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COOPERATIVE MODELS OF BARGAINING

1. *INTRODUCTION.*

The axiomatic theory of bargaining originated in a fundamental paper by J.F. Nash (1950). There, Nash introduced an idealized representation of the ***bargaining problem*** and developed a methodology that gave the hope that the undeterminateness of the terms of bargaining could finally be resolved.

Two agents have access to any of the alternatives in some set S. If they agree on a particular alternative, that is what they get. Otherwise they end up at a prespecified alternative in S, called the disagreement point, d. Both S and d are given directly in utility space. Nash's objective was to develop a theory that would help predict how the agents would establish a compromise among their conflicting preferences. He specified a natural class of bargaining problems (S,d), to which he confined his analysis, and he searched for solutions, that is, rules to associate with each (S,d) in the class a point of S, to be interpreted as this compromise. He achieved this by formulating a list of properties, or axioms, that he thought solutions should satisfy and establishing the existence of a unique solution satisfying all the properties. It is after this first ***axiomatic characterization of a solution*** that much of the subsequent work has been modelled.

Alternatively, solutions are meant to provide a recommendation that an impartial arbitrator could make. There, the axioms may embody normative objectives.

Although criticisms were raised early on against some of the properties Nash used, the solution he identified, now called the Nash solution, remained the dominant solution until the mid–seventies. Then, other solutions were introduced and given appealing characterizations. These results spurred a revival of the theory which since underwent a considerable development. In this chapter we review its current state.

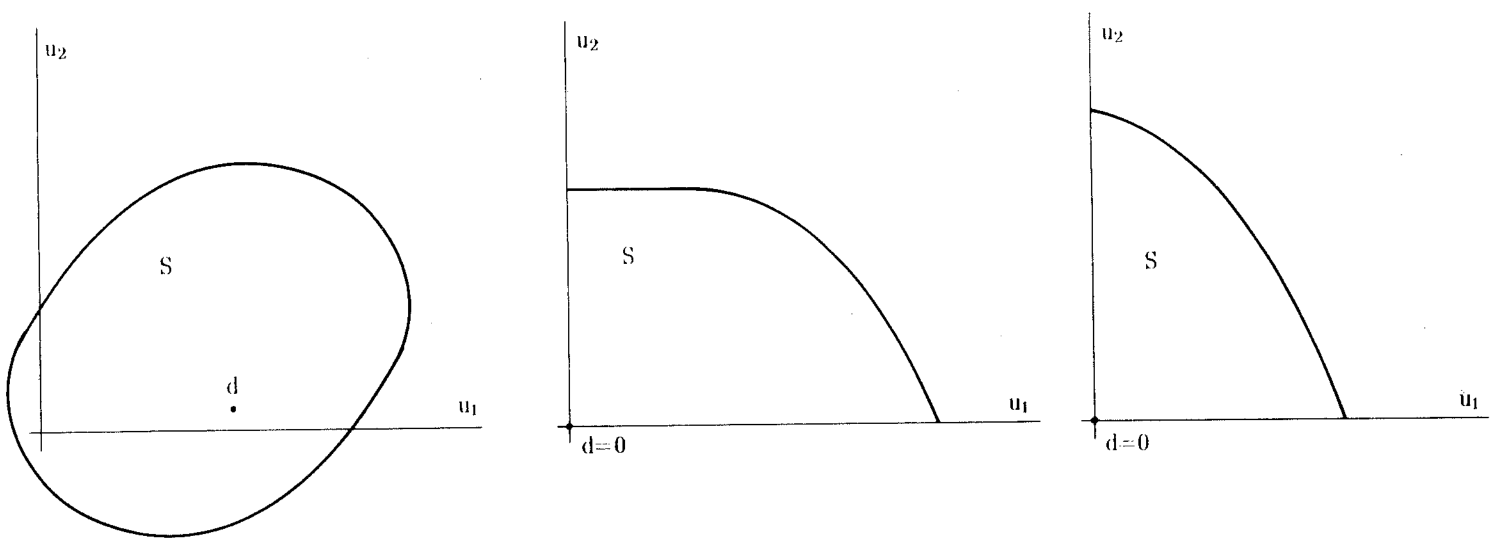
***Remark* 1.** The most complete earlier survey is Roth (1979c). Partial surveys are Schmitz (1977), Kalai (1985), Thomson (1985a), and Peters (1987). Thomson and

Lensberg (1988) analyze the case of a variable number of agents. Thomson (1988) is intended as an up–to–date and detailed review.

2. *DOMAINS. SOLUTIONS.*

An n–person bargaining problem, or simply a ***problem***, is a pair (S,d) where S is a subset of the n–dimensional euclidean space, and d is a point of S. Σ_d^n is the class of problems such that (Fig.1a):

- S is convex, bounded, and closed (it contains its boundary),
- there is at least one point of S strictly dominating d.



An element of Σ_d^2. An element of Σ_0^2. A strictly comprehensive element of Σ_0^2.

(a) (b) (c)

Figure 1

Each point of S gives the utility levels, measured in some von Neumann–Morgenstern scales, reached by the agents through the choice of one of the physical alternatives, or randomization among those alternatives, available to them. Convexity of S is due to the possibility of randomization; boundedness holds if utilities are bounded; closedness is assumed for mathematical convenience. The existence of at least one $x \in S$ with $x>d$ is postulated to avoid the somewhat degenerate case when only some of the agents stand to gain from the agreement. In addition, we will usually assume that

- (S,d) is ***d–comprehensive***: If $x \in S$ and $x \geq y \geq d$, then[1] $y \in S$.

This property of (S,d) follows from the natural assumption that utility is freely disposable (above d). It is sometimes useful to consider problems satisfying the slightly stronger condition that the part of their boundary that dominates d not contain a segment parallel to an axis. Along that part of the boundary of such a ***strictly d–comprehensive*** problem, "utility transfers" from one agent to another are always possible. Let ∂S be the boundary of S.

In most of the existing theory the choice of the 0 of the utility scales is assumed not to matter, and for convenience, we choose scales so that d=0 and ignore d in the notation altogether. However, in some sections, the disagreement point plays a central role; it is then explicitly reintroduced. When d=0, we simply say that a problem is comprehensive instead of d–comprehensive.

Finally, we often require that

- $S \subset \mathbb{R}^n_+$.

This is on the grounds that alternatives at which any agent receives less than what he is guaranteed at d=0 should play no role in the determination of the compromise. (This requirement is given a formal statement later on.)

In summary, we usually consider the class Σ_0^n of problems S as represented in Fig. 1b – c. We occasionally consider ***degenerate*** problems, that is, problems whose feasible set contains no point strictly dominating the disagreement point. Let Γ_d^n and Γ_0^n be the classes

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[1]Vector inequalities: given $x, x' \in \mathbb{R}^n$, $x \geqq x'$ means $x_i \geqq x'_i$ for all i; $x \geq x'$ means $x \geqq x'$ and $x \neq x'$; $x > x'$ means $x_i > x'_i$ for all i.

of degenerate problems associated with Σ_d^n and Σ_0^n.

Sometimes, we assume that utility can be disposed of in any amount: if $x \in S$ then any $y \in \mathbb{R}^n$ with $y \leqq x$ is also in S. We denote by $\Sigma_{d,-}^n$, $\Sigma_{0,-}^n$, $\Gamma_{d,-}^n$, $\Gamma_{0,-}^n$ the classes of such ***fully comprehensive problems*** corresponding to Σ_d^n, Σ_0^n, Γ_d^n, Γ_0^n.

Given $A \subset \mathbb{R}_+^n$, $cch\{A\}$ denotes the "convex and comprehensive hull" of A: it is the smallest convex and comprehensive subset of $\mathbb{R}_+^n$ containing A.

Remark **2.** Other classes of problems have been discussed in the literature. In some studies, no disagreement point is given (Harsanyi 1955, Myerson 1977, Thomson 1981c). In others, an additional reference point is specified; if it is in S, it can be interpreted as a status quo (Brito, Buoncristiani and Intriligator 1977 choose it on the boundary of S), or as a first step towards the final compromise (Gupta and Livne 1988); if it is outside of S, it represents a vector of claims (Chun and Thomson 1988).

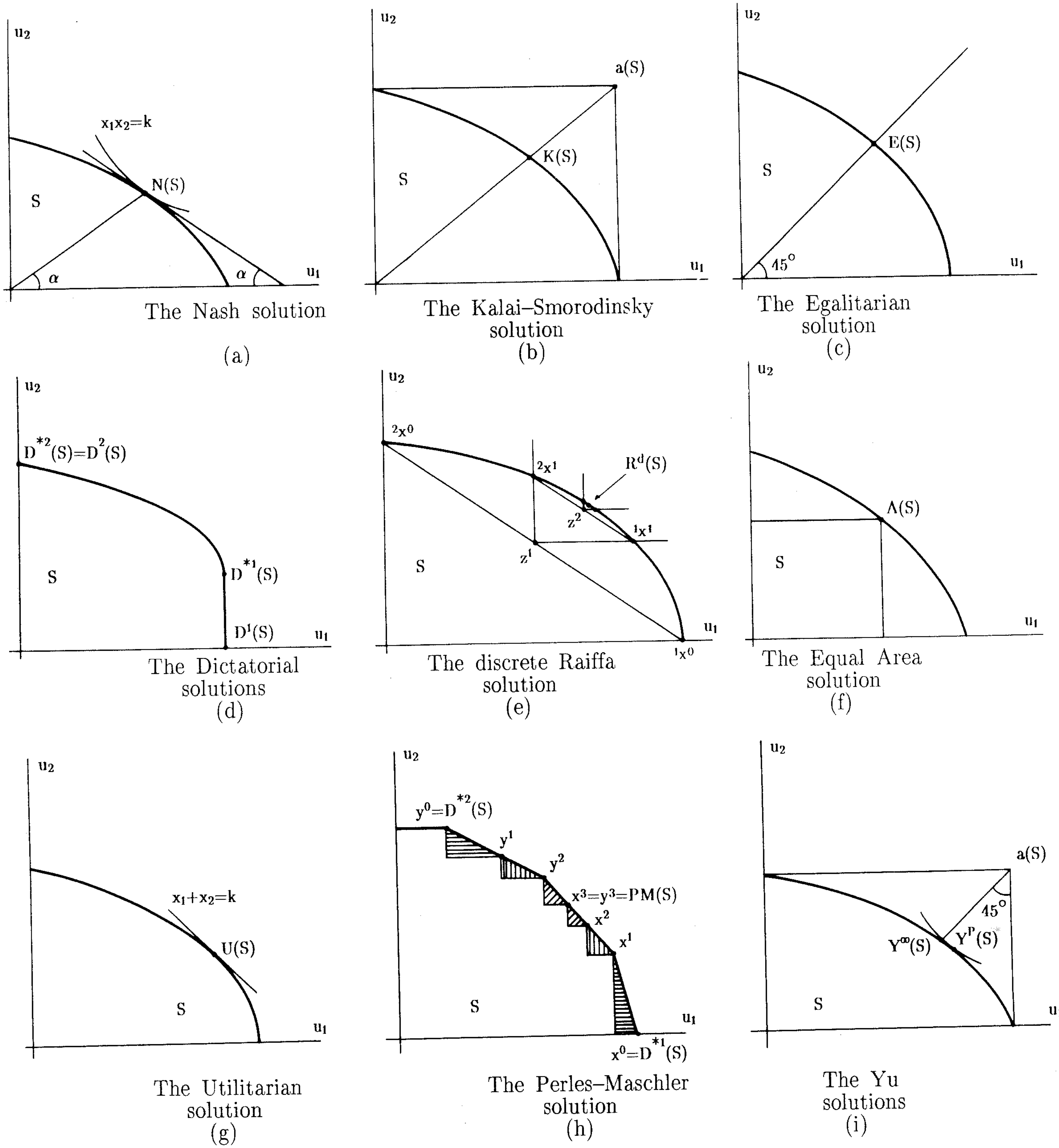
A ***solution*** defined on some domain of problems associates with each element (S,d) of the domain a unique point of S interpreted as a prediction, or a recommendation, for that problem.

Remark **3.** Some authors have considered multivalued solutions (Thomson 1981a; Peters, Tijs and de Koster 1983), and others probabilistic solutions (Peters and Tijs 1984b).

Three solutions play a central role in the theory as it appears today. We introduce them first, but we also present several others so as to show how rich and varied the class of available solutions is. Their definitions, as well as the axioms to follow shortly, are stated for an arbitrary $S \in \Sigma_0^n$, (or $(S,d) \in \Sigma_d^n$).

Nash solution N (Fig. 2a): N(S) is the maximizer of the product Πx_i over S. (N(S,d) is the maximizer of $\Pi(x_i - d_i)$ for $x \in S$ with $x \geqq d$.)

Kalai–Smorodinsky solution K (Fig. 2b): K(S) is the maximal point of S on the segment connecting the origin to ***a(S), the ideal point of S***, defined by $a_i(S) \equiv \max\{x_i | x \in S\}$ for all i.



The Nash solution (a)

The Kalai–Smorodinsky solution (b)

The Egalitarian solution (c)

The Dictatorial solutions (d)

The discrete Raiffa solution (e)

The Equal Area solution (f)

The Utilitarian solution (g)

The Perles–Maschler solution (h)

The Yu solutions (i)

Figure 2

($K(S,d)$ is the maximal point of S on the segment connecting d to $a(S,d)$ where $a_i(S,d) \equiv \max\{x_i \mid x \in S, x \geqq d\}$ for all i.)

Egalitarian solution E (Fig. 2c): E(S) is the maximal point of S of equal coordinates. ($E_i(S,d)-d_i \equiv E_j(S,d)-d_j$ for all i,j.)

Dictatorial solutions D^i and D^{*i} (Fig. 2d): $D^i(S)$ is the maximal point x of S with $x_j=0$ for all $j \neq i$. (Similarly, $D^i_j(S,d)=d_j$ for all $j \neq i$.) If n=2, $D^{*i}(S)$ is the point of PO(S) of maximal i^{th} coordinate. (If S is strictly comprehensive, $D^i(S)=D^{*i}(S)$). If n>2, the maximizer of x_i in PO(S) may not be unique, and some rule has to be formulated to break possible ties (a lexicographic rule is often suggested).

The (discrete) Raiffa solution R^d (Fig. 2e): $R^d(S)$ is the limit point of the sequence $\{z^t\}$ defined by: ${}^ix^0=D^i(S)$ for all i; for all $t \in \mathbb{N}$, $z^t=(\Sigma^i x^{t-1})/n$, and ${}^ix^t \in$ WPO(S) is such that ${}^ix^t_j = z^t_j$ for all $j \neq i$. (On Σ^n_d, start from the $D^i(S,d)$ instead of the $D^i(S)$. A continuous version of the solution is obtained by having z(t) move at time t in the direction of $(\Sigma^i x(t))/n$ where ${}^ix(t) \in$ WPO(S) is such that ${}^ix_j(t)=z_j(t)$ for all $j \neq i$.)

Equal Area solution A (Fig. 2f): For n=2. A(S) is the point $x \in$ PO(S) such that the area of S to the right of the vertical line through x is equal to the area of S above the horizontal line through x. (There are several possible generalizations for $n \geqq 3$. On Σ^n_d, ignore points that do not dominate d.)

Utilitarian solution U (Fig. 2g). U(S) (or U(S,d)) is a maximizer in $x \in S$ of Σx_i.

This solution, which has played a major role in other contexts, presents some difficulties here. First, the maximizer may not be unique. To circumvent this difficulty a tie–breaking rule has to be specified; for n=2 it is perhaps most natural to select the midpoint of the segment of maximizers (if n>2, this rule can be generalized in several different ways). A second difficulty, is that as defined here for Σ^n_d, the solution does not depend on d. A partial remedy here is to search for a maximizer of Σx_i among the points of S that dominate d. In spite of these limitations, the utilitarian solution is often used. At the very least, it has the merit of being a useful limit case.

Perles–Maschler solution PM (Fig. 2h): For n=2. If ∂S is polygonal, PM(S) is the common limit point of the sequences $\{x^t\}$, $\{y^t\}$, defined by: $x^0=D^{*1}(S)$, $y^0=D^{*2}(S)$; for each $t \in \mathbb{N}$, x^t, $y^t \in PO(S)$ are such that $x_1^t \le y_1^t$, the segments $[x^{t-1},x^t]$,$[y^{t-1},y^t]$ are contained in PO(S) and the products $-(x_1^{t-1}-x_1^t)(x_2^{t-1}-x_2^t)$ and $-(y_1^{t-1}-y_1^t)(y_2^{t-1}-y_2^t)$ are equal and maximal. (Equality of the products implies that the triangles of Fig. 2h are matched in pairs of equal areas.) If ∂S is not polygonal, PM(S) is defined by approximating S by a sequence of polygonal problems.

The solution can be given the following equivalent definition when ∂S is smooth. Consider two points moving along ∂S from $D^{*1}(S)$ and $D^{*2}(S)$ so that the product of the components of their velocity vectors in the u_1 and u_2 directions remain constant: The two points will meet at PM(S). The differential system describing this movement can be generalized to arbitrary n; it generates n paths on the boundary of ∂S that meet in one point that can be taken as the desired compromise. (On Σ_d^n, start from the $D^i(S,d)$ instead of the $D^i(S)$.)

Yu solutions Y^p (Fig. 2i): Given $p \in]1,\infty[$, $Y^p(S)$ is the point of S for which the p–distance to the ideal point of S, $(\Sigma|a_i(S)-x_i|^p)^{1/p}$, is minimal. Also, $Y^\infty(S)$ is the maximal point of S such that $a_i(S)-x_i=a_j(S)-x_j$ for all i,j. (On Σ_d^n, use a(S,d) instead of a(S).)

***Remark* 4.** Versions of the Kalai–Smorodinsky solution appear in Raiffa (1953), Crott (1971), Butrim (1976), but the first axiomatization is in Kalai and Smorodinsky (1975). A number of variants have been discussed, in particular by Kalai and Rosenthal (1978) and Salonen (1985, 1987). The Egalitarian solution cannot be traced to a particular source but egalitarian notions are certainly very old. The Equal Area solution is analyzed in Dekel (1982), Ritz (1985), and Anbarci and Bigelow (1988); the Yu solutions in Yu (1973) and Freimer and Yu (1976); the Raiffa solution in Raiffa (1953) and Luce and Raiffa (1957). The utilitarian solution dates back to the mid 19th century. The 2–person Perles–Maschler solution appears in Perles–Maschler (1981) and its n–person extension in Kohlberg, Maschler and Perles (1983).

3. *THE MAIN CHARACTERIZATIONS.*

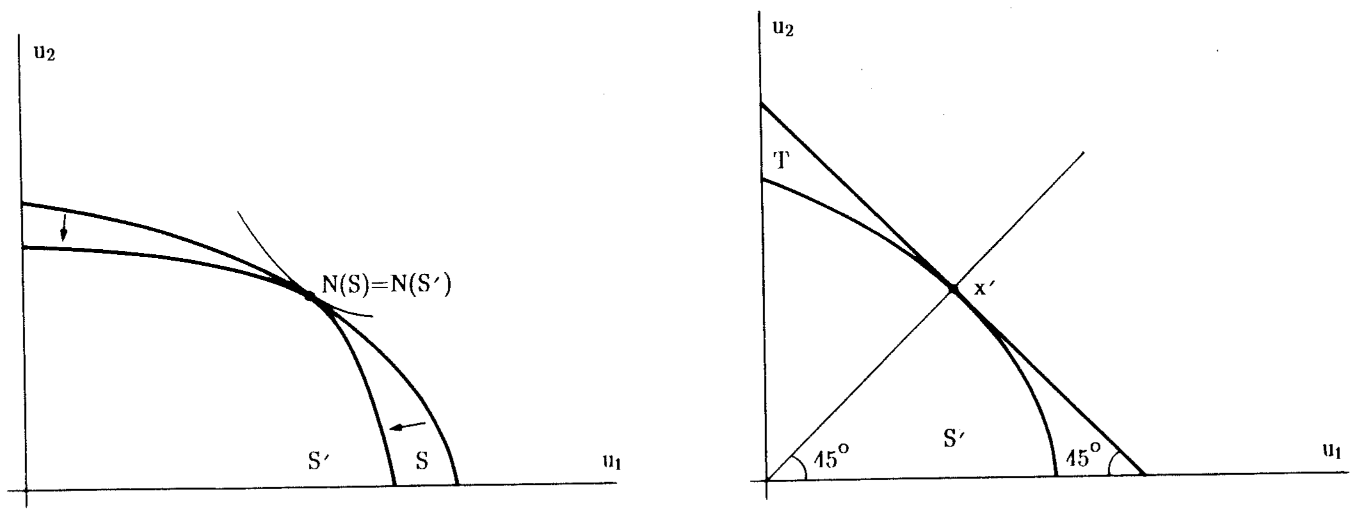
Here we present the classic characterizations of the three solutions that occupy center stage in the theory as it stands today.

3.1 *The Nash solution.*

We start with Nash's fundamental contribution. Nash considered the following axioms:

Pareto–optimality (p.o): $F(S) \in PO(S) \equiv \{x \in S \mid \exists x' \in S \text{ with } x' \geq x\}$.

All gains from cooperation should be exhausted.



N satisfies ***contraction independence***

(a)

Characterization of N

(b)

The Nash solution.

Figure 3

symmetry (sy): If S is invariant under all exchanges of agents, $F_i(S) = F_j(S)$ for all i,j.

If the agents cannot be differentiated on the basis of the information contained in the mathematical description of S, then the solution should treat them the same.

Let $\Lambda_0^n:\mathbb{R}^n \rightarrow \mathbb{R}^n$ be the class of independent person by person, positive linear transformations ("scale transformations"): $\lambda \in \Lambda_0^n$ if there is $a \in \mathbb{R}^n_{++}$ such that for all $x \in$

$\mathbb{R}^n$, $\lambda(x)=(a_1x_1,\ldots,a_nx_n)$. Given $\lambda \in \Lambda_0^n$ and $S \subset \mathbb{R}^n$, $\lambda(S)\equiv\{x' \in \mathbb{R}^n \mid \exists x \in S \text{ with } x'=\lambda(x)\}$.
***scale invariance* (*s.inv*):** $\lambda(F(S))=F(\lambda(S))$.

The solution should be independent of which particular members in the families of utility functions representing the agents' preferences are chosen to describe the problem.

***contraction independence* (*c.i*):** If $S'\subset S$ and $F(S) \in S'$, then $F(S')=F(S)$.

If an alternative is thought to be the best compromise for some problem, then it should still be thought best for any subproblem that contains it.

In the proof of the first result we use the fact that for n=2 if x=N(S), then S has at x a line of support whose slope is the negative of the slope of the line connecting x to the origin (Fig. 2a).

Theorem 1 (Nash 1950): The Nash solution is the only solution on Σ_0^n satisfying ***p.o***, ***sy***, ***s.inv***, and ***c.i***.

Proof (for n=2): It is easy to verify that N satisfies the four axioms (that N satisfies ***c.i*** is illustrated in Fig. 3a). Conversely, let F be a solution on Σ_0^2 satisfying the four axioms. To show that F=N, let $S \in \Sigma_0^2$ be given and let $x\equiv N(S)$. Let $\lambda \in \Lambda_0^2$ be such that $x'\equiv\lambda(x)$ be on the 45° line. Such a λ exists since $x>0$, as is easily checked. The problem $S'\equiv\lambda(S)$ is supported at x' by a line of slope -1 (Fig. 3b). Let $T\equiv\{y \in \mathbb{R}_+^2 \mid \Sigma y_i \leq \Sigma x_i'\}$. T is a symmetric problem and $x' \in PO(T)$. By ***p.o*** and ***sy***, $F(T)=x'$. Clearly, $S'\subset T$ and $x' \in S'$, so that by ***c.i*** $F(S')=x'$. The desired conclusion follows by ***s.inv***.

Q.E.D.

Each of the axioms used by Nash has been the object of some criticism. To the extent that the theory is intended to predict how real–world conflicts are resolved, ***p.o*** is certainly not desirable, since such conflicts often result in dominated compromises. Likewise, we might want to take into account differences between agents pertaining to aspects of the environment that are not explicitly modelled and differentiate among them even though they enter symmetrically in the problem at hand; then, we violate ***sy***. Axiom ***s.inv*** prevents basing compromises on interpersonal comparisons of utility, but such comparisons are made in a wide variety of situations. Finally, if the contraction described in the hypotheses of ***c.i*** is skewed against a particular agent, why should the compromise be prevented from moving against him? In fact, is seems that solutions should be allowed to be responsive to the geometry of S, at least to its main features. (It is precisely considerations of this kind that underlie the characterizations of the Kalai–Smorodinsky and Egalitarian solutions reviewed later.)

Remark 5. Nash's theorem has been considerably refined by subsequent writers. Without ***p.o***, only one other solution becomes admissible: it is the trivial ***disagreement solution***, which associates with every problem its disagreement point, here the origin (Roth 1977a, 1980). Dropping ***sy***, we obtain the following family: given $\alpha \in \Delta^{n-1}$, the ***weighted Nash solution with weights α*** is defined by maximizing over S the product $\Pi x_i^{\alpha_i}$ (Harsanyi and Selten 1972); the Dictatorial solutions and some generalizations also become admissible (Peters 1983b). Without ***s.inv***, many other solutions, such as the Egalitarian solution, are permitted. The same is true if ***c.i*** is dropped; however, let us assume that a function is available that summarizes the main features of each problem into a ***reference point*** to which agents find it natural to compare the proposed compromise in order to evaluate it. By replacing in ***c.i*** the hypothesis of identical disagreement points (implicit in our choice of domains) by the hypothesis of identical reference points, variants of the Nash solution, defined by maximizing the product of utility gains from that reference point, can be obtained under weak assumptions on the reference function. (Roth 1977b, Thomson 1981a.) Axiom ***c.i*** bears a close relation to the axioms of revealed preference of demand theory (Lensberg 1987, Peters and Wakker 1987).

We close this section with the statement of a few interesting properties satisfied by the Nash solution but by many others as well.

The first is a consequence of our choice of domains: N(S) always weakly dominates the disagreement point, here the origin. On Σ_d^n, the property would of course not necessarily be satisfied, so we write it for that domain.

***individual rationality* (*i.r*)**: $F(S,d) \in \boldsymbol{I(S,d)} \equiv \{x \in S \mid x \geqq d\}$.

In fact, the Nash solution (and again many others) satisfies the following stronger condition: all agents should strictly gain from the compromise.

***strong individual rationality* (*st.i.r*)**: $F(S,d) > d$.

The requirement that the compromise depend only on I(S,d) is implicitly made in much of the literature:

***independence of non–individually rational alternatives* (*i.n.i.r*)**: $F(S,d)=F(I(S,d),d)$.

Most solutions satisfy this requirement. A solution that does not, although it satisfies *st.i.r*, is the Kalai–Rosenthal solution (Remark 7).

Another property of interest is that small changes in problems do not lead to wildly different solution outcomes.

***continuity* (*cont*)**: If $S^\nu \to S$ in the Hausdorff topology, and $d^\nu \to d$, then $F(S^\nu,d^\nu) \to F(S,d)$.

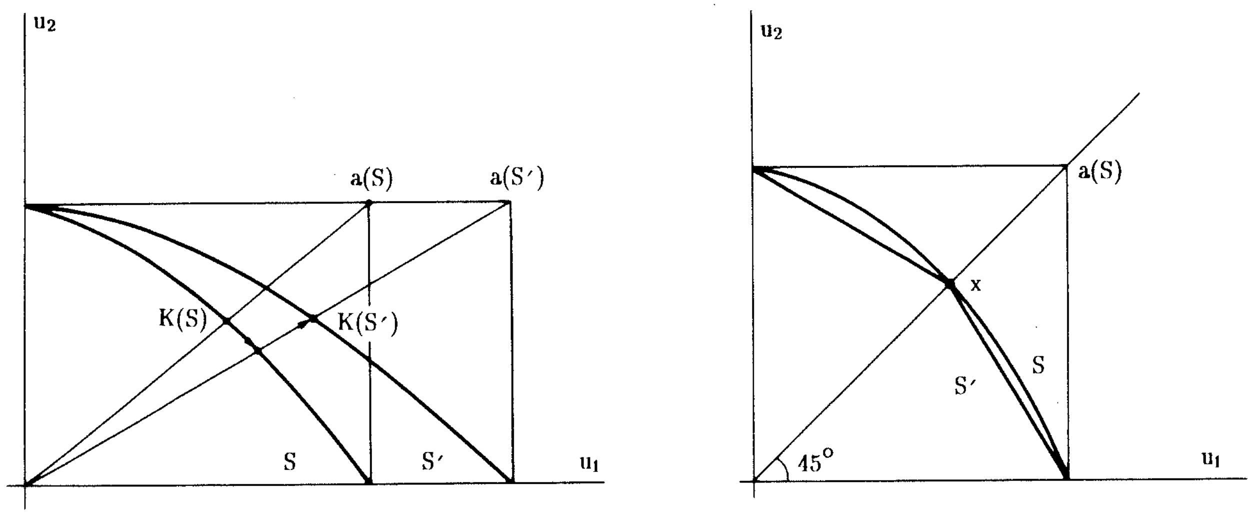
All of the solutions of Section 2 satisfy ***cont***, except for the dictatorial solutions D^{*i} and the Utilitarian and Perles–Maschler solutions.

> ***Remark* 6**: Other continuity properties of solutions are formulated and studied by Jansen and Tijs (1983). A property related to ***cont***, which takes into account closeness of Pareto–optimal boundaries, is used by Peters (1986a) and Livne (1987a).

3.2 ***The Kalai–Smorodinsky solution.*** We now turn to the second one of our three central solutions, the Kalai–Smorodinsky solution. Just like the Egalitarian solution, examined last, the appeal of this solution lies mainly in its monotonicity properties. Recall that $a_i(S) \equiv \max\{x_i \mid x \in S\}$.

***individual monotonicity* (*i.mon*)**: For n=2. If $S' \supset S$, and $a_j(S')=a_j(S)$ for $j \neq i$, then $F_i(S') \geqq F_i(S)$.

If the range of utility levels attainable by agent j remains the same as S expands to S′, while for each such level, the maximal utility level attainable by agent i increases, then agent i should not lose.



K satisfies ***individual monotonicity***
(a)

Characterization of K.
(b)

The Kalai–Smorodinsky solution.

Figure 4

Theorem 2 (Kalai–Smorodinsky 1975): The Kalai–Smorodinsky solution is the only solution on Σ_0^2 satisfying ***p.o***, ***sy***, ***s.inv***, and ***i.mon***.

Proof. It is clear that K satisfies the four axioms (that K satisfies ***i.mon*** is illustrated in Fig. 4a). Conversely, let F be a solution on Σ_0^2 satisfying the four axioms. To see that F=K, let $S \in \Sigma_0^2$ be given. By ***s.inv***, we can assume that a(S) has equal coordinates (Fig. 4b). This implies that x≡K(S) itself has equal coordinates. Then let $S' \equiv \text{cch}\{(a_1(S),0),x,(0,a_2(S))\}$. S′ is a symmetric problem and $x \in PO(S')$ so that by ***p.o*** and ***sy***, F(S′)=x. By ***i.mon*** applied twice, we conclude that $F(S) \geqq x$, and since $x \in PO(S)$, that F(S)=x=K(S).

Q.E.D.

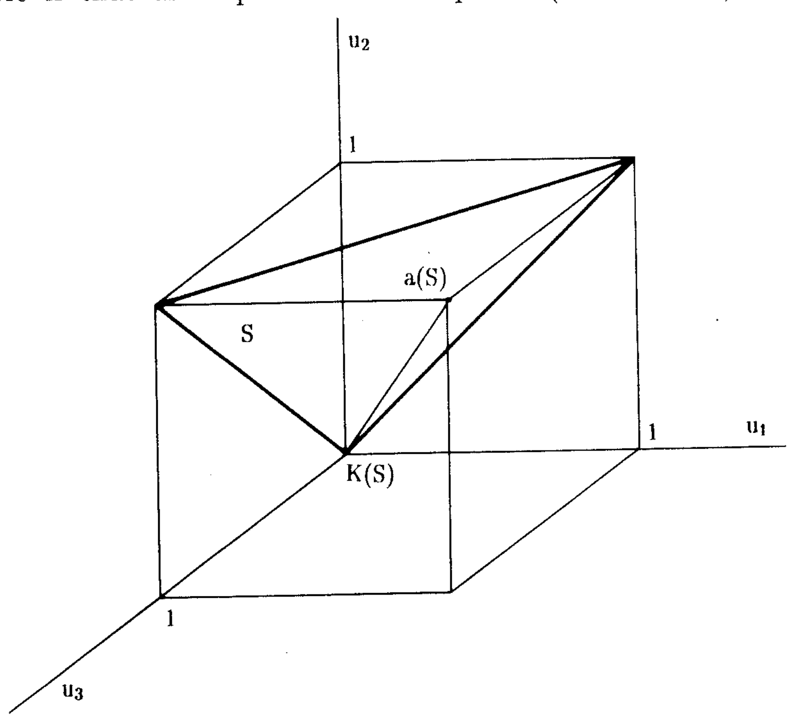
Before presenting variants of this theorem, we first note several difficulties concerning its possible generalization to classes of not necessarily comprehensive n–person problems for n > 2. On such domains the Kalai–Smorodinsky solution often fails to yield Pareto–optimal points, as is shown by the example S = convex hull{(0,0,0),(1,1,0),(0,1,1)} of Fig. 5: there K(S)(=(0,0,0)) is in fact dominated by all points of S (Roth, 1979d). However, by requiring comprehensiveness of the admissible problems, the solution satisfies the following natural weakening of ***p.o***:

weak Pareto–optimality (***w.p.o***): F(S) ϵ ***WPO(S)***$\equiv\{x \ \epsilon\ S \mid \nexists x' \ \epsilon\ S,\ x' > F(S)\}$.

The other difficulty in generalizing Theorem 2 to n>2 is that there are several ways of generalizing ***i.mon*** to that case, not all of which permit the result to go through. One possibility is to write "for all j≠i" in its earlier statement. The following will also do (Roth 1979d, Thomson 1980):

restricted monotonicity (***r.mon***): If $S' \supset S$ and $a(S')=a(S)$, then $F(S') \geqq F(S)$.

To emphasize the importance of comprehensiveness, we note that ***w.p.o***, ***sy***, and ***r.mon*** are incompatible if that assumption is not imposed (Roth 1979d).



A difficulty with the Kalai–Smorodinsky solution for n≧2.

If S is not comprehensive, K(S) may be strictly dominated by all points of S.

Figure 5

Remark* 7:** A lexicographic (see Section 3.3) extension of K that satisfies ***p.o has been characterized by Imai (1983). Deleting ***p.o*** from Theorem 2, a large family of solutions becomes admissible. Without ***sy***, the following generalizations of K are permitted: Given $\alpha \in \Delta^{n-1}$, the ***weighted Kalai–Smorodinsky solution with weights α***, K^{α}: $K^{\alpha}(S)$ is the maximal point of S in the direction of the α–weighted ideal point $a^{\alpha}(S) \equiv (\alpha_1 a_1(S),\ldots,\alpha_n a_n(S))$. These solutions satisfy ***w.p.o*** but not ***p.o***. There are other solutions satisfying only ***w.p.o***, ***s.inv***, and ***i.mon***; they are normalized versions of the "monotone path solutions", discussed later on in connection with the Egalitarian solution (Peters and Tijs 1984a; 1985b). Salonen (1985, 1987) characterizes two variants of the Kalai–Smorodinsky solution. These results, as well as the characterization by Kalai and Rosenthal (1987) of their variant of the solution, and the characterization by Chun (1988a) of a variant of the Yu solution for $p=\infty$, are also close in spirit to Theorem 2.

3.3 ***The Egalitarian solution.*** The Egalitarian solution performs the best from the viewpoint of monotonicity and the characterization that we will offer is based on this fact. However, the price paid for these strong monotonicity properties is that this solution involves interpersonal comparisons of utility (it violates ***s.inv***). Note that it satisfies ***w.p.o*** only, although $E(S) \in PO(S)$ for all strictly comprehensive S.

***strong monotonicity* (*st.mon*)**: If $S' \supset S$, then $F(S') \geqq F(S)$.

All agents should benefit from expanding opportunities; this is irrespective of whether the expansion may be biased in favor of one of them, (for instance, as described in the hypotheses of ***i.mon***). Of course, if that is the case, nothing prevents the solution outcome from "moving more" in favor of that agent.

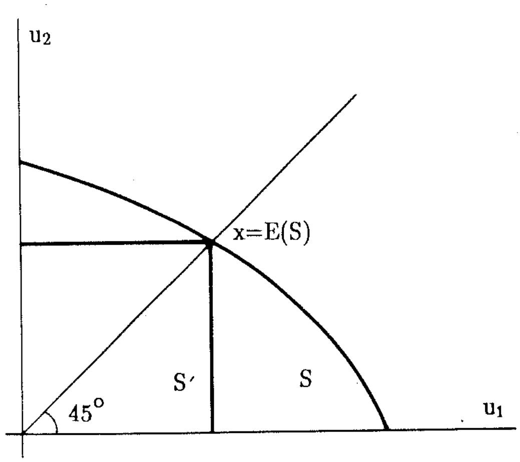
Theorem* 3** (Kalai 1977): The Egalitarian solution is the only solution on Σ_0^n satisfying ***w.p.o, ***sy***, and ***st.mon***.

Proof (for n=2): Clearly, E satisfies the three axioms. Conversely, to see that if a solution F on Σ_0^2 satisfies the three axioms, then F=E, let $S \in \Sigma_0^2$ be given, $x \equiv E(S)$, and $S' \equiv \text{cch}\{x\}$ (Fig. 6a). By ***w.p.o*** and ***sy***, $F(S')=x$. Since $S \supset S'$, ***st.mon*** implies $F(S) \geqq x$. Note that $x \in WPO(S)$. If, in fact, $x \in PO(S)$, we are done. Otherwise, we conclude by a continuity argument involving a sequence of strictly comprehensive problems approaching S from above.

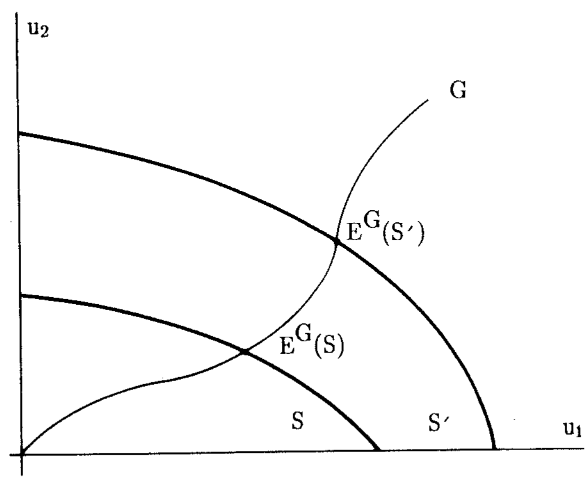
Q.E.D.

It is clear that comprehensiveness of S is needed to obtain weak Pareto–optimality of E(S) even if n=2. Without comprehensiveness, ***w.p.o*** and ***st.mon*** are incompatible (Luce and Raiffa 1957).

Remark 8: Deleting ***w.p.o*** from Theorem 3, we obtain solutions defined as follows: Given $k \in [0,1]$, $E^k(S) = kE(S)$. However, there are other solutions satisfying only ***sy*** and ***st.mon*** (Roth 1979a, 1979b). Without ***sy*** the following solutions become admissible: Given $\alpha \in \Delta^{n-1}$, the ***weighted Egalitarian solution with weights*** α, E^α: $E^\alpha(S)$ is the



Characterization of the Egalitarian solution
(a)



Monotone Path solutions.
(b)

Egalitarian and Monotone Path solutions

Figure 6

maximal point of S in the direction α (Kalai (1977). The axioms ***w.p.o*** and ***st.mon*** essentially characterize the following more general class: Given a strictly monotone path G in $\mathbb{R}^n_+$, the ***monotone path solution relative to G***, E^G: $E^G(S)$ is the maximal point of S along G (Fig. 6b, Thomson and Myerson 1980).

It is clear that ***st.mon*** is crucial in Theorem 3 and that without it, a very large class of solutions would become admissible. However, this axiom can be replaced by other interesting conditions that still produce characterizations of the Egalitarian solution (Kalai 1977b).

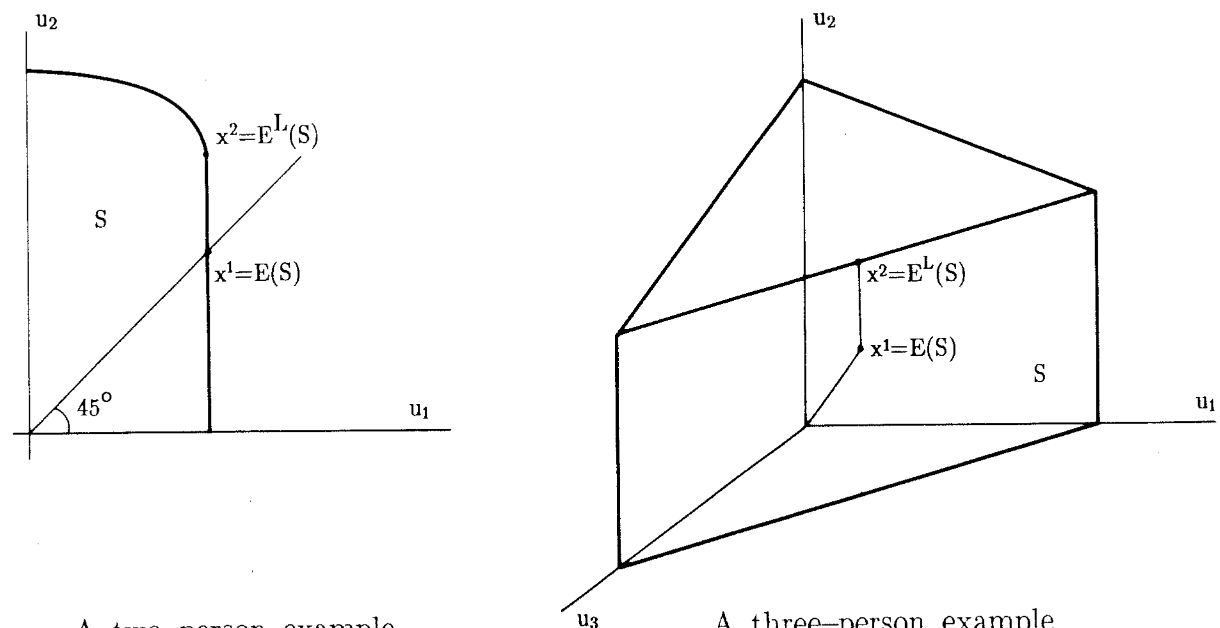
decomposability (***dec***): If $S' \supset S$ and $S'' \equiv \{x'' \in \mathbb{R}^n_+ \mid \exists x' \in S'$ such that $x' = x'' + F(S)\} \in \Sigma^n_0$, then $F(S') = F(S) + F(S'')$.

Imagine that opportunities expand over time from S to S′. The axiom says that F(S′) can be indifferently computed in one step, ignoring the initial problem S altogether, or in two steps, by first solving S and then taking F(S) as starting point for the distribution of the additional opportunities.

Remark **9:** The weakening of ***dec*** obtained by restricting its application to cases where F(S) is proportional to F(S″) can be used together with ***p.o***, ***sy***, ***i.n.i.r***, ***s.inv***, and ***cont*** to characterize the Nash solution (Chun 1988b).

As already noted, the Egalitarian solution does not satisfy ***p.o***, but there is a natural extension of the solution that does (a similar operation can be used to define a version of K that satisfies ***p.o.*** on Σ^n_d for all n). Given $z \in \mathbb{R}^n$, $\tilde{z} \in \mathbb{R}^n$ is obtained from z by writing its coordinates in increasing order. Given $x, y \in \mathbb{R}^n$, ***x is lexicographically greater than y*** if $\tilde{x}_1 > \tilde{y}_1$ or $[\tilde{x}_1 = \tilde{y}_1$ and $\tilde{x}_2 > \tilde{y}_2]$, or, more generally, for some k $[\tilde{x}_1 = \tilde{y}_1, \ldots, \tilde{x}_k = \tilde{y}_k$, and $\tilde{x}_{k+1} > \tilde{y}_{k+1}]$. Now, given $S \in \Sigma^n_0$, its ***lexicographic solution*** outcome $E^L(S)$ is the point of S that is lexicographically maximal. It can be reached by the following simple operation (Fig. 7): let x^1 be the maximal point of S with equal coordinates; if $x^1 \in PO(S)$, then $x^1 = E^L(S)$; if not, identify the greatest subset of the agents whose utilities can be simultaneously increased from x^1 without hurting the remaining agents. Let x^2 be the maximal point of S at which these agents experience equal gains. Repeat this operation from x^2 to obtain x^3, etc.,..., until a point of PO(S) is obtained.

This algorithm produces a well–defined solution satisfying ***p.o*** even on the class of problems that are not necessarily comprehensive. Given S in that class, apply it to its comprehensive hull and note that taking the comprehensive hull of a problem does not affect its set of Pareto–optimal points.



A two–person example
(a)

A three–person example
(b)

The Lexicographic Egalitarian solution.

Figure 7

***Remark* 10:** For characterizations of E^L based on monotonicity considerations, see Imai (1983) and Chun and Peters (1987a). Lexicographic extensions of the Monotone Path solutions are defined, and characterized by similar techniques for n=2, by Chun and Peters (1987b).

4. *OTHER PROPERTIES. THE ROLE OF THE FEASIBLE SET.*

Here, we change our focus, concentrating on properties of solutions.

4.1. ***Symmetry and related properties.*** The symmetry axiom used so far applies to problems that are "fully symmetric", that is, are invariant under all permutations of agents. But some problems may only exhibit partial symmetries that one may want solutions to respect.

anonymity (an): Let $\pi:\{1,\ldots,n\} \to \{1,\ldots,n\}$ be a bijection. Given $x \in \mathbb{R}^n$, let $\pi(x)\equiv(x_{\pi(1)},\ldots,x_{\pi(n)})$. Then, $F(\pi(S)) = \pi(F(S))$.

Consider also the following (Thomson 1988).

responsiveness to asymmetries (res.a): If for all $x \in S$ such that $x_i \geqq x_j$, there exists $y \in S$ with $y_i=x_j$, $y_j=x_i$ and $y_k=x_k$ for all $k \neq i$, j, then $F_j(S) \geqq F_i(S)$.

Most solutions satisfy these properties (the Dictatorial solutions obviously do not satisfy ***an***; the Perles–Maschler solution does not satisfy ***res.a***).

4.2. ***Mid–point domination.*** A minimal amount of cooperation among the agents should allow them to do at least as well as the average of their preferred positions. Accordingly, consider the following two requirements (Sobel 1981, Salonen 1985, respectively), which correspond to two natural definitions of "preferred positions".

mid–point domination (m.p.d): $F(S) \geqq [\Sigma D^i(S)]/n$.

strong mid–point domination (st.m.p.d): $F(S) \geqq [\Sigma D^{*i}(S)]/n$.

Many solutions satisfy ***m.p.d*** (notable exceptions are the Egalitarian and Utilitarian solutions), yet we have (compare with Theorem 1):

Theorem 4 (Moulin 1983). The Nash solution is the only solution on Σ_0^n satisfying ***m.p.d*** and ***c.i.***

Few solutions satisfy ***st.m.p.d*** (the Perles–Maschler solution is an example; Salonen 1985 defines a version of the Kalai–Smorodinsky solution that does too).

4.3 ***Invariance properties.*** The theory presented so far is a cardinal theory, in that it depends on utility functions, although the extent of this dependence varies, as we have seen. Are there solutions that are invariant under all monotone increasing, and independent agent by agent, transformations of utilities? Let $\tilde{\Lambda}_0^n$ be the class of these transformations: $\lambda \in \tilde{\Lambda}_0^n$

if there is for each i, a continuous and monotone increasing function $\lambda_i:\mathbb{R} \rightarrow \mathbb{R}$ such that given $x \in \mathbb{R}^n$, $\lambda(x)=(\lambda_1(x_1),\ldots,\lambda_n(x_n))$.

Ordinal invariance (ord.inv): For all $\lambda \in \tilde{\Lambda}_0^n$, $F(\lambda(S)) = \lambda(F(S))$.

Since convexity of S is not preserved under transformations in $\tilde{\Lambda}_0^n$, we consider the domain $\tilde{\Sigma}_0^n$ obtained from Σ_0^n by dropping this requirement.

Theorem* 5** (Shapley 1969; Roth 1979). There is no solution on $\tilde{\Sigma}_0^2$ satisfying ***st.i.r and ***ord.inv.***

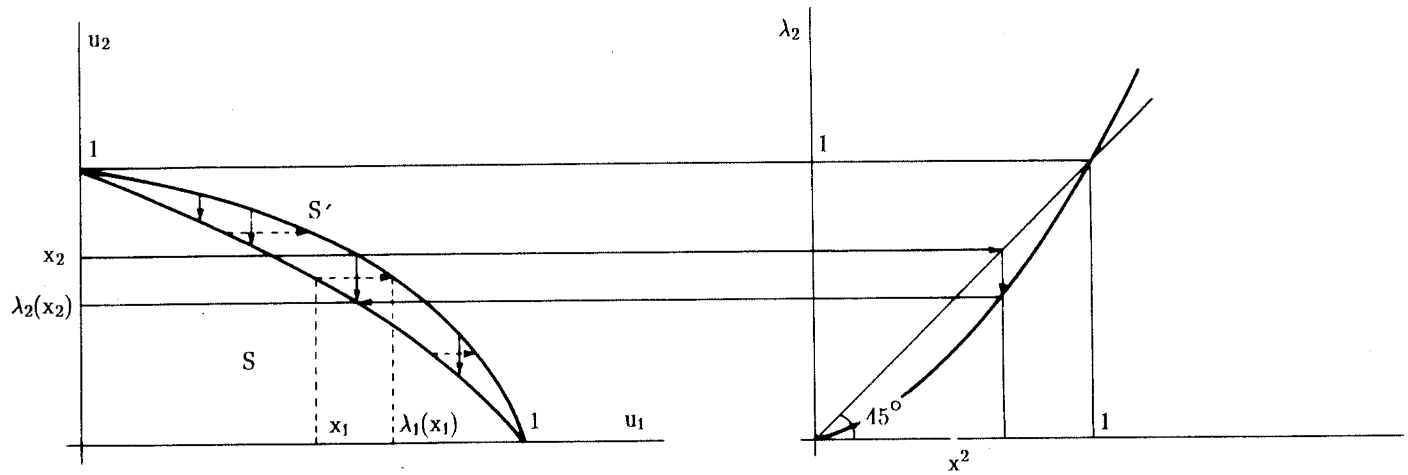
Proof. Let F be a solution on $\tilde{\Sigma}_0^2$ satisfying ***ord.inv*** and let S and S′ be as in Fig. 8a. Let λ_1 and λ_2 be the two transformations from [0,1] to [0,1] defined by following the horizontal and vertical arrows of Fig. 8 respectively. (The graph of λ_2 is given in Fig. 8b; somewhat more explicitly, $\lambda_2(x_2)$, the image of x_2 under λ_2, is obtained by following the arrows from Fig. 8a to Fig. 8b). S is globally invariant under the transformation $\lambda \equiv (\lambda_1,\lambda_2)$, with only three fixed points, the origin and the endpoints of PO(S). Since none of these points is positive, F does not satisfy ***st.i.r.***

Q.E.D.

Theorem* 6** (Shapley 1984; Shubik 1982). There are solutions on the subclass of $\tilde{\Sigma}_0^3$ of strictly comprehensive problems satisfying ***p.o and ***ord.inv.***

Proof. Given $S \in \tilde{\Sigma}_0^3$, let F(S) be the limit point of the sequence $\{x^t\}$ where: x^1 is the point of intersection of PO(S) with $\mathbb{R}^{\{1,2\}}$ such that the arrows of Fig. 9a lead back to x^1; x^2 is the point of PO(S) such that $x_2^2=x_2^1$ and a similarly defined sequence of arrows leads back to x^2; this operation being repeated forever (Fig. 9b). The solution F satisfies ***ord.inv*** since at each step, only operations that are invariant under ordinal transformations are performed.

Q.E.D.



S is globally invariant under the composition of the two transformations defined by the horizontal and the vertical arrows respectively.
(a)

An explicit construction of the transformation to which agent 2's utility is subjected.
(b)

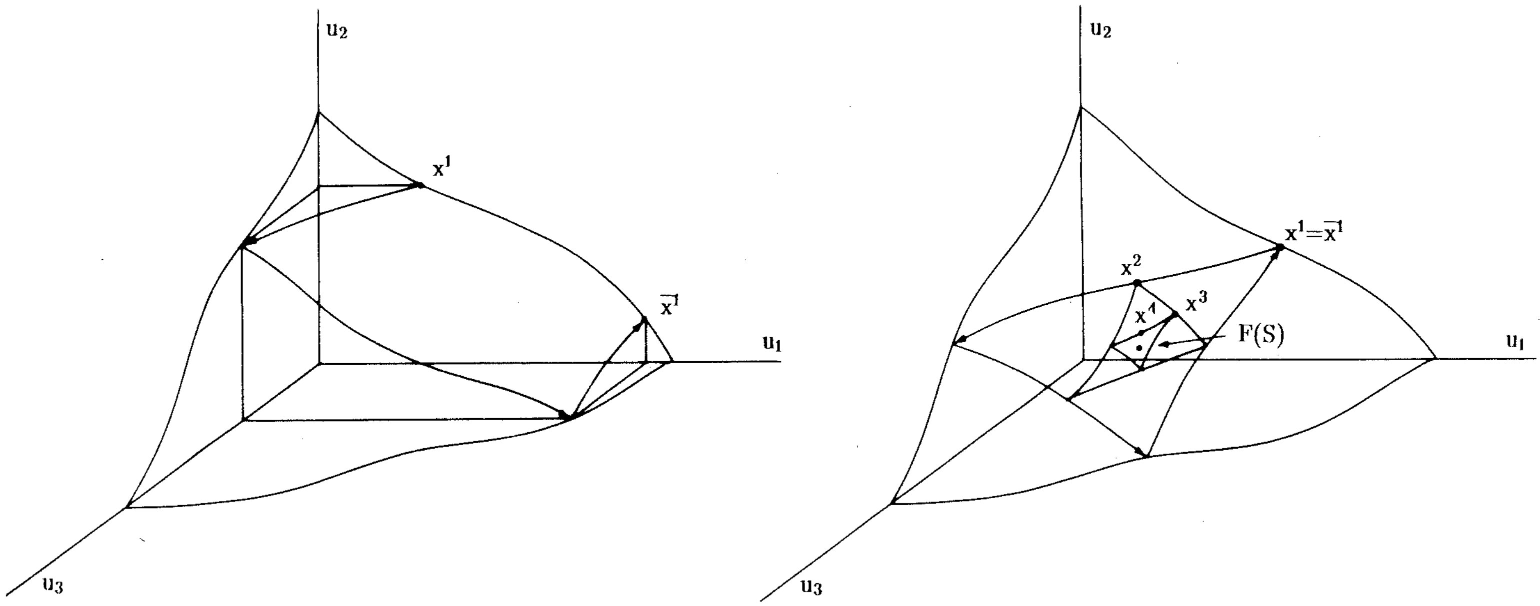
Strong individual rationality and Ordinal invariance are incompatible on $\tilde{\Sigma}_0^2$.

Figure 8

***Remark* 11:** There are other solutions satisfying these properties and yet other such solutions on the class of smooth problems (Shapley 1984).

Instead of allowing the utility transformations to be independent across agents, require now that they be the same for all agents:

***weak ordinal invariance* (*w.ord. inv*):** For all $\lambda \in \tilde{\Lambda}_0^n$ such that $\lambda_i = \lambda_j$ for all i, j, $F(\lambda(S)) = \lambda(F(S))$.



The fixed point argument defining x^1.

(a)

The solution outcome of S is the limit point of the sequence $\{x^t\}$.

(b)

A solution on $\tilde{\Sigma}_0^3$ satisfying ***Pareto–optimality***, ***strong individual rationality***, and ***ordinal invariance***.

Figure 9

This is a significantly weaker requirement than ***ord.inv.*** Indeed, we have:

Theorem 7 (Roth 1979c; Nielsen 1983). The Lexicographic Egalitarian solution is the only solution on the subclass of $\tilde{\Sigma}_0^2$ of problems whose Pareto–optimal boundary is a connected set to satisfy ***w.p.o***, ***sy***, ***c.i***, and ***w.ord.inv.***

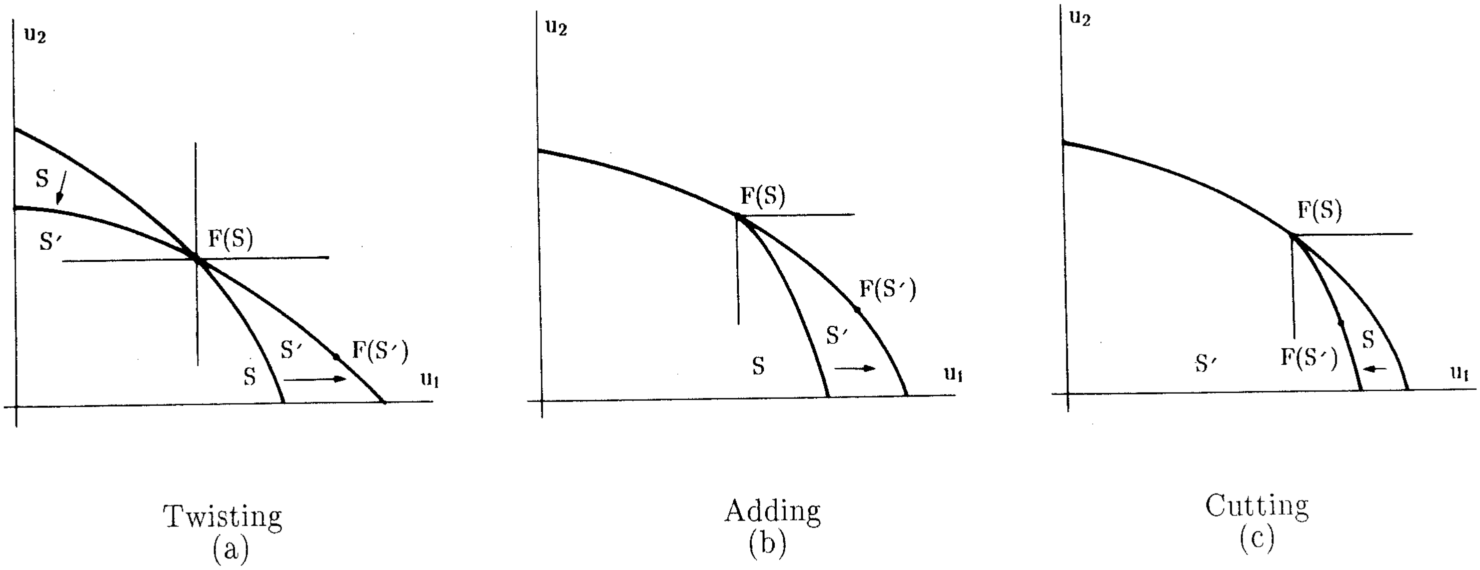
4.4. ***Independence and monotonicity properties.*** Here we formulate a variety of conditions describing how solutions should respond to changes in the geometry of S.

One of the most important conditions we have seen is ***c.i.*** A significantly weaker condition which applies only when the solution outcome of the initial problem is the only Pareto–optimal point of the final problem is:

weak contraction independence (***w.c.i***): If $S'=\mathrm{cch}\{F(S)\}$, then $F(S)=F(S')$.

Dual conditions to ***c.i*** and ***w.c.i***, requiring invariance of the solution outcome under expansions of S, provided it remains feasible, have also been considered. Useful variants of these conditions are obtained by restricting their application to smooth S The Nash and utilitarian solutions can be characterized with the help of these conditions (Thomson 1981b,c). The smoothness restriction implies that utility transfers are possible at the same rate in both directions along the boundary of S. If S is not smooth at F(S), an agent who had been willing to concede along ∂S up to F(S) might have been willing to concede further if the same rate at which utility could be transferred from him to the agent had been available. It is then natural to think of such compromises as somewhat artificial. A number of other conditions that explicitly exclude kinks or corners have been formulated (Chun and Peters 1987a, 1987b; Peters 1986; Chun and Thomson 1987).

A difficulty with the two monotonicity properties used earlier, ***i.mon*** and ***st.mon***, as well as with the independence conditions, is that they preclude the solution from being sensitive to certain changes in S that intuitively seem quite relevant. What would be desirable are conditions pertaining to changes in S that are defined ***relative to the compromise initially established.*** Consider the next conditions (Thomson and Myerson 1980), written for n=2, which involve twisting the boundary of a problem around its solution outcome (***tw***), only adding (***add***), or only subtracting (***cut***), alternatives on one side of the solution outcome (Fig. 10).



Three monotonicity conditions.

Figure 10

Twisting (***tw***): If $x \in S' \backslash S$ implies $[x_i \geqq F_i(S)$ and $x_j \leqq F_j(S)]$ and $x \in S \backslash S'$ implies $[x_i \leqq F_i(S)$ and $x_j \geqq F_j(S)]$, then $F_i(S') \geqq F_i(S)$.

Adding (***add***): If $S' \supset S$, and $x \in S' \backslash S$ implies $x_i \geqq F_i(S)$, then $F_i(S') \geqq F_i(S)$.

Cutting (***cut***): If $S' \subset S$, and $x \in S \backslash S'$ implies $x_i \geqq F_i(S)$, then $F_i(S') \leqq F_i(S)$.

The main solutions satisfy ***tw***, which is easily seen to imply ***add*** and ***cut***. However, the Perles–Maschler solution does not even satisfy ***add***. Axiom ***tw*** is crucial to understanding the responsiveness of solutions to changes in agents' risk aversion (Section 5.3).

Finally, we have the following strong axiom of solidarity. Note that no assumptions are made on the way S relates to S′.

Domination (***dom***): Either $F(S') \geqq F(S)$ or $F(S) \geqq F(S')$.

A number of interesting relations exist between all of these conditions. In light of ***w.p.o*** and ***cont***, ***dom*** and ***st.mon*** are equivalent (Thomson and Myerson 1980) and so are ***add*** and ***cut*** (Livne 1986a). Axiom ***c.i*** implies ***tw*** and so do ***p.o*** and ***i.mon*** together (Thomson and Myerson 1980). Many solutions (Nash, Perles–Maschler, Equal area) satisfy ***p.o*** and ***tw*** but not ***i.mon***. Finally, ***w.p.o***, ***sy***, ***s.inv*** and ***tw*** together imply ***m.p.d*** (Livne 1985a).

The axioms ***tw***, ***i.mon***, ***add***, and ***cut*** can be extended to the n–person case in several alternative ways.

4.6 ***Uncertain feasible set.*** Suppose that bargaining takes place today but that the feasible set will be known only tomorrow: It may be S^1 or S^2 with equal probabilities. The agents' expected utilities today from waiting until the uncertainty is resolved is $x^1 \equiv [F(S^1)+F(S^2)]/2$ whereas if they were to solve the "expected problem" $(S^1+S^2)/2$ they would be $F[(S^1+S^2)/2]$. Since x^1 is in general not Pareto–optimal in $(S^1+S^2)/2$, it would be preferable for them to agree on a compromise today. We require of F that it gives all agents the incentive to solve the problem today: x^1 should dominate $F[(S^1+S^2)/2]$. Slightly more generally, we formulate:

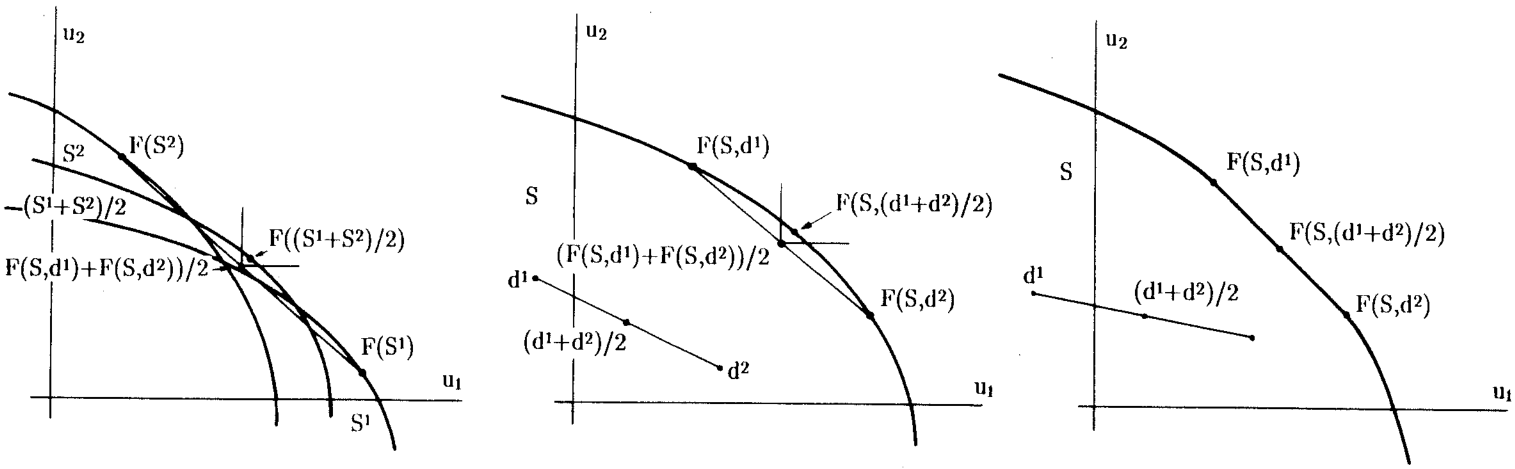
concavity (***cav***): For all $\lambda \in [0,1]$, $F(\lambda S^1+(1-\lambda)S^2) \geqq \lambda F(S^1)+(1-\lambda)F(S^2)$.

Alternatively, we could imagine that the feasible set is the result of the addition of two component problems and formulate the closely related condition:

super–additivity (***sup.add***): $F(S^1+S^2) \geqq F(S^1)+F(S^2)$.

Neither the Nash nor Kalai–Smorodinsky solution satisfies these conditions, but the Egalitarian solution does. Are the conditions compatible with ***s.inv***? Yes. However, only one solution satisfies them together with a few other minimal requirements.

Theorem 8 (Perles and Maschler 1981): The Perles–Maschler solution is the only solution on $\Sigma_0^2 \cup \Gamma_0^2$ satisfying ***p.o***, ***sy***, ***s.inv***, ***sup.add***, and ***cont*** on the subclass of Σ_0^2 of strictly comprehensive problems. (Recall that Γ_0^2 is the class of problems satisfying all the properties required of the elements of Σ_0^2, but violating the requirement that there exists $x \in S$ with $x > 0$.)



(S–)concavity (a) | d–concavity (b) | weak d–concavity (c)

Concavity conditions.

Figure 11

Remark 12: Deleting ***p.o*** from Theorem 8, the solutions PM^{λ} defined by $PM^{\lambda}(S) \equiv \lambda PM(S)$ for $\lambda \in [0,1]$ become admissible. Without ***sy***, we obtain a two–parameter family (Maschler and Perles 1981). ***Cont*** is indispensable (Maschler and Perles 1981) and so are ***s.inv*** (again, consider E) and obviously ***sup.add***. Theorem 8 does not extend to n > 2: In fact, ***p.o***, ***sy***, ***s.inv***, and ***sup.add*** are incompatible on Σ_0^3 (Perles 1982).

Deleting ***s.inv*** from Theorem 8, a joint characterization of Egalitarianism and Utilitarianism can be obtained (note however the important change of domains). In fact, ***cav*** can then be replaced by the following strong condition.

linearity **(*lin*)**: $F(S^1+S^2) = F(S^1)+F(S^2)$.

Theorem 9 (Myerson 1981): The Egalitarian and Utilitarian solutions are the only solutions on $\Sigma_{0,-}^n$ satisfying ***w.p.o***, ***sy***, ***c.i***, and ***cav***. The Utilitarian solutions are the only solutions on

$\Sigma^n_{0,-}$ satisfying ***p.o***, ***sy***, and ***lin*** (provided appropriate tie–breaking rules are applied in the case of the Utilitarian solution).

On the domain $\Sigma^2_{0,-}$, the following weakening of ***lin*** (and ***sup.add***) is compatible with ***s.inv***.

weak linearity (***w.lin***): If $F(S^1)+F(S^2) \in PO(S^1+S^2)$ and S^1 and S^2 are smooth at $F(S^1)$ and $F(S^2)$ respectively, then $F(S^1+S^2)=F(S^1)+F(S^2)$.

The significance of the smoothness restriction has been discussed earlier (Section 4.4).

Theorem **10** (Peters 1986): The weighted Nash solutions are the only solutions on $\Sigma^2_{d,-}$ satisfying ***p.o***, ***st.i.r***, ***s.inv***, ***cont***, and ***w.lin***.

Remark **13:** The Nash solution can be characterized by an alternative weakening of ***lin*** (Chun 1988b). Randomization between all the points of S and its ideal point, and all the points of S and its solution outcome have been considered by Livne (1988) and used by him to formulate invariance conditions that can be used to characterize the Kalai–Smorodinsky and continuous Raiffa solutions.

4.7. ***Separability.*** In Section 4.6 we defined the "addition" of two problems; here, we formulate a notion of "multiplication". Then, we require invariance of the solutions under multiplication.

Given $x, y \in \mathbb{R}^2_+$, let $x*y \equiv (x_1y_1,x_2y_2)$; given $S, T \in \Sigma^2_0$, let $S*T \equiv \{z \in \mathbb{R}^2_+ | z = x*y$ for some $x \in S$ and $y \in T\}$. The domain Σ^2_0 is not closed under the operation $*$.

separability (***sep***): If $S*T \in \Sigma^2_0$, then $F(S*T) = F(S)*F(T)$.

Theorem **11** (Binmore 1984): The Nash solution is the only solution on Σ^2_0 satisfying ***p.o***, ***sy***, and ***sep***.

5. *OTHER PROPERTIES. THE ROLE OF THE DISAGREEMENT POINT.*

In our exposition so far, we have ignored the disagreement point altogether. Here, we study its role in detail, and, for that purpose, we reintroduce it in the notation: a bargaining problem is now a pair (S,d) as originally specified in Section 2. We consider first increases in one of the coordinates of the disagreement point; then, situations when it is uncertain. In each case, we study how responsive solutions are to these changes. We also study how solutions respond to changes in the agents' risk aversion.

5.1. ***Disagreement point monotonicity.*** Here, we formulate monotonicity properties of solutions with respect to changes in d (Thomson 1987). To that end, fix S.

disagreement point monotonicity (d.mon): If $d'_i \geqq d_i$ and for all $j \neq i$, $d'_j = d_j$, then $F_i(S,d') \geqq F_i(S,d)$.

If agent i's fallback position improves, it is natural to expect that he will gain, or at least not lose.

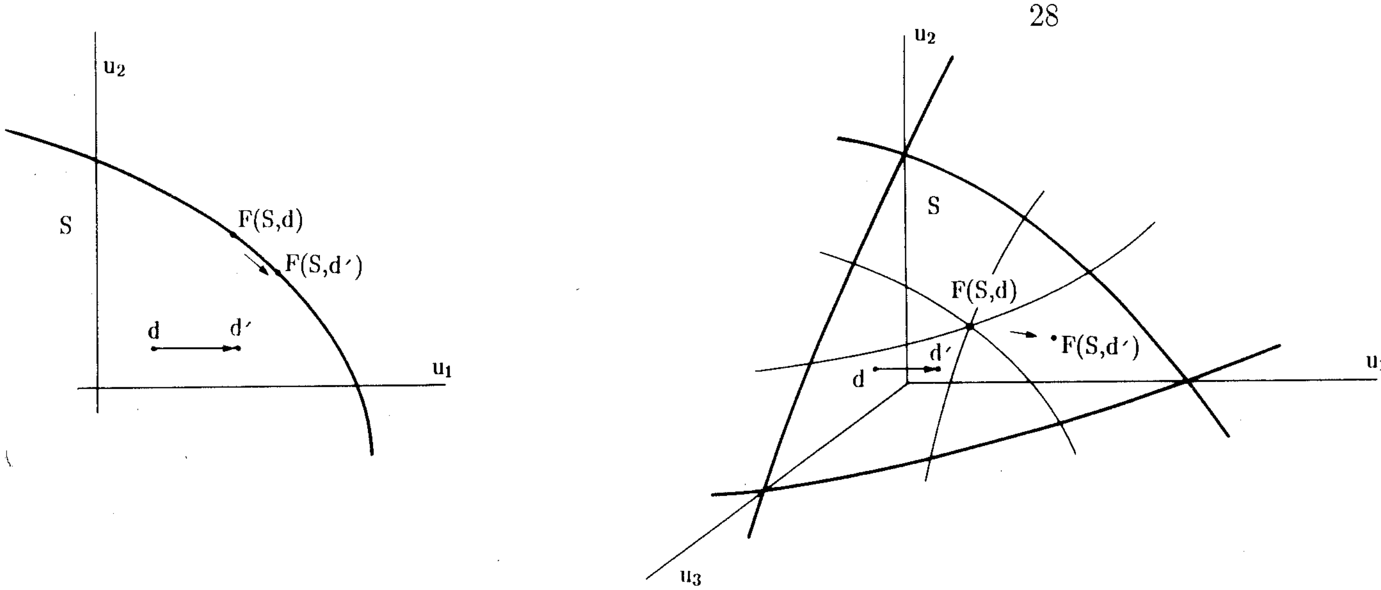
This property is satisfied by all of the solutions that we have encountered. Even the Perles–Maschler solution, which is very poorly behaved with respect to changes in the feasible set, as we saw earlier, satisfies ***d.mon.***

Strong disagreement point monotonicity (st.d.mon): Under the same hypotheses as ***d.mon***, $F_i(S,d') \geqq F_i(S,d)$ and in addition for all $j \neq i$, $F_j(S,d') \leqq F_j(S,d)$.

The gain achieved by agent i should be at the expense (in the weak sense) of all the other agents.

Most solutions, in particular the Nash and Kalai–Smorodinsky solutions and their variants, violate ***st.d.mon.*** However, the Egalitarian solution does satisfy the property and so do the monotone path solutions.

5.2 ***Disagreement point concavity.*** Next, we imagine that there is uncertainty about d. The situation is very similar to that described when we introduced ***cav.*** Suppose that the disagreement point will take one of two positions d^1 and d^2 with equal probabilities and that this uncertainty will be resolved tomorrow. Waiting until tomorrow and solving then whatever problem has come up results in the expected payoffs today $x^1 \equiv [F(S,d^1)+F(S,d^2)]/2$, which is typically Pareto dominated in S. Taking as new disagreement point the expected



Weak monotonicity for n=2

(a)

Strong monotonicity for n=3

(b)

Disagreement point monotonicity conditions

Figure 12

cost of conflict, and solving the problem $(S,(d^1+d^2)/2)$, results in the payoffs $x^2 \equiv F(S,(d^1+d^2)/2)$. If $x^1 \geqq x^2$, the agents will agree to solve the problem today. If neither x_1 dominates x_2 nor x_2 dominates x_1, a conflict may result as to whether to wait or not. The following requirement prevents any such conflict:

disagreement point concavity (d.cav): For all $\lambda \in [0,1]$, $F(S,\lambda d^1+(1-\lambda)d^2) \geqq \lambda F(S,d^1)+(1-\lambda)F(S,d^2)$.

Of all the solutions seen so far, only the weighted Egalitarian solutions satisfy this requirement. Axiom ***d.cav*** is indeed very strong as indicated by the next result which is a characterization of a family of solutions that further generalize the Egalitarian solution: Given $\delta:\Gamma^n \rightarrow \Delta^{n-1}$, the ***directional solution relative to*** δ, E^δ: $E^\delta(S,d)$ is the maximal point of S of the form $d+t\delta(S)$.

Theorem* 12** (Chun and Thomson 1987). The directional solutions are the only solutions on $\Sigma^n_{d,-}$ satisfying ***w.p.o, ***cont***, and ***d.cav***.

This result is somewhat of a disapointment since it says that ***d.cav*** is incompatible with full optimality and permits ***s.inv*** only when $\delta(S)$ is a unit vector (then the resulting directional solution is a dictatorial solution). The following weakening of ***d.cav*** allows recovering full optimality and ***s.inv.***

weak disagreement point concavity **(*w.d.cav*)**: If $[F(S,d^1),F(S,d^2)] \subset PO(S)$ and PO(S) is smooth at $F(S,d^1)$ and $F(S,d^2)$, then for all $\lambda \in [0,1]$, $F(S,\lambda d^1+(1-\lambda)d^2)=\lambda F(S,d^1)+(1-\lambda)F(S,d^2)$.

The boundary of S is linear between $F(S,d^1)$ and $F(S,d^2)$ and it seems natural to require that the solution should respond linearly to linear movements of d between d^1 and d^2. This "partial" linearity of the solution is required however only when the compromise is not forced by sudden changes in the rates at which utility can be transferred.

Theorem **13** (Chun and Thomson 1988). The Nash solution is the only solution on $\Sigma^n_{d,-}$ satisfying ***p.o***, ***sy***, ***s.inv***, ***cont***, and ***w.d.cav.***

A condition related to ***w.d.cav*** is the following:

star-shaped inverse **(*star*)**: $F(S,\lambda d+(1-\lambda)F(S,d)) = F(S,d)$ for all $\lambda \in]0,1]$.

This says that a move of the disagreement point in the direction of the desired compromise does not call for a revision of this compromise.

Theorem **14** (Peters and vanDamme 1988). The weighted Nash solutions are the only solutions on $\Sigma^n_{d,-}$ satisfying ***st.i.r***, ***i.n.i.r***, ***s.inv***, ***d-cont***, and ***star.***

Remark **14.** Conditions related to ***d.cav***, ***w.d.cav*** and ***star*** have been explored. Chun (1987b) shows that a requirement of disagreement point ***quasi***–concavity can be used to characterize a family of solutions that further generalize the directional solutions. Characterizations of the Kalai–Rosenthal solution are given in Peters (1986c) and Chun (1987b). Finally, the continuous Raiffa solution for n=2 is characterized by Livne (1987b), Peters (1986c) and Peters and van Damme (1988). They use the fact for that solution that the shape of the sets of disagreement points leading to the same compromise for each fixed S has differentiability, and certain monotonicity, properties.

Livne (1988) considers situations where the disagreement point is uncertain but information can be obtained about it and characterizes a version of the Nash solution.

5.3 ***Risk–sensitivity.*** Here we investigate how solutions respond to changes in the agents' risk aversion. Other things being equal, is it preferable to face a more risk–averse opponent? To study this issue we need explicitly to introduce the set of underlying physical alternatives. Let C be a set of ***certain options*** and L the set of ***lotteries*** over C. Given two von Neumann–Morgenstern utility functions u_i and u_i': $L \to \mathbb{R}$, ***u_i' is more risk–averse than u_i*** if they represent the same ordering on C and for all $c \in C$, the set of lotteries that are u_i'–preferred to c is contained in the set of lotteries that are u_i–preferred to c. If $u_i(C)$ is an interval, this implies that there is an increasing concave function k: $u_i(C) \to \mathbb{R}$ such that $u_i' = k(u_i)$. An ***n–person concrete problem*** is a list (C,e,u), where C is as above, $e \in C$, and $u=(u_1,...,u_n)$ is a list of von Neumann–Morgenstern utility functions defined over C. The ***abstract problem associated with (C,e,u)*** is the pair $(S,d) \equiv (\{u(\ell) | \ell \in L\},u(e))$.

risk–sensitivity (ri.sens): Given (C,e,u) and (C′,e′,u′), which differ only in that u_i' is more risk–averse than u_i, and such that the associated (S,d), (S′,d′) belong to Σ_d^n, $F_i(S,d) \geqq u_i(\ell')$ where $u'(\ell')=F(S',d')$.

strong risk–sensitivity (st.ri.sens): Under the same hypotheses as ***ri.sens***, $F_i(S,d) \geqq u_i(\ell')$ and in addition, $F_j(S,d) \leqq u_j(\ell')$ for all $j \neq i$.

(C,e,u) is basic if the associated (S,d) satisfies $PO(S) \subset u(C)$. Let $B(\mathscr{C}_d^n)$ be the class of basic problems. If (C,e,u) is basic and u_i' is more risk–averse than u_i, then (C,e,u_i',u_{-i}) also is basic.

Theorem* 15** (Kihlstrom, Roth and Schmeidler 1981, Nielsen 1984). The Nash solution satisfies ***ri.sens on $B(\mathscr{C}_d^n)$ but it does not satisfy ***st.ri.sens.*** The Kalai–Smorodinsky solution satisfies ***st.ri.sens*** on $B(\mathscr{C}_0^n)$.

There is an important logical relation between risk–sensitivity and scale invariance.

Theorem 16 (Kihlstrom, Roth and Schmeidler 1981). If a solution on $B(\mathscr{E}_d^2)$ satisfies ***p.o*** and ***ri.sens***, then it satisfies ***s.inv.*** If a solution on $B(\mathscr{E}_0^n)$ satisfies ***p.o*** and ***st.ri.sens***, then it satisfies ***s.inv.***

For n=2, interesting relations exist between ***ri.sens*** and ***tw*** (Tijs and Peters 1985) and between ***ri.sens*** and ***m.p.d.*** (Sobel 1980).

Remark **15:** For further studies of the risk–sensitivity of solutions see de Koster, Peters, Tijs and Wakker (1983), Peters and Tijs (1981, 1983, 1985a).

For the class of non–basic problems, two cases should be distinguished. If the disagreement point is the image of one of the basic alternatives, what matters is whether the solution is appropriately responsive to changes in the disagreement point.

Theorem 17 (Roth and Rothblum 1982, Thomson 1988). Suppose $C = \{c_1, c_2, e\}$. Suppose F is a solution on Σ_d^2 satisfying ***p.o***, ***s.inv***, and ***d.mon.*** Then, if u_i is replaced by a more risk–averse utility u_i', agent j gains if $u_i(\ell) \geqq \min\{u_i(c_1), u_i(c_2)\}$ and not otherwise.

Remark **16:** The n–person case is studied by Roth (1988). Situations when the disagreement point is obtained as a lottery are considered by Safra and Zilcha (1988). An application to insurance contracts appears in Kihlstrom and Roth (1982).

6. *VARIABLE NUMBER OF AGENTS.*

Most of the axiomatic theory of bargaining has been written under the assumption of a fixed number of agents. Recently however, the model has been enriched by allowing the number of agents to vary. Axioms specifying how solutions could or should respond to such changes have been formulated and new characterizations of the main solutions as well as of new solutions generalizing them have been developed. A detailed account of these developments can be found in Thomson and Lensberg (1988).

We need extra notation. There is an infinite set of "potential agents", indexed by the positive integers. Any finite group may be involved in a problem. Let $\mathscr{P}$ be the set of all such groups. Given $Q \in \mathscr{P}$, $\mathbb{R}^Q$ is the utility space pertaining to that group, and Σ_0^Q the class of subsets of $\mathbb{R}^Q_+$ satisfying all of the assumptions imposed earlier on the elements of Σ^n_0. Let $\Sigma_0 \equiv \cup \Sigma^Q_0$. A ***solution*** is a function $F: \Sigma_0 \to \cup \mathbb{R}^Q_+$ associating with every $Q \in \mathscr{P}$ and every $S \in \Sigma^Q_0$ a point of S. All of the axioms stated earlier for solutions defined on Σ^n_0 can be reformulated so as to apply to this more general notion by simply writing that they hold for every $P \in \mathscr{P}$. As an illustration, the optimality axiom is written as:

PARETO–OPTIMALITY (P.O): Given $Q \in \mathscr{P}$ and $S \in \Sigma^Q_0$, $F(S) \in PO(S)$.

Some axioms, such as ***an***, take a slightly more complicated form however.

ANONYMITY (AN): Given $P, P' \in \mathscr{P}$ with $|P|=|P'|$, $S \in \Sigma^P_0$ and $S' \in \Sigma^{P'}_0$, if there exists a bijection $\gamma: P \to P'$ such that $S'=\{x' \in \mathbb{R}^{P'} \mid \exists x \in S \text{ with } x'_i=x_{\gamma(i)}\ \forall i \in P\}$ then $F_i(S')=F_{\gamma(i)}(S)$ for all $i \in P$.

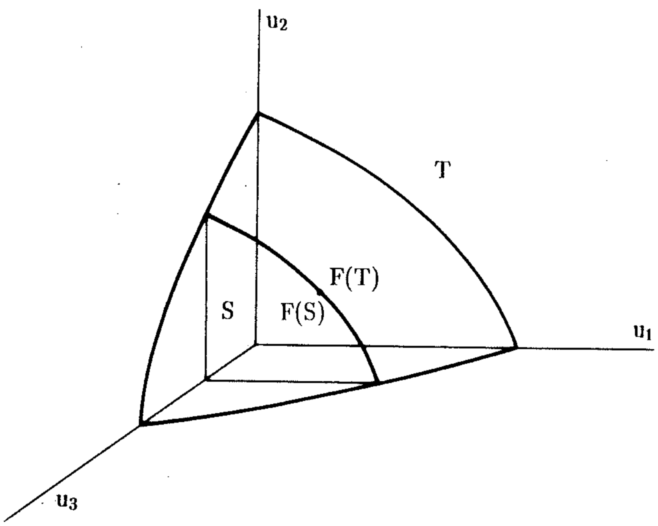
Two conditions specifically concerned with the way solutions respond to changes in the number of agents have been central to the developments repeated in the section. One is an independence axiom, and the other a monotonicity axiom.

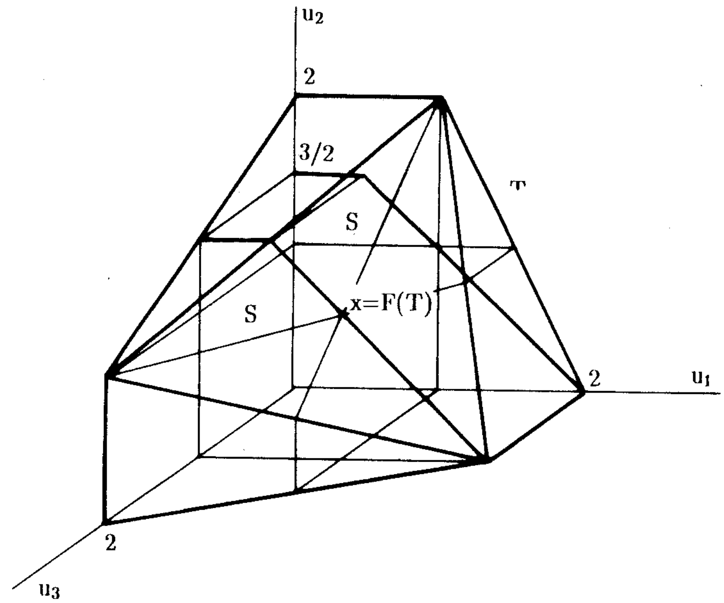
6.1. *The Nash solution.* We start with the independence axiom. Given $Q \in \mathscr{P}$ and $T \in \Sigma^Q_0$, consider some point $x \in T$ as a candidate compromise for T. For x to be acceptable to all, it should be acceptable to all subgroups of Q. Assume then that it has been accepted by the subgroup P′, and that the members of P′ are indifferent through which one of the points of T their utilities $x_{P'}$ are achieved. Then, from the viewpoint of the members of $P \equiv Q \backslash P'$, it is as if they really had access to all of these points. If this set is a well–defined member of Σ^P_0, does the solution recommend for them the utilities x_P? If yes, the solution is ***consistent*** (Fig. 12a). Given $P, Q \in \mathscr{P}$ with $P \subset Q$, $T \in \Sigma^Q_0$ and $x \in T$, let $t^x_P(T) \equiv \{y \in \mathbb{R}^P \mid (y, x_{Q/P}) \in T\}$.

POPULATION CONSISTENCY (POP.CONS): Given $P, Q \in \mathscr{P}$ with $P \subset Q$, if $S \in \Sigma^P_0$ and $T \in \Sigma^Q_0$ are such that $S=t^x_P(T)$, where $x=F(T)$, then $x_P=F(S)$.

POP.CONS is satisfied by the Nash solution (Harsanyi, 1959) but not by the Kalai–Smorodinsky solution nor by the Egalitarian solution. Violations are usual for the

Kalai–Smorodinsky solution but rare for the Egalitarian solution; indeed, on the class of





The axiom of *POP.CONS*
(a)

Characterization of N
(b)

Population Consistency and the Nash solution.

Figure 13

strictly comprehensive problems, the Egalitarian solution does satisfy the condition, and if this restriction is not imposed, it still satisfies the slightly weaker condition obtained by requiring $x_P \leqq F(S)$ instead of $x_P = F(S)$. Let us call this weaker condition Weak Population Consistency (W.*POP.CONS*). The Lexicographic Egalitarian solution E^L satisfies *POP.CONS*.

Theorem 18 (Lensberg 1988). The Nash solution is the only solution on Σ_0 satisfying *P.O*, *AN*, *S.INV*, and *POP.CONS*.

Proof (Fig. 13b): It is straightforward to see that N satisfies the four axioms. Conversely, let F be a solution on Σ_0 satisfying the four axioms. We only show that F coincides with N on Σ_0^P if $|P|=2$. Let $S \in \Sigma_0^P$ be given. By ***S.INV***, we can assume that S is normalized so that $N(S)=(1,1)$.

In a first step, we assume that $PO(S) \supset [(3/2,1/2),(1/2,3/2)]$. Let $Q \in \mathscr{P}$ with $P \subset Q$ and $|Q|=3$ be given. Without loss of generality, we take $P=\{1,2\}$ and $Q=\{1,2,3\}$. (In the Figure, $S \equiv cch\{(2,0),(1/2,3/2)\}$). Now, we translate S by the third unit vector, we replicate the result twice by having agents 2, 3 and 1, and then agents 3, 1 and 2 play the roles of agents 1, 2, and 3 respectively; finally, we define $T \in \Sigma_0^Q$ to be the convex and comprehensive hull of the three sets so obtained. Since T (= cch{(1/2,1,0),(0,1/2,1),(1,0,1/2)}) is invariant under rotations of the agents, by ***AN***, F(T) has equal coordinates; then, by ***P.O***, F(T)=(1,1,1). But, since $t_P^{(1,1,1)}(T) = S$, by ***POP.CONS*** F(S)=(1,1)=N(S), and we are done.

In a second step, we only assume that PO(S) contains a non–degenerate segment centered at N(S). Then, we may have to introduce more than one additional agent and repeat the same construction by having the problem faced by agents 1 and 2 replicated many times, but if the order of replication is sufficiently large, T is indeed such that $t_P^{(1,\dots,1)}(T)=S$ and we conclude as before. If S does not contain a non–degenerate segment centered at N(S), a continuity argument is required.

Q.E.D.

Remark **17:** The above proof requires having access to groups of arbitrarily large cardinalities, but the Nash solution can still be characterized when the number of potential agents is bounded above, by adding ***CONT*** (Lensberg 1988). Unfortunately, two problems may be close in the Hausdorff topology and yet sections of those problems through two points that are close by, parallel to a given coordinate subspace, may not be close to each other. A weaker notion of continuity recognizing this possibility can however be used to obtain a characterization of the Nash solution, even if the number of potential agents is bounded above (Thomson 1985b). Just as in the classic characterization of the Nash solution, ***P.O*** turns out to play a very minor role here: without it, the only additional admissible solution is the disagreement solution (Lensberg and Thomson 1988).

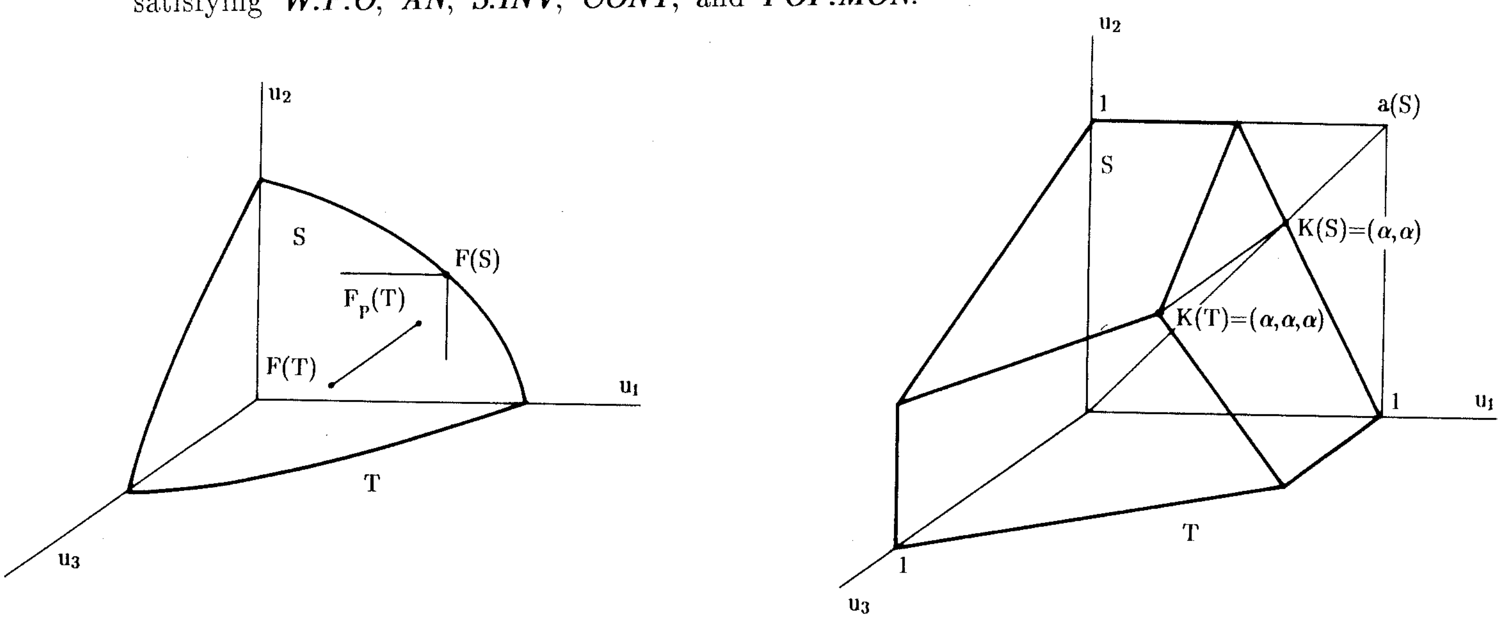
Deleting ***SY*** and ***S.INV*** from Theorem 18, the following solutions become admissible: For each i, let $f_i:\mathbb{R}_+ \to \mathbb{R}$ be an increasing function such that for each P, the function $f^P:\mathbb{R}_+^P$

$\to \mathbb{R}$ defined by $f^P(x)=\sum_{i\epsilon P} f_i(x)$ be strictly quasi–concave. Then, given $P \epsilon \mathcal{P}$ and $S \epsilon \Sigma_0^P$, $F^f(S)\equiv\text{argmax}\{f^P(x)\,|\,x \epsilon S\}$. The ***separable additive*** solutions F^f are the only ones to satisfy *P.O*, *CONT*, and *POP.CONS* (Lensberg 1987; Young 1988 proves a variant of this result).

6.2. ***The Kalai–Smorodinsky solution.*** Instead of allowing some of the agents to leave with their payoffs, we will now imagine them not to be there at all. When some agents leave the scene without this affecting the opportunities of the agents that remain, do all of these gain? The Nash solution does not satisfy this requirement but both the Kalai–Smorodinsky and Egalitarian solutions do.

POPULATION MONOTONICITY (*POP.MON*): Given $P, Q \epsilon \mathcal{P}$ with $P\subset Q$, if $S \epsilon \Sigma_0^P$ and $T \epsilon \Sigma_0^Q$ are such that $S=T_P$, then $F(S)\geqq F_P(T)$.

Theorem **19** (Thomson 1983c). The Kalai–Smorodinsky solution is the only solution on Σ_0 satisfying ***W.P.O***, ***AN***, ***S.INV***, ***CONT***, and ***POP.MON***.



The axiom of *POP.MON*.

(a)

Characterization of the Kalai–Smorodinsky solution.

(b)

Population Monotonicity and the Kalai–Smorodinsky solution.

Figure 14

Proof (Fig. 14b): It is straightforward to see that K satisfies the five axioms. Conversely, let F be a solution on Σ_0 satisfying the five axioms. We only show that F coincides with K on Σ_0^P if $|P|=2$. So let $S \in \Sigma_0^P$ be given. By ***S.INV***, we can assume that S is normalized so that a(S) has equal coordinates. (In the Figure S = cch{(1,0),(1/2,1)}.) Let $Q \in \mathscr{P}$ with $P \subset Q$ and $|Q|=3$ be given. Without loss of generality, we take P={1,2} and Q={1,2,3}. Now, we construct $T \in \Sigma_0^Q$ by replicating S in the coordinates subspaces $\mathbb{R}^{\{2,3\}}$ and $\mathbb{R}^{\{3,1\}}$, and taking the comprehensive hull of the resulting problems and of the point $x \in \mathbb{R}^Q$ of coordinates all equal to the common value of the coordinates of K(S). Since all agents enter symmetrically in the definition of T and $x \in PO(T)$, it follows from ***AN*** and ***W.P.O*** that x=F(T). Now, note that $T_P=S$ and $x_P=K(S)$ so that by ***POP.MON***, $F(S) \geqq K(S)$. Since $|P|=2$, $K(S) \in PO(S)$ and equality holds.

To prove that F and K coincide for problems of cardinality greater than 2, one has to introduce more agents and ***CONT*** becomes necessary.

Q.E.D.

Remark 18: Each of the axioms of Theorem 19 is indispensible. Solutions in the spirit of the solutions E^α described in remark 19 satisfy all of them except ***W.P.O.*** Without ***AN***, we obtain certain generalizations of the weighted Kalai–Smorodinsky solutions. For the role of ***S.INV***, see the next result.

6.3. ***The Egalitarian solution.*** All of the axioms used in the next theorem have already been discussed.

Theorem 20 (Thomson 1983d). The Egalitarian solution is the only solution on Σ_0 satisfying ***P.O***, ***SY***, ***C.I***, ***CONT***, and ***POP.MON***.

Proof It is easy to verify that E satisfies the five axioms (see Fig. 15b for ***POP.MON***). Conversely, let F be a solution on Σ_0 satisfying the five axioms. To see that F=E, let $P \in \mathscr{P}$ and $S \in \Sigma_0^P$ be given. Without loss of generality, suppose E(S) = (1,...,1) and let

$\beta \equiv \text{argmax}\{\Sigma x_i | x \in S\}$ (Fig. 15b). Now, let $Q \in \mathcal{P}$ be such that $Q \supset P$ and $|Q| \geqq \beta+1$; finally, let $T \in \Sigma_0^Q$ be defined by $T=\{x \in \mathbb{R}_+^Q | \Sigma x_i \leqq |Q|\}$. (In Fig. 15b, $|P|=2$ and $|Q| = 3$.) By ***W.P.O*** and ***SY***, $F(T)=(1,...,1)$. Now, let $T' \equiv \text{cch}\{S,F(T)\}$. Since $T' \subset T$ and $F(T) \in T'$, it follows from ***C.I*** that $F(T')=F(T)$. Now, $F_P(T') \leqq F(S)$, so that by ***POP.MON***, $F(S) \geqq y_P$. If $E(S) \in PO(S)$ we are done. Otherwise we conclude by ***CONT***.

Q.E.D.

Remark 19: Without ***W.P.O***, the ***Truncated Egalitarian solutions*** become admissible: Let $\alpha \equiv \{\alpha^P | P \in \mathcal{P}\}$ be a list of non–negative numbers such that for all $P, Q \in \mathcal{P}$ with $P \subset Q$, $\alpha^P \geqq \alpha^Q$; then, given $P \in \mathcal{P}$ and $S \in \Sigma^P$, let $E^\alpha(S) \equiv \alpha^P(1,...,1)$ if this points belongs to S and $E^\alpha(S)=E(S)$ otherwise (Thomson 1984b). The monotone path solutions encountered earlier, appropriately generalized, satisfy all the axioms of Theorem 20, except for ***SY***: Let $G \equiv \{G^P | P \in \mathcal{P}\}$ be a list of monotone paths such that $G^P \subset \mathbb{R}_+^P$ for all $P \in \mathcal{P}$ and for all $P, Q \in \mathcal{P}$ with $P \subset Q$, the projection of G^Q onto $\mathbb{R}^P$ be contained in G^P. Then, given $P \in \mathcal{P}$ and $S \in \Sigma_0^P$, $E^G(S)$ is the maximal point of S along the path G^P (Thomson, 1983a).

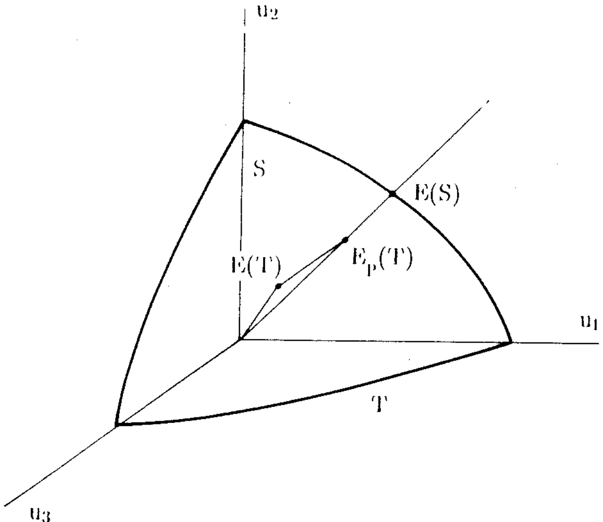
The next result involves considerations of both consistency and monotonicity.

Theorem 21 (Thomson 1984c): The Egalitarian solution is the only solution on Σ_0 satisfying ***W.P.O***, ***SY***, ***CONT***, ***POP.MON***, and ***W.POP.CONS***.

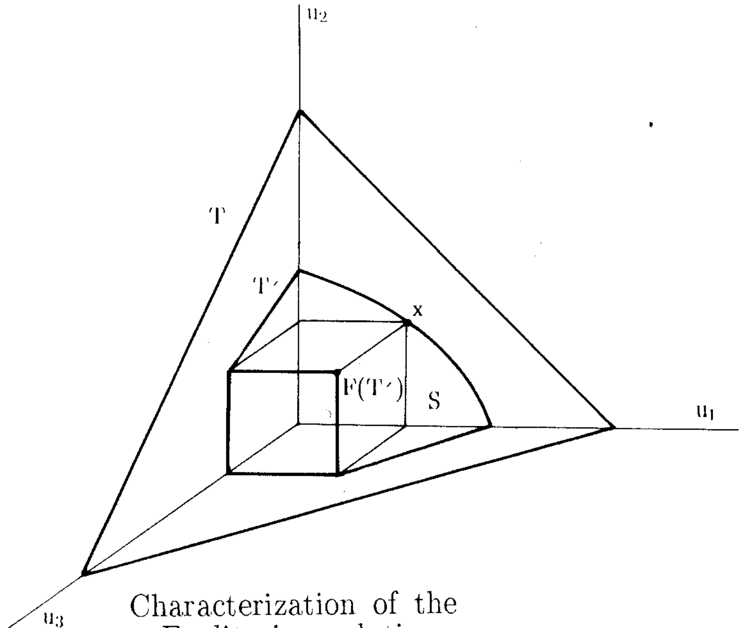
In order to recover full optimality, the extension of ***i.mon*** to the variable population case can be used.

Theorem 22 (Lensberg 1985a, 1985b): The Lexicographic Egalitarian solution is the only solution on Σ_0 satisfying ***P.O***, ***SY***, ***I.MON***, and ***POP.CONS***.

6.4 ***Opportunities and Guarantees.*** Consider a solution F satisfying ***W.P.O***. When new agents come in without opportunities enlarging, as described in the hypotheses of ***POP.MON***,



The Egalitarian solution does not satisfy ***POP.CONS.***
(a)



Characterization of the Egalitarian solution.
(b)

Variable population axioms and the Egalitarian solution.

Figure 15

one of the agents originally present will lose. We propose here a way of quantifying these losses and of ranking solutions on the basis of the extent to which they prevent agents from losing too much. Formally, let P, Q ϵ $\mathscr{P}$ with P$\subset$Q, S ϵ Σ_0^P, and T ϵ Σ_0^Q with S=T_P. Given i ϵ P, we study the ratio $F_i(T)/F_i(S)$ of agent i's final to initial utility: let $\alpha_F^{(i,P,Q)}$ ϵ $\mathbb{R}$ be the greatest number such that $F_i(T)/F_i(S)>\alpha$ for all S, T as described earlier. This is the ***guarantee offered to i by F when he is initially part of P and P expands to Q***: agent i's final utility is guaranteed to be at least $\alpha_F^{(i,P,Q)}$–times his initial utility. If F satisfies ***AN***, then the number depends only on |P| and |Q\P|, denoted m and n respectively, and we can write it as α_F^{mn}. Let $\alpha_F \equiv \{\alpha_F^{mn} | m, n \in \mathbb{N}\}$ be the ***guarantee structure of F***.

We now proceed to compare solutions on the basis of their guarantee structures. Solutions offering greater guarantees are of course preferable.

***Theorem* 23** (Thomson and Lensberg 1983): The guarantee structure α_K of the

Kalai–Smorodinsky solution is given by $\alpha_K^{mn}=1/(n+1)$ for all m.n $\in \mathbb{N}$. If F satisfies ***W.P.O*** and ***AN***, then $\alpha_K \geqq \alpha_F$. In particular, $\alpha_K \geq \alpha_N$. In fact,

$$\alpha_N^{mn}=\frac{m(n+2)-\sqrt{mn(mn+4m-4)}}{2(m+n)} \text{ for all m. n } \in \mathbb{N}.$$

Remark **20**: Theorem 23 says that the Kalai–Smorodinsky solution is best in a large class of solutions. However, it is not the only one to offer maximal guarantees and to satisfy ***S.INV*** and ***CONT***.

However, solutions could be compared in other ways. In particular, protecting individuals may be costly to the group to which they belong. To analyze the trade–off between protection of individuals and protection of groups, we introduce the coefficient $\beta_F^{mn} \equiv \inf\{\sum_{i\in P} \frac{F_i(T)}{F_i(S)} | S \in \Sigma_0^P,\ T \in \Sigma_0^Q,\ P\subset Q,\ S=T_P,\ |P|=m, |Q\backslash P|=n\}$, and we define $\beta_F \equiv \{\beta_F^{mn} | m,\ n \in \mathbb{N}\}$ as the ***collective guarantee structure of F***. Using this notion, we find that our earlier ranking of the Kalai–Smorodinsky and Nash solutions is reversed.

Theorem **24** (Thomson 1983b): The collective guarantee structure β_N of the Nash solution is given by $\beta_N^{mn}=n/(n+1)$ for all m,n $\in \mathbb{N}$. If F satisfies ***W.P.O*** and ***AN***, then $\beta_N \geqq \beta_F$. In particular, $\beta_N \geq \beta_K$. In fact, $\beta_K^{mn}=\alpha_K^{mn}$ for all m,n $\in \mathbb{N}$.

Remark **21**: The Nash solution is not the only one to offer maximal collective guarantees and to satisfy ***S.INV*** and ***CONT***.

Solutions can alternatively be compared on the basis of the opportunities for gains that they offer to individuals (and to groups). Solutions that limit the extent to which individuals (or groups) can gain in spite of the fact that there may be more agents around while opportunities have not enlarged may be deemed preferable. Once again, the Kalai–Smorodinsky solution performs better than any solution satisfying ***W.P.O*** and ***AN*** when the focus is on a single individual, but the Nash solution is preferable when groups are considered. However, the rankings obtained here are less discriminating (Thomson 1987b).

Finally, we compare agent i's percentage loss $F_i(T)/F_i(S)$ to agent j's percentage loss $F_j(T)/F_j(S)$, where both i and j are part of the initial group P: Let $\epsilon_F^{mn} \equiv \inf\{\frac{F_j(T)/F_j(S)}{F_i(T)/F_i(S)} | S \in \Sigma_0^P,\ T \in \Sigma_0^Q,\ P \subset Q,\ S=T_P,\ |P|=m,\ |Q \backslash P|=n\}$ and $\epsilon_F \equiv \{\epsilon_F^{mn} | (m,n) \in (\mathbb{N} \backslash 1) \times \mathbb{N}\}$, and solutions are required to prevent agents from being too differentially affected, then we find

Theorem 25 (Chun and Thomson 1988). The relative guarantee structures ϵ_K and ϵ_E of the Kalai–Smorodinsky and Egalitarian solutions are given by $\epsilon_K^{mn} = \epsilon_E^{mn} = 1$ for all $(m,n) \in (\mathbb{N} \backslash 1) \times \mathbb{N}$. The Kalai–Smorodinsky solution is the only solution on Σ_0 to satisfy *W.P.O*, *AN*, *S.INV* and to offer maximal relative guarantees. The Egalitarian solution is the only solution on Σ_0 to satisfy *W.P.O*, *AN*, *C.I* and to offer maximal relative guarantees.

6.5. ***Replication and Juxtaposition.*** Now, we consider the somewhat more special situation where the preferences of the new agents are required to bear some simple relation to those of the agents originally present, such as when they are exactly opposed or exactly in agreement. There are several ways in which opposition or agreement of preferences can be formalized. And to each such formulation corresponds a natural way of writing that a solution respects the special structure of preferences.

Given a group P of agents facing the problem $S \in \Sigma_0^P$, introduce for each $i \in P$, n_i additional agents "of the same type" as i and let Q be the enlarged group. Given any group P′ with the same composition as P (we write comp(P′) = comp(P)), define the problem $S^{P'}$ faced by P′ to be the copy of S in $\mathbb{R}^{P'}$ obtained by having each member of P′ play the role played in S by the agent in P of whose type he is. Then, to construct the problem T faced by Q, we consider two extreme cases. One case formalizes a situation of maximal compatibility of interests among all the agents of a given type:

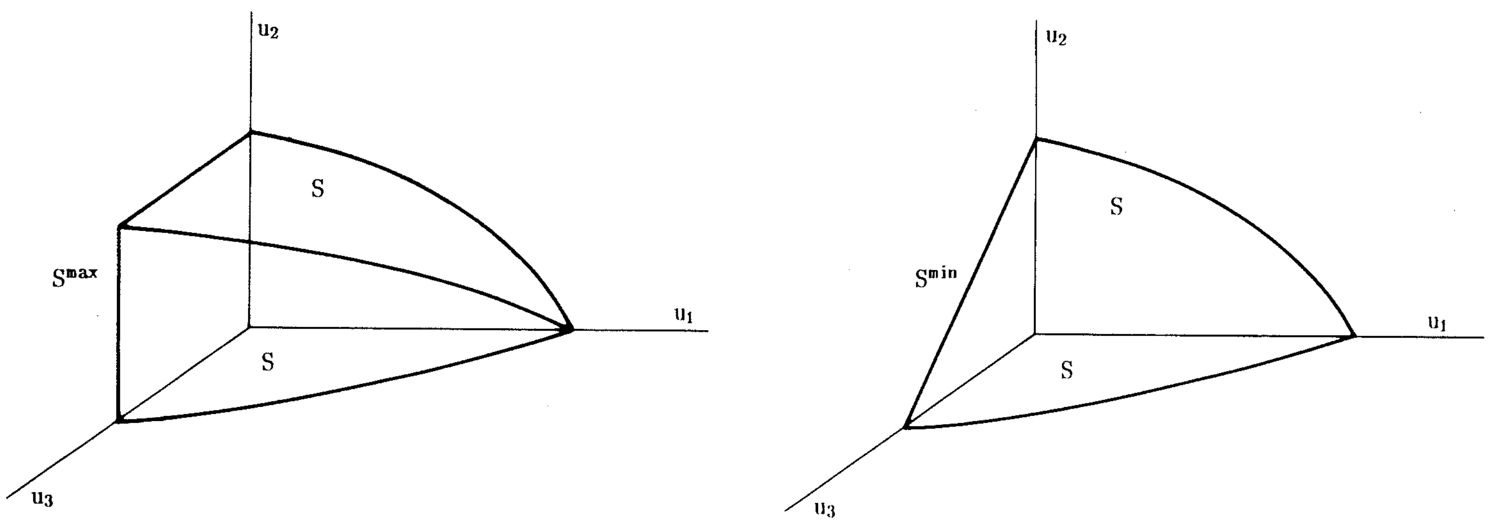
$$S^{max} \equiv \cap\{S^{P'} \times \mathbb{R}^{Q \backslash P'} | P' \subset Q,\ \text{comp}(P') = \text{comp}(P)\}.$$

The other formalizes the opposite:

$$S^{min} \equiv \text{cch}\{S^{P'} | P' \subset Q,\ \text{comp}(P') = \text{comp}(P)\ \}.$$

These two notions are illustrated in Fig. 16 for an initial group of 2 agents and one additional agent (agent 3) being introduced to replicate agent 2.

***Theorem* 26** (based on Kalai 1977a): In S^{max}, all of the agents of a given type receive what the agent they are replicating receives in S if either the Kalai–Smorodinsky or Egalitarian solutions is used. However, if the Nash solution is used, all of the agents of a given type receive what the agent they are replicating would have received in S under the application of the weighted Nash solution with weights proportional to the orders of replication of the different types.



Maximal compatilibity

(a)

Minimal compatibility

(b)

Two notions of replication.

Figure 16

***Theorem* 27** (Thomson 1984a): In S^{min}, the sum of what the agents of a given type receives under the replication of the Nash, Kalai–Smorodinsky, and Egalitarian solutions is equal to what the agent they are replicating receives in S under the application of the corresponding weighted solution for weights proportional to the order of replication.

7. *APPLICATIONS TO ECONOMICS.*

Solutions to abstract bargaining problems, most notably the Nash solution, have been used to solve concrete economic problems, in particular, management–labor conflicts, on numerous occasions; in that application, S is the image in utility space of the possible divisions of a firm's profit, and d the image of a strike. Problems of fair division have also been analyzed in that way; given some bundle of infinitely divisible goods Ω, S is the image in utility space of the set of possible distributions of Ω, and d is the image of the 0 allocation (or of equal division). Alternatively, each agent may start out with a share of Ω, his initial endowment, and choosing d to be the image of the initial allocation may be more natural.

Under standard assumptions on utility functions, the resulting problem (S,d) satisfies the properties typically required of admissible problems in the axiomatic theory of bargaining. Conversely, given $S \in \Sigma_0^n$, it is possible to find exchange economies whose associated feasible set is S (Billera and Bixby 1973).

When concrete information about the physical alternatives is available, it is natural to use it in the formulation of properties of solutions. For instance, expansions in the feasible set are often the result of increases in resources or improvements in technologies. The counterpart of ***st.mon***, (which says that such an expansion would benefit all agents) would be that all agents benefit from greater resources or better technologies. How well–behaved are solutions in this domain? The answer is that when there is only one good, solutions are better behaved than on abstract domains, but as soon as the number of goods is greater than 1, the same behavior should be expected of solutions on both domains (Chun and Thomson 1988).

The axiomatic study of solutions to concrete allocation problems is currently an active area of research. The transposition of the axioms that have been found most useful in the abstract theory of bargaining has been attempted. They have resulted in characterizations of the Walrasian solution (Binmore 1987) and of egalitarian–type solutions (Roemer 1986a, 1988).

8. *STRATEGIC CONSIDERATIONS.*

Analyzing a problem (S,d) as a strategic game requires additional structure: Strategy spaces and an outcome function have somehow to be associated with (S,d). This can be done in a variety of ways. We limit ourselves to describing formulations that remain close to the abstract model of the axiomatic theory and this brief section is only meant to facilitate the transition to chapters in this volume devoted to strategic models.

Consider the following game: each agent demands a utility level for himself; the outcome is the vector of demands if it is in S and d otherwise. The set of Nash (1951) equilibrium outcomes of this ***game of demands*** is $PO(S) \cap I(S,d)$, a typically large set, so that this approach does not help in reducing the set of outcomes significantly. However, if S is known only approximately (replace its characteristic function by a ***smooth*** function), then as the degree of approximation increases, the set of equilibrium outcomes of the resulting ***smoothed game of demands*** shrinks to N(S,d) (Nash 1950, Harsanyi 1956, Zeuthen 1930, Crawford 1979, Anbar and Kalai 1978).

If bargaining takes place over time, agents take time to prepare and communicate proposals, and the worth of an agreement reached in the future is discounted, a ***sequential game of demands*** results. Its equilibria (here some perfection notion has to be used, see Chapter 1) can be characterized in terms of the weighted Nash solutions when the time period becomes small: it is $N^{\delta}(S,d)$ where δ is a vector proportional to the logarithms of the agents′ discount rates (Rubinstein 1982, Binmore 1987; see Chapter 11 for an extensive analysis of this model. Livne 1987a contains an axiomatic analysis of this model).

Imagine now that agents have to justify their demands: there is a family $\mathcal{F}$ of "reasonable" solutions such that agent i can demand $\bar{u}_i$ only if $\bar{u}_i = F_i(S,d)$ for some $F \in \mathcal{F}$. Then strategies are in fact solutions in $\mathcal{F}$. Let F^1 and F^2 be the strategies chosen by agents 1 and 2. If $F^1(S,d)$ and $F^2(S,d)$ differ, eliminate from S all alternatives at which agent 1 gets more than $F^1_1(S,d)$ and agent 2 gets more than $F^2_2(S,d)$; one could argue that the truncated set S^1 is the relevant set over which to bargain; so repeat the procedure: compute $F^1(S^1,d)$ and $F^2(S^1,d)$ ... If $F^1(S^\nu,d)$ and $F^2(S^\nu,d)$ converge to a common point, take that as the solution outcome of this induced ***game of solutions***. For natural families $\mathcal{F}$, convergence does occur for all F^1 and $F^2 \in \mathcal{F}$, and the only equilibrium outcome of the game so defined is N(S,d) (Van Damme 1986, Chun 1985 studies a variant of the procedure).

Thinking now of solutions as normative criteria, note that in order to compute the desired outcomes, the utility function of the agents will be necessary. Since these functions are typically unobservable, there arises the issue of ***manipulation***. To the procedure is associated a game of misrepresentation, where strategies are utility functions. What are the equilibria of this game? In the game so associated with the Nash solution when applied to a one–dimensional division problem, each agent has a dominant strategy which is to pretend that his utility function is linear. The resulting outcome is equal division (Crawford and Varian 1979). If there is more than one good and preferences are known ordinally, a dominant strategy is a least concave representation of one's preferences (Kannai 1977). When there are an increasing number of agents, only one of whom manipulates, the gain that he can achieve by manipulation does not go to zero although the impact on each of the others vanishes; only the first of these conclusions holds, however, when it is the Kalai–Smorodinsky solution that is being used (Thomson 1988). In the multi–commodity case, Walrasian allocations are obtained at equilibria, but there are others (Sobel 1981, Thomson 1984) .

Rather than feeding in agents' utility functions directly, one could of course think of games explicitly designed so as to take into account strategic behavior. Given some objective, embodied in some bargaining solution, does there exist a game whose equilibrium outcome always yields the desired utility allocations? If so, the solution is ***implementable***. The Kalai–Smorodinsky solution is implementable by stage games (Moulin 1984). Recent results indicate that much can be achieved by such games (Moore and Repullo 1988; see Chapter 10 for a review of implementation questions).

BIBLIOGRAPHY

Anbar, D. and E. Kalai (1978), "A One–Shot Bargaining Problem," **International Journal of Game Theory 7**, 13–18.

Anbarci, N. (1988), "Three Essays in Axiomatic and Non–Cooperative Bargaining Theory," University of Iowa Ph.D. Thesis.

_______________ and F. Bigelow (1988), "An Area Monotonic Solution to the Cooperative Bargaining Problem," Working Paper, University of Missouri.

Arrow, K. (1965), **Aspects of the Theory of Risk–Bearing,** Yrjö Jahnsson Foundation, Helsinki.

Binmore, K.G. (1987), "Nash Bargaining Theory I, II, III," in **The Economics of Bargaining** (K.G. Binmore and P. Dasgupta, eds.), Basil Blackwell.

_______________ (1984), "Bargaining Conventions," **International Journal of Game Theory 13**, 193–200.

Brito, D.L., A.M. Buoncristiani and M.D. Intriligator (1977), "A New Approach to Nash's Bargaining Problem," **Econometrica 45**, 1163–1172.

Butrim, B.I. (1976), "A Modified Solution of the Bargaining Problem," (in Russian), **Zhurnal Vychistitel'noi Matematiki Matematicheskoi Fiziki 16**, 340–350.

Chun, Y. (1984), "Note on 'The Nash Bargaining Solution is Optimal,'" University of Rochester Mimeo.

_______________ (1987a), "The Role of Uncertain Disagreement Points in 2–Person Bargaining," Southern Illinois University Discussion Paper No. 87–17.

_______________ (1987b), "Axioms Concerning Uncertain Disagreement Points for 2–Person Bargaining Problems," University of Rochester Working Paper No. 99.

_______________ (1988a), "The Equal–Loss Principle for Bargaining Problems," **Economics Letters 26**, 103–106.

_______________ (1988b), "Nash Solution and Timing of Bargaining," **Economics Letters 28**, 27–31.

_______________ (1988c), "Lexicographic Egalitarian Solution and Uncertainty in the Disagreement Point," forthcoming in **Zeitschrift für Operations Research.**

_______________ and H.J.M. Peters (1987a), "The Lexicographic Egalitarian Solution," Southern Illinois University Discussion Paper No. 87–32.

_______________ and _______________ (1987b) "Lexicographic Monotone Path Solutions," forthcoming in **O.R. Spektrum.**

_______________ and W. Thomson (1987a), "Bargaining with Uncertain Disagreement Points, University of Rochester DP No. 77.

_______________ and _______________ (1987b), "Bargaining Solutions and Relative Guarantees," University of Rochester DP # 115, forthcoming in **Mathematical Social Sciences.**

_______________ and _______________ (1988a), "Monotonicity Properties of Bargaining Solutions When Applied to Economics," **Mathematical Social Sciences 15**, 11–27.

_______________ and _______________ (1988b), "Nash Solution and Uncertain Disagreement Points," University of Rochester Discussion Paper No. 134.

_______________ and _______________ (1988c), "Bargaining Problems With Claims," University of Rochester, forthcoming Discussion Paper.

Crawford, V. (1979), "A Note on the Zeuthen–Harsanyi Theory of Bargaining," U.C. San Diego Discussion Paper 79–4.

_______________ and H. Varian (1979), "Distortion of Preferences and the Nash Theory of Bargaining," **Economics Letters 3**, 203–206.

Crott, H.W. (1971), Experimentelle Untersuchung zum Verhandlungs verhaltern in **Kooperativen Spielen, Zeitschrift für Sozialpsychologie 2**, 61–74.

van Damme, E. (1986), "The Nash Bargaining Solution is Optimal," **Journal of Economic Theory 38**, 78–100.

Dekel, E. (1982), Untitled Mimeo, Harvard University.

Freimer M. and P.L. Yu (1976), "Some New Results on Compromise Solutions for Group Decision Problems," **Management Science 22**, 688–693.

Green, J. (1983), "A Theory of Bargaining with Monetary Transfers," Harvard Institute of Economic Research Discussion Paper.

Gupta, S. and Z. A. Livne (1988), "Resolving a Conflict Situation with a Reference Outcome: An Axiomatic Model," **Management Science 34**, 1303–1314.

Harsanyi, J.C. (1955), "Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility," **Journal of Political Economy 63**, 309–321.

_______________ (1956), "Approaches to the Bargaining Problem Before and After the Theory of Games: A Critical Discussion of Zeuthen's, Hicks', and Nash's Theories," **Econometrica 24**, 144–157.

_______________ (1958), "Notes on the Bargaining Problem," **Southern Economic Journal 24**, 471–476.

_______________ (1959), "A Bargaining Model for the Cooperative n–Person Game," in **Contributions to the Theory of Games IV, Annals of Mathematical Studies No. 40,** (A.W. Tucker and R.d. Luce, eds.), Princeton University Press, Princeton, NJ, 1959.

_______________ (1961), "On the Rationality Postulates Underlying the Theory of Cooperative Games," **Journal of Conflict Resolution 5**, 179–196.

_______________ (1963), "A Simplified Bargaining Model for the n–person Cooperative Game," **International Economic Review 4**, 194–220.

_______________ (1965), "Bargaining and Conflict Situations in the Light of a New Approach to Game Theory," **American Economic Review 55**, 447–457.

_______________ (1977), **Rational Behavior and Bargaining Equilibrium in Games and Social Situations**, Cambridge: Cambridge University Press.

_______________ and R. Selten (1972), "A Generalized Nash Solution for Two–Person Bargaining Games with Incomplete Information," **Management Science 18**, 80–106.

Imai, H. (1983), "Individual Monotonicity and Lexicographic Maxmin Solution," **Econometrica 51**, 389–401, erratum in **Econometrica 51**, 1603.

Jansen, M.J.M. and S.H. Tijs (1983), "Continuity of Bargaining Solutions," **International Journal of Game Theory 12**, 91–105.

Kalai, E. (1977a), "Nonsymmetric Nash Solutions and Replications of Two–Person Bargaining," **International Journal of Game Theory 6**, 129–133.

_______________ (1977b), "Proportional Solutions to Bargaining Situations: Interpersonal Utility Comparisons," **Econometrica 45,** 1623–1630.

_______________ (1985), "Solutions to the Bargaining Problem," Chapter 3 in **Social Goals and Social Organization**; Essays in memory of E. Pazner (L. Hurwicz, D. Schmeidler and H. Sonnenschein, eds.), Cambridge University Press, 77–105.

_______________ and R.W. Rosenthal (1978), "Arbitration of Two–Party Disputes Under Ignorance," **International Journal of Game Theory 7**, 65–72.

_______________ and M. Smorodinsky (1975), "Other Solutions to Nash's Bargaining Problem," **Econometrica 43,** 513–518.

Kannai, Y. (1977), "Concaviability and Construction of Concave Utility Functions," **Journal of Mathematical Economics 4**, 1–56.

Kihlstrom, R.E and A.E. Roth (1982), "Risk Aversion and the Negotiation of Insurance Contracts," **Journal of Risk and Insurance 49**, 372–387.

_______________ , _______________ and D. Schmeidler (1981), "Risk Aversion and Solutions to Nash's Bargaining Problem," in **Game Theory and Mathematical Economics**, (O. Moeschlin and D. Pallaschke, eds.), North–Holland, 65–71.

Kohlberg, E., M. Maschler and M.A. Perles (1983), "Generalizing the Super–Additive Solution for Multiperson Unanimity Games," Discussion Paper to appear.

de Koster, R, H.J.M. Peters, S.H. Tijs and P. Wakker (1983), "Risk Sensitivity, Independence of Irrelevant Alternatives and Continuity of Bargaining Solutions," **Mathematical Social Sciences 4**, 295–300.

Lensberg, T. (1985a), "Stability, Collective Choice and Separable Welfare," Ph.D. Dissertation, Norwegian School of Economics and Business Administration, Bergen, Norway.

________________ (1985b), "Bargaining and Fair Allocation" in **Cost Allocation: Methods, Principles, Applications** (H.P. Young, ed), North–Holland, 101–116.

________________ (1987), "Stability and Collective Rationality," **Econometrica 55**, 935–961.

________________, (1988), "Stability and the Nash Solution," **Journal of Economic Theory 45**, 330–341.

________________ and W. Thomson (1988), "Characterizing the Nash Bargaining Solution Without Pareto–Optimality," **Social Choice and Welfare 5**, 247–259.

Livne, Z. A.(1986a), "The Bargaining Problem: Axioms Concerning Changes in the Conflict Point," **Economics Letters 21**, 131–134.

________________, (1987a), "Bargaining Over the Division of a Shrinking Pie: An Axiomatic Approach," **International Journal of Game Theory, 16**, 223–242.

________________, (1987b), "Axiomatic Characterizations of the Raiffa Solution to the Nash Bargaining Problem," Columbia University mimeo.

________________ (1988), "The Bargaining Problem with an Uncertain Conflict Outcome," **Mathematical Social Sciences 15**, 287–302.

Luce, R.D. and H. Raiffa (1957), **Games and Decisions: Introduction and Critical Survey**, New York: Wiley.

Maschler, M. and M.A. Perles (1981), "The Present Status of the Super–Additive Solution," in **Essays in Game Theory and Mathematical Economics in honor of Oskar Morgenstern** (R. Aumann et al. eds.), 103–110.

Moulin, H. (1983), "Le Choice Social Utilitariste," Ecole Polytechnique DP.

________________ (1984), "Implementing the Kalai–Smorodinsky Bargaining Solution," **Journal of Economic Theory 33**, 32–45.

Myerson, R.B. (1977), "Two–Person Bargaining Problems and Comparable Utility," **Econometrica 45**, 1631–1637.

________________(1981), "Utilitarianism, Egalitarianism, and the Timing Effect in Social Choice Problems," **Econometrica 49**, 883–897.

Nash, J.F. (1950), "The Bargaining Problem," **Econometrica 28**, 155–162.

________________ (1953), "Two–Person Cooperative Games," **Econometrica 21**, 129–140.

________________ (1961), "Noncooperative Games," **Annals of Mathematics 54**, 286–295.

Nielsen, L.T. (1983), "Ordinal Interpersonal Comparisons in Bargaining," **Econometrica 51**, 219–221.

________________ (1984), "Risk Sensitivity in Bargaining with More Than Two Participants," **Journal of Economic Theory 32**, 371–376.

O'Neill, B.(1981), "Comparison of Bargaining Solutions, Utilitarianism and the Minmax Rule by Their Effectiveness," Northwestern University Discussion Paper.

Perles, M.A. (1982), "Nonexistence of Super–Additive Solutions for 3–person Games," **International Journal of Game Theory 11**, 151–161.

________________ and M. Maschler (1981), "A Super–Additive Solution for the Nash Bargaining Game," **International Journal of Game Theory 10**, 163–193.

Peters, H.J.M. (1986a), "Simultaneity of Issues and Additivity in Bargaining," **Econometrica 54**, 153–169.

________________ (1986b), **Bargaining Game Theory**, Ph.D. Thesis, Maastricht, The Netherlands.

________________ (1986c), "Characterizations of Bargaining Solutions by Properties of Their Status Quo Sets," University of Limburg, Department of Economics, Research Memorandum 86–012.

______________ (1987a), "Some Axiomatic Aspects of Bargaining," **Axiomatics and Pragmatics of Conflict Analysis** (J.H.P. Paelinck and P.H. Vossen, eds.), Gower Press, Aldershot, U.K, 112–141.

________________ (1987b), "Nonsymmetric Nash Bargaining Solutions," Chapter 12 in **Surveys in Game Theory and Related Topics** (H.J.M. Peters and O.J. Vrieze, eds) CWI Tract 39, Amsterdam.

________________ and E. Van Damme (1988), "Characterizing the Nash and Raiffa Solutions by Disagreement Point Axioms," Mimeo, May 1988, University of Limburg.

________________ and S. Tijs (1981), "Risk Sensitivity of Bargaining Solutions," **Methods of Operations Research 44**, 409–420.

_____________ and ________________ (1983), "Risk Properties of n–Person Bargaining Solutions," Nijmegen University Discussion Paper 8323.

________________ and ________________ (1984a), "Individually Monotonic Bargaining Solutions for n–Person Bargaining Games," **Methods of Operations Research 51**, 377–384.

________________ and ________________ (1984b), "Probabilistic Bargaining Solutions," **Operations Research Proceedings**, Springer–Verlag, Berlin, 548–556.

______________ and ______________ (1985a) "Risk Aversion in n–Person Bargaining," **Theory and Decision** 18, 47–72.

______________ and ______________ (1985b), "Characterization of All Individually Monotonic Bargaining Solutions," **International Journal of Game Theory** 14, 219–228.

______________, ______________ and R. de Koster (1983), "Solutions and Multisolutions for Bargaining Games," **Methods of Operations Research** 46, 465–476.

______________ and P. Wakker (1987), "Independence of Irrelevant Alternatives and Revealed Group Preferences," University of Limburg Working Paper 87–014.

Raiffa, H. (1953), "Arbitration Schemes for Generalized Two–Person Games," in **Contributions to the Theory of Games II**, Annals of Mathematics Studies, No 28, (H.W. Kuhn and A.W. Tucker, eds), Princeton: Princeton University Press, 361–387.

Ritz, Z. (1985), "An Equal Sacrifice Solution to Nash's Bargaining Problem," University of Illinois, Mimeo.

Roemer, J., (1986a), "Equality of Resources Implies Equality of Welfare," **Quarterly Journal of Economics** 101, 751–784.

______________ (1986b), "The Mismarriage of Bargaining Theory and Distributive Justice," **Ethics** 97, 88–110.

______________, (1988) "Axiomatic Bargaining Theory on Economic Environments," **Journal of Economic Theory** 45, 1–31.

Rosenthal, R.W. (1976), "An Arbitration Model for Normal–Form Games," **Mathematics of Operations Research** 1, 82–88.

______________ (1978), "Arbitration of Two–Party Disputes Under Uncertainty," **Review of Economic Studies** 45, 595–604.

Roth, A.E. (1977a), "Individual Rationality and Nash's Solution to the Bargaining Problem," **Mathematics of Operations Research** 2, 64–65.

______________ (1977b), "Independence of Irrelevant Alternatives, and Solutions to Nash's Bargaining Problem," **Journal of Economic Theory** 16, 247–251.

______________ (1978), "The Nash Solution and the Utility of Bargaining," **Econometrica** 46, 587–594, 983.

______________ (1979a), "Proportional Solutions to the Bargaining Problem," **Econometrica** 47, 775–778.

______________ (1979b), "Interpersonal Comparisons and Equal Gains in Bargaining," Mimeo.

______________ (1979c), **Axiomatic Models of Bargaining**, Springer–Verlag, Berlin and New York, No. 170.

______________ (1979d), "An Impossibility Result Concerning n–Person Bargaining Games," **International Journal of Game Theory** 8, 129–132.

________________ (1980), "The Nash Solution as a Model of Rational Bargaining," in **Extremal Methods and Systems Analysis**, (A.V. Fiacco and K.O. Kortanek, eds.), Springer–Verlag.

________________ (1988), "Risk Aversion in Multi–Person Bargaining Over Risky Agreements," University of Pittsburg mimeo.

________________ and U. Rothblum (1982), "Risk Aversion and Nash's Solution for Bargaining Games with Risky Outcomes," **Econometrica 50**, 639–647.

Safra, Z. and I. Zilcha (1988), "Risk Aversion in Nash Bargaining Problem With Risky Outcomes and Risky Disagreement Outcome," Tel Aviv University Discussion Paper No. 18–88.

Salonen, H. (1985), "A Solution for Two–Person Bargaining Problems," **Social Choice and Welfare 2**, 139–146.

________________ (1987), "Partially Monotonic Bargaining Solutions," **Social Choice and Welfare 4**, 1–8.

Schelling, T.C. (1959), "For the Abandonment of Symmetry in Game Theory," **Review of Economics and Statistics 41**, 213–224.

Schmitz, N. (1977), "Two–Person Bargaining Without Threats: A Review Note," **Proceedings of the OR Symposium – Aachen 29**, 517–533.

Shapley, L.S. (1969), "Utility Comparison and the Theory of Games," in **La Dècision** (G.Th. Guilbaud, ed.), Editions du CNRS, Paris,251–263.

________________ (1984), Oral Presentation at the I.M.A., Minneapolis.

Shubik, M. (1982), **Game Theory in the Social Sciences**, MIT University Press.

Sobel, J. (1981), "Distortion of Utilities and the Bargaining Problem," **Econometrica 49**, 597–619.

Thomson, W. (1980), "Two Characterizations of the Raiffa Solution," **Economics Letters 6**, 225–231.

________________ (1981a), "A Class of Solutions to Bargaining Problems," **Journal of Economic Theory 25**, 431–441.

________________ (1981b), "Independence of Irrelevant Expansions," **International Journal of Game Theory 10**, 107–114.

________________ (1981c), "Nash's Bargaining Solution and Utilitarian Choice Rules," **Econometrica 49**, 535–538.

________________ (1983a), "Truncated Egalitarian and Monotone Path Solutions," University of Minnesota Discussion Paper (January).

________________ (1983b), "Collective Guarantee Structures," **Economics Letters 11**, 63–68.

_______________ (1983c), "The Fair Division of a Fixed Supply Among a Growing Population," **Mathematics of Operations Research 8**, 319–326.

_______________ (1983d), "Problems of Fair Division and the Egalitarian Principle," **Journal of Economic Theory 31**, 211–226.

_______________ (1984a), "Two Aspects of the Axiomatic Theory of Bargaining," University of Rochester Discussion Paper, No. 6.

_______________ (1984b), "Truncated Egalitarian Solutions," **Social Choice and Welfare 1**, 25–32.

_______________ (1984c), "Monotonicity, Stability and Egalitarianism," **Mathematical Social Sciences 8**, 15–28.

_______________ (1985a), "Axiomatic Theory of Bargaining with a Variable Population: A Survey of Recent Results," in **Game Theoretic Models of Bargaining**, (A.E. Roth, ed), Cambridge University Press, Chapter 11, 233–258.

_______________ (1985b), "On the Nash Bargaining Solution," University of Rochester Mimeo.

_______________ (1986), "Replication Invariance of Bargaining Solutions," ***International Journal of Game Theory* 15**, 59–63.

_______________ (1987a), "Monotonicity of Bargaining Solutions with Respect to the Disagreement Point," **Journal of Economic Theory**, **42**, 50–58.

_______________ (1987b), "Individual and Collective Opportunities," **International Journal of Game Theory 16**, 245–252.

_______________ (1988), "Bargaining Theory: The Axiomatic Approach," University of Rochester lecture notes.

_______________ and T. Lensberg (1983), "Guarantee Structures for Problems of Fair Division," **Mathematical Social Sciences 4**, 205–218.

_______________ and _______________ (1988), **The Theory of Bargaining with a Variable Number of Agents**, Cambridge University Press, forthcoming.

_______________ and R.B. Myerson (1980), "Monotonicity and Independence Axioms," **International Journal of Game Theory 9**, 37–49.

Tijs S. and H.J.M. Peters (1985), "Risk Sensitivity and Related Properties for Bargaining Solutions," Chapter 10 in **Game Theoretic Models of Bargaining**, (A.E. Roth, ed), Cambridge University Press, 215–231.

Young, P. (1988), "Consistent Solutions to the Bargaining Problem," University of Maryland mimeo.

Yu, P.L. (1973), "A Class of Solutions for Group Decision Problems," **Management Science 19**, 936–946.

Zeuthen, F. (1930), **Problems of Monopoly and Economic Welfare**, London: G. Routledge.