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A Test of the Harris Ergodicity of Stationary Dynamical Economic Models*

The First Order, Univariate Case

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Abstract

A formal statistical test of Harris ergodicity is developed for known univariate first order dynamical systems. The analysis is conducted by examining the properties of a density of states generated by a law of motion on the state space. The distribution of the test statistic is derived under the null hypothesis that the law of motion is Harris ergodic, and the test is shown to be consistent against the alternative that the law is not ergodic. The test can be easily performed using standard statistical and mathematical computer packages.

Keywords: Ergodic Theory, Harris Ergodicity, Testing Dynamic Economic Models.

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1. Introduction

The verification of the dependence properties of laws of motion arising from general nonlinear optimization problems has heretofore proved to be intractable. Yet, assumptions concerning the ergodic or mixing properties of a process underlie most of what is known concerning statistical inference in dynamic models, and it is now understood how the failure of ergodicity can affect standard inference procedures adversely.[1] The purpose of this paper is to introduce a framework for the statistical testing of the hypothesis of ergodicity in nonlinear models.

We are motivated in part by the growing body of evidence gleaned from investigations of the properties of laws of motion arising from various classes of dynamic optimization models that such processes may indeed fail to possess the ergodic property. The work of Arthur (1985), for example, concerns the dynamics of microeconomic allocation decisions between objects with increasing returns or decreasing supply costs. Describing his law of motion (here, of market shares) by a sequence $\{x_n\}$, and given two samples from the set of possible events $\{t_i\}$ and $\{t_i'\}$ with corresponding time paths $\{x_n\}$ and $\{x_n'\}$, then under such conditions, $|x_n - x_n'|$ does not converge to zero as $n \uparrow \infty$, violating a necessary condition for ergodicity. Such considerations also enter the work of David (1986). The more general analysis of Majumdar, Mitra, and Nyarko (1986) implies that the decision variables from a general class of dynamic programs can have an arbitrarily large number of ergodic subclasses associated with them. They focus on the process $x_{t+1} = h[F(x_t, e_{t+1})]$ where $h(.)$ is an optimal investment policy function, $F(.,.)$ is a gross output function and $\{e_t\}$ is a sequence of shocks to the production process. If $F(.,.)$ is not strictly concave, the set into which x_t eventually enters depends very much on the initial conditions. Failure of the ergodic property also is investigated in Chamberlain and Wilson (1984) for consumption and by Domowitz and Muus (1984) in the context of Euler equations for investment with similar results.

Dynamic models can be used for structural estimation, to justify assumptions that then are used for seminonparametric or nonparametric estimation, or for calibration and simulation exercises. Verification of the statistical properties of such models is important in all three cases, and may even be considered a goal of the last. The work cited above is rather specialized, and cannot be applied generally in any attempt to establish the dependence properties of the laws of

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[1] The role of dependence conditions in time series based theories of statistical inference is reviewed in Domowitz (1985), including the part played by complementary stationarity assumptions. Most theory allowing deviations from stationarity relies on dependence concepts stronger than the ergodic property, and we do not deal with deviations from the ergodic property arising from nonstationarity, e.g. explosiveness. Inference in nonergodic environments is discussed in Domowitz and Muus (1987).

motion. One of the few attempts in the economic literature to construct a model and demonstrate the ergodic property of the law(s) of motion is that of Richard Day and his coauthors. To help further motivate the tests developed below, we present a brief summary of the Keynesian model of the business cycle fluctuations of Day and Shafer (1987) and Day and Lin (1987).

In many respects, the model is quite standard. Prices and wages are assumed fixed, while real output, Y, and the real interest rate, r, vary. Assuming temporary market clearing and an exogenously-determined money supply, M, the LM curve is derived in the usual way as

$$r = L^m(Y; M) \tag{1.1}$$

while the aggregate demand curve is denoted $D(r, y, G, \nu, \tau)$, where G is government expenditure, τ is the income tax rate, and ν is a parameter reflecting the intensity of endogenous investment demand. A $\nu > 0$ implies some induced investment over the autonomous component. Aggregate demand is then

$$AD(Y; G, \nu, \tau, M) = D[L^m(Y; M), Y; G, \nu, \tau] \tag{1.2}$$

Assuming either an adjustment story or dependence of demand on past income,

$$Y_{t+1} = AD(Y_t; G, \nu, \tau, M) \tag{1.3}$$

describes the law of motion for income.

The general case is worked out by Day and Lin (1987), but the basic story is simple. As income increases enough, the real interest rate rises sharply in response, possibly resulting in a reduction of aggregate demand. Aggregate demand will be nonlinear with a range of negative slope, and cycles, even persistent ones, are possible. Interest rates rise during expansions, potentially depressing investment. A recession may then be brought about by the fall in the aggregate demand. The transactions demand for money goes down, and if the interest rate falls sharply enough, a boost in investment starts the recovery. Day and Shafer (1987) show that for large enough ν (enough induced investment), a variety of potentially unstable cycles is the result, motivating the desire to check the ergodic property of the law of motion for aggregate income.

Given some specific functional forms, the authors then find a trapping set for Y, i.e., values $\hat{Y}$ and Y' such that if Y starts in the interval $[\hat{Y}, Y']$, it stays in that interval forever. They suggest simulations as a means to determine the ergodicity of income given its complicated law of motion. This scheme is in the spirit of many economic models defying closed-form solutions in which the case a calibration procedure is used to simulate data out of the model and check whether they resemble *real world* data.

It is apparent that we need a rigorous statistical theory for such a proof-by-simulation methodology. In this paper we concentrate on tests for the dependence properties of such laws of motion within known model structure. Dependence properties are defined in terms of probability measures, virtually mandating an analysis of laws of motion in terms of densities. Although it might seems counter-intuitive, even a single dimensional system containing one object whose dynamics are deterministic can generate a density of states, and we exploit this idea in Section 2 to characterize the probabilistic properties of a known law of motion in a state space. The laws we consider are restricted in two important ways. We examine only univariate processes here. The problem is not in the theory; the basic results extend directly to vector valued laws of motion as well as to more general spaces. Numerical computation of the proposed statistic becomes difficult in higher dimensional state spaces, however, and we match theory with practice in considering only univariate systems. Partially due to this, and to greatly simplify the analysis, we introduce stochastic elements only through a distribution over possible initial conditions, rather than consider separate innovations processes, for example. This source of randomness alone is sufficient to analyze a law of motion and construct the tests. It also accords well with much dynamic simulation methodology, and the analysis is indeed designed for such conditions.

Harris ergodicity and the weaker concept of ergodicity itself are introduced and illustrated in Section 3. The key insight concerns the limiting behavior of the density of states when (Harris) ergodicity obtains, namely the convergence of the density to the uniform law. A test based on this result is constructed in Section 4, following the usual development of the Kolmogorov-Smirnov test. The consistency of the test is examined in Section 5, in which it is shown that the test of Harris ergodicity has power one against the alternative of non-ergodicity. Some suggestions for generalizing the test to the case of a single time series of observations (e.g. data observed in practice rather than simulated) are taken up in the conclusion.

2. Laws of Motion on a Space of Densities

We take a law of motion to be a mapping $S: X \mapsto X$ that defines $x_{t+1} = S(x_t)$. X is referred to as the state of the system. We assume that the process $\{x_t\}$ is a stationary univariate Markov process. Such maps arise naturally in the context of standard discrete-time optimal control problems such as

$$\max_{y_t} \sum_{t=0}^{\infty} u_t(x_t, y_t) \tag{2.1}$$

$$s.t. \qquad x_{t+1} = g(x_t, y_t)$$

in which $u(.,.)$ is the (expected) return or utility in period t, and x and y are the state and control variables, respectively. Under suitable conditions, there exists a unique solution, $y_t = h(x_t)$, called the optimal policy rule, which implies a law of motion for x

$$x_{t+1} = g(x_t, h(x_t)) \equiv S(x_t) \tag{2.2}$$

The analysis of such laws is one of the primary research objectives of rational-expectations macroeconomics, in particular.[2]

We take $S(.)$ to be a measurable map on a complete normalized measure space $(X, \Im, \mu)$, $\mu(X) = 1$, and we further assume that $S(.)$ is non-singular.[3] The essential element of randomness that we require is introduced by putting a density over possible initial conditions, x_0, as suggested by Bellman (1973). Denote this density by $f(x)$, and we consider only such densities with no point mass with respect to μ.

For any non-singular law of motion S, there exists a unique operator $P: D \mapsto D$ defined by

$$\int_A Pf(x)d\mu(x) = \int_{S^{-1}(A)} f(x)d\mu(x) \tag{2.3}$$

for all $f \in D$ where D is the space of all densities on X; i.e. the space of nonnegative L^1 functions on X that integrate to one. P is known as the ***Frobenius-Perron operator corresponding to S*** (e.g. Lasota and Mackey (1985), pp. 36-37), which we abbreviate to the F-P operator, suppressing the dependence on S where no confusion results. It is easy to verify that P is a Markov operator on D, and that for iterates on S, $S_n = S \circ \ldots \circ S$, the corresponding F-P operator is $P_n = P^n$.

The important point here is that the operator P describes the evolution of the initial density f, through the state-space transformation S. The density of X after n periods is simply $P^n f(x)$.[4] If we take $g_n = P^n f(x)$, it follows that $g_n(y)$ is the induced law of motion on the space of densities, D. Different interpretations are possible. If we think of many individuals in the economy, initially distributed according to $f(.)$, each changing his state by passing through S, the overall distribution of individuals moves according to P. We also might conceive of a likelihood over the entire state space, and given S, we put a probability on where a representative agent will be after n periods; thus updating our prior in Bayesian terms, or applying the Chapman-Kolmogorov updating equations.

A simple example which we can carry throughout the remainder of the paper may help to clarify the concepts. Consider a non-overlapping generations model where each individual lives for

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[2] See, for example, the collection of papers contained in Lucas and Sargent (1981), and the references therein. Much current effort is devoted to teasing these derived laws of motion out of intrinsically nonlinear problems that defy closed form solutions.

[3] $S: X \mapsto X$ is nonsingular if for all $B \in \Im$ with $\mu(S(B)) = 0$ we have $\mu(B) = 0$.

[4] Additional details may be found in Lasota and Mackey (1985), Chapter 3.

only one period and is succeeded by his son the moment he dies. The only product in that economy is mangos, and a mango seed takes one period (generation) to grow into a tree and bear one fruit, after which it dies instantly. A son inherits all the trees that his father planted, chooses the number of mangos to eat and the number of seeds to plant. If a person decides not to plant a seed, then that causes him some pain, either because of a bequest motive or because he would have to eat the seed, which is not a pleasant experience. Notice that the family line can look like an infinitely lived individual with infinite discounting of future utility, but the dynamics of the system still stem from the bequest motive. The model, thus could be written as follows:

$$\max_{L_t} \; u(C_t, L_t) \tag{2.4}$$
$$s.t. \quad L_t \leq C_t$$
$$and \quad C_{t+1} = g(C_t, L_t)$$

where the first constraint is that the number of seeds (units of labor) thrown cannot exceed the number of mangos eaten. The second constraint specifies the technology of growing trees, and drives the dynamics of the system. To further simplify the system, we specify the Cobb-Douglas utility function $u(C, L) = CL$ which together with the first constraint will obviously give us $C_t = L_t; \; \forall t$. We then specify the technology that drives the system and study the behavior of consumption (our state variable) in the state space as well as the density space. The discontinuities in the technology can easily be explained by a soil corrosion argument.

$$C_{t+1} = 2L_t \;\; mod \;\; 1 \tag{2.5}$$

and, thus, our law of motion is

$$C_{t+1} = 2C_t \;\; mod \;\; 1 \tag{2.6}$$

In this economy, $S: [0,1] \mapsto [0,1]$, and we can specify the measure to be Lebesgue measure. Then taking the set A to be the interval $[0, x]$, and totally differentiating both sides of definition (2.3) with respect to x, we get

$$Pf(x) = \frac{d}{dx} \int_{S^{-1}([0,x])} f(s)ds \tag{2.7}$$

and substituting in

$$S^{-1}([0,x]) = [0, \frac{x}{2}] \cup [\frac{1}{2}, \frac{1+x}{2}]$$

we get

$$Pf(x) = \frac{d}{dx} \sum_{i=0}^{1} \int_{\frac{i}{2}}^{\frac{(i+x)}{2}} f(s).ds$$
$$= \frac{1}{2} \sum_{i=0}^{2-1} f(\frac{i+x}{2}) \tag{2.8}$$

and clearly, by a simple inductive argument, we get the density space representation of the law of motion after n periods

$$P^n f(x) = \frac{1}{2^n} \sum_{i=0}^{2^n-1} f(\frac{i+x}{2^n}) \tag{2.9}$$

The state space trajectory of the diadic law of motion (2.6) is given in Figure 1a. The corresponding evolution of the law of motion in the density space for initial density $f(x) = 2x$ is illustrated in Figure 1b. Note that the density appears to be converging to the uniform law as n grows. It happens that such a property is the key to our test of ergodicity, and to this we now turn.

3. Ergodic and exact laws of motion

We begin with the formal definition of the concepts to be tested

Definition

Let $(X, \Im, \mu)$ be a complete normalized measure space, and let a measurable non-singular transformation $S: X \mapsto X$ be given. Then S is called ***ergodic*** if for all $A \in \Im$ such that $S^{-1}(A) = A$, $\mu(A) = 0$, or $\mu(X \setminus A) = 0$. S is said to be ***exact*** or ***Harris ergodic***[5] if $\lim_{n\uparrow\infty} \mu(S^n(A)) = 1$ for all $A \in \Im$ with $\mu(A) > 0$. ○

It can be quite difficult in practice to isolate processes that are ergodic but not Harris ergodic, a point to which we return in Section 5. In fact, in the general theory of (infinite state space) Markov chains, ergodic behavior is characterized as having the Harris ergodic property (e.g. Nummelin, 1984, Proposition 6.3 and the discussion on p.76). Aperiodic, irreducible Markov chains are the simplest and most familiar example of Harris ergodic processes. Mild conditions can be found under which reflecting random walks and renewal processes are Harris ergodic (Nummelin (1984), pp. 12, 21, 46, 117). The following result due to Lasota and Mackey (1985) characterizes ergodicity and exactness in the space of densities D.

Theorem 3.1 (Lasota and Mackey (1985))

Let $(X, \Im, \mu)$ be a complete normalized measure space, and let $S: X \mapsto X$ be a measure preserving transformation[6]*, and let P be the F-P operator corresponding to S. Then*

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[5] It is easy to see the equivalence between the definition of exactness in Lasota and Mackey together with their result quoted in part ii. of Theorem 3.1 below and the definition of Harris ergodicity e.g. in Nummelin (1984), p.114.

[6] S is said to be measure preserving if $\mu(S^{-1}(A)) = \mu(A)$ for all $A \in \Im$.

(i.) S is ergodic if and only if the sequence $\{P^n f\}$ is Cesàro convergent to 1 for all $f \in D$.

(ii.) S is exact if and only if $\{P^n f\}$ is strongly convergent[7] to 1 for all $f \in D$. ○

Given this result, it follows that our law of motion (2.6) for consumption is exact, and hence also ergodic (since strong convergence implies Cesàro convergence). From equation (2.9), we have

$$\lim_{n\uparrow\infty} P^n f(x) = \lim_{n\uparrow\infty} \frac{1}{2^n} \sum_{i=0}^{2^n-1} f(\frac{i+x}{2^n})$$

$$= \int_0^1 f(t)dt = 1 \tag{3.1}$$

uniformly in x, satisfying part (ii.) of Theorem 3.1 above, and the process is therefore exact and hence ergodic (measure preservation of S is trivial to check).

Before we proceed to issues of statistical verification of the ergodic property, there is a technical issue that merits special attention, namely the assumption that S preserves the chosen measure. It is not difficult to find mappings that are not measure preserving. In fact, we can change our mango economy slightly so as to slow down the repulsion from 0, say

$$C_{t+1} = \begin{cases} C_t/(1-C_t), & C_t \in [0,1/2) \\ 2C_t - 1, & C_t \in [1/2,1] \end{cases} \tag{3.2}$$

This law of motion does not preserve Lebesgue measure. Looking at its behavior in the state space, we see the same behavior that we saw in the law of motion (2.6). It can be shown, however, that due to the slower speed of repulsion from 0 (which is also the source of the non-measure preservation), for all $\epsilon > 0$,

$$\lim_{n\uparrow\infty} \int_\epsilon^1 P^n f(t)d\mu(t) = 0 \tag{3.3}$$

i.e. all the mass will in the limit be collected in an arbitrarily small neighborhood of 0.[8]

Since a measure typically will be chosen for convenience, there obviously is no guarantee that the law of motion arising from an economic model will preserve that measure. For the purposes of this paper, we shall ignore this problem and construct the test under the standard assumption of ergodic theory of the measure preservation of S. It is clear to us, however, that the test in the later sections of this paper can be generalized to the non-measure preserving case. To achieve a characterization of ergodicity and exactness in the density space without assuming measure-preservation, the following result has been constructed and proved in El-Gamal (1988a).

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[7] i.e. in L^1 norm.

[8] This is referred to as the paradox of the weak repeller in Lasota and Mackey (1985).

Theorem 3.2

Given $(X, \Im, \mu)$ a complete normalized measure space, and a nonsingular $S: X \mapsto X$ with the corresponding F-P operator P, let $Pf = f$. Then

(i.) S is ergodic if and only if,

$$\lim_{n\uparrow\infty} \frac{1}{n}\sum_{k=0}^{n-1} P^k g = f \qquad \forall g \in D$$

(ii.) S is Harris Ergodic (Exact) if and only if

$$\lim_{n\uparrow\infty} P^n g = f \qquad \forall g \in D$$

○

The proof of part (i.) is provided in El-Gamal (1988a), and the proof of part (ii.) is identical replacing the Cesàro average everywhere by $P^n f$, and replacing Cesàro convergence with strong convergence. The more general assumption that we shall need to deal with is that of stationarity. The test as it stands in the later sections of this paper, even when generalized to the non-measure preserving case, will still have power 1 against the alternative of non-stationary ergodic. Examples of processes that are non-staionary and ergodic are very hard to develop, but they still need to be dealt with before our test is complete. At this stage of our research, we limit ourselves to the case where we assume the stationarity of the law of motion. Sufficient conditions for the existence of a staionary density (which is needed to put Theorem 3.2 to work) are available. For example, if the F-P operator is a constrictive Markov operator, we know that it will have a stationary density.[9] An operational test of the stationarity assumption is not part of this paper, however.

We may now proceed to consider the general strategy and limitations behind the construction of a test. As noted above, we allow only measure-preserving laws of motion here. For simplicity, and since the statistic to be developed is to be used for laws of motion on compact trapping sets as suggested by the analysis of Day and Shafer (1987), we restrict attention to the case

$$S: ([0,1], \Im[0,1], \mu) \mapsto ([0,1], \Im[0,1], \mu) \tag{3.4}$$

where $\Im[0,1]$ is the σ-field generated by the Borel sets of $[0,1]$, and μ is Lebesgue measure. According to Theorem 3.1, a test of the ergodicity of S is simply a test that the Cesàro limit starting from all densities over the initial condition is uniform, while a test for Harris ergodicity involves only direct convergence of $P^n f$, for all f in D. It is the latter property which we can exploit directly, and,

[9] See Proposition 5.4.1, Lasota and Mackey (1985) for definitions and proofs.

therefore, the test in the next section is designed for the null of Harris ergodicity. The reason for choosing the null of Harris ergodicity instead of ergodicity is a technical one. In our development in Section 5 of the Kolmogorov Smirnov asymptotic distribution of our statistic, we need to invoke a central limit theorem. That step was straight forward for the case of Harris ergodicity since we got to invoke a vector central limit theorem for the i.i.d. case. For ergodicity, our test will contain an average over time, and we need to impose more technical conditions whose relationship with exactness and ergodicity is not known. (see discussion of the choice of the null and alternative hypotheses in Section 5). Finally, a test of correct size can be constructed using only a single density over initial conditions, but the statement *for all* f *in* D must be taken seriously if the power of the test is to be examined. We defer this issue to Section 5 and now proceed to an operational test under the null of Harris ergodicity.

4. An Operational Test of Harris Ergodicity

We start by choosing an initial density f, and generate n independent identically distributed data points $x_1, \ldots x_n$ from f. We can then run those n initial conditions through the law of motion $S: X \mapsto X$ r times. As $r \uparrow \infty$, we have the result of Theorem 3.1 above which tells us that the observations $S^r(x_1), \ldots, S^r(x_n)$ defined to be $y_1, \ldots, y_n$ should be distributed independent identically distributed according to the density function $P^r f$ which for large r is arbitrarily close to the uniform.[10] We then test for the smallness of the difference between the empirical distribution generated and the uniform distribution. Let the statistic $U_n(s)$ (this is the standard notation for a *uniform process*, see Shorack and Wellner (1986), and Pollard (1984)) evaluated at a scalar point $s \in [0,1]$ be

$$U_n(s) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [I_{y_i < s} - s] \tag{4.1}$$

A natural null to test, therefore would be that $\|U_n\|$ is small (where $\|.\|$ denotes the sup-norm). To develop this test along the lines of the Kolmogorov-Smirnov test, we need to show that the process U_n converges in distribution to the Brownian bridge U. The Brownian bridge process U is defined to be a process with continuous sample paths such that for each finite subset T of $[0,1]$, the random vector defined by the finite dimensional projection of U on T has a multivariate normal distribution with mean zero, and variance covariance:

$$cov(U(s), U(t)) = s(1-t) \quad 0 \le s \le t \le 1 \tag{4.2}$$

[10] How *large* an r we need will depend on the value of n which we are driving to infinity for the asymptotic distribution theory to be useful. We will need to turn to this issue before our test can be considered complete.

(Pollard, 1984, p.95, and Shorack and Wellner, 1986, p.30). The existence of the Brownian bridge is a well known result in stochastic processes, and its proofs are available in most standard texts on empirical processes. Once we prove that the test statistic U_n converges to the Brownian bridge U, we can use the well known result about the Brownian bridge (from which the Kolmogorov-Smirnov test was developed, see Shorack and Wellner, 1986, p. 142) that:

$$Pr\{\|U\| > x\} = 2\sum_{k=1}^{\infty}(-1)^{k+1}e^{-2k^2x^2} \tag{4.3}$$

and we can therefore employ the standard Kolmogorov-Smirnov test available in all major statistical and mathematical computer packages.

The observations $y_1, ..., y_n$ are i.i.d. and uniform under the null, and we can use the usual development of the Kolmogorov-Smirnov test. As in Shorack and Wellner (1986, Chapter 3), and Pollard (1984, Chapter 5), we notice that the random variables U_n is distributed binomial with mean zero and variance covariance matrix

$$cov(U_n(s), U_n(t)) = min(s,t) - ts \tag{4.4}$$

which gives us the desired variance covariance structure for the process U_n. It remains to show that the process converges to a normal process which has the same first and second moments and will, therefore, be a Brownian bridge. The normality of the vector process defined by the finite dimensional projection of U_n on the set $T = \{s_1, ..., s_k\}$ defined by:

$$U_n = \begin{pmatrix} \frac{1}{\sqrt{n}}\sum_{i=1}^{n}[I_{y_i<s_1} - s_1] \\ \frac{1}{\sqrt{n}}\sum_{i=1}^{n}[I_{y_i<s_2} - s_2] \\ \vdots \\ \frac{1}{\sqrt{n}}\sum_{i=1}^{n}[I_{y_i<s_k} - s_k] \end{pmatrix} \tag{4.5}$$

follows directly from the standard vector central limit theorem for the i.i.d. case. We therefore have the asymptotic normality of the process U_n and by the above argument, it converges to the Brownian bridge and we can use the Kolmogorov-Smirnov test to test for the Harris ergodicity of the model.

We performed a simulation to test the exactness of the mango economy driven by the diadic technology analytically studied in the last section. We drew 1000 i.i.d. observations from some initial density, and then we let each of them run through the diadic transformation for 10 replications. The results are shown in Table 1. The idea is to consider the class of initial densities that can be written as histograms with 10 equally spaced intervals. The first ten items in the table show that for all the initial densities that have a uniform mass on one of those 10 intervals, the process converges very quickly to the uniform [0,1] density. It follows, therefore, that any convex combination of those initial densities (i.e. any histogram on those 10 intervals) has to also converge to the uniform since the Frobenius-Perron operator P operating on those densities is a linear operator. The last two items on the table are to suggest that we can run the same procedure on finer and finer grids if we wish, depending on the class of initial densities we wish to consider.

5. The asymptotic power of the test

In Section 3 above, we characterized the ergodicity and Harris ergodicity properties in terms of Cesàro and strong convergence of $P^k f$ to the uniform for all starting densities f in some class of densities D. In Section 4, we suggested a test of Harris ergodicity based on that result. The suggested test can give us the warranted size under the null for any initial density. We would like to obtain power 1, however, against the alternative. We would like to know that the probability of failure to reject the null of Harris ergodicity under the alternative is zero. In this section, we will suggest a method of choosing initial densities $f \in D$ such that the power 1 criterion is met.

We formulate the null and alternative hypotheses as
H_0: *S is Harris ergodic*
against
H_A: *S is non ergodic.*

Note that our alternative is not simply ***not Harris ergodic***, and so our null and alternative sets are not exhaustive. Although this formulation seems a bit uncommon, non-exhaustive sets of hypotheses have been an important subject in the literature; see Lehmann (1986, Chapter 9) for discussion. Such partitions are appropriate in cases where a decision can be made based on the null or alternative, but between the two may lie cases for which the relative advantages or disadvantages of rejection are not clear. This is the case here, where we are greatly interested in the failure of the ergodic property for purposes of statistical inference, say, but where we have constructed the test against a weaker null for technical reasons. Given the nature of the hypotheses, it is impossible to measure the size of the ***indifference*** set $H_I = \{S \textit{ ergodic but not Harris ergodic}\}$. This set is not empty, but examples of elements in H_I are a bit arcane. For example, consider the Lebesgue measure-preserving transformation, $x_{t+1} = x_t + \theta \mod 1$. If θ is rational, the process is nonergodic and falls within H_A. If, however, θ is irrational, $\{x_t\}$ is ergodic but not Harris ergodic. The set H_I might be of some interest if processes with other dependence properties used in the statistical literature such as strong or uniform mixing were contained therein (see Domowitz (1985) for the role of such conditions). It is our conjecture that strong mixing processes are indeed Harris ergodic, but, although a Harris ergodic process is mixing it is not necessarily strong mixing; the mango economy is an example of this. Thus, strong mixing processes are not necessarily contained in H_I. The theoretical lack of a proper ***nesting*** of dependence concepts precludes a more general characterization of H_I.

One traditional way to construct a test would be to consider the value of the statistic $\|U_n(f)\|$ which we know is distributed according to the Kolmogorov-Smirnov distribution, and compute the statistic

$$\max_{f \in D} \|U_n(f)\| \tag{5.1}$$

A natural way to perform such a test would be to parametrize a particular class of densities D by a set of parameters $\theta \in \Theta$ where Θ is usually taken to be a compact space. In such circumstances (e.g. see Davies (1977)) even though the distribution of the *max* cannot be computed, we may be able to find an upper bound on the probability of the *max* being larger than sum number. Unfortunately, such techniques will usually (as in Davies (1977)) require the numerical computation of a multiple integral depending on the dimensionality of Θ. Numerical computation of integrals over a space of high dimensionality is highly unreliable, and such a procedure will usually produce only probability bounds for a statistic (as in Davies (1977)). We therefore suggest a different procedure for getting the consistency (power 1 against the alternative of non-ergodicity) of our test. The suggested technique was strongly inspired by, and its derivation is closely similar to, the work of Bierens (1988).

We consider the class of densities $f \in D$ that can be written in the form

$$f(x) = c_0 + c_1 x + \ldots + c_k x^k$$

to be general enough for analysis.[11] To constrain the elements of D to be densities, we require that

(i.) $c_0 \geq -\sum_{i:c_i<0} c_i$. (non-negativity)

(ii.) $\int_X f(x) = 1$. ($D \subset$ density space)

The idea of the test is to randomly select an element of the set D. Then, we wish to show

---

[11] See for example Gallant and Tauchen (1987), in which such representations are studied asymptotically but implemented for very small k. Notice, however, that the rate at which k goes to infinity need not depend on the rate at which n does in our context since we are not employing any rigorous argument for filling the density space with those polynomial densities. The admissible class of initial density in this paper is simply taken to be a *large* set.

that , given the procedure that we shall use to randomly select $\tilde{f} \in D$, under the null

$$\|U_n(\tilde{f})\| \xrightarrow{d} \text{K-S} \tag{5.2}$$

and under the alternative

$$\|U_n(\tilde{f})\| \xrightarrow{p} \infty \tag{5.3}$$

where K-S is the law given by (4.3). Let us first define the method that we shall use to randomly choose elements of the above density space.

To set up the algorithm, first notice that if we set

$$p_0 = c_0 + \sum_{i:c_i<0} c_i \tag{5.4}$$

and, for $i = 1, ..., k$,

$$p_i = \frac{c_i}{(i+1)} \qquad c_i \geq 0 \tag{5.5}$$

$$p_i = \frac{-ic_i}{(i+1)} \qquad c_i < 0 \tag{5.6}$$

then the vector $p_0, p_1, ..., p_k$ is a probability vector. This has been shown by Ahrens and Dieter (1974), and used also in Devroye (1986) to generate numbers from a particular $\tilde{f}(x) = \sum_{i=0}^{k} c_i x^i$. The method is to generate a discrete random variable Z from the multinomial distribution with the probability vector $p_0, ..., p_k$, and then generate x as

(i.) $x = U^{\frac{1}{Z+1}}$ if $c_Z \geq 0$.

(ii.) $x = U_1{}^{\frac{1}{Z+1}} U_2$ otherwise.

then x is distributed according to $\tilde{f}$ (Devroye (1986), pp. 71-73). This sets up our algorithm for selecting random coefficients $c_i, i = 1, \ldots, k$.

Algorithm A

(i.) Choose k large.

(ii.) Generate $U_1, \ldots, U_k$ i.i.d. U[0,1].

(iii.) Generate the order statistics $U_{(0)}, \ldots, U_{(k+1)}$ by sorting $U_1, \ldots, U_k$, and setting $U_{(0)} = 0$, and $U_{(k+1)} = 1$.

(iv.) Generate $p_i = U_{(i+1)} - U_{(i)}, i = 0, \ldots, k$. This is a random vector on the $k + 1^{st}$ dimensional unit simplex. We now use those to find the coefficients of the polynomial.

(v.) Compute $c_0, \ldots, c_k$

(a.) *for* $i = 1, \ldots, k$, *generate U (i.i.d. U[0,1])*

(b.) $c_i = (i+1)p_i$ *if* $U > 0.5$.

(c.) $c_i = \frac{-(i+1)}{i} p_i$ *otherwise.*

(d.) $c_0 = p_0 - \sum_{i:c_i<0} c_i$.

(vi.) *Generate a discrete variate Z from the multinomial distribution with probability vector* $p_0, \ldots, p_k$. ∎

(vii.) *Now generate x from* $\tilde{f}(x) = \sum_{i=0}^{k} c_i x^i$, *generate* U, U_1, U_2, *i.i.d. U[0,1],*

(a.) $x = U^{\frac{1}{Z+1}}$ *if* $c_Z \geq 0$.

(b.) $x = U_1^{\frac{1}{Z+1}} U_2$ *otherwise.*

(viii.) *Repeat steps (vi.), and (vii.) to generate as many data points as necessary from* $\tilde{f}(x) = \sum_{i=0}^{k} c_i x^i$.
○

Given this algorithm for selecting $\tilde{f} \in D$, we shall prove the following theorem.

Theorem 5.1

Let $\tilde{f}$ be generated by the above defined algorithm A, then, equations (5.2), and (5.3) hold under the null of Harris ergodicity and the alternative of non-ergodicity respectively. ○

To prove this theorem, we will first need the following result. Again, generalizing the proof methodology of Bierens (1988), let the set β (for bad), be
$\beta = \{\tilde{f} \in D : P^n \tilde{f} \to 1$ when P is not ergodic $\}$.

Then we have the following lemma.

Lemma 5.1

Under the conditions of Theorem 5.1,

$$Pr_A\{\beta\} = 0$$

where Pr_A is the probability under algorithm A as $k \uparrow \infty$. ○

Proof of Lemma 5.1

We wish to show that $Pr\{\beta\} = 0$. Now, we know that under the alternative of non-ergodicity, there exists a set B with $\mu(B) > 0$ such that $S^{-1}(B) = B$. This implies that all the mass that will be in the asymptote in the set B must start there from the initial condition. Moreover, we know that in the asymptote, the mass in B will be $\int_B 1d\mu = \mu(B)$. This suggests to us that

$$Pr\{\beta\} \leq Pr\{\int_B (c_0 + c_1x + \ldots + c_kx^k)dx \geq \mu(A)\} \tag{5.7}$$

Since we are considering maps S on [0,1] that are non-singular, we can without loss of generality consider equation (5.7) where the upper bound on the R.H.S. is for $int(B)$, the interior of B, and hence, we can without loss of generality only consider sets in (5.7) such that $B = \cup_{i\in I}(a_i, b_i)$, and we then get

$$L.H.S.(5.7) \leq Pr\{(c_0 \sum_{i\in I}(b_i - a_i) + \frac{c_1}{2}\sum_{i\in I}(b_i - a_i)^2 +$$

$$\ldots + \frac{c_k}{(k+1)}\sum_{i\in I}(b_i - a_i)^k) \geq \sum_{i\in I}(b_i - a_i)\} \tag{5.8}$$

but now, notice that each of the elements

$$\sum_{i\in I}(b_i - a_i)^l \leq [\sum_{i\in I}(b_i - a_i)]^l \quad l \geq 1 \tag{5.9}$$

since each of the elements inside the sum is in $(0, 1)$, we can therefore use the inequality (5.9) to get yet another upper bound for the $R.H.S.(5.8)$, and dividing both sides of the inequality inside the expression by $\sum_{i\in I}(b_i - a_i)$, we get

$$R.H.S.(5.8) \leq Pr\{\sum_{l=0}^{k} \frac{c_l}{(l+1)}[\sum_{i\in I}(b_i - a_i)]^l \geq 1\} \tag{5.10}$$

and putting $[\sum_{i\in I}(b_i - a_i)]^l = \sigma^l < 1$,

$$R.H.S.(5.10) = Pr\{\sum_{l=0}^{k} \frac{c_l}{(l+1)}\sigma^l \geq 1\} \tag{5.11}$$

but referring back to the definitions of the probability vector $p_0, p_1, \ldots, p_k$ in (5.5) and (5.6), we get

$$R.H.S.(5.11) = Pr\{p_0 + \sum_{i:c_i<0} \frac{(i+1)}{i}p_i - \sum_{i:c_i<0} \frac{p_i}{i}\sigma^i + \sum_{i:c_i\geq 0} p_i\sigma^i \geq 1\} \tag{5.12}$$

collecting terms, and substituting $\sum_{i=0}^{k} p_i = 1$, we get

$$R.H.S.(5.12) = Pr\{\sum_{i:c_i\geq 0} p_i\sigma^i + \sum_{i:c_i<0} \frac{(i+1-\sigma^i)}{i}p_i \geq \sum_{i:c_i\geq 0} p_i + \sum_{i:c_i<0} p_i\} \tag{5.13}$$

and collecting terms,

$$R.H.S.(5.13) = Pr\{\sum_{i:c_i<0} \frac{p_i}{i}(1-\sigma^i) \geq \sum_{i:c_i\geq 0} p_i(1-\sigma^i)\} \stackrel{k\uparrow\infty}{\longrightarrow} 0 \tag{5.14}$$

under algorithm A; and the desired result is proven. ∎

Now, the proof of Theorem 5.1 follows trivially and is identical (after adaptation to our problem) to the proof of Theorem 6 in Bierens (1988).

Proof of Theorem 5.1

Under the null, we consider the characteristic function of $\|U_n(\tilde{f})\|$, and we can easily see that by bounded convergence,

$$Ee^{it\|U_n(\tilde{f})\|} = E[E[e^{it\|U_n(\tilde{f})\|}|\tilde{f}]] \rightarrow Ee^{it\text{K-S}}$$

Under the alternative, we know that for r large $\exists\epsilon > 0$s.t.

$$Pr_A\{\frac{1}{\sqrt{n}}\|U_n\| > \epsilon\} = Pr_A\{D \setminus \beta\} = 1$$

and therefore, the desired result is proven. ∎

We performed a simulation similar to that in Table 1, where we randomly chose a density using algorithm A, and we generated 1000 initial conditions from that density. In Table 2, we show the p-value from the K-S test for equality of the evolving density after a number of iterations through the diadic transformation. Notice that the first element of the table with 0 iterations corresponds to an infinite number of iterations on the identity operator, and we can clearly see the p-value of 0.000 for that law of motion, rejecting its Harris ergodicity.

Concluding Remarks

We have established a framework suitable for tests of dependence properties such as ergodicity and Harris ergodicity. A test for the latter has been given for cases in which simulation methodology is appropriate. Amongst other things, three important points stand out as requiring further research.

The luxury of simulating a model is the availability of cross- sectional information. Indeed, implementation of the Birkhoff ergodic theorem (e.g. Walters (1982), p.34), were that otherwise possible (we do not think it is), would require such cross-sectional information under any circumstances. The advantage of the framework proposed here is that, conceptually at least, the test could be designed for only a single time series of observations. The idea is first to estimate the F-P operator from available data; techniques for doing so are well known (e.g. Roussas (1969), and Yakowitz (1979)), and have been shown to be consistent even for deterministic systems (El-Gamal (1988a)). The algorithm used in this paper can then be used to generate the analogous test of Harris ergodicity, comparing $P^n f$ to unity in the measure preserving case or to the density specified in Theorem 3.2 (ii), if the mapping does not preserve Lebesgue measure (the estimation of that density under the assumption of stationarity is also a standard procedure in the literature, and its asymptotic results have been shown to hold for deterministic systems in El-Gamal (1988a)). The problems inhibiting the development of the final stages of such a test at the current stage of research are purely technical, not conceptual (as is widely believed). The problems center around a demonstration of the consistency of an estimator of $P^n f$ for an arbitrary f. This constitutes a topic of current research by the authors. The second problem that we need to deal with is the relative rates at which r (the number of replications), n (the number of draws from the initial density), and k (the number of terms in the polynomial representation of the initial density), should go to infinity. We do not have those rates at this stage, and we leave the task of finding them as a subject for future research. Again, we can afford the luxury of doing so because of the simulation nature of the test, which allows us to control the values of all three parameters without being constrained by data.

The third point concerns a test of ergodicity as the null hypothesis, using the results on Cesàro convergence to construct the statistic. The strategy behind the Harris ergodicity test is not directly applicable; a central limit theorem for averages of densities would require dependence conditions far beyond the null hypothesis. Another one of our options would be to impose a condition of statistical stability on our law of motion S (definition in Lasota and Mackey (1985, p. 95)), or any of a host of other technical assumptions that will guarantee the closing of the gap between ergodicity and Harris ergodicity. Such assumptions, however, are typically untestable and will therefore be in defiance of the spirit of this paper, namely testing all technical assumptions that are used in an econometric investigation. We are currently attempting to adapt very new theory on large deviations for dependent variables in order to circumvent this problem. In this instant, the problems are neither conceptual nor technical, but purely numerical.

Table 1*

Testing The Exactness of The Diadic Transformation

Initial density	p-value
Unif(0.0,0.1)	0.5314
Unif(0.1,0.2)	0.4524
Unif(0.2,0.3)	0.2142
Unif(0.3,0.4)	0.5543
Unif(0.4,0.5)	0.5878
Unif(0.5,0.6)	0.5408
Unif(0.6,0.7)	0.4409
Unif(0.7,0.8)	0.2213
Unif(0.8,0.9)	0.5543
Unif(0.9,1.0)	0.5878
Unif(0.1,0.11)	0.8654
Unif(0,0.001)	0.6861

* The same 1000 seeds were used to compute all p-values

Figure 1.a

Trajectory Of The Diadic Transformation

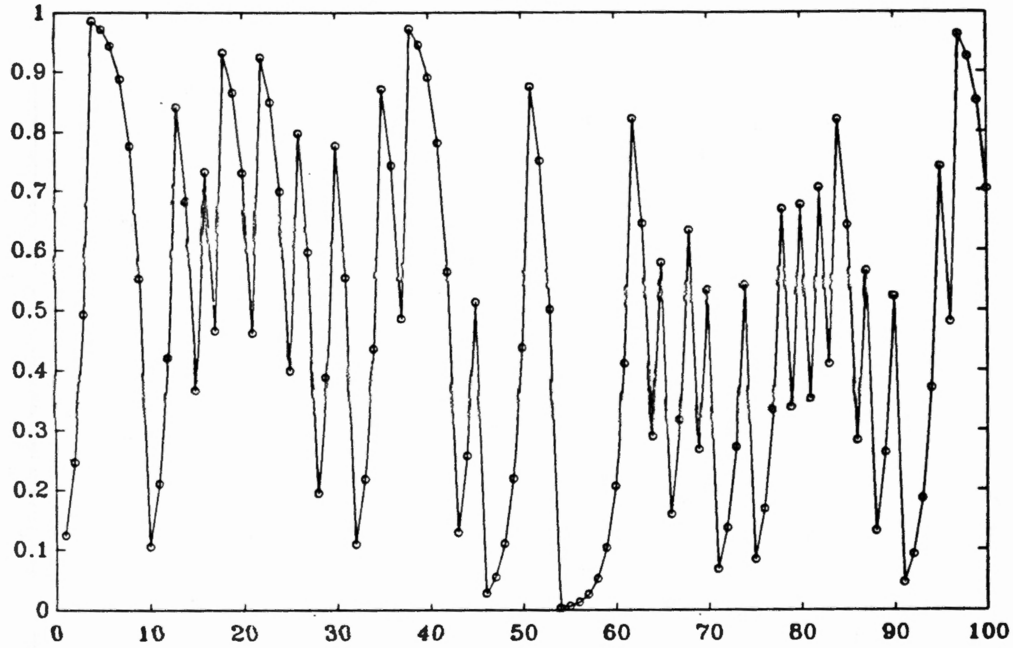

Figure 1.b

Evolution Of The Density

(Lasota and Mackey (1985), p.9)

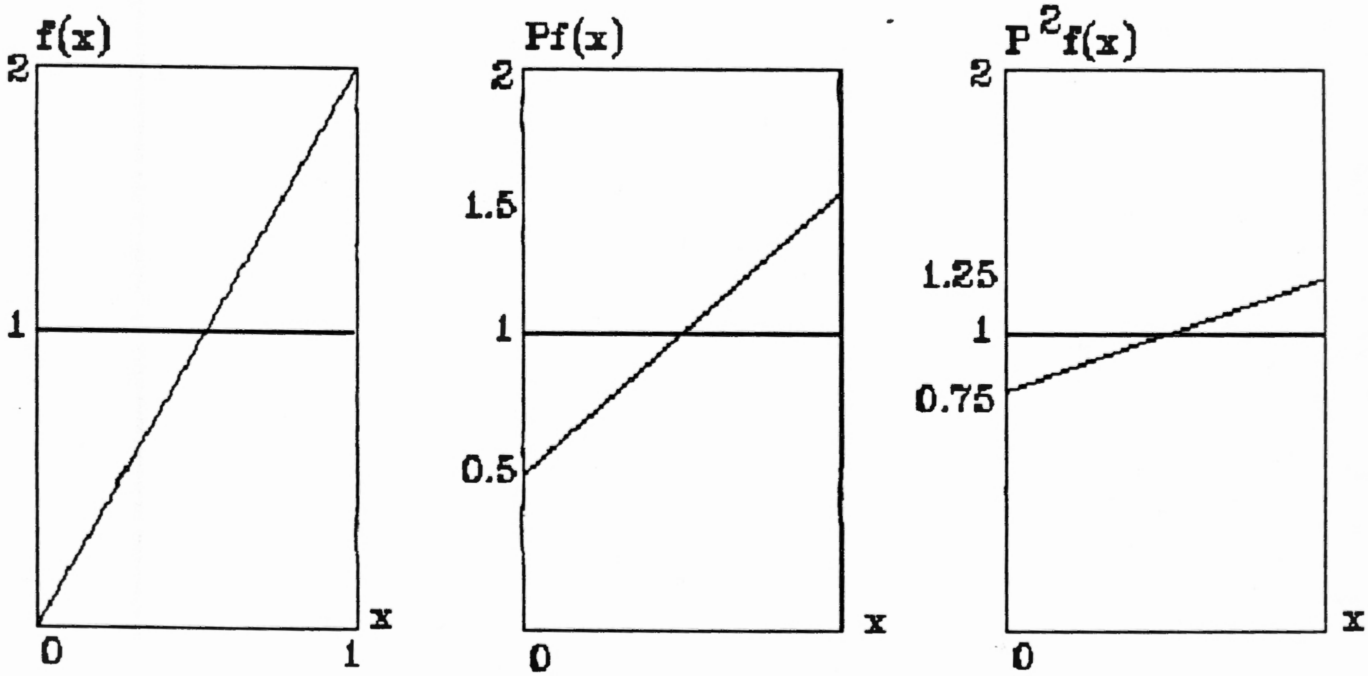

Table 2*

Testing The Exactness of The Diadic Transformation (randomized test)

Number of Iterations	p-value
0	0.520E-03
1	0.157E-03
2	0.756E-02
3	0.430E-02
4	0.845
5	0.938
6	0.785
7	0.309
8	0.298
9	0.953
10	0.965
=========	=============

* The same 1000 seeds were used to compute all p-values.

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