumptions of Theorem 3 part ii. satisfy all the conditions of the above theorem, and by the definition of f_n, the desired result follows directly. One point has to be dealt with, however, and that is the satisfaction of Doeblin's condition which is clearly violated in our case. This problem can easily be solved by considering the estimator of the process x_t^m defined by the transition density

$$p_m(x'|x) = \frac{m}{\sqrt{2\pi}} e^{-m^2(x'-S_m(x))^2/2}$$

which obviously satisfies Doeblin's condition. Using Kushner's (1984) Theorem 6, p.156-158 reproduced below at the end of the proof of Theorem 4, we can see that the sequence of corresponding stationary densities f_m converges to f pointwise, and the results of Theorem 3 follow by considering f_n to be the estimator of $f_{M(n)}$ where $M(n)$ is chosen for each n to be large enough so that the difference between the warranted density and the approximating one converges to zero after normalization by $\sqrt{n}$. ■

This theorem gives us asymptotic results for the kernel estimator of the stationary density in terms of the consistency a.e. and asymptotic normality around its mean. We would like to have similar results for the conditional density function defining the Frobenius-Perron operator. We start with the weaker results in Theorem 4 which are mainly due to Roussas (1969). The results of Roussas, however, require the continuity of $j(.,.)$ and thus, the continuity of $p(.|.)$. In our deterministic law, however, the transition function $G(x'|x) = \int_{-\infty}^{x'} p(x''|x)dx''$ is a heavyside function which has the value 0 for all $x' < S(x)$ and then jumps to the value 1. The conditional density, thus, is a Dirac delta function at the value $S(x)$. In the proof of Theorem 4 below, one uses the result of Roussas for continuous j and then takes a sequence of stochastic Markov processes that weakly converge to our deterministic process. This technique of defining deterministic systems as the weak limit of stochastic processes will be of great interest in the analysis of our same methods of estimation for some stochastic systems.

*Theorem 4:**

Given assumptions **1.i, 2.i, 3.i,iii** *and defining the estimate of* $G(.|.)$

$$G_n(x'|x) = \int_{-\infty}^{x'} p_n(x''|x)dx''$$

we have the result: $\forall x \in X, \delta > 0, \exists A$ *with* $\mu(A) < \delta$ *and*

$$\sup_{x' \notin A} |G_n(x'|x) - G(x'|x)| \longrightarrow 0 \quad \textit{in probability } \forall x', x$$

* This theorem with the extra assumption of the continuity of $j(.,.)$ and a stronger result of convergence of the sup for $x \in X$ is stated in a different form without proof in Prakasa- Rao (1983), and proved in Roussas (1969), the proof provided here uses results from both sources and replaces the assumption of continuity of $j(.,.)$ by continuity of S except at a finite number of points.

Proof of Theorem 4 :

We start by proving the result for j continuous. By the definition of f_n, j_n, & p_n, we have:

$$p_n(x''|x) - p(x''|x) = \frac{1}{f_n(x)}[j_n(x,x'') - j(x,x'')] + \frac{1}{f(x)f_n(x)}[f(x) - f_n(x)]j(x,x'')$$

but by the definition of G

$$G_n(x'|x) - G(x'|x) = \int_{-\infty}^{x'} (p_n(x''|x) - p(x''|x))dx''$$

and using the above equality together with the obvious fact that

$$\int_{-\infty}^{x'} j(x,x'')dx'' \le \int_{-\infty}^{\infty} j(x,x'')dx'' = f(x)$$

and using the triangle inequality, and taking sup over all x' ,we get:

$$\sup_{x'} |G_n(x'|x) - G(x'|x)| \le \frac{1}{f_n(x)} \sup_{x'} \int_{-\infty}^{x'} |j_n(x,x'')|dx'' + \frac{1}{f_n(x)}|f(x) - f_n(x)|$$

Taking plims of both sides, and using the facts that the LHS is non-negative, and that the first term on the RHS converges in probability to 0 $\forall x$ by Roussas (1969) lemma 3.3, and the second term converges in probability to 0 by its L^1 convergence to 0 (Theorem 3 above), the LHS converges to 0 in probability.

Now, we need to replace the assumption of the continuity of $j(.,.)$ by that of the continuity of S except at finitely many points. Start by considering a sequence of stochastic processes with the transition density

$$p_m(x'|x) = \frac{m}{\sqrt{2\pi}} e^{-m^2(x'-S_m(x))^2/2}$$

where S_m is a sequence of continuous functions converging to S. It is clear that that sequence of normal transition densities converges to the Dirac delta function at $S(x)$ which is the true $p(x'|x)$. Moreover, for all finite m, p_m are continuous. Assuming for the moment that f is the continuous stationary density for all the p_m's (this delicate point of the proof will be treated shortly), then the sequence of joint densities j_m defined by $j_m(x,x') = p_m(x'|x)f(x)$ are also continuous. By the constructed convergence, $\exists M$ such that

$$\sup_{x' \notin A} |G_M(x'|x) - G(x'|x)| \le \epsilon/2$$

Now, taking the sequence of estimators G_n to estimate G_M instead of G, we can use the triangle inequality to write

$$\sup_{x' \notin A} |G_n(x'|x) - G(x'|x)| \le$$

$$\sup_{x' \notin A} |G_n(x'|x) - G_M(x'|x)| + \sup_{x' \notin A} |G_M(x'|x) - G(x'|x)|$$

but , by the above result from Roussas, the first term on the RHS converges in probability to zero. And by construction, the second term of the RHS converges to zero completely, we conclude that the LHS has to converge in probability to zero which is the desired result.

One detail in the above proof remains to be checked; the sequence of stochastic processes (which are still ergodic), each have a stationary density f_m, and we need to be able to show that the sequence of those stationary densities converges pointwise to f. The following result is due to Kushner (1984, thm. 6, pp. 156-158), it is stated here without proof. Since it is easier in the approximation literature to consider a sequence of stochastic processes x^ϵ weakly converging as $\epsilon \downarrow 0$ to a deterministic diffusion process x, we consider the limit to be the deterministic Markov process that agrees with x at all integer values of time. Consider the following assumptions:

A.0 The limiting diffusion process $x(.)$ corresponding to our deterministic law of motion at all integer time points gives a unique solution on $[0, \infty)$ for all initial conditions x_0 to the Itô equation:

$$dx = b(x)dt + \sigma(x)dw$$

A.1 The diffusion limit $x(.)$ has a unique stationary density f which defines a unique invariant measure.

A.2 The pair $\{x_t^\epsilon, \xi_t^\epsilon\}$ is Markov $\forall \epsilon > 0$ and let $x_0^\epsilon \Rightarrow x_0$ where the initial condition $x_0^\epsilon, \xi_0^\epsilon$ has the stationary distribution defined by the measure $\nu^\epsilon(.)$, then $x^\epsilon(.) \Rightarrow x^\epsilon(.)$, the process with x_0 as its initial condition. Let $\mu^\epsilon(.)$ be the marginal measure for $\{x_t^\epsilon, \xi_t^\epsilon\}$.

A.3 $\exists \epsilon_0 > 0$, and for all nonrandom initial conditions (x, ξ)=$(x_0^\epsilon, \xi_0^\epsilon$ and $\epsilon > 0$, there exists $T_\epsilon(x, \xi)$, a transient time which may $\uparrow \infty$ as $\epsilon \downarrow 0$, such that

$$\{x_t^\epsilon : 0 < \epsilon \leq \epsilon_0, \ t \geq T_\epsilon(x, \xi), x, \xi\}$$

is tight.

Theorem: (thm. 6, Kushner (1984), pp. 156-158)

Given A.0, A.1 and A.3; $\{\mu^\epsilon(.)\}$ is tight. Moreover, if A.2 holds, then $\mu^\epsilon(.) \Rightarrow \mu(.)$.

We thus have the convergence of the stationary densities to f, and the proof of Theorem 4 is completed.

■

The results of Theorems 3 and 4 give us all the necessary ingredients for statistical inference. With the consistency and asymptotic normality of the stationary density estimate (Theorem 3), and the consistency of the transition function (Theorem 4), we can forecast the probability of the system being in some set at some point in time. Moreover, with the estimate of the transition function, we can forecast the evolution of densities starting from an initial density other than the stationary one, and hence examine the short term effects of redistribution policies.

8. Unoptimized small sample behavior of the estimators

It is a common practice to test the small sample behavior of estimators whose asymptotic robustness has been established. In this section, we start by estimating the stationary density, the cross-period joint density, and the transition density for the trajectory of the diadic transformation depicted in Figure 3.a. In order to qualitatively evaluate the performance of the estimator under less than perfect circumstances, the estimation was performed using the Kernel which takes the value 0.5 on the interval [-1,1], and vanishes outside that interval. This is the grand father of all kernel estimators that was referred to as the naive density estimator by Fix and Hodges (1951). There was no attempt to use the literature on the optimal choice of the smoothing factor, it was taken to be $n^{-0.75}$. The estimated stationary density is shown in Figure 5.a, and the estimate is within one standard deviation of its true value (the uniform density) using the asymptotic normality result of Theorem 3, part ii. The joint density is depicted in Figure 5.b, and it clearly outlines the diadic transformation. The transition density also clearly shows the diadic transformation although the peaks are more ambiguous because of the irregularity of the estimated stationary density. For a sample of only 100 observations, however, and with unoptimised choices of the kernel and smoothing parameters, the estimator is relatively successful.

A second simulation was performed and the three densities of interest were estimated for the trajectory of the diadic transformation with shocks that is depicted in Figure 6. The reason for performing this simulation could be motivated by the proof technique used for Theorem 4* where one considered a sequence of stochastic systems converging to the deterministic system of interest. More generally, one would be interested in testing whether the techniques developed here can be successfully used in stochastic systems where the deterministic component dominates the stochastic component. The trajectory in Figure 6 was generated using $x_{t+1} = (2x_t + r) \bmod 1$ where r is distributed uniformly on [0,0.001]. The estimated stationary density, joint density, and transition density are shown, consecutively, in Figures 7.a, 7.b, 7.c, and it is clear that the dominance of the diadic transformation over the shock component was depicted.

9. Conclusion

* See last step in proof of Theorem 4 above

In this paper, we have mathematically developed the law of motion for the hyperstate space, and characterized that law of motion for deterministically chaotic processes. The simple example of the non-overlapping generations mango economy was used to illustrate the various concepts.

By characterizing the chaotic behavior in the density space, we have justified the estimation and forecasting of the density of the state variable. By estimating and forecasting those densities, one extracts more information from the data than one would using mechanisms that only estimate its conditional expectation. By further estimating the transition function (F-P operator) in the hyper-state space, one can analyze short term effects of redistributive policies. In other words, if the policy makers were to change the distribution of the state variable in the economy without altering the individuals' maximization problem, we can forecast the evolution of that distribution using the estimated transition function.

For the estimation mechanism, we suggested the kernel estimator for the law of motion and the stationary density of the process. Consistency and asymptotic normality results were proved, and then, a small sample simulation was performed for the mango economy example which yielded satisfactory results.

The theory about the relation between the law of motion in the state space and the F-P operator holds for any complete measure space. Generalization of the results of the estimators to the multivariate case where the space X is a subspace of R^d is direct. The consistency result for the stationary density in the multivariate case are available in the literature of kernel estimation. The asymptotic normality result can also be achieved using the multi-variate version of the central limit theorem used in the proof of Theorem 3, part ii. For the consistency result of the transition function, one can adjust the results of Yakowitz (1979) in the same manner used in the proof of Theorem 4 above. The proof strategies are generally the same as those used in this section, and results are potentially stronger (Yakowitz proves convergence of his estimator with probability one which is stronger than the result of Roussas (1969) that was adapted for Theorem 4).

Another generalization that was alluded to in the proof of Theorem 4, and the simulation in Figures 6 and 7 is the usefulness of our simple techniques for some stochastic systems. We can utilize the fact that a stochastic control problem could give a dynamic solution whose behavior in the hyper-state space is very similar to that of a deterministic system by estimating the law of motion for the latter.

A test for chaos can easily be developed using the result of Theorem 2 since, under the null, the estimated F-P operator should have the estimated stationary density as the unique stationary

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For instance, see Devroye and Györfi (1985), thm. 1, p. 148, proved on pp. 156-159 density. Necessary and sufficient conditions for that uniqueness are not hard to find, and thus, we have all the necessary components for a test of ergodicity of a time series. If it is maintained that the data generating process was deterministic, then that also constitutes a test of chaos. Progress towards the development of that test has been made in El-Gamal 1988, and Domowitz and El-Gamal 1988. Further progress along that line of research is expected soon.

Figure 1

The Diadic Transformation

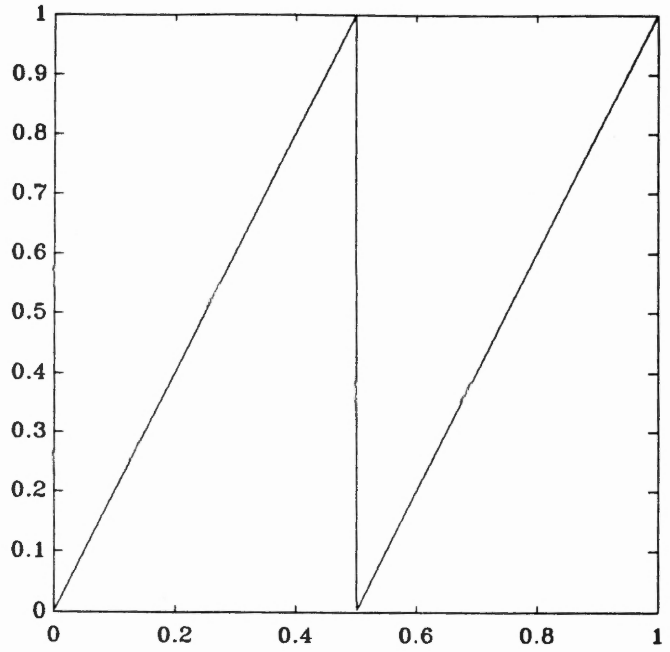


Figure 2

The Weak Repeller

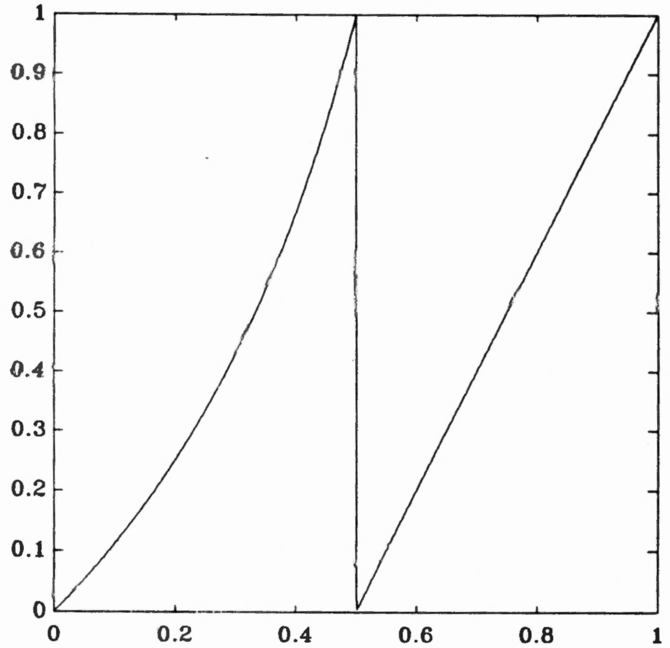


Figure 3.a

Trajectory Of The Diadic Transformation

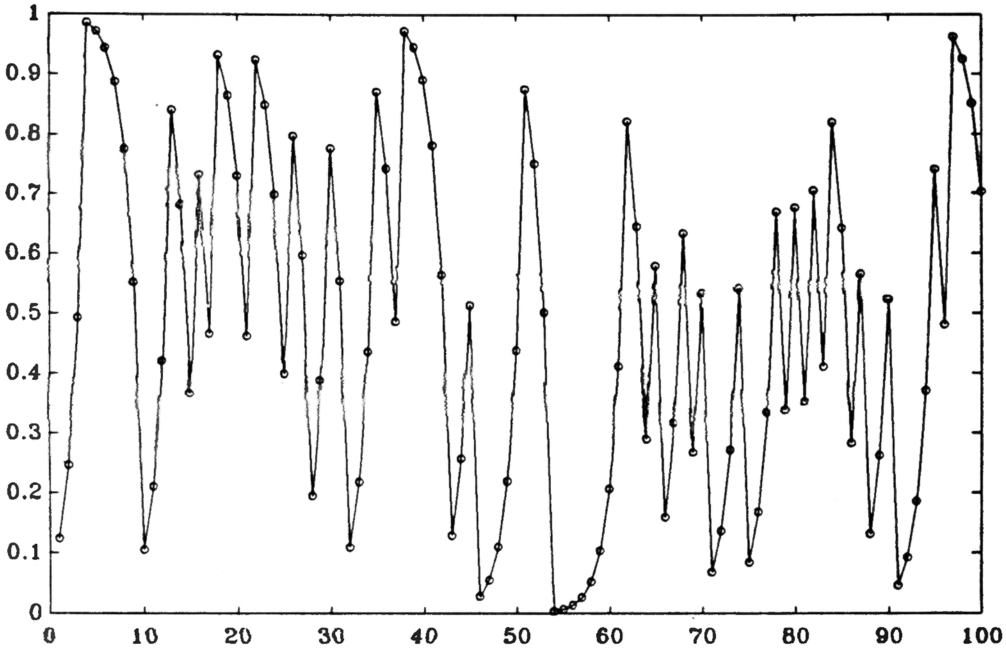


Figure 3.b

Evolution Of The Density

(Lasota and Mackey (1985), p.9)

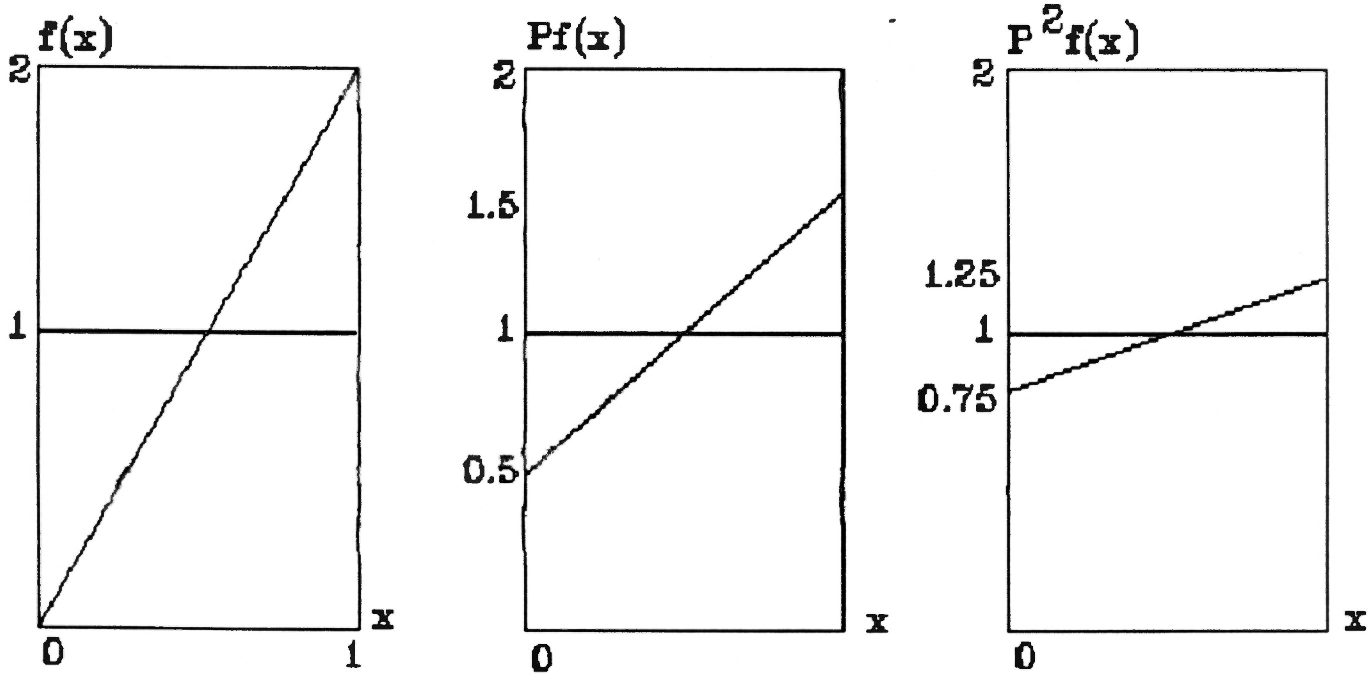


Figure 4.a

Trajectory Of The Weak Repeller

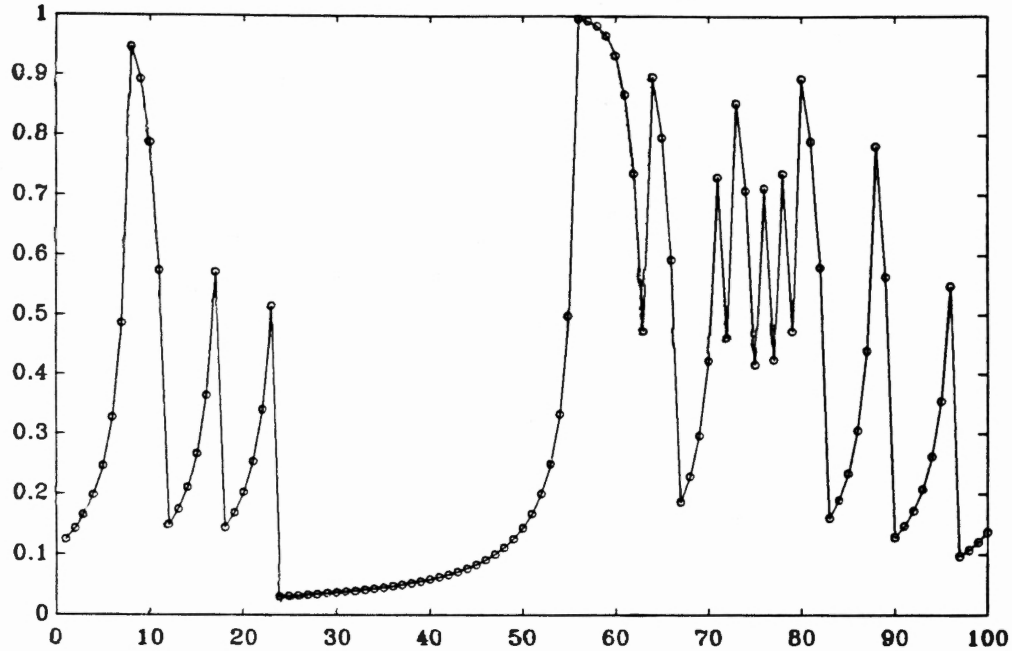


Figure 4.b

Evolution Of The Density

(Lasota and Mackey (1985), p.12)

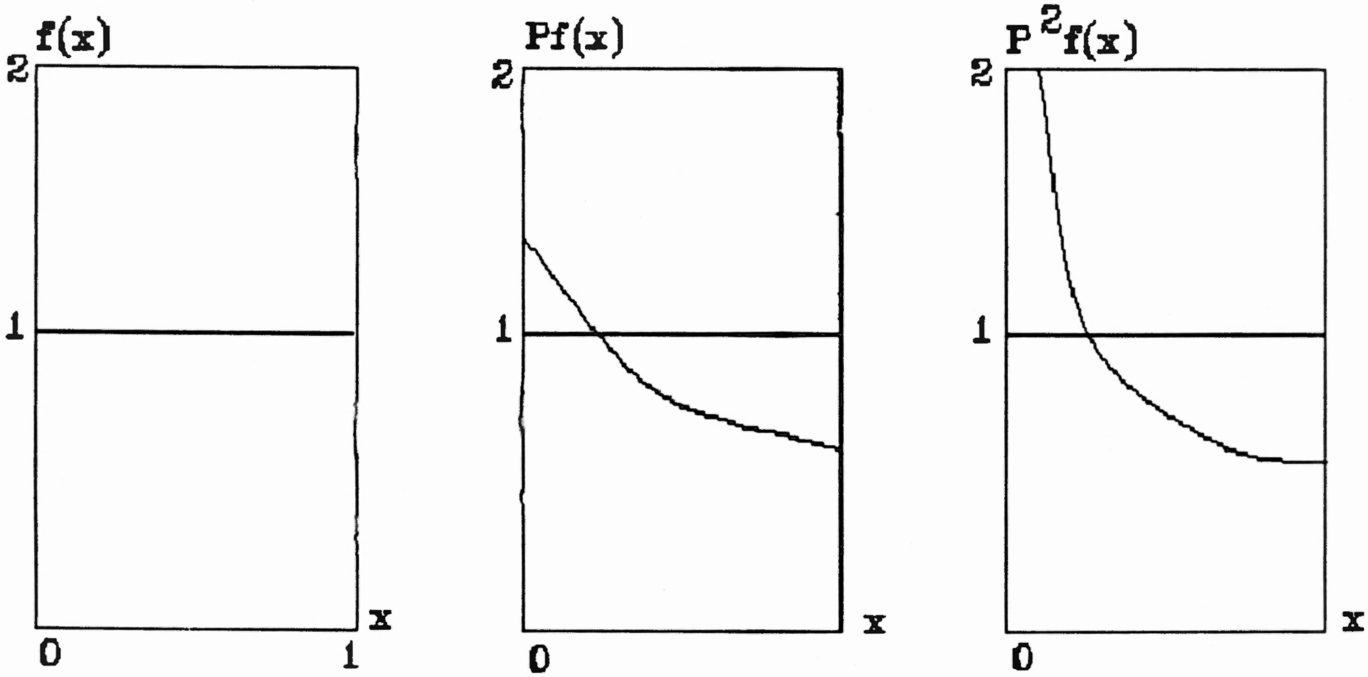


Figure 5.a

Estimated Stationary Density

For The Diadic Transformation

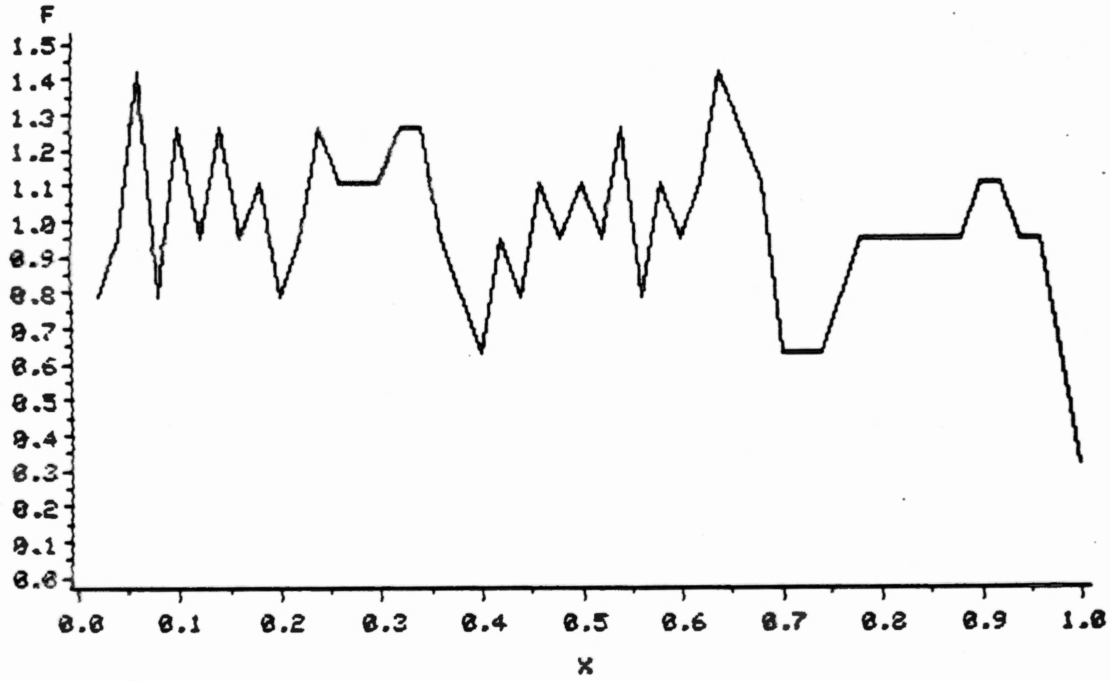


Figure 5.b

Estimated Joint Density

For The Diadic Transformation

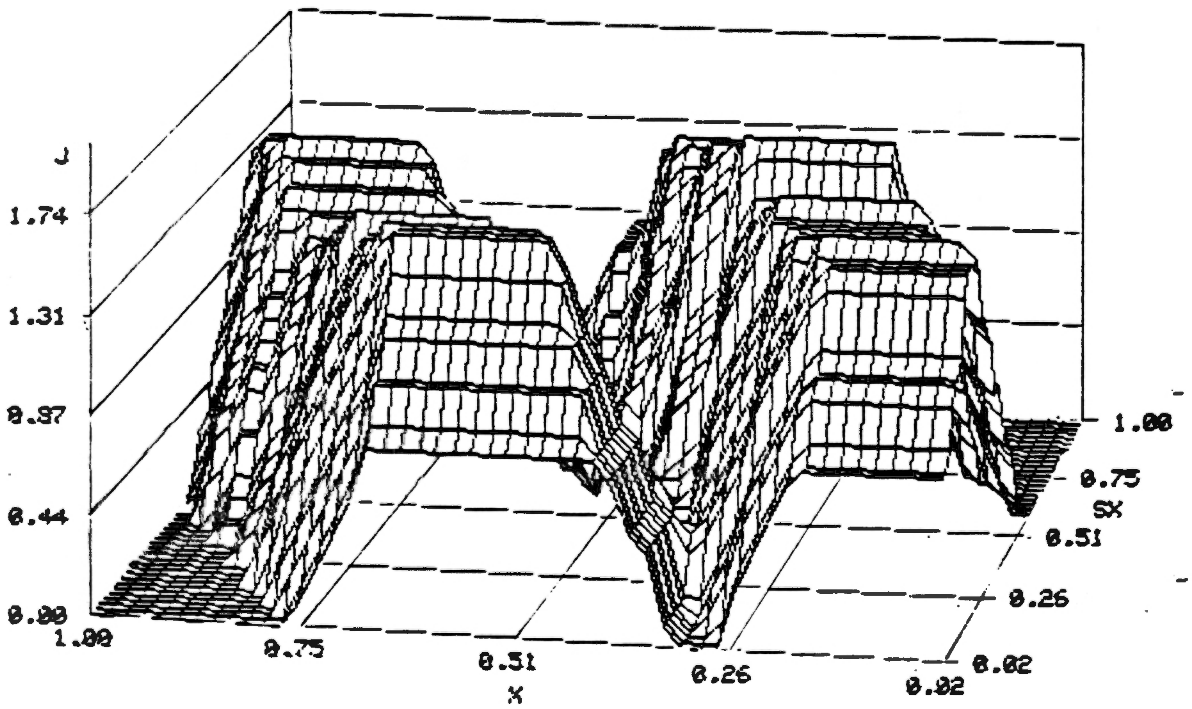


Figure 5.c

Estimated F-P Operator (Conditional Density)

For the Diadic Transformation

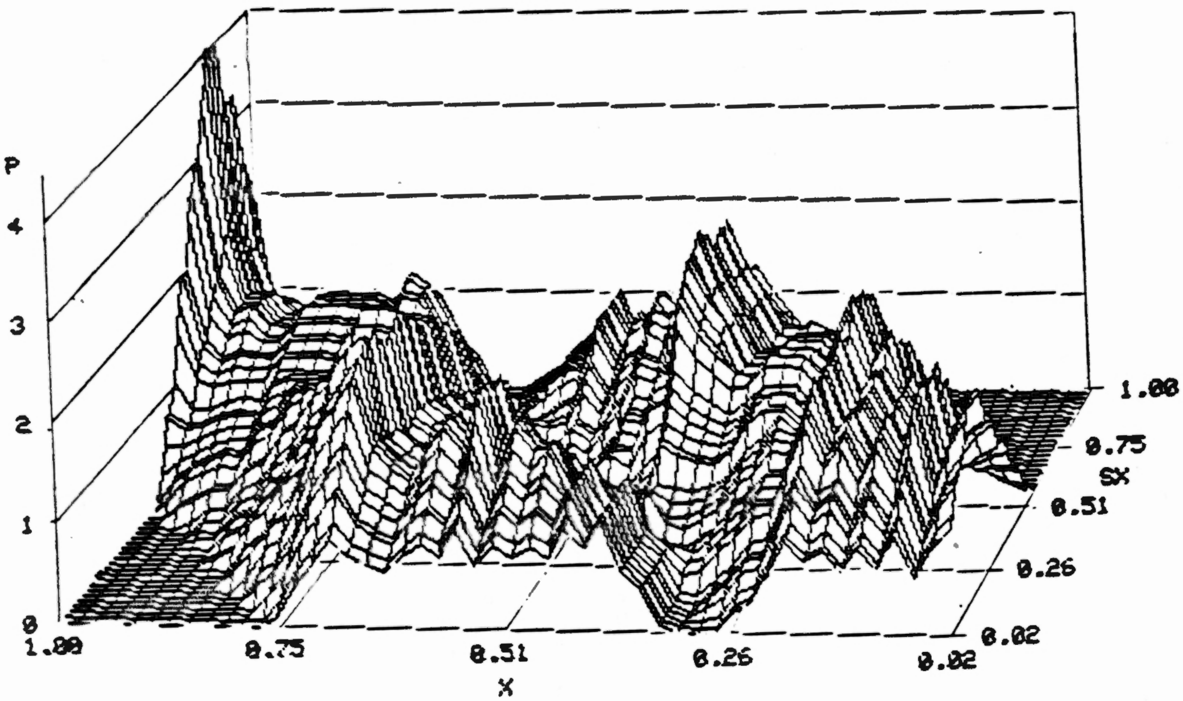


Figure 6
Trajectory of The Diadic Transformation
With Shocks

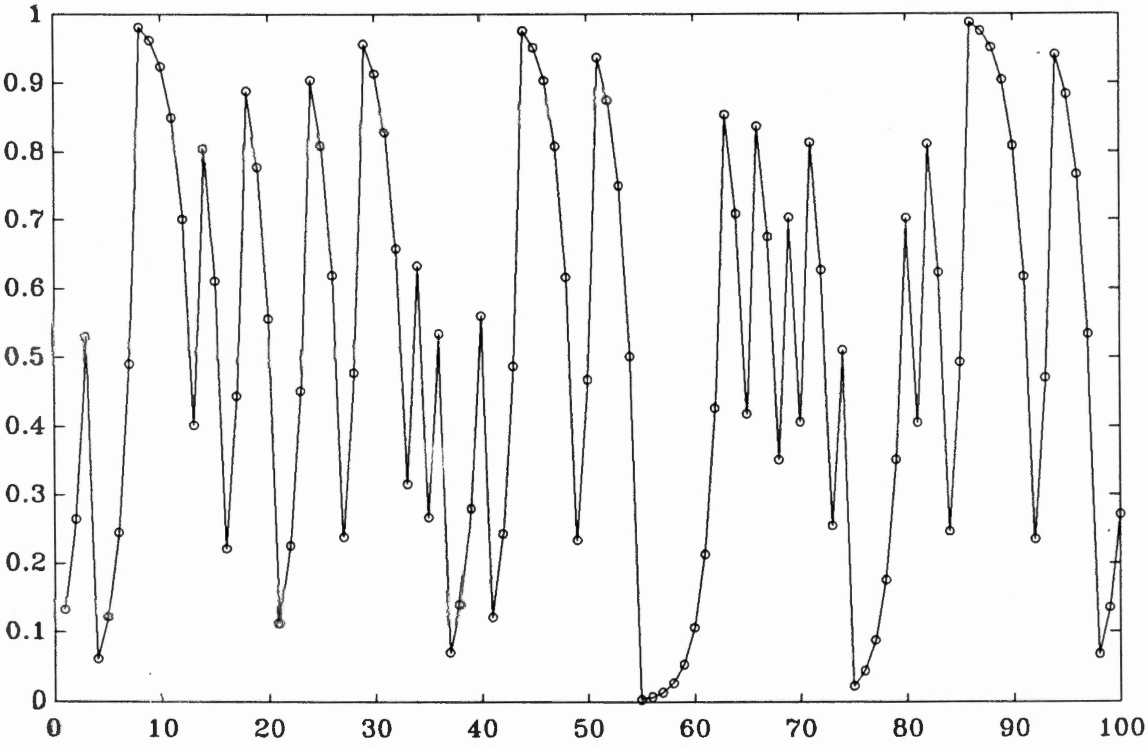


Figure 7.a

Estimated Stationary Density

For The Diadic Transformation With Shocks

Figure 7.b

Estimated Joint Density

For The Diadic Transformation With Shocks

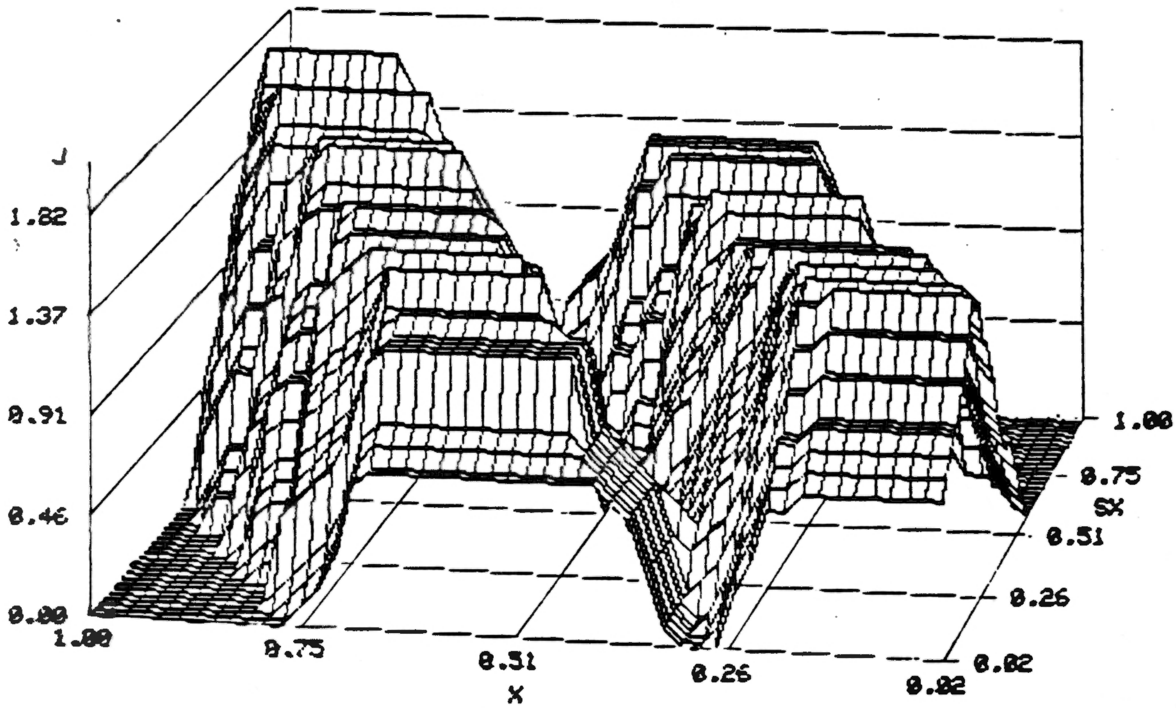
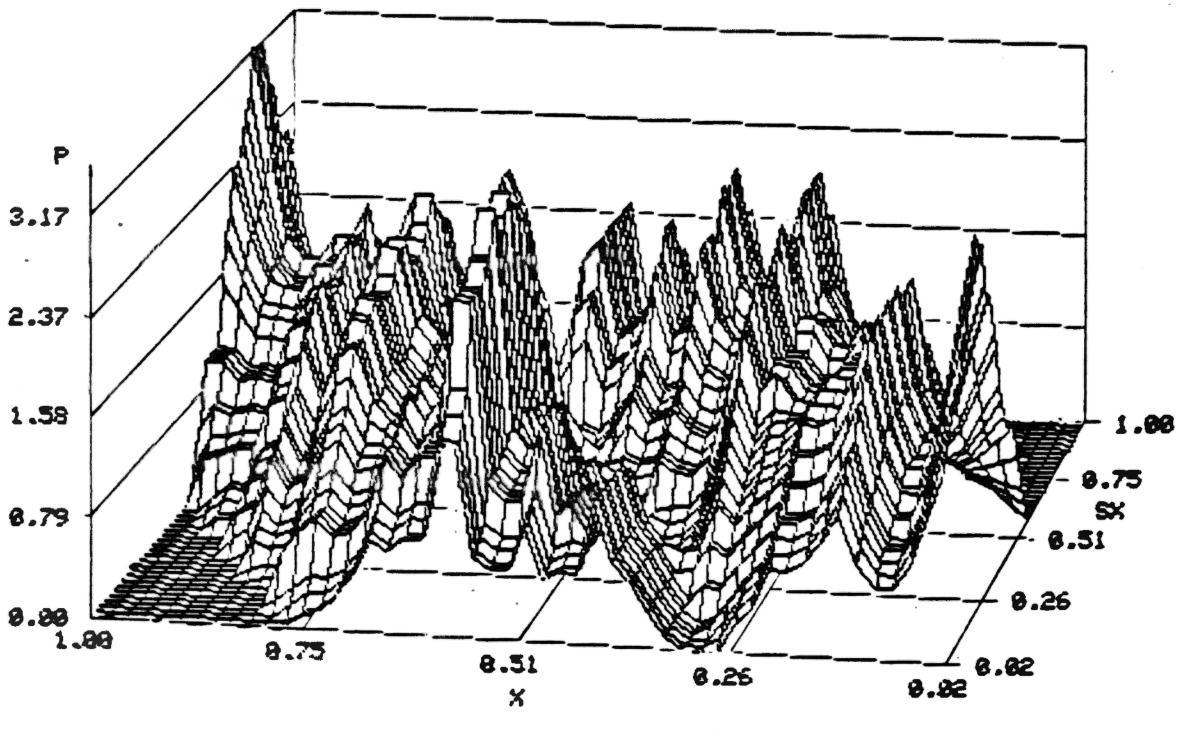


Figure 7.c

Estimated F-P Operator (Conditional Density)

For the Diadic Transformation With Shocks



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