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A CHARACTERIZATION OF STATIONARY PERFECT EQUILIBRIUM IN DYNAMIC GAMES*

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ABSTRACT

This paper characterizes the stationary subgame–perfect equilibria of the dynamic game that results when two or more agents simultaneously exploit a productive asset. Previous studies have relied almost exclusively on the use of parametrized examples to examine this problem. Our analysis is conducted in a general framework with no functional form restrictions.

We show that despite the behavioral complexity (subgame perfection in stationary strategies) and the lack of any restrictions on these strategies, stationary equilibria exhibit remarkably regular dynamics. The state sequence is (eventually) monotone from any initial state, and always converges to a steady state. When this convergence does not take place in finite time, the steady state is strictly smaller than that resulting from a first–best solution, i.e., overexploitation, or a "tragedy of the commons" occurs.

Turning to welfare questions, we demonstrate that stationary equilibria are strictly (Pareto–) suboptimal from every initial state. Under a weaker welfare criterion – that of productive efficiency borrowed from the capital theory literature – the results are less unambiguous.

Finally, we also provide a complete characterization of differentiable equilbria, and show that they possess all the salient characteristics previously derived in parametrized examples.

1. Introduction and Summary

The dominant paradigm within which strategic behavior is studied in economic theory is that of repeated games. However, by requiring that the environment of the game be unchanging, this framework does not lend itself to meaningful analysis of situations in which players' current actions affect future environments. This paper studies equilibria in a more general strategic structure, that of dynamic games. A dynamic game introduces a state variable to represent the environment of play which moves through time in response to players' actions.

The flavor of the problem we consider in this paper is perhaps best caught by the following "fishing game". Every season two commercial fisheries fish simultaneously and independently in the same pool of water. The fish is sold commercially and the next fishing season finds the two fisheries back in the same waters and so on hereafter. The commonality of the fishing grounds implies an externality in the fisheries′ actions: the size of the total catch in the current fishing season determines the number of fish in the waters next season.[1] A unilateral decrease in the first fishery′s catch this season is likely to imply more fish for both firms in the next season. Whether or not the first fishery would decide on such a reduction depends of course on the subsequent reaction of the second fishery. It is natural to assume that the fisheries behave non–cooperatively and then ask: what are the economic consequences of such behavior? Does the fish population oscillate as in some predator–prey models or is monotonic growth or decline inevitable? How does the non–cooperative outcome compare with an alternative institutional structure, a cartelization of the fisheries? Is there always over–exhaustion of the resource, i.e. a "tragedy of the commons?" In general, what are the welfare properties of such non–cooperative equilibria? These and other questions are examined in this paper.[2] Specifically, we provide a complete characterization of stationary subgame–perfect equilibria in dynamic resource games.

This analysis falls under two heads: positive and normative. The positive questions relate to the dynamic behavior of the state trajectory in equilibrium, while the normative questions examine, under two criteria, the welfare properties of equilibria. The benchmark dynamics and payoffs are those arising from the set of Pareto–optimal solutions.

Regarding the positive questions, we show that the state path in any stationary equilibrium is eventually monotone.[3] Limit points of these paths are steady states under equilibrium play. If convergence to a steady state does not occur in finite time, then the steady state must lie (strictly) below the so–called "golden rule", (the unique steady–state of any first–best solution) and a "tragedy of the commons" phenomenon occurs. Sufficient conditions are provided on the equilibrium strategies under which a "tragedy" is always witnessed.[4]

Turning to the normative questions, we show that consumption paths (hence, payoffs) generated by stationary equilibria are strictly suboptimal from every initial state, except (possibly) the golden rule itself. This suboptimality follows from either over–consumption, a "tragedy of the commons," or, more surprisingly, from under–consumption, a situation, so to speak of "anorexia in the commons," a high steady state sustained since any increase in consumption by one player provokes a sharp increase in consumption by the others.[5] We further examine a weaker welfare criterion, related to a similar criterion in the capital theory literature, the efficiency of equilibrium consumption paths. Paradoxically enough, paths that reflect a "tragedy of the commons" are efficient. "Anorexia in the commons" may be not only suboptimal, but worse even inefficient.

Finally, we provide a complete characterization of differentiable equilibria, such as those exhibited by the example in Levhari and Mirman (1980). We show that these equilibira invariably result in a "tragedy of the commons" from all initial states, and are thus uniformly suboptimal but generate uniformly efficient consumption paths.

Two of their results merit further comment. Given the complexity of behavior (i.e., subgame–perfection in stationary strategies) with no restrictions whatsoever on the shape of these strategies, it is somewhat surprising that equilibrium state paths involve such regular behavior. This is especially so considering that the investment function in equilbrium need not be non–decreasing in stock levels for this result to obtain.

Secondly, the uniform suboptimality of stationary equilibria appears to indicate that they might be of use as punishment phases in trigger–strategy equilibria meant to support more efficient paths. Although there is an obvious analogy here with the use of inefficient stage game Nash equilibria in the analysis of repeated games, the question is more complex: the degree of suboptimality varies with the stock level, hence for a given set of parameters, cooperation may be enforceable from some initial states, but not others. Indeed, Benhabib and Radner (1988) demonstrate this when payoffs are linear in consumption for both players with restrictions on the rate at which players may consume. For models with concave payoffs, such as ours, this appears considerably more difficult to prove, since the bang–bang characteristic of solutions to a dynamic optimization problem with linear payoffs does not hold.

Let us now indicate the related literature. In dynamic resource games, Lancaster (1973) and Levhari and Mirman (1980) have studied parametrized models with functional forms postulated for the one–period reward and transition functions. In each case, a Nash equilibrium is computed that is uniformly suboptimal. Cave (1987) and Benhabib and Radner (1988) examine the possiblility of enforcing cooperation through the threat of reversion to a suboptimal equilibrium in the event of defection. Cave′s analysis is restricted to the Levhari–Mirman example. Benhabib and Radner impose no restrictions on the transition function but limit as mentioned above the one–period payoff functions to being linear in consumption and impose a maximum rate of consumption for the players. Neither restriction is present in our paper. Within this framework, they obtain interesting results, including one on the existence of a form of equilibrium not present in repeated games that they label "switching equilibrium". It is unclear to what extent their analysis generalises when these assumptions are dropped. Certainly, many of their key results appear to depend, in an essential manner, on the linearity of payoffs. Reinganum and Stokey (1985) study a parametrized model in which profits rather than utility from consumption is the criterion. Their focus is on the extent of commitment possible, and its effect on outcomes. Lastly, Sundaram (1987) demonstrates the existence of stationary equilibrium in a general formulation, identical to the one we employ in this paper but with an additional symmetry assumption on the one-period payoffs. It is disturbing to note that there are no existence theorems available when this symmetry assumption is dropped, although the examples in Lancaster (1973) and Levhari–Mirman (1980) do not require it. It is hoped that the analysis provided in this paper will help in this direction.

2.1 Notation and Definitions

The set of reals (resp. non-negative reals, strictly positive reals) is denoted by $\mathbb{R}$ (resp. $\mathbb{R}_+$, $\mathbb{R}_{++}$).

Let ϕ: $\mathscr{D} \subset \mathbb{R} \to \mathbb{R}$, where $\mathscr{D}$ is open. Let $\Phi\ (y;\ x) = [\ \frac{\phi(y) - \phi(x)}{y - x}\ ]$ for $x \in \mathscr{D}$. The Dini Derivates of ϕ at $x \in \mathscr{D}$ are four quantities defined as follows:

$$D^+\phi(x) = \limsup_{y\downarrow x} \Phi(y;\ x) \qquad D_+\phi(x) = \liminf_{y\downarrow x}\Phi(y;\ x)$$

$$D^-\phi(x) = \limsup_{y\uparrow x}\Phi(y;\ x) \qquad D_-\phi(x) = \liminf_{y\uparrow x}\Phi(y;\ x)$$

Note that ϕ is differentiable from the right (resp. left) at x iff $D^+\phi(x) = D_+\phi(x) \neq \pm\infty$ (resp. $D^-\phi(x) = D_-\phi(x) \neq \pm\infty$), and that ϕ is differentiable at x with derivative $\phi'(x) = D^+\phi(x)$ iff $D^+\phi(x) = D_+\phi(x) = D^-\phi(x) = D_-\phi(x) \neq \pm\infty$.

If ϕ reaches a (local) maximum at x, then $D^+\phi(x) \leq 0$, $D_-\phi(x) \geq 0$. This result is important in the sequel. A proof of this, and other features of Dini derivates may be found in Royden (1968, p. 98)

Sometimes we will need to use a modified version of these derivates. Let $\{y_t\}$ be a sequence in $\mathscr{D}$, such that $y_t \downarrow x$. Then, define $D^+\phi(x; y_t) = \limsup_{t\to\infty}\Phi(y_t; x)$, and $D_+\phi(x; y_t) = \liminf_{t\to\infty}\Phi(y_t; x)$. $D^-\phi(x; y_t)$ and $D_-\phi(x; y_t)$ are defined similarly by considering sequences $y_t \uparrow x$.

ϕ is said to be lower–semicontinuous or lsc (resp. upper–semicontinuous or usc) at $x \in \mathscr{D}$ if $\liminf_{n\to\infty}(x_n) \geq \phi(x)$ resp. $\limsup_{n\to\infty}\phi(x_n) \leq \phi(x))$ for all sequences $x_n \to x$. Further, ϕ is lsc (resp. usc) on $\mathscr{D}$ if it is lsc (resp. usc) at all $x \in \mathscr{D}$. Note that ϕ is continuous at x (resp. on $\mathscr{D}$) iff it is both lsc and usc at x (rep. on $\mathscr{D}$).

2.2 The Model

The dynamic game considered in this paper is fully described by the tuple $\{S, N, (A_i(y)), f, (u_i), \delta\}$ where S is the set of states of the system, here the possible values of the stock of the productive asset or resource; a generic element of S will be denoted by y, with y_t denoting the state in period t = 0, 1, 2, ... N = {1, ... ,n} is the player set. Given y, player $i \in N$ picks an action $a_i \in A_i(y)$. This action is made with full knowledge of the game's history and the chosen actions of the other players, and represents player i's planned extraction of the resource. If plans are collectively feasible ($\Sigma_i a_i \leq y$) they are carried out and player i receives an instantaneous reward or utility $u_i(a_i)$. If plans are infeasible ($\Sigma_i a_i > y$) then we assume that each player receives an amount proportional to his bid, so that player i receives an amount $(a_i/\Sigma_j a_j)y$, hence a reward of $u_i((a_i/\Sigma_j a_j)y)$.[6] The transition function f then converts the left over stock $\eta(y, a_1, \ldots, a_n) = \max(0, y - \Sigma_i a_i)$ to the available stock in the next period, and the process is repeated ad infinitum. All players discount future rewards by the factor $\delta \in (0, 1)$.

The formal structure of the game we study is described by the following:

(i) S is a compact interval $[0, \bar{s}]$ of $\mathbb{R}_+$, where $\bar{s} > 0$ will be defined shortly.

(ii) $N = \{1, 2\}$. This restriction of the player set is made purely for notational convenience. The generic player is indexed by i. In all statements referring to i, j will denote the other player.

(iii) $A_i(y) = [0,y]$ for all $y \in S$. Thus each player is allowed to extract all of the available stock.

(iv) $f: \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

(F1) $f(0) = 0$

(F2) f is continuous and strictly increasing on $\mathbb{R}_+$

(F3) f is strictly concave on $\mathbb{R}_+$, and differentiable on $\mathbb{R}_{++}$ with $f'(0+) > 1$, $f'(\infty) < 1$.

(v) $u_i: \mathbb{R}_+ \to \mathbb{R}$ satisfies

(U1) u_i is continuous, strictly concave and strictly increasing on $\mathbb{R}_+$

(U2) u_i is continuously differentiable on $\mathbb{R}_{++}$ with $\lim_{c\downarrow 0} u_i'(c) = +\infty$.

The assumptions (F1) – (F3) imply the existence of $\bar{x} > 0$ such that $f(\bar{x}) = \bar{x}$. Define $\bar{s} = \bar{x}$. Note that f now maps $S = [0,\bar{s}]$ into itself.[7]

In order to describe the play of the game, the notion of a strategy is required. Let y_0 denote an arbitrary initial value of the state variable.

Let $H_t = (y_0, a_0, \dots, y_{t-1}, a_{t-1}, y_t)$ denote a generic history of the game upto period t, where a_s denotes the vector $[a_{1s}, a_{2s}]$, and let H^t denote the set of all possible histories upto t. A strategy for player i, denoted $\bar{g}_i$, is a sequence of functions (g_t^i) where for each t, g_t^i specifies the action a_{it} to be taken by player i as a (Borel–measurable) function of the history H_t, satisfying $g_t^i\,(H_t) \le y_t$. A strategy is Markovian if the information structure required to implement it is Markovian, i.e., if g_t^i depends on H_t only through y_t for each t (so for each t, g_t^i is a Borel–function from S into itself). In addition a Markovian strategy (g_t^i) is stationary if $g_t^i = g_i$ for all t, where g_i is a Borel–function from S into itself satisfying $g_i(y) \le y$ for all $y \in S$.

A pair of strategies $(\tilde{g}_1, \tilde{g}_2)$ determines a unique sequence of histories $\{H_t\}$ from $y_0 \in S$ and hence uniquely defines the total discounted reward for player i that we denote $W_i\ (\tilde{g}_1, \tilde{g}_2)(y_0)$. If G_i denotes the set of all admissible strategies for player i, then $\tilde{g}_1{}^*$ is a <u>best-response</u> of player i to $\tilde{g}_j$ iff

$$W_i(\tilde{g}_i{}^*,\tilde{g}_j)(y) \geq W_i(\tilde{g}_i,\tilde{g}_j)(y) \text{ for all } \tilde{g}_i \in G_i \text{ for all } y \in S$$

The function $W_i\ (\tilde{g}_i{}^*, \tilde{g}_j)$: $S \to \mathbb{R}$ is called player i's <u>value function</u> from responding optimally to $\tilde{g}_j$.

A <u>Nash equilibrium</u> to the dynamic game is a pair of strategies $(\tilde{g}_1, \tilde{g}_2)$ such that for each i, $\tilde{g}_i$ is a best-response to $\tilde{g}_j$. A Nash equilibrium is said to be <u>subgame-perfect</u> if the following condition holds: let $\tilde{g}_i\ (t, H_t)$ denote the <u>continuation</u> of $\tilde{g}_i$ from period t, given a history H_t upto t. Then <u>for each t</u>, <u>and every possible history upto t</u>, $(\tilde{g}_1\ (t,H_t), \tilde{g}_2\ (t, H_t))$ constitutues an equilibrium to the game starting from H_t. This paper confines attention to equilibria that are subgame-perfect.

2.3 <u>Stationary Equilibria in the Dynamic Game</u>

A stationary equilibrium to the dynamic game is simply an equilibrium in stationary strategies. That is, a stationary equilibrium is a pair of Borel functions (g_1, g_2) such that for i = 1, 2, g_i: $S \to S$ satisfies $g_i(y) \in [0, y]$, and g_i is a best-response to g_j. Since stationary equilibria are implementable with a Markovian information structure, they are also referred to as Markov-Perfect equilibria. (Note that, by history-independence, stationary equilbria are subgame-perfect.) Throughout this paper, however, we use the more descriptive term Stationary Perfect Equilibria or SPE.

For the symmetric version of this dynamic game $(u_i = u_j)$, Sundaram (1987) demonstrates the existence of an SPE (g_1, g_2) in which the functions g_i are both lsc on S and satisfy $0 < g_1(y) + g_2(y) < y$ at all $y > 0$. It is disturbing to note however that in the assymmetric game, there are no general results available in this direction, although the examples in Lancaster (1973) and Levhari–Mirman (1980) do not use symmetry. The main problem is that although the lower–semicontinuity of g_i on S is a minimal sufficient condition to admit a best–response, the best–response to a lsc strategy will not, in general, be lsc. Thus, a standard Debreu–Nash fixed–point argument on a map taking strategies into best–responses is infeasible.

2.4 The First Best Outcomes

The benchmark payoffs and dynamics against which the SPE will be compared is that resulting from a solution to the "Planning Problem," i.e., the set of (first–best) outcomes that result when the players operate as a cartel. This set is completely described by the solution to (2.1) – (2.3) as the parameter α ranges over [0,1]

$$\text{(2.1)} \qquad \underset{\{c_{1t},\, c_{2t}\}}{\text{Maximize}} \quad \Sigma_{t=0}^{\infty}\, \delta^t \, [\alpha u_1\,(c_{1t}) + (1-\alpha)\, u_2\,(c_{2t})]$$

subject to

$$\text{(2.2)} \qquad y_0 = y \in S;\ y_{t+1} = f(y_t - c_{1t} - c_{2t})\,, \qquad t \geq 0$$

$$\text{(2.3)} \qquad 0 \leq c_{1t} \leq c_{1t} + c_{2t} \leq y_t\,, \qquad t \geq 0.$$

This optimization problem is a standard one in intertemporal allocation theory; consequently, we confine ourselves to stating, without proof, its important characteristics from this paper's point of view.

Fix $\alpha \in [0, 1]$. The problem (2.1) – (2.3) has a unique stationary solution. This solution is associated with continuous policy functions and a continuous, concave value function. Under the optimal solution, the state sequence $\{y_t\}$ is monotone from any $y_0 \in S$. Further y_t converges to the unique steady-state y_{GR} where y_{GR} and x_{GR} solve $\delta f'(x_{GR}) = 1$, and $f(x_{GR}) = y_{GR}$. Note that y_{GR}, the so-called "golden rule", is independent of the choice of α or y_0. Lastly, convergence to y_{GR} from $y_0 \neq y_{GR}$, $y_0 \neq 0$, is always asymptotic: $y_t \neq y_{GR}$ for any t if $y_0 \neq y_{GR}$.

3. The Ramsey–Euler equations

This section presents the Ramsey–Euler equations for the dynamic game. These equations constiture the first–order (necessary) conditions for a pair of strategies to be best–responses to one another; consequently there are two sets of equations, one corresponding to each player. Repeated use of these equations is made in the sequel to analyze dynamic behavior under the SPE.

Since each player takes the other's action as parametric to his own optimization problem, the environment a player faces is in general not differentiable, and indeed may not even be continuous. The notion of Dini derivates introduced in Section 2 is therefore brought into play to help characterize the first–order conditions. Some additional notation, maintained throughout this paper, will be of great help here. Let $\psi(y) = \max\ \{0,\ y - g_1(y) - g_2(y)\}$ be the "savings" function in equilibrium. Also let $\tilde{S} = \{\tilde{y} \in S \mid \tilde{y} = f(\psi(y))$ for some $y \in S\}$ be the "reachable set".

Lemma 3.1 Let $0 < \tilde{y} \in \tilde{S}$, and $\tilde{y} = f(\psi(y))$. Suppose $\psi(y) > 0$. If there exists a sequence $\{\tilde{y}_n\}$ in $\tilde{S}$ with $\tilde{y}_n \downarrow \tilde{y}$ (resp. $\tilde{y}_n \uparrow \tilde{y}$) such that $g_j(\tilde{y}_n) \to g_j(\tilde{y})$, then the first (resp. second) equation below holds:

$$u_i'(g_i(y)) \geq \delta u_i'(g_i(\tilde{y}))f'(\psi(y))(1 - D_+g_j(\tilde{y};\ \tilde{y}_n))$$

(3.1)

$$u_i'(g_i(y)) \leq \delta u_i'(g_i(\tilde{y}))f'(\psi(y))(1 - D^-g_j(\tilde{y};\ \tilde{y}_n)).$$

The proof of this result may be found in the Appendix. Clearly, $D_+g_j(\tilde{y}; y_n) \geq D_+g_j(\tilde{y})$, and $D^-g_j(\tilde{y}; y_n) \leq D^-g_j(\tilde{y})$. Thus, as an immediate consequence of equations (3.1), we have

Theorem 3.1 Under the hypothesis of lemma 3.1, equations (3.1) hold with $D_+g_j(\tilde{y}; y_n)$ and $D^-g_j(\tilde{y}; y_n)$ replaced respectively with $D_+g_j(\tilde{y})$ and $D^-g_j(\tilde{y})$.

Remark 1 If g_j is continuous from the right (resp. left) at $\tilde{y}$, the first (resp. second) equation in (3.1) clearly holds. However, this condition is stronger than needed.

Remark 2 If g_j is differentiable at $\tilde{y}$, equations (3.1) collapse to the single equation

$$(3.2) \qquad u_i'(*g_i(y)) = \delta u_i'(g_i(\tilde{y}))f'(\psi(y))(1-g_j'(\tilde{y})).$$

Equation (3.2) holds at all $y \in S$ if g_1 and g_2 are C^1 functions, as for instance in the example of Levhari–Mirman (1980).

4. Positive Behavior in Equilibrium: Dynamics under the SPE

This section examines the dynamics and asymptotic behavior of the state under an SPE. Throughout, all statements are with respect to an arbitrary initial stock $y_0 \in S$. For definiteness, we denote by $(y_t(y_0))$ the sequence of values taken by the state under an SPE (g_1, g_2), i.e., $y_t(y_0)$ is defined recursively for $t \geq 0$ by $y_{t+1}(y_0) = f(\psi(y_t(y_0)))$, where $y_0(y_0) = y_0$.

Our first result is a slight generalization of Theorem 6.1 in Sundaram (1987). It states that the equilibrium state sequence $\{y_t(y_0)\}$ is (eventually) monotone from any $y_0 \in S$. Throughout this section and the rest of the paper, we denote the value functions under the SPE by V_1 and V_2, i.e., V_i denotes player i's value function from a best–response to g_j.

Theorem 4.1: Let $\{y_t(y_0)\}$ be the state sequence from $y_0 \in S$ under the SPE (g_1, g_2). Let $T = \min \{\tau | y_\tau(y_0) = 0\}$. Then, either $y_{t+1}(y_0) \geq y_t(y_0)$ for $t = 1, ..., T-1$, or $y_{t+1}(y_0) \leq y_t(y_0)$ for $t = 1, ..., T-1$.

Note If $T = \infty$, then Theorem 4.1 says that either $y_{t+1}(y_0) \leq y_t(y_0)$ for all $t \geq 1$, or $y_{t+1}(y_0) \geq y_t(y_0)$ for all $t \geq 1$.

The proof of this theorem follows from modifications of arguments considered in the proof of Theorem 6.1 in Sundaram (1987). Consequently, we only sketch the outline of the proof here. For simplicity in notation, we drop the index y_0.

"Proof" If $y_{t+1} = y_t$ for some $t = 1, ..., T-2$, then $y_{t+k} = y_t$ for all $k \geq 0$. Suppose $y_{t+1} > y_t$. Note that $\psi(y_{t+1}), \psi(y_t) > 0$. Then, we must have $V_i(y_{t+1}) > V(y_t)$ for $i = 1, 2$. For if not, some player could consume more at time t, keeping the state at y_t gaining more immediate utility and maintaining at least the same continuation value. But for i to have a greater continuation value from y_{t+1} than y_t means that the net stock (i.e., after j's consumption) must be greater at y_{t+1} than at y_t, i.e. we must have

$$(4.1) \qquad y_{t+1} - g_j(y_{t+1}) > y_t - g_j(y_t).$$

For if not, any action feasible at y_{t+1} is also feasible at y_t, and it cannot be the case that $V_i(y_{t+1}) > V_i(y_t)$. Now a standard argument, from the theory of optimal growth, exploiting the strict concavity of u_i in its agrument (see Sundaram (1987, lemma II.4) for details) shows that if (4.1) holds, then it follows that

(4.2) $$\psi(y_{t+1}) \geq \psi(y_t),$$

and therefore by the monotonicity of f, $y_{t+2} \geq y_{t+1}$.

A similar argument works when $y_{t+1} < y_t$ for some t = 1, ..., T–2.

Q.E.D.

A result implicitly used in this proof is invoked several times in the course of this section. For ease of reference, we state it here:

Lemma 4.2: Suppose for y, y′ in S, it is the case that $y - g_j(y) > y' - g_j(y') > 0$. Then,

(i) $V_i(y) > V_i(y')$, and

(ii) $\psi(y) \geq \psi(y')$.

Further, for $y, y' \in \tilde{S}$, with $y > y' > 0$, (i) and (ii) always hold. Lastly, if (ii) holds strictly, then it must be the case that $y - g_j(y) > y' - g_j(y')$.

The dynamic behavior of $\{y_t(y_0)\}$ is trivial if the stock is extinct in finite time. Therefore, in the rest of this section we concentrate on the non–trivial alternative $y_t(y_0) > 0$ for all t.

Since S is compact and $\{y_t(y_0)\}$ is monotone it converges to a limit denoted $\bar{y}(y_0)$. Let A* represent the set of all such limit points:

$$A^* = \{y \in S | y = \bar{y}(y_0) \text{ for some } y_0 \in S\}$$

A steady state of the SPE is a value $y \in S$ such that $f(\psi(y)) = y$. Let A^0 represent the set of all steady–states.

$$A^0 = \{y \in S | y = f(\psi(y))\}.$$

Since $f(0) = 0$, so $0 \in A^0$ and A^0 is non-empty. Obviously $A^0 \subset A^*$ since $y_t(y) = y$ for all $y \in A^0$. Somewhat surprisingly, it turns out that the requirement:

(LSC) g_1 and g_2 are lsc functions on S

implies that the reverse containment also holds, so that <u>from every initial state, the system converges to a steady state</u>.

Why (LSC)? Using arguments similar to those used in Sundaram (1987, Theorem 3.1) it is easilty shown that the minimal sufficient condition for a strategy g_i to admit a best–response is that g_i be lsc on S.[8] Under weaker conditions, g_i may fail to admit <u>any</u> best response. Thus, (LSC) does not constitute a strong restriction to place on (g_1, g_2).

<u>Theorem 4.3</u>: <u>If the SPE satisfies</u> (LSC), <u>then</u> $A^* \subset A^0$.

<u>Proof</u>: Fix an arbitrary initial state $y_0 \in S$, and for ease of notation denote by $\{y_t\}$ and $\bar{y}$ respectively, the sequence $\{y_t(y_0)\}$ and the limit $\bar{y}(y_0)$. If y_t is constant beyond some point, then obviously $\bar{y} \in A^0$. So assume that y_t converges asymptotically to $\bar{y}$.

Let $y_t \to \bar{y}$. If $\psi(y_t) \to \psi(\bar{y})$, then certainly $\bar{y} \in A^0$ by the continuity of f.

Suppose, per absurdem, that this were not the case. We must have $\psi(\bar{y}) > \lim_{t\to\infty} \psi(y_t)$ by the fact that ψ is usc. (Note that the limit on the right is well–defined since by the monotonicity of $\{y_t\}$ and f.) This yields in turn, using lemma 4.2, the existence of an i and a T such that for all $t \geq T, V_i(\bar{y}) > V_i(y_t)$, and $V_i(\bar{y}) > \lim_{t\to\infty} V_i(y_t)$.[9] Let $\bar{V} = \lim_{t\to\infty} V_i(y_t)$ and define $\epsilon > 0$ by $V_i(\bar{y}) = \bar{V} + \epsilon/\delta$, and c_t by the solution to $\bar{y} = f(y_t - g_j(y_t) - c_t)$. Since $\bar{y} = \lim_{t\to\infty} f(y_{t+1}) = \lim_{t\to\infty} f(y_t - g_1(y_t) - g_2(y_t))$, c_t is well–defined and arbitrarily close to $g_i(y_t)$ for large t. Further, t can be chosen to satisfy

(i) $u_i(c_t) - u_i(g_i(y_t)) > -\epsilon/2$, and

(ii) $$V_i(y_{t+1}) - \bar{V} < \epsilon/2.$$

But then,

$$\begin{aligned} u_i(c_t) + \delta V_i(f(y_t - g_j(y_t) - c_t)) &= u_i(c_t) + \delta V_i(\bar{y}) \\ &> u_i(g_i(y_t)) + \delta\bar{V} + \epsilon/2 \\ &> u_i(g_i(y_t)) + \delta V_i(y_{t+1}) \\ &= V_i(y_t), \end{aligned}$$

contradicting the hypothesis that g_i represents a BR to g_j. This establishes the theorem.

Q.E.D.

Note: From now on, throughout the paper, except where stated to the contrary, the SPE will be assumed to satisfy condition (LSC).

The rest of this section is devoted to examining a generic element of A* in relation to y_{GR}. This analysis will simplify tremendously the process of obtaining the answers to the two normative questions posed in the Introduction.

To this end, define $y \in A^*$ to be right–stable (resp. left–stable) if there exists $y_0 \in \tilde{S}$ such that $y_0 > y$, $\bar{y}(y_0) = y$ (resp. $y_0 < y$, $\bar{y}(y_0) = y$). Also define $y \in A^*$ to be right–asymptotically–stable or r.a.s. (resp. left–asymptotically–stable or l.a.s.) if there is $y_0 \in \tilde{S}$ such that $y_0 > y_t(y_0) > y$ for all t, $\bar{y}(y_0) = y$ (resp. $y_0 < y_t(y_0)$, for all t, $\bar{y}(y_0) = y$).

An immediate consequence of the above definitions is the following:

Lemma 4.4: If $y \in A^*$ is left–asymptotically–stable then

(4.1) $$\delta f'(\psi(y))\,(1-D^{-}g_i(y)) \geq 1,\ i = 1,\ 2.$$

Similarly if $y \in A^*$ is right–asymptotically–stable,

(4.2) $$\delta f'(\psi(y))\,(1-D_{+}g_i(y)) \leq 1,\ i = 1,\ 2.$$

Proof: Lemma 3.1 shows that the first (resp. second) equation in (3.1) requires only the right– (resp. left–) continuity of g_j at $\tilde{y}$ along a sequence. Now, observe that the left–asymptotic–stability of y requires the left–continuity of g_i, i = 1, 2, and the right–stability of y requires the right continuity of g_i, i = 1, 2, along some sequence as a consequence of Theorem 4.3 and the lower semicontinuity of the g_i. Together with equations (3.1) and the fact that y is a steady–state, this establishes (4.1) – (4.2).

Q.E.D.

Lemma 4.4 gives us a handle on comparing $y \in A^*$ and y_{GR}. For if $y \in A^*$ satisfied $D^-g_i(y) \geq 0$ for some i, then by (4.1) $\delta f'(\psi(y)) \geq 1 = \delta f'(x_{GR})$, so $\psi(y) \leq x_{GR}$ by the concavity of f and $y \leq y_{GR}$ by the monotonicity of f. We derive and strengthen these inequalities now.

Lemma 4.5: If $\bar{y} \in A^*$ is l.a.s., then $\bar{y} < y_{GR}$.

Proof: For ease of notation, let $\bar{x} = \psi(\bar{y})$. Define the function H by $H(y) = f(\psi(y)) - y$ for $y \in S$. Since $\bar{y}$ is l.a.s., so $H(y_t(y_0)) > 0$ along a path $y_t(y_0)$, and of course $H(\bar{y}) = 0$. Therefore, employing the notion of Dini Derivates again, we have

$$\liminf_{z \uparrow \bar{y}} \frac{H(\bar{y}) - H(z)}{\bar{y} - z} \leq \liminf_{t \to \infty} \frac{H(\bar{y}) - H(y_t)}{\bar{y} - y_t} \leq 0,,$$

where $y_t = y_t(y_0)$; so, from the definition of H,

(4.3) $$f'(\bar{x})\ (1-D^-g_1(\bar{y}) - D^-g_2(\bar{y})) \leq 1.$$

Compare (4.1) – (4.3). Some manipulation readily shows that together they imply that it must be the case that $D^-g_i(\bar{y}) > 0$ for some i. Therefore, we have $\delta f'(\psi(\bar{y})) > 1 = \delta f'(x_{GR})$, so $\bar{y} < y_{GR}$ by the concavity and strict monotonicity of f.

Q.E.D.

The complementary result to lemma 4.8 – namely, that if $\bar{y}$ is right–asymptotically–stable, then $\bar{y} < y_{GR}$ involves a rather cumbersome proof, and may be found in the Appendix to this paper:

Lemma 4.6: If $\bar{y} \in A^*$ is right–asymptotically–stable then $\bar{y} < y_{GR}$.

What about those steady states to which convergence is always non–asymptotic? In general, we find that we are unable to preclude the possibility that such steady states may be above the golden–rule y_{GR}. We can however show some basic properties of this subset of A^0. Partition A^0 into two sets A_1 and A_2 defined by

$$A_1 = \{y \in A^0 | y \text{ is r.a.s. or l.a.s.}\}$$

$$A_2 = A^0 \backslash A_1.$$

Then, one can show

Lemma 4.7: If A_2 is non–empty, it contains only a finite number of points. Furthermore, there is a critical $\tilde{y}^* \in \tilde{S}$ in this case such that for $y_0 \in \tilde{S}$, $\bar{y}(y_0) < y_{GR}$ (resp. $\bar{y}(y_0) \geq y_{GR}$) if and only if $y_0 \leq y^*_0$ (resp. $y_0 > y^*_0$).

Proof: The second part of the lemma is trivial to establish: let $y^*_0 = \sup\{y_0 \in A_1\}$ and apply lemma 4.2.

Suppose A_2 is non–empty and non–finite. Since $A_2 \subset S$, and S is compact, there is a limit point of A_2 in S. Employing appropriately modified versions of Theorem 4.3 and lemmas 4.5 – 4.6 it is straightforward to show that this limit point is a steady state and must lie strictly below y_{GR}. But then it cannot be the limit of a sequence of points all of which are at least as large (by hypothesis) as y_{GR}.

Q.E.D.

A "tragedy of the commons" is said to occur when the dynamic game converges to a steady state $\bar{y}$ that lies strictly below y_{GR}, i.e., when competition (Nash play) results in over–exhaustion of the stock relative to a first–best outcome (the planning solution).

In summary, this section has established the following: all SPE generate monotone state trajectories. If policy functions are lower–semicontinuous, then the limit–points of these trajectories form steady–states of the game under the SPE. In this case, therefore, the set of limit–points and steady–states coincide completely. Whenever the convergence to a steady–state is asymptotic, the steady–state lies strictly below the unique steady state of the first–best solution, and a strict "tragedy of the commons" obtains. This result may not hold if convergence is not asymptotic. Indeed non–asymptotic convergence may well lead to the reverse phenomenon of under–exhaustion under competition. In the event that there are steady–states to which convergence is non–asymptotic, very specific dynamic behavior is called for under the SPE: convergence to such states must take place in finite time.

5. Normative behavior under the SPE I: Optimality

We define the SPE to be (Pareto–) Optimal from an initial state $y_0 \in S$, if there is no alternate feasible sequence of actions $(c_{1t}, c_{2t})_{t=0}^{\infty}$ for the players yielding rewards $R_i = \Sigma_{t=0}^{\infty} \delta^t u_i(c_t)$ that satisfies $R_i \geq V_i(y_0)$ with strict inequality for at least one i.

The result below provides an (almost) complete characterization of optimality under the SPE. ("Almost" because one initial state, y_{GR}, poses a problem; see details below.)

Theorem 5.1: The SPE (g_1, g_2) yields a strictly sub–optimal payoff vector $(V_1(y), V_2(y))$ from each non–zero initial state y (barring possibly y_{GR}).

Proof: The theorem follows from an embarrassingly simple application of lemmas 4.8 –

4.9. If (g_1, g_2) is to result in an optimal payoff vector $(V_1(y), V_2(y))$ from some $y \in S$, $0 \neq y \neq y_{GR}$, then (by the uniqueness of the first–best solutions) it must be the case that the path $\{y_t(y)\}$ generated by (g_1, g_2) from any y coincides with some first–best path from y. But first–best paths are necessarily asymptotically convergent (i.e., $y_{t+1} \neq y_t$ for any t) and always converge to y_{GR}. Whereas if $\{y_t(y)\}$ is an asymptotically–convergent path under the SPE then by lemmas 4.8 and 4.10, we have $\lim_{t\to\infty} y_t(y) = \bar{y}(y) < y_{GR}$. Clearly both conditions cannot hold simultaneously.

Q.E.D.

Remark: The initial state 0 is irrelevant since it is an absorbing state under both the SPE and the first–best problem. The initial state y_{GR} however poses a special problem. For each $\alpha \in (0, 1)$, y_{GR} constitutes an absorbing state under optimal actions in the first–best solution. Since non–asymptotic convergence to y_{GR} is involved, Section 4 does not throw any light on this case. It is just possible that y_{GR} is a steady state under the SPE. We have been unable to rule this out.

6. Normative behavior under the SPE II: Efficiency

Optimality is a rather strong condition to require of Nash equilibria, in general. In this section we examine the SPE against a weaker criterion, that of productive efficiency, borrowed from the economic growth and capital accumulation literature. Let $0 < y_0 \in S$ be an arbitrary initial state, and $\{y_t(y_0)\}_{t=0}^{\infty}$ the state path from y_0 under the SPE. For simplicity of notation, let $c_{it} = g_i(y_t(y_0))$. $i = 1, 2$. Then, the path $\{c_{1t}, c_{2t}\}$ is defined to be production–efficient (or simply efficient) from y_0 if there is no other feasible path $(c'_{1t}, c'_{2t})_{t=0}^{\infty}$ from y_0 such that (i) $c'_{it} \geq c_{it}$, $i = 1, 2$, $t = 0, 1, 2, \ldots$, and (ii) $c'_{it} > c_{it}$ for some i and t. It is inefficient otherwise. In the single–player case, the dynamic game reduces to the aggregative model of economic growth. Efficiency of programs has been the object of considerable study in this paradigm (see, e.g., Cass (1972), or Mitra (1979)). We shall shortly have occasion to draw on the results from this literature.

Note that efficiency of the SPE from y_0 could alternatively (and equivalently) have been defined as follows: define c_{1t}, c_{2t} as above and let $c_t = c_{1t} + c_{2t}$, $t = 0. 1, \ldots$. Then, $\{c_{1t}, c_{2t}\}_{t=0}^{\infty}$ is efficient from y_0 if and only if there does not exist a sequence $\{c_t'\}$ with $y_0' = y_0$, such that $c_t' \geq c_t$ for all t and $c_t' > c_t$ for some t. Thus, checking efficiency in the dynamic game is equivalent to checking efficiency of the sequence $\{c_t\}$.

Define $\{c_t\}$ as above. Let $x_t = y_t - c_t$, $t \geq 0$, and define the sequence $\{p_t\}$ by $p_0 = f'(f^{-1}(y_0))$, $p_{t+1} = p_t/f'(x_t)$ for $t \geq 0$. Then, a well-known criterion for efficiency (see, e.g., Mitra (1979)) states that $\{c_t\}$ is efficient from y_0 if $\lim_{t\to\infty} p_t x_t = 0$. This immediately leads to:

<u>Lemma 6.1</u>: <u>Let</u> $\bar{y} = \lim_{t\to\infty} y_t$. If $\bar{y} \leq y_{GR}$, <u>then</u> $\{c_t\}$ <u>is efficient</u>.

<u>Proof</u>: Since $y_t \to \bar{y}$, so $x_t \to \bar{x} = \psi(\bar{y})$ and $f'(x_t) \to f'(\bar{x})$. Since $\bar{y} \leq y_{GR}$, so $\bar{x} \leq x_{GR}$, and $f'(\bar{x}) \geq f'(x_{GR}) = 1/\delta > 1$. Therefore $\lim_{t\to\infty} (\pi_{s=1}^{t} f'(x_s)) = \infty$. Since $p_{t+1} = p_t/f'(x_t)$, so $p_{t+1} = p_0/(\pi_{s=1}^{t} f'(x_s))$. and $p_t \to 0$.

Q.E.D.

<u>Corollary 6.2</u>: <u>Let</u> $z^* = \sup \{\bar{y} | \bar{y} \in A^0\}$. <u>If</u> $z^* \leq y_{GR}$, <u>then the SPE is efficient from every initial state</u>.

We obtain, therefore, the following results on efficiency under the SPE:

<u>Theorem 6.3</u>: <u>Under any of the following conditions, the SPE is efficient from every initial state</u>:

(i) g_i <u>is continuous and non-decreasing on</u> S <u>for some</u> i.

(ii) ψ <u>is strictly increasing on</u> S, <u>or at least</u>

(iii) ψ is strictly increasing at z^*.

Proof: (i) implies that the Dini–derivates of g_i are non–negative everywhere, so at any steady–state $\bar{y} \in A^0$, we have $\delta f'(\psi(\bar{y})) \geq 1 = \delta f'(x_{GR})$, so $x_{GR} \geq \psi(\bar{y})$, or $y_{GR} \geq f(\psi(\bar{y})) = \bar{y}$. Similarly, (ii) results in all sequences (barring those originating at a steady–state) being asymptotically convergent, so lemmas 4.8 and 4.10 imply $\bar{y} < y_{GR}$ for all $\bar{y} \in A^0$. Finally, (iii) ensures convergence to z^* is asymptotic, so $z^* = \sup A^0$, y_{GR}. Apply lemma 6.1 in all three cases.

Q.E.D.

Recall from Section 4 that $A^0 = A_1 \cup A_2$ where A_2 is the set of steady–states to which all convergence is non–asymptotic (i.e., in a finite number of steps). If A_2 is non–empty and contains a point $\bar{y}$ strictly larger than y_{GR}, than all initial states that converge to $\bar{y}$ (including $\bar{y}$ itself) could result in inefficient paths. This follows from a straightforward application of the Phelps–Koopmans theorem (see, e.g., Burmeister and Dobell (1970)).

7. Differentiable Equilibria

Many computable equilibria, such as those obtained by Levhari and Mirman (1980), involve strategy functions that are continuously differentiable in their argument. In this section we show that the added hypothesis of differentiability results in a marked strengthening of the statements of the previous three sections, notably Section 6.

So suppose (g_1, g_2) are continuously differentiable on S. Our first result is:

Lemma 7.1: $\psi'(y) \geq 0$ at all $y \in S$. Further, ψ is strictly increasing everywhere on S.

Proof: Since (g_1, g_2) are differentiable everywhere, the Ramsey–Euler equations (3.1) may be written as

(7.1) $$u_i'(g_i(y)) = \delta u_i'(g_i(\tilde{y}))\, f'(\psi(y))\, (1 - g_j'(\tilde{y})),$$

i, j = 1, 2, $i \neq j$, where $\tilde{y} = f(\psi(y))$. Suppose $\psi(y) = \psi(y')$ for $y, y' \in S$. Then $\tilde{y} \equiv$

$f(\psi(y)) = f(\psi(y')) \equiv \tilde{y}'$, so (i) $g_i(\tilde{y}) = g_i(\tilde{y}')$, (ii) $f'(\psi(y)) = f'(\psi(y'))$, and (iii) $g_j'(\tilde{y}) = g_j'(\tilde{y}')$. It follows from (7.1) that $g_i(y) = g_i(y')$, i = 1, 2. Since $\psi(y) = \psi(y')$ by hypothesis, $y - g_1(y) - g_2(y) = y' - g_1(y') - g_2(y')$, and therefore $g_i(y) = g_i(y')$ yields $y = y'$. Thus, ψ is one–to–one on S. Since ψ is continuous on S by hypothesis, it follows that ψ is strictly monotone on S, so $\psi' \geq 0$ on S or $\psi' \leq 0$ on S. Since $\psi(0) = 0$ and $\psi \geq 0$, it must be the case that $\psi' \geq 0$ on S.

Q.E.D.

Since ψ is strictly increasing on S, all convergence to steady states is asymptotic. Therefore, any steady state of a differentiable equilibrium lies strictly below the golden–rule y_{GR}.[10] The results of the previous sections now imply:

<u>Theorem 7.2</u>: <u>Differentiable SPE are</u> (i) <u>efficient from every initial state, and</u> (ii) <u>strictly suboptimal from every non–zero initial state</u>.

<u>Appendix</u>

1. <u>The Ramsey–Euler equations</u>

Suppose $\{\tilde{y}_n\}$ is a sequence in $\tilde{S}$ such that $\tilde{y}_n \downarrow \tilde{y}$ and $g_j(\tilde{y}_n) \to g_j(\tilde{y})$. since $\tilde{y}_n$, $\tilde{y} \in \tilde{S}$, there are y_n, $y \in S$ such that $\tilde{y}_n = f(\psi(y_n))$, $\tilde{y} = f(\psi(y))$. For ease of notation, let $x_n = \psi(y_n)$, $x = \psi(y)$. Note that x_n, $x > 0$, and (since $\tilde{y}_n \to \tilde{y}$ and f is continuous) $x_n \to x$. Define

$$W(x_n) = [u_i(y - g_j(y) - x_n) + \delta u_i(f(x_n) - g_j(f(x_n)) - \psi(\tilde{y}))].$$

Then, W is the 2–period reward accumulating to i if he deviates from the suggested action $\psi(y)$ in the first period but restores investment levels to $\psi(\tilde{y})$ in the second. By the continuity of g_j along the sequence $\tilde{y}_n$ and the continuity of f and u_i, $W(x_n) \to W(x)$ as $n \to \infty$. Further, it is clear that $W(x_n) \leq W(x)$ for all n since g_i is a best–response to g_j. Since $\tilde{y}_n \downarrow \tilde{y}$, by the monotonicity of f, we must have $x_n \downarrow \tilde{x}$. Therefore, for all n,

$$\frac{W(x_n) - W(x)}{x_n - x} \leq 0$$

so $D^+W(x; x_n) \leq 0$. This translates to the first equation in (3.1). The second equation is analagously proved by considering $\tilde{y}_n \uparrow \tilde{y}$.

Proof of Lemma 4.7

If $\bar{y}$ is r.a.s, then there is $y_0 \in \tilde{S}$ such that $y_t(y_0) > \bar{y}$ for all t and $y_t(y_0) \downarrow \bar{y}$. Observe that it must then be the case for some i along some subsequence of $y_t(y_0)$ (henceforth denoted y_t in this proof) that $g_i(y_t) \geq g_i(\bar{y})$. Else, from some point on (say T), $g_i(y_t) < g_i(\bar{y})$ for $t \geq T$. Since $y_t > \bar{y}$, player i is clearly better off consuming a larger amount c_t that lands the state directly in $\bar{y}$. (Indeed, it follows that $g_i(y_t) \geq g_i(\bar{y})$ must hold along some (possibly different) subsequences for <u>both</u> i.) Fix i.

Some new notation would help in the inequalities to follow.

Define $x_t = \psi(y_t)$, $\hat{y}_t = y_t - g_j(y_t)$. and

$$W(y_t, x_t) = u_i(\hat{y}_t - x_t) + \delta u_i(\hat{y}_{t+1} - x_{t+1})$$
$$W(y_t, \bar{x}) = u_i(\hat{y}_t - \bar{x}) + \delta u_i(\hat{y} - x_{t+1})$$

where $\hat{y} = \bar{y} - g_j(\bar{y})$. $W(y_t, x_t)$ describes the two–period reward to player i along the path prescribed by g_i from y_t, while $W(y_t, \bar{x})$ describes the two–period reward i would obtain by straying from the path in t, but restoring investment levels in (t+1), where straying changes the investment level in t to $\bar{x} < x_t$. Since g_i is a BR to g_j, we have $W(y_t, x_t) - W(y_t, \bar{x}) \geq 0$ for all t. To simplify notation further at this point, let

$$A_t = \left[\frac{f(x_t) - f(\bar{x})}{\bar{x} - x_t} - \frac{g_j(f(x_t)) - g_j(f(\bar{x}))}{\bar{x} - x_t} \right]$$

$$= \left[\frac{y_{t+1} - \bar{y}}{\bar{x} - x_t} - \frac{g_j(y_{t+1}) - g_j(\bar{y})}{\bar{x} - x_t} \right]$$

For the mean–value theorem, we note the existence of α_t, β_t, such that $u_i'(\alpha_t)\ (\bar{x} - x_t) = u_i(\hat{y}_t - x_t) - u_i(\hat{y}_t - \bar{x})$. and $\delta u_i'(\beta_t)A_t(\bar{x} - x_t) = \delta\ u_i(\hat{y}_{t+1} - x_{t+1}) - \delta u_i(\hat{y} - x_{t+1})$, where α_t lies between $(\hat{y}_t - x_t)$ and $(\hat{y}_t - \bar{x})$, and β_t lies between $(\hat{y}_{t+1} - x_{t+1})$ and $(\hat{y} - x_{t+1})$. Since $\psi(y_t) \downarrow \psi(\bar{y})$, so by the lower semicontinuity of g_i, $i = 1, 2$, $(\hat{y} - x_t) \to (\hat{y} - \bar{x})$, $(\hat{y}_t - \bar{x}) \to (\hat{y} - \bar{x})$, and $(\hat{y}_{t+1} - x_{t+1})$, $(\hat{y} - x_{t+1}) \to (\hat{y} - \bar{x})$, as $t \to \infty$. Note that since $(\hat{y}_t - x_t)$ and $(\hat{y} - x_t) \to (\hat{y} - \bar{x})$, so α_t and $\beta_t \to (\hat{y} - \bar{x})$.

Summing up the following obtains:

$$W(y_t, x_t) - W(y_t, \bar{x})$$
$$= (u_i'(\alpha_t) + \delta u_i'(\beta_t)A_t)\ (\bar{x} - x_t) \geq 0.$$

so since $x_t > \bar{x}$, $x_t \to \bar{x}$.

$$u_i'(\alpha_t) + \delta u_i'(\beta_t)A_t \leq 0$$

or

$$-\delta A_t \geq \frac{u_i'(\alpha_t)}{u_i'(\beta_t)} \to 1 \text{ as } t \to \infty .$$

Therefore, for all $\epsilon > 0$, there is $T(\epsilon)$ such that for all $t \geq T(\epsilon)$, $-\delta A_t \geq (1-\epsilon)$. Employing the definition of A_t and invoking the earlier argument that (at least subsequentially) $g_i(y_t) - g_i(\bar{y}) \geq 0$, we obtain[11]

(A.1) $$\frac{f(\bar{x}) - f(x_t)}{\bar{x} - x_t} \geq \frac{1-\epsilon}{\delta} + \frac{g_j(\bar{y}) - g(y_{t+1})}{\bar{x} - x_{t+1}}$$

This implies, taking the liminf of both sides as $t\to\infty$, that

$$f'(\bar{x}) \geq \frac{(1-\epsilon)}{\delta} + D_+ g_i(\bar{y};\ y_t)f'(\bar{x})$$

or transposing terms and using the fact that this equation holds for all $\epsilon > 0$,

$$\delta f'(\bar{x})(1 - D_+ g_i(\bar{y};\ y_t)) \geq 1.$$

But $y_t \downarrow \bar{y}$, so the conditions of lemma 3.1 are met, which implies that

$$\delta f'(\bar{x})(1 - D_+ g_i(\bar{y};\ y_t)) \leq 1,$$

so we have

(A.2) $$\delta f'(\bar{x})(1 - D_+ g_i(\bar{y};\ y_t)) = 1.$$

Once again, let $H(y_t) = f(\psi(y_t)) - y_t$. Since $y_t \downarrow \bar{y}$ so $H(y_t) < 0$ and $H(y_t) \to H(\bar{y}) = 0$. Therefore,

$$\frac{H(y_t) - H(\bar{y})}{y_t - \bar{y}} < 0,$$

which implies $D^+H(\bar{y};\ y_t) \leq 0$. This is just

$$\text{(A.3)} \qquad f'(\bar{x})(1 - D_+g_1(\bar{y};\ y_t) - D_+g_2(\bar{y};\ y_t)) \leq 1.$$

Since i was fixed arbitrarily at the start of the proof, (A.2) holds for both i. But this is possible in conjunction with (A.3) only if $D_+g_i(\bar{y};\ y_t) > 0$ for some i. Therefore, from (A.2), $\delta f'(\bar{x}) > 1$, or $\bar{x} < x_{GR}$ and $\bar{y}$, y_{GR}.

Q.E.D.

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FOOTNOTES

[1]There is of course a second source of externality possible. If both fisheries sell in the same market, the total catch also determines the market price. Our focus in this paper however is only on the former (the sequential externality). The assumptions we make are equivalent to assumiong that the fisheries sell in different markets and are monopolists in those markets, with some additional structure on the shape of the demand curve.

[2]It should be noted that the "fishing game" is a convenient expositional device. The class of dynamic games analyzed in the sequel can equally well apply (with appropriate modifications where necessary) to common property extraction of an exhaustible resource like oil, the "splitting of the pie" between agents in a firm or the economy and so on. Indeed, in a striking and novel example, Lancaster (1973) views a modern firm as a productive asset exploited by its owners and (unionized) workers, which is one interpretation of the framework we study.

[3]By "eventually" we mean that the sequence of states is monotone beyond some finite time. The remarkably regular behavior implied by this result is substantially more general than the result on monotonicity of capital accumulation paths in aggregative (classical or nonclassical) growth models. In the latter, the observed monotonicity of the state path is an immediate consequence of the non–decreasing nature of the optimal investment function, which in turn derives from the strict concavity of the one–period reward function (see, e.g., Dechert and Nishimura (1983)). Our result hold regardless of whether the equilibrium investment function is non–decreasing or not! See section 4 for details.

[4]These conditions include as special cases the examples computed by several authors. In the event these conditions are not met, it is possible – although we have been unable to show this via an example – that steady state(s) may lie above the

golden rule, and under–consumption occurs in equilibrium. Intuitively, this could occur because players get "locked into" sustaining too high a level of the resource: equilibrium strategies are shaped such that a small drop in the stock level causes a large increase in consumption by one or both players, causing the stock level to plummet. The main difficulty with constructing an example is that computable equilibria typcially possess well–behaved (e.g., differentiable) strategies. It is shown in this paper that such strategies always result in over–consumption. On the other hand, since a stationary equilibrium of a two–person dynamic game involves the simultaneous solution to two dynamic programming problems, with the environment of each being specified by the solution to the other, examples with highly discontinuous equilibrium strategies seem exceptionally difficult to construct.

[5]Of course "anorexia" really means an absence of appetite" whereas players in a high steady state equilibrium under consume for strategic reasons. As often, style edges out substance!

[6]The exact form of the allocation mechanism is irrelevant from the point of view of this paper. Indeed, any function ξ: $S \times A_1 \times \ldots \times A_n \to A_1 \ldots \times A_n$ that (i) allocates to each player his planned action when plans are individually and collectively feasible, i.e., $\xi_i(y, a_1, \ldots, a_n) = a_i$, if $a_i \in A_i(y)$ and $\Sigma_i a_i \leq y$, and (ii) exhausts the resource through some distribution when they are not, i.e., satisfies $\Sigma_i\ \xi_i(y, a_1, \ldots, a_n) = y$ if $\Sigma a_i > y$, would do al well for the results, while complicating notation.

[7]There is no loss of generality in this assumption. If the initial value of the state y_0 is greater than $\bar{s}$ then the analysis could be carried through by assuming $S = [0, y_0]$ since f still maps S into itself.

[8]Indeed, this condition remains minimally sufficient, under much more general conflict–resolution rules than the one we employ. For example, if the function ξ of footnote 6 is continuous, g_i needs to be lsc to ensure the existence of a best–response.

[9]Actually, obtaining the strict inequality requires further work. Lemma 4.2 yields for i = 1 , 2, $V_i(\bar{y}) \geq \lim_{t\to\infty} V_i(y_t)$. Since ψ is discontinuous at $\bar{y}$ and g_1, g_2 are lsc on S, so one of g_1 or g_2 (say g_j) is discontinuous at $\bar{y}$. From a simple modification of the proof of Theorem 5.1 in Sundaram (1987), it can be shown that V_i is continuous from the left (right) at y iff g_j is continuous from the left (right) at y; this implies that V_i is also discontinuous at $\bar{y}$. Since ψ jumps up at $\bar{y}$, so g_j jumps down at $\bar{y}$, hence V_i jumps up at $\bar{y}$. And since g_j is discontinuous along the sequence $\{y_t\}$, so is V_i. Q.E.D.

[10]Although this follows from lemmas 4.8 – 4.9, a simple and direct proof may also be given. Since ψ is monotone and continuous, there is a largest steady-state z*. Defining $H(y) = f(\psi(y)) - y$ for $y \in S$, we note that $H'(z^*) \leq 0$ or $f'(\psi(z^*))\ (1 - g_1'(z^*) - g_2'(z^*)) \leq 1$. Combining this with equation (7.1) implies $g_i'\ (z^*) > 0$, and the result follows.

[11]Note that (A.1) already implies (by taking limits as $t \to \infty$) that $\delta f'(\bar{x}) \geq 1$, since $\epsilon > 0$ is arbitrary, and the second term on the RHS is non-negative for all t.

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