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Majumdar, Mukul and Rangarajan Sundaram

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SYMMETRIC STOCHASTIC GAMES OF RESOURCE EXTRACTION: THE EXISTENCE OF NON-RANDOMIZED STATIONARY EQUILIBRIUM***

Mukul K. Majumdar*

and

Rangarajan Sundaram**

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*Department of Economics, Cornell University, 460 Uris Hall, Ithaca, NY 14853-7601 USA.

**Department of Economics, University of Rochester, Harkness Hall, Rochester, NY 14627 USA.

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ABSTRACT

We consider a class of symmetric stochastic games with a continuum of states and actions. By imposing special structures on the law of motion we prove the existence of a Nash equilibrium in non-randomized stationary strategies.

1. Introduction

1.1 Stochastic Games: A Description

A two-person discounted stochastic game (see, e.g., Parthasarathy (1973), or Parthasarathy (1982) for related references) is described by a tuple $\{S, A_1(s), A_2(s), q, r_1, r_2, \beta\}$ having the following interpretation: S, a non-empty Borel subset of a Polish space, is the set of all states of the system; $A_i(s)$, a non-empty Borel subset of a Polish space, is the set of actions available to player $i(=1,2)$, when the state is $s \in S$. It is typically assumed that for each $i = 1,2$, $A_i(s) \subset A_i$ for all $s \in S$, where the A_i's are themselves Borel subsets of Polish spaces. q defines the law of motion of the system by associating (Borel-measurably) with each triple $(s, a_1, a_2) \in S \times A_1 \times A_2$ a probability measure $q(\cdot|s, a_1, a_2)$ on the Borel subsets of S. r_1 and r_2 are bounded measurable functions on $S \times A_1 \times A_2$; the function r_i is the instantaneous reward function for player i. Lastly, β is the discount factor the players employ. Periodically, the players observe a state $s \in S$ and pick actions $a_i \in A_i(s)$, $i = 1,2$; this coice of actions is made with full knowledge of the game's history. As a consequence of the chosen actions, two things happen: firstly the players receive awards of $r_1(s, a_1, a_2)$ and $r_2(s, a_1, a_2)$ respectively. Secondly, the system moves to a new state s' according to the distribution $q(\cdot|s, a_1, a_2)$. The process is then repeated from the states s', and so on <u>ad infinitum</u>. The objective of each player is to maximize expected payoffs over the infinite duration of the game.

Let $h_t = \{s_0, a_{10}, a_{20}, \ldots, s_{t-1}, a_{1,t-1}, a_{2,t-1}, s_t\}$ denote a generic <u>history</u> of the game up to period t, and let H_t denote the set of all possible histories up to t. Let $P(A_i(s))$ and $P(A_i)$ be the set of all probability distributions on $A_i(s)$ and A_i respectively, $i = 1,2$. A <u>strategy</u> Σ_i for player i is a sequence of functions $\{\sigma_{it}\}$, where for each t, σ_{it} specifies an action for player i by associating (Borel measurably) with each history h_t, an element of $P(A_i(s_t))$. A strategy Σ_i for player i is (non-randomized) <u>stationary</u> if there is a Borel function $\sigma_i: S \to A_i$ such that $\sigma_i(s) \in A_i(s)$ for all $s \in S$, and $\sigma_{it}(h_t) = \sigma_i(s_t)$ for all h_t and for all t. We shall refer to the function σ_i as a <u>policy function</u>, and when talking about a non-randomized stationary strategy, we also refer to it by the associated policy function.

A pair (Σ_1, Σ_2) of strategies for players 1 and 2 respectively, associates with each initial states s, a t^{th}-period expected reward $r_{it}(\Sigma_1, \Sigma_2)(s)$ for player i determined by the functions r_1 and r_2. The total expected reward for player i, denoted $I_i(\Sigma_1, \Sigma_2)(s)$ is then

$$I_i(\Sigma_1, \Sigma_2)(s) = \sum_{t=0}^{\infty} \beta^t r_{it}(\Sigma_1, \Sigma_2)(s) .$$

A strategy Σ_1^* is <u>optimal</u> for player 1 (or, constitutes a <u>best-response</u> (BR) to Σ_2) if $I_1(\Sigma_1^*, \Sigma_2)(s) \geq I_1(\Sigma_1, \Sigma_2)(s)$ for all Σ_1 and s. Similarly, a BR to Σ_1 is defined for player 2. A <u>Nash</u>

<u>equilibrium</u> (or, simply, equilibrium) to the stochastic game is a pair of strategies (Σ_1^*, Σ_2^*) such that for i = 1,2, Σ_i^* is a BR to Σ_j^*, $j \neq i$.

1.2 Summary of the main results

This paper considers a special class of stochastic games allowing for a continuum of states <u>and</u> actions. The sets of states and actions are required to satisfy certain restrictions, as is the stochastic process that determines the law of motion q. The special structure is motivated by models in the economic theory of non-cooperative extraction of common-property resources.[1] While a brief explanation of this link is provided in subsection 2.2, a detailed explanation (in the context of a deterministic game) may be found in Chapter 2 of this thesis.

The imposition of a certain symmetry in the payoff functions (equation (R1) below) in addition to the restrictions mentioned above enables us to prove the following strong results: there is an equilibrium in (non-randomized) <u>stationary</u> strategies to the class of games considered in the paper. Further, the policy functions associated with the equilibrium can be chosen to be <u>lower-semicontinuous</u> functions,[2] with slopes bounded above by

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[1]This problem has been studied in a deterministic framework quite extensively, but by using specific functional forms - see e.g., see Levhari and Mirman (1980).

[2]A real-valued function f is lower-semicontinuous or lsc [resp. upper-semicontinuous, or usc] at a point x in its domain if for all $x_n \to x$, it is the case that liminf $f(x_n) \geq f(x)$ [resp. lim

1/2. The sharpness of this result is to be contrasted with the available results in the literature, where existence is typically shown in randomized strategies that cannot be easily characterized. The price paid for obtaining this result is that the model is more restrictive than the standard models in for example, in for example, Nowak (1985), Parthasarathy (1973), or Himmelberg et al (1976).

2. The Model

2.1 Notation and Definitions

The set of all real numbers (resp. non-negative reals, strictly positive reals) is denoted by $\mathfrak{R}$ (resp. $\mathfrak{R}_+$, $\mathfrak{R}_{++}$). The n-fold Cartesian product of $\mathfrak{R}$, $\mathfrak{R}_+$, and $\mathfrak{R}_{++}$ are denoted by $\mathfrak{R}^n$, $\mathfrak{R}^n_+$, and $\mathfrak{R}^n_{++}$, respectively. For any set X, 2^X denoted the set of all non-empty subsets of X.

Let $S = A_1 = A_2 = [0,1]$. Define the feasible action correspondences for player i = 1,2 by $A_i(s) = [0,s]$. Clearly, the feasible action correspondences are continuous on S.[3]

Before proceeding to describe the formal structure of the game, we present an informal interpretation of its components. The non-negative number s denotes the available stock of a common-

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sup $f(x_n) \le f(x)$]. Note that f is continuous at x iff it is both usc and lsc at x.

[3]For the definition of a continuous correspondence, (as well as for those of upper-semicontinuous (usc) and lower-semicontinuous (lsc) correspondences), the reader is referred to Debreu (1959).

property resource, while a_i represents player i's planned extraction of the resource. (Both players are assumed to know s and the other player's plan.) If plans are feasible (i.e., if $a_1 + a_2 \leq s$) then they are carried out and player i received a reward ("utility" in intertemporal-economics parlance) of $u(a_i)$. If plans are infeasible ($a_1 + a_2 > s$) then we assume ad hoc that each player extracts half the available stock of the resource and receives a reward of $u(s/2)$. We shall have more to say about this ad hoc assumption shortly.

Given (s, a_1, a_2) the function $h(s, a_1, a_2) = \max\{0, s - a_1 - a_2\}$ determines the 'investment' level, the amount left over after extraction by the players. This investment is transformed stochastically into next-period's available stock s', for example, through a 'renewal' function f, and the realization of a random variable r, as $s' = f(h(s, a_1, a_2), r)$. The functions f and h, combined with the distribution of r yields a (conditional) probability distribution of s' given (s, a_1, a_2). We denote this conditional distribution by q and, rather than impose assumptions on f and r, impose restrictions directly of q.

Departing from standard practice we define the transition mechanism q as a (conditional) probability distribution function on $\mathfrak{R}_+$, given $(s, a_1, a_2) \in \mathfrak{R}^3_+$, so that if **s** denotes next period's realization given (s, a_1, a_2), then $q(s'|s, a_1, a_2) = \Pr\{\mathbf{s} \leq s'|s, a_1, a_2\}$. It will follow from the restrictions we place on q that if $s \in S$, $a_i \in A_i(s)$, then $q(1|s, a_1, a_2) = 1$, so next period's stock is also in S w.p.1.

For simplicity denote the vector (s, a_1, a_2) by $y \in \mathfrak{R}^3_+$ and $h(s, a_1, a_2)$ by $h(y)$. The restrictions on q are (i) a 'boundedness' condition that with each investment level today is associated an upper bound on the stock available tomorrow, (ii) strictly positive investments today yield strictly positive stocks tomorrow and (iii) no free production.

(Q1) (i) For each $y \in \mathfrak{R}^3_+$, there is $s(y) \in \mathfrak{R}_+$ such that $q(s(y)|y =$ 1.

(ii) If $h(y) > 0$, then $\inf\{s':q(s'|y) > 0\} \in \mathfrak{R}_{++}$. Further, in this case, $q(\cdot|y)$ is continuous on $\mathfrak{R}$.

(iii) If $h(y) = 0$, then $q(0|y) = 1$.

We also assume that higher investments yield probabilistically higher stock levels.

(Q2) If $h(y) > h(\bar{y})$, then $q(s'|y) \leq q(s'|\bar{y})$ for all $s' \in \mathfrak{R}_+$.

The next two assumptions are concerned with reproductivity of the resource. Assumption (Q3) requires the existence of a maximum sustainable stock (set equal to unity by a suitable choice of measurement-units), while (Q4) implies that for a positive but sufficiently small level of investment, with probability one, the stock tomorrow is no less than the investment today (usually referred to as a "productivity" or Inada condition). Formally:

(Q3) **If $h(y) \geq 1$, then $q(h(y)|y) = 1$.**

(Q4) **There is $\eta \in (0,1)$ such that if $0 < h(y) < \eta$, then $q(h(y)|y) = 0$.**

Finally, the standard weak continuity of the law of motion q:

(Q5) **If $y^n \to y$, then the sequence of distribution functions $q(\cdot|y^n)$ converges weakly to the distribution function $q(\cdot|y)$.**

Example. Let λ be uniformly distributed on [1,2], and let $f(x) = \frac{1}{2}\sqrt{x}$, $x \geq 0$. Define $q(s'|y) = \Pr\{\lambda\ f(h(y)) \leq s'\}$. If $h(y) \geq 1$, $\lambda f(h(y)) = \frac{\lambda}{2}\sqrt{h(y)} \leq h(y)$, so $q(h(y)|y) = 1$. If $h(y) < \eta = \frac{1}{4}$, then $q(h(y)|y) = \Pr\{\lambda f(h(y)) \leq h(y)\} = \Pr\{\frac{\lambda}{2}\sqrt{h(y)} \leq h(y)\} = \Pr\{\lambda \leq 2\sqrt{h(y)}\} = 0$. Similarly, the other conditions are verified.

Next, let $u:\Re_+ \to \Re_+$ be a function satisfying the following condition:

(U1) **u is strictly concave, strictly increasing and continuous on $\Re_+$; u is continuously differentiable on $\Re_{++}$ and satisfies $\lim_{c \downarrow 0} u'(c) = \infty$**

Example. $u(c) = c^\alpha$, $\alpha \in (0,1)$.

The reward functions, like the transition mechanism, are defined for all $(s, a_1, a_2) \in \Re^3_+$, and are given by:

(R1) $r_i(s, a_1, a_2) = u(a_i)$, if $s - a_1 - a_2 \geq 0$

$= u(s/2)$, otherwise

Assumption (R1) forms the basic symmetry assumption on the game, crucial in showing the existence of equilibrium. In what follows, (Q1) through (Q5), (U1) and (R1) are always assumed.

2.2 <u>The Main Theorem</u>

Note that by assumption (Q3), for any $s \in s$, and any $a_i \in A_i(s)$, it is the case that $s' \in S$ w.p.1. Further if $s \in [0,\bar{y}]$ for any $\bar{y} > 1$, and if $0 \leq a_1, a_2 \leq s$, then $s' \in [0,\bar{y}]$ w.p.1. The first of these observations implies that the game is well-defined. The second observation is important to show existence of equilibrium as we shall explain in section 4.

Futhermore, owing to the <u>ad hoc</u> nature of the infeasibility rule, trivial equilibria always exist, i.e., equilibria in which players plan to extract more than the available stock in some period after some history.[4] Indeed, it is easily checked that the non-randomized strategies (γ_1, γ_2) defined by $\gamma_1(s) = s = \gamma_2(s)$ in each period constitutes an equilibrium to the game. The main result in this paper is the demonstration of existence of a

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[4]We refer to such equilibria as "trivial" since they depend in an essential way on the infeasibility rule employed. Our desire to find non-trivial equilibria is also motivated in part by the question: under what conditions is it the case that resources are not driven to extinction in finite time? For more on this and related questions, see Clemhout and Wan (1985) and the references cited therein.

non-trivial equilibrium in non-randomized stationary strategies (γ_1^*, γ_2^*), i.e., an equilibrium in which at each $s > 0$, it is the case that $(\gamma_1^*(s) + \gamma_2^*(s)) < s$. This equilibrium is independent of the infeasibility rule employed, as will be demonstrated in section 4. Formally, we can state the main:

Existence Theorem. Under (Q1) through (Q5), (U1) and (R1), the stochastic game has an equilibrium (γ_1^*, γ_2^*) in non-randomized stationary strategies satisfying:

(i) $0 < \gamma_1^*(s) + \gamma_2^*(s) < s$ for all $s > 0$;

(ii) (γ_1^*, γ_2^*) are lower semicontinuous on S;

(iii) for $i = 1,2$ and for all s, s' in S $s \neq s'$,

$$\frac{\gamma_i^*(s) - \gamma_i^*(s')}{s - s'} \leq \frac{1}{2}$$

The next two sections contain an outline of the proof. Some informal remarks on the strategy we adopt may be useful: first, the game is transformed to a "generalized game" in the sense of Debreu (1952) by making action spaces dependent. This makes the outcome independent of the ad hoc infeasibility rule. Most of sections 3 and 4 is concerned with establishing an equilibrium to the generalized game. That the equilibrium satisfies condition (i) [hence, is independent of the ad hoc infeasibility rule, and that the resource is not extinct in finite time] is shown in Lemma

4.9 and leads to the conclusion that the equilibrium of the generalized game is also the equilibrium of the original game.

3. The Best-Response Map

For simplicity of notation we omit player subscripts from what follows. Let $\gamma:S \rightarrow S$ be a measurable function satisfying $\gamma(s) \in [0,s]$ for each $s \in S$. Each such γ defines a non-randomized stationary strategy for player i. Given γ define player j's ($j \neq i$) feasible action correspondence by $A_j(\gamma)(s) = [0,\ s - \gamma(s)]$. Now note that in maximizing total expected payoff, player j faces a stationary environment: the functions u, q, and γ are invariant with time (q is now simply $q(\cdot|s,\ \gamma(s),\ a)$ from the point of view of player j). Thus in seeking the optimal solution to such a problem, by Lemma 2 in Blackwell (1965) player j can restrict attention to non-randomized strategies. Let $G(\gamma)$ represent the set of all such strategies. Each strategy $\tilde{\gamma} \in G(\gamma)$ must safisfy (by the dependence of the action spaces), the condition $\tilde{\gamma}\ (h_t) \leq [s_t - \gamma(s_t)]$ where $h_t = (s_0,\ a_{10},\ a_{20},\ldots,s_{t-1},\ a_{1,t-1},\ a_{2,t-1},\ s_t)$ is the history of the game up to period t, and the actions $(a_{1\tau},\ a_{2\tau})$ for $\tau \leq t$ are determined by γ and $\tilde{\gamma}$. Each strategy $\tilde{\gamma} \in G(\gamma)$ also yields an expected payoff to player j that we shall denote by $W_\gamma(\tilde{\gamma})(s)$, where $s_0 = s$ (the subscript γ of W denotes the dependence of player j's actions - hence his expected payoffs - on γ). A strategy $\tilde{\gamma}^* \in G(\gamma)$ is <u>optimal</u> and $\tilde{\gamma}^*$ constitutes a <u>generalized best-response</u> (GBR) to γ if $W_\gamma(\tilde{\gamma}^*)(s) \geq W_\gamma(\tilde{\gamma})(s)$, for all $s \in S$, for all $\tilde{\gamma} \in G(\gamma)$. That is, a GBR is a strategy $\tilde{\gamma} \in G(\gamma)$ that solves for all $s \in S$.

$$\text{(P)} \qquad \max_{\{\tilde{\gamma} \in G(\gamma)\}} W_{\gamma}(\tilde{\gamma})(s), \text{ given } q, \gamma.$$

If such a $\tilde{\gamma}^*$ exists (of course, it need not always), then $W_g(\tilde{\gamma}^*)$ is referred to as player j's <u>value function</u> from optimally responding to γ. Conditions to ensure that a GBR exists are presented below.

<u>Theorem 3.1.</u> <u>Suppose</u> $\gamma:S \to S$ <u>is a lower-semicontinuous (lsc) function on S satisfying</u> $\gamma(s) \in [0,s]$ <u>for each</u> $s \in S$ <u>and further, for all</u> $s_1 \neq s_2$, $[\gamma(s_1) - \gamma(s_2))/(s_1 - s_2)] \leq 1$. <u>Then, problem (P) is well-defined</u>: <u>there is a Borel function</u> $\hat{\gamma}^*:S \to S$ <u>such that</u> $\hat{\gamma}^*$ <u>is optimal in</u> $G(\gamma)$, <u>i.e., player i has a stationary</u> GBR <u>to</u> γ. <u>Furthermore, the value function</u> $W_{\gamma}(\hat{\gamma}^*)$ (<u>henceforth denoted by</u> V_{γ}) <u>is upper-semicontinuous</u> (usc) <u>on</u> S.

The proof of this result is in the Appendix.

4. <u>The Existence of Equilibrium</u>

It follows from Theorem 3.1 that <u>if</u> we could show that lsc policy functions γ possessed lsc GBR functions $\hat{\gamma}$, an equilibrium to the generalized game could be obtained by using a standard Debreu-Nash fixed-point argument on the space of lsc functions (endowed with a suitable topology). Unfortunately, it is easy to show the existence of lsc functions that do <u>not</u> possess lsc GBR functions.[5]

We employ therefore a completely different approach, one in which the symmetry in the payoff functions is exploited to provide the equilibrium. As the first step in the process, we expand S to a larger space $\bar{S} = [0, \bar{y}]$ for $\bar{y} > 1$. The equilibrium is constructed on $[0,\bar{y})$, and it is shown below (Lemma (4.8)) that the restriction of the equilibrium strategies to S is an equilibrium on S. Note that by (Q1) - (Q5), if $s \in \bar{S}$, then $s' \in \bar{S}$ w.p.1. Consider the following space of functions on $\bar{S}$:

$$\Psi = \{ \psi:\bar{S} \to \bar{S} \mid \psi \text{ is usc and non-decreasing on } \bar{S}$$

$$\psi(\bar{y}) = \bar{y}, \text{ and } \psi(s) \in [0,s] \text{ for all } s \in \bar{S} \}$$

Each $\psi \in \Psi$ defines a (non-randomized stationary) strategy $\gamma(\psi)$ for player 1 by the rule

$$\gamma(\psi)(s) = \frac{1}{2} (s - \psi(s)).$$

Since ψ is usc, non-decreasing, so $\gamma(\psi)$ satisfies the conditions of Theorem 3.1 (which of course is not affected by expanding the state and actions spaces to $\bar{S}$ from S) and there exists a GBR denoted by $\hat{\gamma}(\psi)$. Define $\hat{\psi}:\bar{S} \to \bar{S}$ by

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[5]A trivial example is the following: let $\gamma(s0 = s$ for $s \in [0,1)$ and $\gamma(1) = 0$. The unique GBR is $\hat{\gamma}(s) = 0$ for $s \in [0,1)$ and $\hat{\gamma}(1) = 1$, which is not lsc at $s = 1$.

$$\hat{\psi}(s) = s - \gamma(\psi)(s) - \hat{\gamma}(\psi)(s).$$

In Lemma 4.3 below, it is shown that there exists a $\hat{\gamma}(\psi)$, a unique GBR to each $\gamma(\psi)$ such that $\hat{\psi}$ defined thus is in Ψ. This defines a map from Ψ into itself. Consider a fixed-point of this map. At such a point, $\hat{\psi} = \psi$, so from the above equations, some manipulation yields $\hat{\gamma}(\psi) = \gamma(\psi)$ or $\gamma(\psi)$ <u>is GBR to itself on</u> $\bar{S}$. Lemmata 4.8-4.10 then conclude the proof by showing that it is in fact the case that $\gamma(\psi)$ is a <u>best-response</u> to itself when the state space is restricted to S. By the symmetry of the payoffs (equation R1) the argument is complete.

These ideas underlie the following results but rather than invoke the functions $\gamma(\psi)$ and $\hat{\gamma}(\psi)$, notation is simplified as follows: player 2's actions in response to $\gamma(\psi)$ are now interpreted as the investment level he chooses given player 1's action, so that if he takes an action $a > 0$, his instantaneous reward is given by $u(s - \gamma(\psi)(s) - a)$. Define $R_\psi(s) = \frac{1}{2}(s + \psi(s))$ for $s \in S$, $\psi \in \Psi$. Note that the conditional distribution over $\bar{S}$ of next period's state s' depends now only on a. Abusing notation we denote this distribution by $q(\cdot|a)$. Finally, let V_ψ denote player 2's value function from a GBR to $\gamma(\psi)$. We rewrite the Bellman Optimality equation in this notation as:

$$(4.1) \qquad V_\psi(s) = \max_{a \in [0, R_\psi(s)]} \{u(R_\psi(s) - a) + \beta \int V_\psi(s')dq(s'|a)\}$$

Let $\hat{V}$ denote the function V_ψ when $\psi(s) = s$ for all $s \in \bar{S}$. Then, clearly, for any $\psi \in \Psi$, $V_\psi \le \hat{V}$. Define

$$\Omega = \{v:\bar{S} \to \Re_+ \mid v \text{ is usc and non-decreasing on } \bar{S},$$
$$v(0) = \frac{u(0)}{1-\beta}\ ,\ v(\bar{y}) = \frac{u(\bar{y})}{1-\beta}\ ,\ v \le \hat{V}\}$$

Endow Ψ, Ω with the topology of weak-convergence (i.e., pointwise convergence to continuity points of the limit function - see e.g., Billingsley (1968)). We can then show

<u>Lemma 4.1:</u> Ψ <u>and</u> Ω <u>are convex metric spaces. Further</u>, Ψ <u>has the fixed-point property</u>.

<u>Proof</u>: Convexity is obvious. To see compactness of Ψ consider the set N of finite measures ν on the Borel sets of $\bar{S}$ satisfying $\nu(\bar{S}) = \bar{y}$ for all $\nu \in N$. Since $\bar{S}$ is compact metric, a well known result establishes that N endowed with the topology of weak convergence (weak topology, for short) is also a compact metric space (see, e.g., Parthasarathy (1967)). If Ψ_0 denotes the set of distribution functions corresponding to measures in N, it follows that Ψ_0 is also a compact metric space under the weak topology. Since Ψ is a closed subset of Ψ_0 it also has this property. That it possesses the fixed-point property follows from the Schauder-Tychonoff theorem (see, e.g., Smart (1974)), whose conditions are easily seen to be met.

Ω is similarly a compact metric space if we can show it to be closed in the weak topology. Since $\hat{V}$ corresponds to the value function of a one-person dynamic programming problem with (weakly-)continuous transition and continuous payoffs, it is straightforward to show that $\hat{V}$ is itself a continuous function. By the assumptions on q, $\hat{V}(0) = u(0)/(1-\beta)$. Since $v \leq \hat{V}$ for all $v \in \Omega$, the result readily follows. ||

Now observe that for fixed ψ, the feasible action correspondence $[0, R_\psi(s))]$ is increasing in s, <u>i.e.</u>, any action feasible at s_1 is also feasible at s_2 if $s_2 > s_1$. Since u is increasing in its argument, it is immediate by the upper-semicontinuity of V_ψ that

<u>Lemma 4.2</u>. <u>For each</u> ψ, V_ψ <u>is non-decreasing and right-continuous on</u> $\bar{S}$.

Now for each ψ redefine the value of V_ψ at $\bar{\bar{y}}$ by setting $V_\psi(\bar{\bar{y}}) = \frac{u(\bar{\bar{y}})}{1-\beta}$. Thus defined, V_ψ still satisfies the conditions of lemma 4.2, <u>therefore</u> $V_\psi \in \Omega$ <u>for each</u> $\psi \in \Psi$.

As the second step in the proof we shall now construct a map from Ψ into itself. To this end, we define for $\psi \in \Psi$ and $v \in \Omega$ a map $F_{\psi,v}: \bar{S} \to 2^{\bar{S}}$ by $F_{\psi,v}(\bar{\bar{y}}) = \bar{\bar{y}}$, and for $0 \leq s < \bar{\bar{y}}$,

$$F_{\psi,v}(s) = \underset{a \in [0,R_\psi(s)]}{\text{argmax}} \{u(R_\psi(s)-a) + \beta \int v(s')dq(s'|a)\}$$

If $v = V_\psi$, the we shall write F_ψ for $F_{\psi,v}$.

By Lemma 2.1 and Theorem 2.1 in Parthasarathy (1973), $F_{\psi,v}$

is well-defined and a measurable correspondence, and further admits a measurable selection. (This is a consequence of the fact that $v \in \Omega$ implies $\int v(s')dq(s'|a)$ is usc as a function of a; see Appendix, lemma A.1). In fact, we can show that

<u>Lemma 4.3</u>: <u>There is a unique selection</u> $\hat{\psi}$ <u>from</u> $F_{\psi,v}$ <u>such that</u> $\psi \in \Psi$.

<u>Proof</u>: The lemma is proved in 3 steps:

<u>Claim 1</u>: If $s_1 > s_2$, $a_1 \in F_{\psi,v}(s_1)$, $a_2 \in F_{\psi,v}(s_2)$, then $a_1 \geq a_2$.

This is proved by a standard argument in intertemporal economics that relies upon the strict concavity of u. See the Appendix for details.

<u>Claim 2</u>: If $s_n \downarrow s$, $a_n \in F_{\psi,v}(s_n)$, $a_n \to a$, then $a \in F_{\psi,v}(s)$.

<u>Proof</u>: Suppose, contrary to the claim, it was the case that $a \notin F_{\psi,v}(s)$. Since the latter is non-empty it contains $\hat{a} \leq s - \gamma(\psi)(s)$ such that

$$(4.2) \qquad u(R_\psi(s) - \hat{a}) + \beta \int v(s')dq(s'|\hat{a}) > u(R_\psi(s) - a) + \beta \int v(s')dq(s'|a).$$

Since $a_n \downarrow a$, assumption (Q5) implies the weak convergence of $q(\cdot|a_n)$ to $q(\cdot|a)$. Since v is usc, $\limsup_{n\to\infty} \int v(s')dq(s'|a_n) \leq \int v(s')dq(s'|a)$, so combining this with equation (4.2) and the fact the right-continuity of ψ and continuity of u together imply $u(R_\psi(s_n) - a_n) \to u(R_\psi(s) - a)$, we obtain the existence of $\alpha > 0$ such that for large n

$$(4.3) \qquad u(R_\psi(s_n) - a_n) + \beta \int v(s')dq(s'|a_n) + 2\alpha < u(R_\psi(s) - \hat{a}) + \beta \int v(s')dq(s'|\hat{a}).$$

Using the additional fact that $u(R_\psi(s_n) - \hat{a}) \to u(R_\psi(s) - \hat{a})$ (4.3) in turn implies that for all sufficiently large n

$$(4.4) \qquad u(R_\psi(s_n) - a_n) + \beta \int v(s')dq(s'|a_n) + \alpha < u(R_\psi(s_n) - \hat{a}) + \beta \int v(s')dq(s'|\hat{a}).$$

But $\hat{a} \le s - \gamma(\psi)(s) \le s_n - \gamma(\psi)(s_n)$, so $\hat{a}$ is feasible at s_n. Equation (4.4) therefore contradicts the optimality of a_n for all large n.

Note that by claim 2, $\max\{F_{\psi,v}(s)\}$ is well-defined at each $s \in [0,\bar{\bar{y}})$. Defining $\hat{\psi}(s) = \max\{F_{\psi,v}(s)\}$ for $s \in S$, we see that claims 1 and 2 together imply that $\hat{\psi}$ is right-continuous and non-decreasing. Therefore $\hat{\psi}$ is usc on S, and $\hat{\psi} \in \Psi$, since $F_{\psi,v}(\bar{\bar{y}}) = \bar{\bar{y}}$. The last step in the proof of Lemma 4.3 is

<u>Claim 3</u>: $\hat{\psi}$ is the only usc selection from $F_{\psi,v}$.

<u>Proof</u>: Suppose there were another usc selection $\bar{\psi}$. Note that $\bar{\psi}$ is non-decreasing, hence right-continuous. Since $\hat{\psi} \ne \bar{\psi}$, there is $s \in \bar{S}$ such that $\hat{\psi}(s) \ne \bar{\psi}(s)$, so $\hat{\psi}(s) > \bar{\psi}(s)$. Let $s_n \downarrow s$. Then $\bar{\psi}(s_n) \downarrow \bar{\psi}(s)$, so for large enough n, $\hat{\psi}(s) > \bar{\psi}(s_n) \in F_{\psi,v}(s_n)$, but

$s < s_n$ and $\hat{\psi}(s) \in F_{\psi,v}(s)$, so this contradicts claim 1. ||

Note that if $v = V_{\psi}$, then Lemma 4.3 implies that for each $\psi \in \Psi$, there is a GBR $\hat{\gamma}(\psi)$ to $\gamma(\psi)$ such that the resulting 'savings' function $\hat{\psi}(s) = s - \gamma(\psi)(s) - \hat{\gamma}(\psi)(s)$ is in Ψ. Thus, Lemma 4.3 defines a map from Ψ into itself. A fixed-point ψ^* of this map yields a pair of functions (γ^*,γ^*) defined by $\gamma^*(s) = 1/2(s - \psi^*(s))$ sucht that γ^* is a GBR to itself on $[0,\bar{y}]$. Since Ψ possesses the fixed-point property (Lemma 4.1), the continuity of the map $B:\Psi \to \Psi$, $B(\psi)(s) = \hat{\psi}(s) = \max\{F_{\psi}(s)\}$ will provide us with the desired fixed-point. A few preliminary results are needed first:

<u>Lemma 4.4</u>: <u>Let</u> $\hat{\psi}$ <u>be the unique selection from</u> $F_{\psi,v}$ <u>satisfying</u> $\hat{\psi} \in \Psi$. If $\hat{\psi}$ <u>is continuous at</u> $s \in [0,\bar{y}]$, <u>then</u> $F_{\psi,v}$ <u>is single-valued at</u> s.

<u>Proof</u>: Suppose not. Let $\hat{\psi}(s) > a \in F_{\psi,v}(s)$. Let $s_n < s$, $s_n \to s$. By continuity of $\hat{\psi}$ at s, $\hat{\psi}(s_n) \to \hat{\psi}(s)$. So for large n, $\hat{\psi}(s_n) > a$, but $a \in F_{\psi,v}(s)$, and $s > s_n$, a contradiction to claim 1 in Lemma 4.3. ||

<u>Lemma 4.5</u>. <u>Let</u> $\psi_n \to \psi \in \Psi$, <u>and</u> $s_n \to s \in \bar{S}$.

<u>Then</u>,

(i) $\limsup_{n\to\infty} \psi_n(s_n) \le \psi(s)$

(ii) <u>if ψ is continuous at</u> s, <u>then</u> $\lim_{n\to\infty} \psi_n(s_n) = \psi(s)$.

<u>Proof</u>: See Appendix.

<u>Lemma 4.6</u>. <u>Suppose</u> $v_n \to v \in \Omega$ <u>and</u> $\psi_n \to \psi \in \Psi$. <u>Suppose also that</u> $s \in \bar{S}$ <u>is a continuity point of</u> ψ. <u>Then</u>,

$$\int v_n(s')dq(s'|\psi_n(s)) \to \int v(s')dq(s'|\psi(s)).$$

<u>Proof</u>: By the generalized Dominated convergence theorem (see Hildenbrand (1974)), it suffices to show that (i) $q(\cdot|\psi_n(s))$ converges weakly to $q(\cdot|\psi(s))$, (ii) $\{v_n\}$ is a uniformly integrable sequence, and (iii) $v_n \to v$ in distribution. Since, by hypothesis, s is a continuity point of ψ, so $\psi_n(s) \to \psi(s)$, and (i) follows from assumption (Q5). Since $v_n(s') \le (1-\beta)^{-1} u(\bar{\bar{y}})$ for all $s' \in \bar{S}$, (ii) is immediate. Let μ_n be the measure on $\bar{S}$ corresponding to $q(\cdot|\psi_n(s))$, and μ that corresponding to $q(\cdot|\psi(s))$. Then, we need to show that $\mu_n v_n^{-1}$ converges weakly to μv^{-1}. Since μ_n converges weakly to μ, it suffices by Billingsley (1968), Theorem 5.5) to show that $\mu(E') = 0$ where $E' = \{s' \in \bar{S}|$ there is $s_n' \to s'$ such that $v_n(s_n')$ does not converge to $v(s')\}$. Let $E = \{s' \in \bar{S}|v$ is discontinuous at $s'\}$. Clearly $E' \subset E$ (apply lemma 4.5). Further, E' is measurable by Billingsley (1968, p.226). Note that $0 \notin E'$, since $s_n' \to 0$ implies by Lemma 4.5 that $\limsup_{n\to\infty} v_n(s_n') \le v(0) = (1-\beta)^{-1} u(0)$, while since $v_n \in \Omega$, $v_n(s_n') \ge v_n(0) = (1-\beta)^{-1} u(0)$, so $\liminf_{n\to\infty} v_n(s_n') \ge (1-\beta)^{-1} u(0) = v(0)$. We identify two cases: (i) $\psi(s) = 0$, so $q(s'|\psi(s)) = 1$ for all $s' \ge 0$. Since $0 \notin E'$,

clearly $\mu(E') \leq \mu(E) = 0$ since in this case $\mu(A) = 0$ if $0 \notin A$ for any Borel set A. Case (ii) $\psi(s) > 0$. By Q1(ii), $q(\cdot|\psi(s))$ is continuous, and its induced measure μ contains no atoms, so (since E is countable), $\mu(E') \leq \mu(E) = 0$ in this case also. ||

<u>Lemma 4.7</u>: <u>Suppose</u> k_1, k_2 <u>are non-decreasing, right continuous functions on</u> $\hat{S} = [0,\bar{y}]$. <u>Suppose also that</u> $\hat{D}$ <u>is dense in</u> $\hat{S}$ <u>and</u> $k_1 = k_2$ <u>on</u> $\hat{D}$. <u>Then</u>, $k_1 = k_2$ on $\hat{S}$.

<u>Proof</u>: Straightforward.

We are now ready for

<u>Lemma 4.8</u>: $B:\Psi \to \Psi$ <u>is a continuous map when</u> Ψ <u>is endowed with the weak topology</u>.

<u>Proof</u>: Recall that sequential arguments suffice. Let ψ_n be a sequence in Ψ converging (weakly) to $\psi \in \Psi$. Let $\hat{\psi}_n = B(\psi_n)$ and for notational simplicity denote V_{ψ_n} by V_n. Since Ψ, Ω, are compact metric, we may assume without loss of generality that $\hat{\psi}_n \to \hat{\psi} \in \Psi$, $V_n \to V \in \Omega$. We are required to show that $\hat{\psi} = B(\psi)$.

As a first step, consider

$$F(s) = \operatorname*{argmax}_{a\in[0,R_\psi(s)]} \{u(R_\psi(s) - a) + \beta \int V(s')dq(s'|a)\}, \quad s \in [0,\bar{y})$$

$$= \bar{y} \quad , \quad s = \bar{y}.$$

By Lemma 4.3, there is a unique $\bar{\psi} \in \Psi$ such that $\bar{\psi}(s) \in F(s)$ for s

$\epsilon\ \bar{S}$. We claim that $\bar{\psi} = \hat{\psi}$. Note that to prove this claim, it suffices by Lemma 4.7 to show that $\bar{\psi} = \hat{\psi}$ on a set dense in $\bar{S}$.

Let D′ be the set of disconitnuity points of any of the following functions: ψ_n, $\hat{\psi}_n$, $\bar{\psi}$, ψ, $\hat{\psi}$, V_n, and V. Since each of these functions is monotone (and right continuous), D′ is at most countable. Hence, $D = \bar{S} - D'$ is dense in $\bar{S}$.

We shall show that $\bar{\psi} = \hat{\psi}$ on D. Let $s \epsilon D$. Consider first the case $\bar{\psi}(s) < R_{\psi}(s)$. Since ψ is continuous at s, $\psi_n(s) \to \psi(s)$, so $R_{\psi_n}(s) \to R_{\psi}(s)$, and therefore, for large n, $R_{\psi_n}(s) > \bar{\psi}(s)$. For all such n,

$$(4.5)\qquad u(R_{\psi_n}(s) - \hat{\psi}_n(s)) + \beta \int V_n(s')dq(s'|\hat{\psi}_n(s))$$

$$\geq u(R_{\psi_n}(s) - \bar{\psi}(s)) + \beta \int V_n(s')dq(s'|\bar{\psi}(s)).$$

By Lemma 4.6, and since $s \epsilon D$, $\int V_n(s')dq(s'|\hat{\psi}_n(s)) \to \int V(s')dq(s'|\hat{\psi}(s))$, and $\int V_n(s')dq(s'|\bar{\psi}(s)) \to \int V(s')dq(s'|\bar{\psi}(s))$, so taking limits in (4.9) yields

$$(4.6)\qquad u(R_{\psi}(s) - \hat{\psi}(s)) + \beta \int V(s')dq(s'|\hat{\psi}(s))$$

$$\geq u(R_{\psi}(s) - \bar{\psi}(s)) + \beta \int V(s')dq(s'|\bar{\psi}(s)).$$

Now suppose $\bar{\psi}(s) = R_{\psi}(s)$. Then, since

$$(4.7)\qquad u(R_{\psi_n}(s) - \hat{\psi}(s)) + \beta \int V(s')dq(s'|\hat{\psi}(s))$$

$$\geq u(0) + \beta \int V_n(s')dq(s'|R_{\psi}(s)),$$

the same arguments imply that taking limits in (4.11) we obtain

$$u(R_\psi(s) - \hat{\psi}(s)) + \beta \int V(s')dq(s'|\hat{\psi}(s))$$

(4.8) $$\geq u(0) + \beta \int V(s')dq(s'|R_\psi(s))$$

$$= u(R_\psi(s) - \bar{\psi}(s)) + \beta \int V(s')dq(s'|\bar{\psi}(s)).$$

Equations (4.6) and (4.8) imply that $\hat{\psi}(s) \epsilon F(s)$ for $s \epsilon D$, if $\bar{\psi}(s) \epsilon F(s)$ for $s \epsilon D$. This implies that $\hat{\psi} = \bar{\psi}$ on D, and by Lemma 4.6, $\hat{\psi} = \bar{\psi}$ on $\bar{S}$.

Now define $V^*:\bar{S} \to \mathbb{R}_+$ by $V^*(\bar{y}) = \frac{u(\bar{y})}{1-\beta}$, and for $s \epsilon [0,\bar{y}]$

$$V^*(s) = \max_{a\epsilon[0,R_\psi(s)]} \{u(R_\psi(s) - a) + \beta \int V(s')dq(s'|a)$$

(4.9)

$$= u(R_\psi(s) - \hat{\psi}(s)) + \beta \int V(s')dq(s'|\hat{\psi}(s)).$$

We claim that $V^* = V$. To see this note that (by Lemma 2.1 and Theorem 2.1 in [15]) V^* is now usc on $\bar{S}$. It is trivial to see that V^* is non-decreasing for $a \epsilon [0,R_\psi(s_1)]$ implies $a \epsilon [0,R_\psi(s_2)]$ whenever $s_1 < s_2$. So clearly $V^* \epsilon \Omega$. As above it suffices to show that $V^* = V$ on D. So let $s \epsilon D$. For each n,

(4.10) $$V_n(s) = u(R_{\psi_n}(s) - \hat{\psi}_n(s)) + \beta \int V_n(s')dq(s'|\hat{\psi}_n(s))$$

and taking limits as $n \to \infty$ yields (since $s \epsilon D$)

(4.11) $$V(s) = u(R_\psi(s) - \hat{\psi}(s)) + \beta \int V(s')dq(s'|\hat{\psi}(s)).$$

From (4.9) and (4.11), $V = V^*$ on D, so $V = V^*$ on $\bar{S}$. Thus, we have shown that for $s \in [0,\bar{\bar{y}}]$

$$V(s) = \max_{a \in [0, R_\psi(s)]} \{u(R_\psi(s) - a) + \beta \int V(s')dq(s'|a)\}$$

(4.12)

$$= u(R_\psi(s) - \hat{\psi}(s)) + \beta \int V(s'|\hat{\psi}(s)).$$

To complete the proof, it is shown by using similar arguments in Strauch (1966) and Maitra (1968) that V is indeed the expected payoff (on $[0,\bar{\bar{y}}]$ from employing the stationary strategy $\hat{\gamma}(\psi)(y) = y - \gamma(\psi)(y) - \hat{\psi}(y) =$
$\frac{1}{2}(y + \psi(y)) - \hat{\psi}(y)$.

Since V satisfies the Bellman Optimality Equations (4.12), and it $\hat{\psi}$ yields a total expected payoff of V, it is indeed the case that $B(\psi) = \hat{\psi}$. ||

Combining Lemmas 4.1 and 4.7, we see the existence of a $\psi^* \in \Psi$ such that $B(\psi^*) = \psi^*$. Therefore, there is a function $\hat{\gamma}^* = \hat{\gamma}(\psi^*)$, such that $\hat{\gamma}^*$ is a GBR to itself on $[0,\bar{\bar{y}}]$ for problem (P). Denote the restrictions of $\hat{\gamma}^*$ to s by γ^*.

<u>Lemma 4.8</u>: γ^* <u>is a GBR to itself on</u> S.

<u>Proof</u>: By our assumptions on q, if the game starts with the state in S, the state stays in S forever. If $\hat{\gamma}^*$ is a GBR to itself on $\bar{S}$, then γ^* must be a GBR to itself on S for what happens in $(\bar{y},\bar{\bar{y}})$

is now irrelevant. ||

The next two results (finally!) establish the existence of a non-trivial Nash equilibrium in non-randomized stationary strategies to the stochastic game of Section 2.

<u>Lemma 4.9</u>: $2\gamma^*(s) < s$ <u>for all</u> $0 < s \leq \bar{y}$.

<u>Proof</u>: See Appendix.

<u>Lemma 4.10</u>: γ^* <u>is a BR to itself on S for the stochastic game of Section 2</u>.

<u>Proof</u>: See Appendix.

The existence of a non-trivial equilibrium is thus established by Lemmas 4.9 and 4.10. To see that it satisfies the other properties outlined in Section 1, note that the ψ^* that, as a fixed-point of B, generated the γ^* is non-decreasing, so for $s_1 \neq s_2 \in S$,

$$\frac{\gamma^*(s_1) - \gamma^*(s_2)}{s_1 - s_2} = \frac{1}{2} \left[\frac{s_1 - s_2 - \gamma^*(s_1) + \gamma^*(s_2)}{s_1 - s_2}\right]$$

$$= \frac{1}{2} \left[1 - \frac{\psi^*(s_1) - \psi^*(s_2)}{s_1 - s_2} \right]$$

$$\geq \frac{1}{2} \quad ,$$

and finally since ψ^* is usc on S and γ^* is defined by $\gamma^*(s) = \frac{1}{2}(s - \psi^*(s))$, γ^* is lsc on S. ||

Appendix

A1: Proof of Theorem 3.1

Theorem 3.1 is established through several lemmata. Let $Z = \{(s,a) | s \in S,\ 0 \leq a \leq s - \gamma(s)\}$.

Lemma A.1: Let $v: S \to \mathfrak{R}_+$ be a bounded, non-negative and non-decreasing function. Let $\tilde{v}(s,a) = \int v(s')dq(s'|s,\gamma(s),a)$ for $(s,a) \in Z$. Then $\tilde{v}: Z \to \mathfrak{R}_+$ is usc on Z.

Proof: Let $(s_n, a_n) \to (s,a) \in Z$. Since α is lsc on S, so $\limsup_{n\to\infty} (s_n - \gamma(s_n) - a_n) \leq (s - \gamma(s) - a)$. Assume wlog that $\gamma(s_n)$ converges to $\hat{a}$. By (Q5), $q(\cdot|s_n, \gamma(s_n), a_n)$ converges weakly to $q(\cdot|s, \hat{a}, a)$. Since $\hat{a} \geq \gamma(s)$, this implies by (Q3) that $q(s'|s, \hat{a}, a) \geq q(s'|s, \gamma(s), a)$ for all $s' \in S$. Together these result in

$$\limsup_{n\to\infty} \tilde{v}(s_n, a_n) = \limsup_{n\to\infty} \int v(s')dq(s'|s_n, \gamma(s_n), a_n)$$

$$\leq \int v(s')dq(s'|s, \hat{a}, a)$$

$$\leq \int v(s')dq(s, \gamma(s), a)$$

$$= \tilde{v}(s, a)$$

where the first inequality obtains since v is usc and the second since v is non-negative and non-decreasing. Of course, these inequalities imply the desired result. ||

<u>Lemma A.2</u>: <u>Let</u> $A(\gamma)(s) = [0, s-\gamma(s)]$ <u>for</u> $s \in S$. <u>Let</u>

$$v^*(s) = \max_{a \in A(\gamma)(s)} \tilde{v}(s,a).$$

<u>Then</u>, (i) v^* <u>is usc on</u> S, <u>and</u> (ii) <u>there is a Borel function</u> $k:S \to S$ <u>such that</u> $k(s) \in A(\gamma)(s)$ <u>for all</u> s, <u>and</u> $v^*(s) = \tilde{v}(s) = \tilde{v}(s, k(s))$.

<u>Proof</u>: Since α is lsc, so $A(\gamma):S \to 2^S$ is a upper-semicontinuous correspondence. Together with lemma A.1, the hypothesis of Lemma 2.1 and Theorem 2.1 in Parthasarathy (1973) are readily seen to be met, from where lemma A.2 follows. ||

Define USC(S) to be the space of all non-negative, non-decreasing, bounded usc functions on S, endowed with the sup-norm topology.

<u>Lemma A.3</u>: USC(S) <u>is a complete metric space</u>.

<u>Proof</u>: By Maitra (1968, lemma 4.2), the space of all bounded usc functions on S is a complete metric space, when endowed with the sup-norm. Trivially, USC(S) is a closed subset of this space. ||

Define an operator T on USC(S) by

$$Tw(s) = \max_{a \epsilon A(\gamma)(s)} \{u(a) + \beta \int w(s')dq(s'|s, \gamma(s), a\}$$

for w ϵ USC(S), s ϵ S. Then,

<u>Lemma A.4</u>: T <u>maps</u> USC(S) <u>into itself and is a contraction</u>.

<u>Proof</u>: By lemma A.1 $\int w(s')dq(s'|s, \gamma(s),a)$ is usc on Z. Trivially so is u. Hence by lemma A.2, Tw is usc on S. Since u, w are non-negative and bounded, so is Tw. Finally, by the assumptions on γ, we have $s_1 < s_2$ implies $A(\gamma)(s_1) \subset A(\gamma)(s_2)$. Since u, w are non-decreasing, this implies that Tw also enjoys this property.

A straightforward application of Blackwell (1965, Theorem 5) utilizing the fact that $\beta \epsilon (0,1)$ shows that T is a contraction. ||

Lemmata A.3, A.4 and the Banach fixed-point theorem (Smart (1974, p. 2)) imply that T has a unique fixed-point $V^* \epsilon$ USC(S), so that

$$\text{(A.1)} \qquad V^*(s) = \max_{a \epsilon A(\gamma)(s)} \{u(a) + \beta \int V^*(s')dq(s'|s, \gamma(s), a)\}$$

Lemma A.2 (ii) indicates the presence of a Borel function $\hat{\alpha}^*$ such that $\hat{\gamma}^*(s) \epsilon A(\gamma)(s)$ at each s ϵ S, and

$$\text{(A.2)} \qquad V^*(s) = u(\hat{\gamma}^*(s) + \beta \int V^*(s')dq(s'|s, \gamma(s), \hat{\alpha}^*(s)).$$

The completion of the proof of Theorem 3.1 now follows the lines of Maitra (1968). Let $W_\gamma(\tilde{\gamma})$ denote expected payoffs to player i from a strategy $\tilde{\alpha} \in G(\gamma)$. Stranch (1966, Theorem 5.1) uses (A.2) to show that $W_\gamma(\hat{\gamma}^*) = V^*$, or V^* is the payoff from using $\hat{\alpha}^*$. From Blackwell (1965, Theorem 6), this in turn implies that $\hat{\gamma}^*$ is an optimal strategy for (P) under the hypothesis of Theorem 3.1. Since $V_\gamma = V^* \in USC(S)$, V_γ is usc on S. ||

<u>A2</u>: <u>Proof of Lemma 4.3, Claim 1</u>

Assume to the contrary that $a_1 < a_2$. Then, $a_1 < R_\psi(s_2)$, while since $s_2 < s_1$, we also have $a_2 < R_\psi(s_1)$, so a_1 (resp. a_2) is feasible at s_2 (resp. s_1). Therefore,

$$\text{(A.3)} \qquad u(R_\psi(s_1) - a_1) + \beta \int v(s')dq(s'|a_1)$$

$$\geq u(R_\psi(s_1 - a_2) + \beta \int v(s')dq(s'|a_2)$$

$$\text{(A.4)} \qquad u(R_\psi(s_2) - a_2) + \beta \int v(s')dq(s'|a_2)$$

$$\geq u(R_\psi(s_2) - a_1) + \beta \int v(s')dq(s'|a_1)$$

Adding (A.3), (A.4), and cancelling common terms,

$$\text{(A.5)} \qquad u(R_\psi(s_1) - a_1) + u(R_\psi(s_2) - a_2)$$

$$\geq u(R_\psi(s_1) - a_1) + u(R_\psi(s_2) - a_1).$$

Some rearrangement readily shows that (A.5) contradicts the <u>strict</u> concavity of u. ||

A3: Proof of Lemma 4.5

Suppose (i) were violated. Then there exists a subsequence (which we continue to denote by n), an integer N, and positive numbers δ and α such that for $n \geq N$

$$\psi_n(s_n) > \psi(s) + 2\alpha$$

and

$$|s_n - s| < \delta,$$

where $\delta > 0$ is chosen so that ψ is continuous at $(s + \delta)$, $\psi(s + \delta) < \psi(s) + \alpha$ and

$$\psi_n(s_n) \leq \psi_n(s + \delta).$$

Combining these inequalities,

$\psi_n(s+ \delta) \geq \psi_n(s_n) > \psi(s) + 2\alpha > \psi(s + \delta) + \alpha$. So $\lim_{n\to\infty} \psi_n(s + \delta) \geq \psi(s + \delta) + \alpha$, while since ψ is continuous at $(s + \delta)$, $\lim_{n\to\infty} \psi_n(s + \delta) = \psi(s + \delta)$, a contradiction. This establishes (i).

A completely analagous argument exploiting the left-continuity of ψ establishes that if ψ is continuous at s, then $\liminf_{n\to\infty} \psi_n(s_n) \geq \psi(s)$, proving (ii). ||

A4: Proof of Lemma 4.9

Suppose contrary to the lemma, there were some $s > 0$ at which $2\gamma^*(s) = s$, or $\gamma^*(s) = s/2$. Then, since γ^* is a GBR to itself,

$$V_{\gamma*}(s) = u(s/2) + \frac{\beta}{1-\beta} u(0).$$

Consider the action $(s/2 - \delta)$ for small $\delta > 0$. By (Q1) $s_\delta = \inf\{s' | q(s'|\delta) > 0\}$ is in $\Re_{++}$ for $\delta > 0$. If δ is chosen less than ζ, then (Q4) implies that $s_\delta \geq \delta$. We claim that there is $\delta \in (0,\zeta)$ such that the action $(s/2 - \delta)$ followed by $\frac{1}{2}$ s' (where s' is <u>any</u> realization of next period's state) is feasible and in expected payoff terms dominates $V_{\gamma*}(s)$.

Feasibility is obvious since $\gamma^*(\bar{s}) \leq \bar{s}/2$ for all $\bar{s} \in S$. Note therefore, that it suffices to show that (since $s_\delta \geq \delta$)

$$\text{(A.6)} \qquad \beta[u(\delta/2) - u(0)] > [u(s/2) - u(s/2 - \delta)]$$

for δ sufficiently small. By the Mean Value Theorem, the LHS of (A.6) is equal to $\beta u'(z_\delta)\delta/2$ for some $z_\delta \in (0, \delta/2)$, while the RHS is equal to $u'(w_\delta)\delta$ for $w_\delta \in (s/2 - \delta, s/2)$. Thus, proving (A.6) is equivalent to showing there exists $\delta > 0$ such that

$$\text{(A.7)} \qquad \beta u'(z_\delta) > 2u'(w_\delta).$$

As $\delta \downarrow 0$, $z_\delta \downarrow 0$, so (U1) implies the LHS of (A.7) tends to infinity while the RHS is bounded. This establishes the claim that there is a strategy that is <u>feasible</u> and <u>dominates</u> γ^* in expected payoff terms, a contradiction establishing the lemma. ||

<u>A5</u>: <u>Proof of Lemma 4.10</u>

If γ^* were not a BR to itself, then there exists s > 0 and an action a such that $a + \gamma^*(s) > s$, and the action a at s provides some player with a greater expected payoff than $\gamma^*(s)$. An argument identical to that used above in establishing lemma 4.9 furnishes a contradiction. ||

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SYMMETRIC STOCHASTIC GAMES OF RESOURCE EXTRACTION:
THE EXISTENCE OF NON-RANDOMIZED STATIONARY EQUILIBRIUM***

Mukul K. Majumdar*

and

Rangarajan Sundaram**

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*Department of Economics, Cornell University, 460 Uris Hall, Ithaca, NY 14853-7601 USA.

**Department of Economics, University of Rochester, Harkness Hall, Rochester, NY 14627 USA.

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ABSTRACT

We consider a class of symmetric stochastic games with a continuum of states and actions. By imposing special structures on the law of motion we prove the existence of a Nash equilibrium in non-randomized stationary strategies.

1. Introduction

1.1 Stochastic Games: A Description

A two-person discounted stochastic game (see, e.g., Parthasarathy (1973), or Parthasarathy (1982) for related references) is described by a tuple $\{S, A_1(s), A_2(s), q, r_1, r_2, \beta\}$ having the following interpretation: S, a non-empty Borel subset of a Polish space, is the set of all states of the system; $A_i(s)$, a non-empty Borel subset of a Polish space, is the set of actions available to player $i(= 1,2)$, when the state is $s \in S$. It is typically assumed that for each $i = 1,2$, $A_i(s) \subset A_i$ for all $s \in S$, where the A_i's are themselves Borel subsets of Polish spaces. q defines the law of motion of the system by associating (Borel-measurably) with each triple $(s, a_1, a_2) \in S \times A_1 \times A_2$ a probability measure $q(\cdot|s, a_1, a_2)$ on the Borel subsets of S. r_1 and r_2 are bounded measurable functions on $S \times A_1 \times A_2$; the function r_i is the instantaneous reward function for player i. Lastly, β is the discount factor the players employ. Periodically, the players observe a state $s \in S$ and pick actions $a_i \in A_i(s)$, $i = 1,2$; this coice of actions is made with full knowledge of the game's history. As a consequence of the chosen actions, two things happen: firstly the players receive awards of $r_1(s, a_1, a_2)$ and $r_2(s, a_1, a_2)$ respectively. Secondly, the system moves to a new state s' according to the distribution $q(\cdot|s, a_1, a_2)$. The process is then repeated from the states s', and so on <u>ad infinitum</u>. The objective of each player is to equilibrium (or, simply, equilibrium) to the stochastic game is a pair of strategies (Σ_1^*, Σ_2^*) such that for i = 1,2, Σ_i^* is a BR to Σ_j^*, $j \neq i$.

1.2 Summary of the main results

This paper considers a special class of stochastic games allowing for a continuum of states and actions. The sets of states and actions are required to satisfy certain restrictions, as is the stochastic process that determines the law of motion q. The special structure is motivated by models in the economic theory of non-cooperative extraction of common-property resources.[1] While a brief explanation of this link is provided in subsection 2.2, a detailed explanation (in the context of a deterministic game) may be found in Chapter 2 of this thesis.

The imposition of a certain symmetry in the payoff functions (equation (R1) below) in addition to the restrictions mentioned above enables us to prove the following strong results: there is an equilibrium in (non-randomized) stationary strategies to the class of games considered in the paper. Further, the policy functions associated with the equilibrium can be chosen to be lower-semicontinuous functions,[2] with slopes bounded above by

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[1]This problem has been studied in a deterministic framework quite extensively, but by using specific functional forms - see e.g., see Levhari and Mirman (1980).

[2]A real-valued function f is lower-semicontinuous or lsc [resp. upper-semicontinuous, or usc] at a point x in its domain if for all $x_n \to x$, it is the case that liminf $f(x_n) \geq f(x)$ [resp. lim

property resource, while a_i represents player i's planned extraction of the resource. (Both players are assumed to know s and the other player's plan.) If plans are feasible (i.e., if $a_1 + a_2 \leq s$) then they are carried out and player i received a reward ("utility" in intertemporal-economics parlance) of $u(a_i)$. If plans are infeasible ($a_1 + a_2 > s$) then we assume ad hoc that each player extracts half the available stock of the resource and receives a reward of $u(s/2)$. We shall have more to say about this ad hoc assumption shortly.

Given (s, a_1, a_2) the function $h(s, a_1, a_2) = \max \{0, s - a_1 - a_2\}$ determines the 'investment' level, the amount left over after extraction by the players. This investment is transformed stochastically into next-period's available stock s', for example, through a 'renewal' function f, and the realization of a random variable r, as $s' = f(h(s, a_1, a_2), r)$. The functions f and h, combined with the distribution of r yields a (conditional) probability distribution of s' given (s, a_1, a_2). We denote this conditional distribution by q and, rather than impose assumptions on f and r, impose restrictions directly of q.

Departing from standard practice we define the transition mechanism q as a (conditional) probability distribution function on $\Re_+$, given $(s, a_1, a_2) \in \Re^3_+$, so that if s denotes next period's realization given (s, a_1, a_2), then $q(s'|s, a_1, a_2) = \Pr\{s \leq s'|s, a_1, a_2\}$. It will follow from the restrictions we place on q that if $s \in S$, $a_i \in A_i(s)$, then $q(1|s, a_1, a_2) = 1$, so next period's

(Q3) **If $h(y) \geq 1$, then $q(h(y)|y) = 1$.**

(Q4) **There is $\eta \in (0,1)$ such that if $0 < h(y) < \eta$, then $q(h(y)|y) = 0$.**

Finally, the standard weak continuity of the law of motion q:

(Q5) **If $y^n \to y$, then the sequence of distribution functions $q(\cdot|y^n)$ converges weakly to the distribution function $q(\cdot|y)$.**

Example. Let λ be uniformly distributed on [1,2], and let $f(x) = \frac{1}{2}\sqrt{x}$, $x \geq 0$. Define $q(s'|y) = Pr\{\lambda\ f(h(y)) \leq s'\}$. If $h(y) \geq 1$, $\lambda f(h(y)) = \frac{\lambda}{2}\sqrt{h(y)} \leq h(y)$, so $q(h(y)|y) = 1$. If $h(y) < \eta = \frac{1}{4}$, then $q(h(y)|y) = Pr\{\lambda f(h(y)) \leq h(y)\} = Pr\{\frac{\lambda}{2}\sqrt{h(y)} \leq h(y)\} = Pr\{\lambda \leq 2\sqrt{h(y)}\} = 0$. Similarly, the other conditions are verified.

Next, let $u:\mathfrak{R}_+ \to \mathfrak{R}_+$ be a function satisfying the following condition:

(U1) **u is strictly concave, strictly increasing and continuous on $\mathfrak{R}_+$; u is continuously differentiable on $\mathfrak{R}_{++}$ and satisfies $\lim_{c \downarrow 0} u'(c) = \infty$**

Example. $u(c) = c^\alpha$, $\alpha \in (0,1)$.

The reward functions, like the transition mechanism, are defined for all $(s, a_1, a_2) \in \mathfrak{R}^3_+$, and are given by:

<u>non-trivial</u> equilibrium in non-randomized stationary strategies (γ_1^*, γ_2^*), i.e., an equilibrium in which at each $s > 0$, it is the case that $(\gamma_1^*(s) + \gamma_2^*(s)) < s$. This equilibrium is independent of the infeasibility rule employed, as will be demonstrated in section 4. Formally, we can state the main:

<u>Existence Theorem</u>. <u>Under</u> (Q1) <u>through</u> (Q5), (U1) <u>and</u> (R1), <u>the stochastic game has an equilibrium</u> (γ_1^*, γ_2^*) <u>in non-randomized stationary strategies satisfying</u>:

(i) $0 < \gamma_1^*(s) + \gamma_2^*(s) < s$ <u>for all</u> $s > 0$;

(ii) (γ_1^*, γ_2^*) <u>are lower semicontinuous on</u> S;

(iii) <u>for</u> $i = 1,2$ <u>and for all</u> s, s' <u>in</u> S $s \neq s'$,

$$\frac{\gamma_i^*(s) - \gamma_i^*(s')}{s - s'} \leq \frac{1}{2}$$

The next two sections contain an outline of the proof. Some informal remarks on the strategy we adopt may be useful: first, the game is transformed to a "generalized game" in the sense of Debreu (1952) by making action spaces dependent. This makes the outcome independent of the ad hoc infeasibility rule. Most of sections 3 and 4 is concerned with establishing an equilibrium to the generalized game. That the equilibrium satisfies condition (i) [hence, is independent of the ad hoc infeasibility rule, and that the resource is not extinct in finite time] is shown in Lemma all $s \in S$, for all $\tilde{\gamma} \in G(\gamma)$. That is, a GBR is a strategy $\tilde{\gamma} \in G(\gamma)$ that solves for all $s \in S$.

$$(\mathbf{P}) \qquad \underset{\{\tilde{\gamma} \in G(\gamma)\}}{\text{Max}} W_{\gamma}(\tilde{\gamma})(s), \text{ given } q, \gamma.$$

If such a $\tilde{\gamma}^*$ exists (of course, it need not always), then $W_g(\tilde{\gamma}^*)$ is referred to as player j's <u>value function</u> from optimally responding to γ. Conditions to ensure that a GBR exists are presented below.

<u>Theorem 3.1.</u> <u>Suppose</u> $\gamma: S \to S$ <u>is a lower-semicontinuous (lsc) function on S satisfying</u> $\gamma(s) \in [0,s]$ <u>for each</u> $s \in S$ <u>and further, for all</u> $s_1 \neq s_2$, $[\gamma(s_1) - \gamma(s_2))/(s_1 - s_2)] \leq 1$. <u>Then, problem (P) is well-defined</u>: <u>there is a Borel function</u> $\hat{\gamma}^*: S \to S$ <u>such that</u> $\hat{\gamma}^*$ <u>is optimal in</u> $G(\gamma)$, <u>i.e., player i has a stationary</u> GBR <u>to</u> γ. <u>Furthermore, the value function</u> $W_{\gamma}(\hat{\gamma}^*)$ (<u>henceforth denoted by</u> V_{γ}) <u>is upper-semicontinuous</u> (usc) <u>on</u> S.

The proof of this result is in the Appendix.

4. <u>The Existence of Equilibrium</u>

It follows from Theorem 3.1 that <u>if</u> we could show that lsc policy functions γ possessed lsc GBR functions $\hat{\gamma}$, an equilibrium to the generalized game could be obtained by using a standard Debreu-Nash fixed-point argument on the space of lsc functions (endowed with a suitable topology). Unfortunately, it is easy to show the existence of lsc functions that do <u>not</u> possess lsc GBR

$$\hat{\psi}(s) = s - \gamma(\psi)(s) - \hat{\gamma}(\psi)(s).$$

In Lemma 4.3 below, it is shown that there exists a $\hat{\gamma}(\psi)$, a unique GBR to each $\gamma(\psi)$ such that $\hat{\psi}$ defined thus is in Ψ. This defines a map from Ψ into itself. Consider a fixed-point of this map. At such a point, $\hat{\psi} = \psi$, so from the above equations, some manipulation yields $\hat{\gamma}(\psi) = \gamma(\psi)$ or $\gamma(\psi)$ is GBR to itself on $\bar{S}$. Lemmata 4.8-4.10 then conclude the proof by showing that it is in fact the case that $\gamma(\psi)$ is a best-response to itself when the state space is restricted to S. By the symmetry of the payoffs (equation R1) the argument is complete.

These ideas underlie the following results but rather than invoke the functions $\gamma(\psi)$ and $\hat{\gamma}(\psi)$, notation is simplified as follows: player 2's actions in response to $\gamma(\psi)$ are now interpreted as the investment level he chooses given player 1's action, so that if he takes an action $a > 0$, his instantaneous reward is given by $u(s - \gamma(\psi)(s) - a)$. Define $R_\psi(s) = \frac{1}{2}(s + \psi(s))$ for $s \in S$, $\psi \in \Psi$. Note that the conditional distribution over $\bar{S}$ of next period's state s' depends now only on a. Abusing notation we denote this distribution by $q(\cdot|a)$. Finally, let V_ψ denote player 2's value function from a GBR to $\gamma(\psi)$. We rewrite the Bellman Optimality equation in this notation as:

$$(4.1) \qquad V_\psi(s) = \max_{a \in [0, R_\psi(s)]} \{u(R_\psi(s) - a) + \beta \int V_\psi(s') dq(s'|a)\}$$

closed in the weak topology. Since $\hat{V}$ corresponds to the value function of a one-person dynamic programming problem with (weakly-)continuous transition and continuous payoffs, it is straightforward to show that $\hat{V}$ is itself a continuous function. By the assumptions on q, $\hat{V}(0) = u(0)/(1-\beta)$. Since $v \leq \hat{V}$ for all $v \in \Omega$, the result readily follows. ||

Now observe that for fixed ψ, the feasible action correspondence $[0, R_\psi(s))]$ is increasing in s, i.e., any action feasible at s_1 is also feasible at s_2 if $s_2 > s_1$. Since u is increasing in its argument, it is immediate by the upper-semicontinuity of V_ψ that

Lemma 4.2. For each ψ, V_ψ is non-decreasing and right-continuous on $\bar{S}$.

Now for each ψ redefine the value of V_ψ at $\bar{y}$ by setting $V_\psi(\bar{y}) = \frac{u(\bar{y})}{1-\beta}$. Thus defined, V_ψ still satisfies the conditions of lemma 4.2, therefore $V_\psi \in \Omega$ for each $\psi \in \Psi$.

As the second step in the proof we shall now construct a map from Ψ into itself. To this end, we define for $\psi \in \Psi$ and $v \in \Omega$ a map $F_{\psi,v}:\bar{S} \to 2^{\bar{S}}$ by $F_{\psi,v}(\bar{y}) = \bar{y}$, and for $0 \leq s < \bar{y}$,

$$F_{\psi,v}(s) = \underset{a \in [0,R_\psi(s)]}{\text{argmax}} \{u(R_\psi(s)-a) + \beta \int v(s')dq(s'|a)\}$$

If $v = V_\psi$, the we shall write F_ψ for $F_{\psi,v}$.

By Lemma 2.1 and Theorem 2.1 in Parthasarathy (1973), $F_{\psi,v}$

fact the right-continuity of ψ and continuity of u together imply $u(R_\psi(s_n) - a_n) \to u(R_\psi(s) - a)$, we obtain the existence of $\alpha > 0$ such that for large n

$$(4.3) \qquad u(R_\psi(s_n) - a_n) + \beta \int v(s')dq(s'|a_n) + 2\alpha < u(R_\psi(s) - \hat{a}) + \beta \int v(s')dq(s'|\hat{a}).$$

Using the additional fact that $u(R_\psi(s_n) - \hat{a}) \to u(R_\psi(s) - \hat{a})$ (4.3) in turn implies that for all sufficiently large n

$$(4.4) \qquad u(R_\psi(s_n) - a_n) + \beta \int v(s')dq(s'|a_n) + \alpha < u(R_\psi(s_n) - \hat{a}) + \beta \int v(s')dq(s'|\hat{a}).$$

But $\hat{a} \leq s - \gamma(\psi)(s) \leq s_n - \gamma(\psi)(s_n)$, so $\hat{a}$ is feasible at s_n. Equation (4.4) therefore contradicts the optimality of a_n for all large n.

Note that by claim 2, $\max\{F_{\psi,v}(s)\}$ is well-defined at each $s \in [0,\bar{y})$. Defining $\hat{\psi}(s) = \max\{F_{\psi,v}(s)\}$ for $s \in S$, we see that claims 1 and 2 together imply that $\hat{\psi}$ is right-continuous and non-decreasing. Therefore $\hat{\psi}$ is usc on S, and $\hat{\psi} \in \Psi$, since $F_{\psi,v}(\bar{y}) = \bar{y}$. The last step in the proof of Lemma 4.3 is

<u>Claim 3</u>: $\hat{\psi}$ is the only usc selection from $F_{\psi,v}$.

<u>Proof</u>: Suppose there were another usc selection $\bar{\psi}$. Note that $\bar{\psi}$ is non-decreasing, hence right-continuous. Since $\hat{\psi} \neq \bar{\psi}$, there is $s \in \bar{S}$ such that $\hat{\psi}(s) \neq \bar{\psi}(s)$, so $\hat{\psi}(s) > \bar{\psi}(s)$. Let $s_n \downarrow s$. Then $\bar{\psi}(s_n) \downarrow \bar{\psi}(s)$, so for large enough n, $\hat{\psi}(s) > \bar{\psi}(s_n) \in F_{\psi,v}(s_n)$, but

(ii) *if* ψ *is continuous at* s, *then* $\lim_{n\to\infty} \psi_n(s_n) = \psi(s)$.

Proof: See Appendix.

Lemma 4.6. *Suppose* $v_n \to v \in \Omega$ *and* $\psi_n \to \psi \in \Psi$. *Suppose also that* $s \in \bar{S}$ *is a continuity point of* ψ. *Then*,

$$\int v_n(s')dq(s'|\psi_n(s)) \to \int v(s')dq(s'|\psi(s)).$$

Proof: By the generalized Dominated convergence theorem (see Hildenbrand (1974)), it suffices to show that (i) $q(\cdot|\psi_n(s))$ converges weakly to $q(\cdot|\psi(s))$, (ii) $\{v_n\}$ is a uniformly integrable sequence, and (iii) $v_n \to v$ in distribution. Since, by hypothesis, s is a continuity point of ψ, so $\psi_n(s) \to \psi(s)$, and (i) follows from assumption (Q5). Since $v_n(s') \le (1-\beta)^{-1} u(\bar{y})$ for all $s' \in \bar{S}$, (ii) is immediate. Let μ_n be the measure on $\bar{S}$ corresponding to $q(\cdot|\psi_n(s))$, and μ that corresponding to $q(\cdot|\psi(s))$. Then, we need to show that $\mu_n v_n^{-1}$ converges weakly to μv^{-1}. Since μ_n converges weakly to μ, it suffices by Billingsley (1968), Theorem 5.5) to show that $\mu(E') = 0$ where $E' = \{s' \in \bar{S}|$ there is $s'_n \to s'$ such that $v_n(s'_n)$ does not converge to $v(s')\}$. Let $E = \{s' \in \bar{S}| v$ is discontinuous at $s'\}$. Clearly $E' \subset E$ (apply lemma 4.5). Further, E' is measurable by Billingsley (1968, p.226). Note that $0 \notin E'$, since $s'_n \to 0$ implies by Lemma 4.5 that $\limsup_{n\to\infty} v_n(s'_n) \le v(0) = (1-\beta)^{-1} u(0)$, while since $v_n \in \Omega$, $v_n(s'_n) \ge v_n(0) = (1-\beta)^{-1} u(0)$, so $\liminf_{n\to\infty} v_n(s'_n) \ge (1-\beta)^{-1} u(0) = v(0)$. We identify two cases: (i) $\psi(s) = 0$, so $q(s'|\psi(s)) = 1$ for all $s' \ge 0$. Since $0 \notin E'$,

$\epsilon\ \bar{S}$. We claim that $\bar{\psi} = \hat{\psi}$. Note that to prove this claim, it suffices by Lemma 4.7 to show that $\bar{\psi} = \hat{\psi}$ on a set dense in $\bar{S}$.

Let D' be the set of disconitnuity points of any of the following functions: ψ_n, $\hat{\psi}_n$, $\bar{\psi}$, ψ, $\hat{\psi}$, V_n, and V. Since each of these functions is monotone (and right continuous), D' is at most countable. Hence, $D = \bar{S} - D'$ is dense in $\bar{S}$.

We shall show that $\bar{\psi} = \hat{\psi}$ on D. Let $s \epsilon D$. Consider first the case $\bar{\psi}(s) < R_\psi(s)$. Since ψ is continuous at s, $\psi_n(s) \to \psi(s)$, so $R_{\psi_n}(s) \to R_\psi(s)$, and therefore, for large n, $R_{\psi_n}(s) > \bar{\psi}(s)$. For all such n,

$$(4.5) \qquad u(R_{\psi_n}(s) - \hat{\psi}_n(s)) + \beta \int V_n(s')dq(s'|\hat{\psi}_n(s))$$

$$\geq u(R_{\psi_n}(s) - \bar{\psi}(s)) + \beta \int V_n(s')dq(s'|\bar{\psi}(s)).$$

By Lemma 4.6, and since $s \epsilon D$, $\int V_n(s')dq(s'|\hat{\psi}_n(s)) \to \int V(s')dq(s'|\hat{\psi}(s))$, and $\int V_n(s')dq(s'|\bar{\psi}(s)) \to \int V(s')dq(s'|\bar{\psi}(s))$, so taking limits in (4.9) yields

$$(4.6) \qquad u(R_\psi(s) - \hat{\psi}(s)) + \beta \int V(s')dq(s'|\hat{\psi}(s))$$

$$\geq u(R_\psi(s) - \bar{\psi}(s)) + \beta \int V(s')dq(s'|\bar{\psi}(s)).$$

Now suppose $\bar{\psi}(s) = R_\psi(s)$. Then, since

$$(4.7) \qquad u(R_{\psi_n}(s) - \hat{\psi}(s)) + \beta \int V(s')dq(s'|\hat{\psi}(s))$$

$$\geq u(0) + \beta \int V_n(s')dq(s'|R_\psi(s)),$$

From (4.9) and (4.11), $V = V^*$ on D, so $V = V^*$ on $\bar{S}$. Thus, we have shown that for $s \in [0,\bar{\bar{y}}]$

$$V(s) = \max_{a \in [0,R_\psi(s)]} \{u(R_\psi(s) - a) + \beta \int V(s')dq(s'|a)\}$$

(4.12)

$$= u(R_\psi(s) - \hat{\psi}(s)) + \beta \int V(s'|\hat{\psi}(s)).$$

To complete the proof, it is shown by using similar arguments in Strauch (1966) and Maitra (1968) that V is indeed the expected payoff (on $[0,\bar{\bar{y}}]$ from employing the stationary strategy $\hat{\gamma}(\psi)(y) = y - \gamma(\psi)(y) - \hat{\psi}(y) = \frac{1}{2}(y + \psi(y)) - \hat{\psi}(y)$.

Since V satisfies the Bellman Optimality Equations (4.12), and it $\hat{\psi}$ yields a total expected payoff of V, it is indeed the case that $B(\psi) = \hat{\psi}$. ||

Combining Lemmas 4.1 and 4.7, we see the existence of a $\psi^* \in \Psi$ such that $B(\psi^*) = \psi^*$. Therefore, there is a function $\hat{\gamma}^* = \hat{\gamma}(\psi^*)$, such that $\hat{\gamma}^*$ is a GBR to itself on $[0,\bar{\bar{y}}]$ for problem (P). Denote the restrictions of $\hat{\gamma}^*$ to s by γ^*.

<u>Lemma 4.8</u>: γ^* <u>is a GBR to itself on</u> S.

<u>Proof</u>: By our assumptions on q, if the game starts with the state in S, the state stays in S forever. If $\hat{\gamma}^*$ is a GBR to itself on $\bar{S}$, then γ^* must be a GBR to itself on S for what happens in $(\bar{y},\bar{\bar{y}})$

$$= \frac{1}{2} \left[1 - \frac{\psi^*(s_1) - \psi^*(s_2)}{s_1 - s_2} \right]$$

$$\geq \frac{1}{2} \quad ,$$

and finally since ψ^* is usc on S and γ^* is defined by $\gamma^*(s) = \frac{1}{2}(s - \psi^*(s))$, γ^* is lsc on S. ||

Appendix

A1: Proof of Theorem 3.1

Theorem 3.1 is established through several lemmata. Let $Z = \{(s,a) | s \in S,\ 0 \leq a \leq s - \gamma(s)\}$.

Lemma A.1: Let $v: S \to \Re_+$ be a bounded, non-negative and non-decreasing function. Let $\tilde{v}(s,a) = \int v(s')dq(s' | s, \gamma(s), a)$ for $(s,a) \in Z$. Then $\tilde{v}: Z \to \Re_+$ is usc on Z.

Proof: Let $(s_n, a_n) \to (s,a) \in Z$. Since α is lsc on S, so $\limsup_{n\to\infty} (s_n - \gamma(s_n) - a_n) \leq (s - \gamma(s) - a)$. Assume wlog that $\gamma(s_n)$ converges to $\hat{a}$. By (Q5), $q(\cdot | s_n, \gamma(s_n), a_n)$ converges weakly to $q(\cdot | s, \hat{a}, a)$. Since $\hat{a} \geq \gamma(s)$, this implies by (Q3) that $q(s' | s, \hat{a}, a) \geq q(s' | s, \gamma(s), a)$ for all $s' \in S$. Together these result in

$$\limsup_{n\to\infty} \tilde{v}(s_n, a_n) = \limsup_{n\to\infty} \int v(s')dq(s' | s_n, \gamma(s_n), a_n)$$

$$\leq \int v(s')dq(s' | s, \hat{a}, a)$$

Define an operator T on USC(S) by

$$Tw(s) = \max_{a \epsilon A(\gamma)(s)} \{u(a) + \beta \int w(s')dq(s'|s, \gamma(s), a\}$$

for $w \epsilon USC(S)$, $s \epsilon S$. Then,

<u>Lemma A.4</u>: T <u>maps</u> USC(S) <u>into itself and is a contraction</u>.

<u>Proof</u>: By lemma A.1 $\int w(s')dq(s'|s, \gamma(s),a)$ is usc on Z. Trivially so is u. Hence by lemma A.2, Tw is usc on S. Since u, w are non-negative and bounded, so is Tw. Finally, by the assumptions on γ, we have $s_1 < s_2$ implies $A(\gamma)(s_1) \subset A(\gamma)(s_2)$. Since u, w are non-decreasing, this implies that Tw also enjoys this property.

A straightforward application of Blackwell (1965, Theorem 5) utilizing the fact that $\beta \epsilon (0,1)$ shows that T is a contraction. ||

Lemmata A.3, A.4 and the Banach fixed-point theorem (Smart (1974, p. 2)) imply that T has a unique fixed-point $V^* \epsilon USC(S)$, so that

$$\text{(A.1)} \qquad V^*(s) = \max_{a \epsilon A(\gamma)(s)} \{u(a) + \beta \int V^*(s')dq(s'|s, \gamma(s), a)\}$$

Lemma A.2 (ii) indicates the presence of a Borel function $\hat{\alpha}^*$ such that $\hat{\gamma}^*(s) \epsilon A(\gamma)(s)$ at each $s \epsilon S$, and

$$\text{(A.2)} \qquad V^*(s) = u(\hat{\gamma}^*(s) + \beta \int V^*(s')dq(s'|s, \gamma(s), \hat{\alpha}^*(s)).$$

A3: Proof of Lemma 4.5

Suppose (i) were violated. Then there exists a subsequence (which we continue to denote by n), an integer N, and positive numbers δ and α such that for $n \geq N$

$$\psi_n(s_n) > \psi(s) + 2\alpha$$

and

$$|s_n - s| < \delta,$$

where $\delta > 0$ is chosen so that ψ is continuous at $(s + \delta)$, $\psi(s + \delta) < \psi(s) + \alpha$ and

$$\psi_n(s_n) \leq \psi_n(s + \delta).$$

Combining these inequalities,

$\psi_n(s+ \delta) \geq \psi_n(s_n) > \psi(s) + 2\alpha > \psi(s + \delta) + \alpha$. So $\lim_{n\to\infty} \psi_n(s + \delta) \geq \psi(s + \delta) + \alpha$, while since ψ is continuous at $(s + \delta)$, $\lim_{n\to\infty} \psi_n(s + \delta) = \psi(s + \delta)$, a contradiction. This establishes (i).

A completely analagous argument exploiting the left-continuity of ψ establishes that if ψ is continuous at s, then $\liminf_{n\to\infty} \psi_n(s_n) \geq \psi(s)$, proving (ii). ||

A4: Proof of Lemma 4.9

Suppose contrary to the lemma, there were some $s > 0$ at which $2\gamma^*(s) = s$, or $\gamma^*(s) = s/2$. Then, since γ^* is a GBR to itself, action a such that $a + \gamma^*(s) > s$, and the action a at s provides some player with a greater expected payoff than $\gamma^*(s)$. An argument identical to that used above in establishing lemma 4.9 furnishes a contradiction. ||

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