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NO–ENVY, CONSISTENCY AND MONOTONICITY IN ECONOMIES WITH INDIVISIBILITIES

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ABSTRACT

We consider the problem of fair allocation in economies with indivisibilities. Our objective is to identify appealing subsolutions of the No–Envy solution. First we propose several such selections motivated by intuitive considerations of fairness. Second we formulate desirable properties of solutions, and look for solutions satisfying them: Given an allocation chosen by a solution for some economy and a subgroup of the agents, consider the problem of fairly distributing the resources that this group has collectively received. The solution is *consistent* if it recommends the same bundle be attributed to each of these agents as initially. We show that there is no proper subsolution of the No–Envy solution that satisfies *consistency*. However, many subsolutions of the No–Envy solution satisfy the converse of this property. A solution is *monotonic* if an allocation recommended by the solution for an economy is also chosen by it for any other economy obtained from the former by enlarging for every agent, the set of allocations to which he prefers that allocation. We show that there is no proper subsolution that satisfies *monotonicity*.

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1. Introduction

We consider the problem of fair allocation in economies with a finite number of indivisible "objects" such that each agent can have at most one (e.g., houses, jobs) and a single infinitely divisible good, thought of as money. A **solution** associates with each such economy a set of feasible allocations, interpreted as desirable for the economy. A fundamental example is the **No-Envy solution**, which selects the set of allocations at each of which no agent prefers the bundle of any other agent to his own.

The no-envy notion is quite appealing, because of both its direct normative significance and its full compatibility with efficiency——here an envy-free allocation is in fact necessarily efficient. However, the set of envy-free allocations may be quite large, and in these situations the No-Envy solution fails to make a precise recommendation. Our objective in this paper is to identify appealing, and preferably "small", subsolutions of the No-Envy solution.

Our approach is two-pronged. First, we propose new solutions, motivated by intuitive considerations of fairness. We introduce two alternative ways of measuring how well each agent is treated in relation to the others. We then construct solutions that treat agents as equally as possible according to these measures.

Our second direction of search is complementary to the first one. That is, we formulate desirable properties of solutions and we look for solutions that satisfy these properties together.

Our main test of good behavior involves variations in the number of agents: Suppose that an allocation has been chosen by a solution as providing a good resolution of the problem of fair distribution in some economy. Then pick any subset of the agents, identify all the resources that they have collectively received, and consider the problem of fairly allocating among them these resources. The solution is **consistent** if it recommends that the same bundle be attributed to each of the agents as initially. If a solution is not consistent, revisions of allocations within subgroups may be necessary when these subgroups are considered in isolation. Consistency is therefore a requirement of internal "stability" or robustness.[1]

We also investigate two weakenings of consistency. First, we only require that for each economy, the solution provide *at least one* allocation for which the conclusion of consistency is met. Alternatively, we apply the requirement only to two-person subgroups.[2]

We show that there is no proper subcorrespondence of the No-Envy solution that satisfies consistency (and a very weak independence property). However, there are many proper subcorrespondences that satisfy its bilateral version.

A "dual" property of consistency is **converse consistency**: Given an economy and given a feasible allocation for it, check whether its restriction to any two-person subeconomy defines a desirable way of distributing among these two agents the total resources they have received

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[1]The consistency property has recently been investigated in a variety of fields, such as the theories of bargaining (e.g. Lensberg,1987), cooperative games in coalitional form (e.g. Peleg,1985), fair division in exchange economies with infinitely divisible goods (Thomson,1988a), taxation (e.g. Young,1987), cost allocation (e.g. Moulin,1985), fair representation (e.g. Balinski and Young, 1982), and two-sided matching (Toda,1988 and Sasaki,1988). (See Thomson,1988b, for a survey of this literature.)

[2]This variant was suggested for bargaining solutions by Harsanyi(1959) and studied by Lensberg(1987). Young(1987) considered a two-person version of consistency for taxation problems.

at the allocation. If whenever the answer is yes for all two-person subeconomies, the allocation itself is desirable for the original economy, say that the solution satisfies **converse consistency**.[3] The No-Envy solution satisfies this condition. Does the property provide us with selections from the No-Envy solution? Yes, but here the problem is that it is too weak to help us to single out a particular subsolution.

Finally, we examine the following **monotonicity** condition: If an allocation is desirable for some economy, then it is also desirable for any economy such that for each agent, the set of allocations to which he prefers that allocation is larger than before.[4] We show that among the subcorrespondences of the No-Envy solution, this solution itself is the only one that satisfies monotonicity and a weak regularity condition.

2. The Model

We extend the model examined by Svensson(1983), Maskin(1987), and Alkan, Demange and Gale(1988) (henceforth ADG) to accommodate a variable number of agents. Of particular relevance to our analysis is ADG.[5]

Let Q be an infinite set of **potential agents**, with members denoted by i, j, $\cdots$, and $\mathcal{A}$ an infinite set of **potential objects**, with members denoted

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[3]Axioms of converse consistency have been analyzed for games in coalitional form by e.g. Peleg(1985), and for the fair division problem in classical economies by Thomson(1988a). Conditions under which the Pareto-optimality of an allocation can be deduced from the Pareto-optimality of its restrictions to subeconomies have been established by e.g. Rader(1968), and recently by Goldman and Starr(1982).

[4]This monotonicity condition has played an important role in a variety of situations since its introduction by Maskin(1977) in connection with the theory of implementation.

[5]Luce and Raiffa(1957), Kolm(1972) and Crawford and Heller(1979) studied versions of this model.

by α, β, $\cdots$. Each agent can consume at most one of these objects. There is also a single infinitely divisible good called **money**.

An **economy** is a list $e \equiv (Q, A, M;\ u_Q)$ where Q is a finite set of agents drawn from $\mathcal{Q}$, A is a finite set of objects drawn from $\mathcal{A}$, and $M \in \mathbb{R}$ is an amount of money. Only for simplicity, we assume that $|Q| = |A|$.[6] Each agent $i \in Q$ is endowed with a preference relation on $A \times \mathbb{R}$, assumed to admit a numerical representation u_i: $u_i(\alpha, m_0)$ is the "utility" to agent i of receiving the object α and m_0 units of money. As in ADG, we let m_0 be positive or negative.[7] Each u_i is assumed to be continuous and increasing in money, and such that for all $\alpha \in A$, $\lim_{m_0 \to \infty} u_i(\alpha, m_0) = \infty$. The symbol $u_Q \equiv (u_i)_{i \in Q}$ denotes a list of such utility functions. Let $\mathcal{E}$ be the class of these economies.

Let $e \equiv (Q, A, M;\ u_Q) \in \mathcal{E}$ be given. **A feasible allocation for** e is a pair $z \equiv (\sigma, m)$ where $\sigma : Q \to A$ is a bijection, and $m \in \mathbb{R}^A$ is such that $\sum_{\alpha \in A} m_\alpha = M$. The mapping σ assigns objects to agents. For each $\alpha \in A$, the coordinate m_α of m designates the amount of money that goes to whomever receives object α. One may think of m_α units of money being combined with object α into a "bundle" that is assigned to an agent.[8] Let $z_i \equiv (\sigma(i), m_{\sigma(i)})$ for all $i \in Q$. With z is associated the vector of utilities $u(z) \equiv (u_i(z_i))_{i \in Q} \in \mathbb{R}^Q$. Given $Q' \subseteq Q$, let $\sigma_{Q'}$ be the restriction of σ to

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[6]Given a finite set X, $|X|$ denotes the number of elements in X. Our analysis can be extended to the case of $|Q| \neq |A|$ by enlarging the model through a standard operation. See Section 6 and the Appendix.

[7]We might want m_0 to be negative when, for example, the cost of providing objects has to be covered by the agents.

[8]This notation is due to Svensson(1983).

Q', $m_{\sigma(Q')}$ the restriction of m to $\sigma(Q')$,[9] and $z_{Q'} \equiv (\sigma_{Q'}, m_{\sigma(Q')})$. Let $\mathcal{Z}(e)$ be the set of feasible allocations for e.

Definition: A **solution** is a correspondence Φ that associates with each economy $e \in \mathcal{E}$ a nonempty subset $\Phi(e)$ of $\mathcal{Z}(e)$.

A solution provides for each economy a set of feasible allocations regarded as desirable for the economy. A familiar example is the following:

The Pareto solution (P): An allocation $z \in \mathcal{Z}(e)$ where $e \equiv (Q, A, M; u_Q) \in \mathcal{E}$, is **Pareto-efficient for** e if there is no $z' \in \mathcal{Z}(e)$ such that $u_i(z'_i) \geqq u_i(z_i)$ for all $i \in Q$ with at least one strict inequality. Let $\mathbf{P}(e)$ be the set of these allocations.

3. The No-Envy Solution and Proposed Refinements

We are interested in solutions satisfying the following fundamental notion of equity: Simply, no agent prefers the bundle of any other agent to his own (Foley, 1967).

The No-Envy solution (N): An allocation $z \in \mathcal{Z}(e)$ where $e \equiv (Q, A, M; u_Q) \in \mathcal{E}$ is **envy-free for** e if for all $i, j \in Q$, $u_i(z_i) \geqq u_i(z_j)$. Let $\mathbf{N}(e)$ be the set of these allocations.

For purposes of comparison, we will often refer to economies where all goods are infinitely divisible. In such **classical economies**, the set of envy-free allocations and the set of Pareto-efficient allocations are not

[9] $\sigma(Q') \equiv \{ \sigma(i) \mid i \in Q' \}$.

usually related by inclusion. In the class of economies considered here, however, we have the following relation (Svensson, 1983 and ADG).

Proposition 1: For all $e \in \mathcal{E}$, $N(e) \subseteq P(e)$.

Proposition 1 states one of the important differences between classical economies and economies with indivisibilities. In our context, the No-Envy solution is fully compatible with efficiency.

However, the set of envy-free allocations may be quite large, and in these situations the No-Envy solution does not make a precise recommendation. In order to achieve some refinement of the set of envy-free allocations, we introduce two alternative measures of how well each agent is treated relative to each other agent.

First, we note that with each two-person economy $(\{i, j\}, \{\alpha, \beta\}, M_0;$ $\{u_i, u_j\})$ can be associated a number measuring the "size" of its set of envy-free allocations. Observing that for any two-person economy, the assignment of objects at envy-free allocations is unique,[10] we determine, for either of the two agents, the difference in the amounts of money he holds at his least and most preferred allocations in the set. This difference, which will be referred to as the "surplus" for the two-person economy, indicates how much freedom we have in solving the distribution problem without generating envy.

Now, given an n-person economy $(Q, I, M;\ u_Q) \in \mathcal{E}$, and given an envy-free

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[10]This is true except for the rare case where there is a feasible allocation at which each agent is indifferent between his bundle and the other's: $\exists\ \tilde{z} \in Z(e)$ such that $u_i(\tilde{z}_i) = u_i(\tilde{z}_j)$ and $u_j(\tilde{z}_j) = u_j(\tilde{z}_i)$. It should be noted that the same statement cannot be made about allocations that are simply required to be Pareto-efficient. Nor can it be made about economies containing more than two agents.

allocation $z \equiv (\sigma, m)$ for that economy, and a pair of agents $\{i,j\} \subseteq Q$, we quantify how fairly agent i is treated at z relative to agent j by the proportion of the surplus in the economy $(\{i,j\},\{\sigma(i),\sigma(j)\},m_{\sigma(i)}+m_{\sigma(j)};\{u_i,u_j\})$ that he receives.

Formally, given $e \equiv (Q,A,M;\ u_Q) \in \mathcal{E}$, $z \equiv (\sigma,m) \in N(e)$, and $i,\ j \in Q$, let $\underline{m}_{ij}(z) \equiv \min\{\ m_0 \in \mathbb{R} \mid u_i(\sigma(i),m_0) \geqq u_i(\sigma(j),m_{\sigma(i)}+m_{\sigma(j)}-m_0)\ \}$, and $\bar{m}_{ij}(z) \equiv \max\{\ m_0 \in \mathbb{R} \mid u_j(\sigma(i),m_0) \leqq u_j(\sigma(j),m_{\sigma(i)}+m_{\sigma(j)}-m_0)\ \}$.
The quantities $\underline{m}_{ji}(z)$ and $\bar{m}_{ji}(z)$ are similarly defined.

Proposition 2: For all $e \equiv (Q,A,M;\ u_Q) \in \mathcal{E}$ *with* $Q \equiv \{i,j\}$, *if there is no* $\tilde{z} \in \mathcal{Z}(e)$ *such that* $u_i(\tilde{z}_i) = u_i(\tilde{z}_j)$ *and* $u_j(\tilde{z}_j) = u_j(\tilde{z}_i)$, *then for all* $z \equiv (\sigma,m)$, $z' \equiv (\sigma',m') \in N(e)$, $\sigma = \sigma'$.

Proof: Suppose that $u_i(z_i) > u_i(z_j)$ and $\sigma \neq \sigma'$. Let $\sigma(i) = \sigma'(j) \equiv \alpha$, and $\sigma(j) = \sigma'(i) \equiv \beta$. If $m_\alpha \leqq m'_\alpha$, then $m_\beta \geqq m'_\beta$, and $u_i(z'_i) \equiv u_i(\beta,m'_\beta) \leqq u_i(\beta,m_\beta) < u_i(\alpha,m_\alpha) \leqq u_i(\alpha,m'_\alpha) \equiv u_i(z'_j)$. Hence $z' \notin N(e)$. If $m_\alpha > m'_\alpha$, then $m_\beta < m'_\beta$, and $u_j(z'_j) \equiv u_j(\alpha,m'_\alpha) < u_j(\alpha,m_\alpha) \leqq u_j(\beta,m_\beta) < u_j(\beta,m'_\beta) \equiv u_j(z'_i)$. Thus we have $z' \notin N(e)$, a contradiction. *Q.E.D.*

By Proposition 2, the two allocations $[(\sigma(i),\underline{m}_{ij}(z)),\ (\sigma(j),\bar{m}_{ji}(z))]$ and $[(\sigma(i),\bar{m}_{ij}(z)),\ (\sigma(j),\underline{m}_{ji}(z))]$ are indeed the "end-points" of the set of envy-free allocations for the economy $(\{i,j\},\{\sigma(i),\sigma(j)\},m_{\sigma(i)}+m_{\sigma(j)};\{u_i,u_j\})$.[11,12] The former allocation is the worst for agent i and the best for agent j in this set, and the latter the opposite. Obviously,

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[11]If $\exists\ \tilde{z} \in \mathcal{Z}(e)$ such that $u_i(\tilde{z}_i) = u_i(\tilde{z}_j)$ and $u_j(\tilde{z}_j) = u_j(\tilde{z}_i)$, then $N(e) = \{\ (\tilde{z}_i,\tilde{z}_j),\ (\tilde{z}_j,\tilde{z}_i)\ \}$ and $\forall\ z \in N(e)$, $\bar{m}_{ij}(z) - \underline{m}_{ij}(z) = \bar{m}_{ji}(z) - \underline{m}_{ji}(z) = 0$.

[12]They are also the two egalitarian-equivalent and efficient allocations (Pazner and Schmeidler(1978a)) for this economy.

$\bar{m}_{ij}(z) - \underline{m}_{ij}(z) = \bar{m}_{ji}(z) - \underline{m}_{ji}(z)$. Let $s_{\{i,j\}}(z)$ be this difference. The number $s_{\{i,j\}}(z)$ is the **surplus for {i, j} at z**. Let

$p_{ij}(z) \equiv (m_{\sigma(i)} - \underline{m}_{ij}(z))/s_{\{i,j\}}(z)$ if $s_{\{i,j\}}(z) \neq 0$, and

$p_{ij}(z) \equiv 1/2$ if $s_{\{i,j\}}(z) = 0$.

Note that $0 \leqq p_{ij}(z) \leqq 1$, $0 \leqq p_{ji}(z) \leqq 1$ and $p_{ij}(z) + p_{ji}(z) = 1$. The number $p_{ij}(z)$ is **agent i's (proportional) share of the surplus for {i, j} at z.**[13] Let $p(z) \equiv (p_{ij}(z))_{\substack{i,j \in Q \\ i \neq j}} \in \mathbb{R}^{|Q|\times(|Q|-1)}$.

Alternatively, we measure the **distance that agent i is from envying agent j at z** by the maximal amount of money that can be added to the bundle of agent j without causing agent i to envy him, and we quantify how well agent i is treated in relation to agent j by this number. Let

$d_{ij}(z) \equiv \max\{\, m_0 \in \mathbb{R} \mid u_i(z_i) \geqq u_i(\sigma(j), m_{\sigma(j)} + m_0) \,\}$ and

$d(z) \equiv (d_{ij}(z))_{\substack{i,j \in Q \\ i \neq j}} \in \mathbb{R}^{|Q|\times(|Q|-1)}$.

Now given an economy $e \equiv (Q, A, M;\ u_Q)$ and an allocation $z \in N(e)$, we keep the complete record of how well each agent is treated *in relation to each other agent*. (This record contains $|Q|\times(|Q|-1)$ such numbers.) By taking the average of the numbers pertaining to a given agent, we obtain a measure of how well this agent is treated *on average* relative to all the other agents. Let

$p_i^a(z) \equiv \dfrac{1}{|Q|-1} \sum_{\substack{j \in Q \\ j \neq i}} p_{ij}(z)$ and $d_i^a(z) \equiv \dfrac{1}{|Q|-1} \sum_{\substack{j \in Q \\ j \neq i}} d_{ij}(z)$,

$p^a(z) \equiv (p_i^a(z))_{i \in Q} \in \mathbb{R}^Q$ and $d^a(z) \equiv (d_i^a(z))_{i \in Q} \in \mathbb{R}^Q$.

We have four distinct records on the basis of which we can evaluate

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[13] If $s_{\{i,j\}}(z) = 0$, then every agent is indifferent between his bundle and the other's at z, and in that sense, both agents are treated equally.

the allocation. Using these records, we propose to treat agents as equally as possible. A most natural way to do this is perhaps to choose allocations whose associated record has maximal minimum coordinate.[14]

A further refinement can be obtained by using the **lexicographic order**: For any two vectors $x, y \in \mathbb{R}^n$, say that x is **lexicographically greater than** y, if $x_1 > y_1$, or $[x_1 = y_1$ and $x_2 > y_2]$, $\cdots$, or $[x_1 = y_1, \cdots, x_{n-1} = y_{n-1}$, and $x_n > y_n]$. First rearrange the coordinates of the record associated with each allocation in increasing order. Then select the allocations whose reordered records are lexicographically maximal. Maximization (or minimization) in the lexicographic ordering is a standard procedure in social choice and game theory to perform selections.[15]

Note that none of the solutions proposed above depends on utility representations, all of them are nonempty for all $e \in \mathcal{E}$, and for two-person economies all choose the allocation at which the two agents are treated exactly equally.[16] These solutions should therefore be contrasted with a solution introduced by ADG, which consists in maximizing over the set of envy-free allocations the minimum coordinate of the associated list of utility levels. This solution depends on utility representations. Our refinements are more in the spirit of the literature inspired by the No-Envy concept, which is a purely ordinal concept.

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[14]Diamantaras and Thomson(1988), following Chaudhuri(1986), measured the distance that an agent is from envying another agent in classical economies by the maximal rate of proportional expansion of that agent's bundle compatible with no-envy, and proposed the subsolution of the No-Envy solution that consists in maximizing the minimum coordinate of the list of such distances for all pairs of agents.

[15]See the lexicographic maximin social choice rule (Sen,1970) and the nucleolus (Schmeidler,1969).

[16]Tadenuma(1989) establishes the single-valuedness of the solution that maximizes the minimum of the average distance.

4. Consistency and its Variants

We now formulate properties of solutions pertaining to certain situations involving variations in the number of agents.

Let $z \equiv (\sigma, m)$ be any one of the allocations that the solution Φ has recommended for an arbitrary economy $e \equiv (Q, A, M;\ u_Q)$, and let Q' be any subset of Q. Together, the members of Q' have received the collection $(\sigma(i))_{i \in Q'}$ of objects and the amount $\sum_{i \in Q'} m_{\sigma(i)}$ of money. If these resources had to be distributed among them, would $(z_i)_{i \in Q'}$ constitute a fair distribution? If the answer is always yes, then the solution is **consistent**. Consistency says that the fairness of a distribution can never be called into question internally by subgroups of agents.

Let $e \equiv (Q, A, M;\ u_Q) \in \mathcal{E}$, $Q' \subseteq Q$ and $z \equiv (\sigma, m) \in \mathcal{Z}(e)$ be given. The **subeconomy of e with respect to Q' and z**, denoted by $t^z_{Q'}(e)$, is the economy $(Q', \sigma(Q'), \sum_{i \in Q'} m_{\sigma(i)};\ u_{Q'})$.

Consistency (CONS): $\forall\ e \equiv (Q, A, M;\ u_Q) \in \mathcal{E}$, $\forall\ z \in \Phi(e)$, and $\forall\ Q' \subseteq Q$, $z_{Q'} \in \Phi(t^z_{Q'}(e))$.

Next, we propose two natural weakenings of Consistency. First, we only require the solution to provide *at least one* allocation for which the conclusion of Consistency is met. Second, we apply the requirement only to *subgroups of cardinality two*.

Weak Consistency (W.CONS): $\forall\ e \equiv (Q, A, M;\ u_Q) \in \mathcal{E}$, $\exists\ z \in \Phi(e)$ such that $\forall\ Q' \subseteq Q$, $z_{Q'} \in \Phi(t^z_{Q'}(e))$.

Bilateral Consistency (B.CONS): $\forall\ e \equiv (Q, A, M;\ u_Q) \in \mathcal{E}$, $\forall\ z \in \Phi(e)$, and

$\forall\ Q' \subseteq Q$ with $|Q'| = 2$, $z_{Q'} \in \Phi(t^z_{Q'}(e))$.

Next we consider a "dual" of Consistency, **Converse Consistency**. Whereas Consistency allowed us to deduce the Φ-optimality of the restriction of an allocation to a subgroup on the basis of the Φ-optimality of the allocation for some original economy, Converse Consistency permits the opposite operation. Given some allocation z, feasible for some economy $e \equiv (Q, A, M;\ u_Q)$, check whether its restriction to any two-person subgroup defines a desirable distribution among them of the objects and the amount of money that they have collectively received. If whenever the answer is yes for all subgroups, the allocation is in fact Φ-optimal for e, then we say that Φ satisfies Converse Consistency.

Converse Consistency (C.CONS): $\forall\ e \equiv (Q, A, M;\ u_Q) \in \mathcal{E}$, $\forall\ z \in \mathcal{Z}(e)$, if $\forall\ Q' \subseteq Q$ with $|Q'| = 2$, $z_{Q'} \in \Phi(t^z_{Q'}(e))$, then $z \in \Phi(e)$.

We also consider the following version, which is weaker in two respects. First, it requires Φ-optimality of the restrictions of the allocation under consideration to *all* proper subeconomies (instead of only subeconomies of cardinality two). Second, it applies only to sufficiently large economies. The larger the economy, the more conditions the allocation has to meet before being declared Φ-optimal for the whole economy.

Weak Converse Consistency (W.C.CONS): There exists a positive integer k such that $\forall\ e \equiv (Q, A, M;\ u_Q) \in \mathcal{E}$ with $|Q| \geqq k$, $\forall\ z \in \mathcal{Z}(e)$, if $\forall\ Q' \subset Q$ with $|Q'| \leqq |Q| - 1$, $z_{Q'} \in \Phi(t^z_{Q'}(e))$, then $z \in \Phi(e)$.

We will also use a very weak independence property. Let $e \equiv (Q, I, M;$

$u_Q) \in \mathcal{E}$ be given. Let $\pi : Q \to Q$ be a permutation of the members of Q, and let Π_Q be the set of these permutations. **An allocation $z' \in \mathcal{Z}(e)$ is obtained by an indifferent permutation from an allocation $z \in \mathcal{Z}(e)$** if there is $\pi \in \Pi_Q$ such that for all $i \in Q$, $z'_i = z_{\pi(i)}$, and $u(z') = u(z)$. This binary relation is an equivalence relation on $\mathcal{Z}(e)$, which we will denote by $\approx$.

In an indifferent permutation, the bundles to be allocated are unchanged. Who receives which bundle may change, but only for the agents that are indifferent between their old and new bundles.

The following property says that if an allocation can be obtained from a Φ-optimal allocation through an indifferent permutation, then it is also Φ-optimal.

Independence of Indifferent Permutations (I.I.P.): $\forall\, e \in \mathcal{E}$, $\forall\, z \in \Phi(e)$, and $\forall\, z' \in \mathcal{Z}(e)$, if $z' \approx z$, then $z' \in \Phi(e)$.

We begin with an investigation of Consistency. First, note that both the Pareto solution and the No-Envy solution satisfy the property. They also satisfy Independence of Indifferent Permutations. In fact, these solutions enjoy the two properties in classical economies as well.[17]

However, no *proper* subcorrespondence of the No-Envy solution satisfies both Consistency and Independence of Indifferent Permutations:

Theorem 1: If a subcorrespondence Φ *of the No-Envy solution* N *satisfies* I.I.P. *and* CONS, *then* Φ = N.

The proof of Theorem 1 relies on the following lemmas. Lemma 1 is due

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[17]For Consistency, this fact was noted and used in Thomson(1988a).

to ADG.

Lemma 1: Let $e \equiv (Q, A, M;\ u_Q) \in \mathcal{E}$, *and* $z \equiv (\sigma, m)$, $z' \equiv (\sigma', m') \in N(e)$ *be given. Also, let*

$$Q_{z>z'} \equiv \{\ i \in Q \mid u_i(z_i) > u_i(z'_i)\ \},\quad A_{m>m'} \equiv \{\ \alpha \in A \mid m_\alpha > m'_\alpha\ \},$$
$$Q_{z<z'} \equiv \{\ i \in Q \mid u_i(z_i) < u_i(z'_i)\ \},\quad A_{m<m'} \equiv \{\ \alpha \in A \mid m_\alpha < m'_\alpha\ \},$$
$$Q_{z\sim z'} \equiv \{\ i \in Q \mid u_i(z_i) = u_i(z'_i)\ \},\quad A_{m=m'} \equiv \{\ \alpha \in A \mid m_\alpha = m'_\alpha\ \}.$$

Then, both σ and σ' *are bijections from* $Q_{z>z'}$ *to* $A_{m>m'}$, $Q_{z<z'}$ *to* $A_{m<m'}$, *and* $Q_{z\sim z'}$ *to* $A_{m=m'}$.

The next lemma is the key to Theorem 1. It states that for any economy e and any allocation $z \in N(e)$, there is an economy e' with one additional agent and one additional object, such that (i) $N(e')$ contains an envy-free allocation z' whose restriction to the original economy coincides with z, and (ii) any other allocation in $N(e')$ is obtained from z' by an indifferent permutation.

Lemma 2: Let $e \equiv (Q, A, M;\ u_Q) \in \mathcal{E}$ *and* $z \equiv (\sigma, m) \in N(e)$ *be given. Then there exists* $e' \equiv (Q', A', M';\ u'_{Q'}) \in \mathcal{E}$ *such that:*

(i) $Q' = Q \cup \{i_0\}$ *for some* $i_0 \in \mathcal{Q}\backslash Q$, *and* $A' = A \cup \{\alpha_0\}$ *for some* $\alpha_0 \in \mathcal{A}\backslash A$.

(ii) $z' \in N(e')$, *where* $z' \equiv (\sigma', m')$ *is defined by* $\sigma'_Q = \sigma$, $\sigma'(i_0) = \alpha_0$, *and* $m'_A = m$, $m'_{\alpha_0} = M' - M \equiv M_0$.

(iii) *For all* $z'' \in N(e')$, $z'' \approx z'$.

(iv) $e = t_Q^{z'}(e')$.

Proof: Let $u'_{Q'}$ be such that

(1) for all $i \in Q$, all $\alpha \in A$ and all $m_0 \in \mathbb{R}$, $u'_i(\alpha, m_0) = u_i(\alpha, m_0)$

(2) for all $i \in Q$, $u'_i(\alpha_0, M_0) = u_i(\sigma(i),\ m_{\sigma(i)})$, and

(3) for all $\alpha \in A$, $u'_{i_0}(\alpha_0, M_0) = u'_{i_0}(\alpha, m_\alpha)$.

Then, it is easily checked that $z' \in N(e')$, and that $e = t_Q^{z'}(e')$. It remains to show that for all $z'' \in N(e')$, $z'' \approx z'$.

Claim 1: $m''_{\alpha_0} \leqq m'_{\alpha_0}$. Suppose that $m''_{\alpha_0} > m'_{\alpha_0}$. Since $z'' \in N(e')$, then for all $i \in Q'$, $u'_i(z''_i) \geqq u'_i(\alpha_0, m''_{\alpha_0}) > u'_i(\alpha_0, m'_{\alpha_0}) = u'_i(\sigma'(i), m'_{\sigma'(i)}) = u'_i(z'_i)$. This is impossible because $z' \in P(e')$ by Proposition 1.

Claim 2: For all $\alpha \in A'$, $m''_\alpha = m'_\alpha$. Otherwise, for some $\beta \in A$, $m''_\beta > m'_\beta$. It follows from Lemma 1, Claim 1 and $\sigma'(i_0) = \alpha_0$ that $u'_{i_0}(z''_{i_0}) \leqq u'_{i_0}(z'_{i_0})$. Hence, $u'_{i_0}(z''_{i_0}) \leqq u'_{i_0}(z'_{i_0}) = u'_{i_0}(\alpha_0, M_0) = u'_{i_0}(\beta, m'_\beta) < u'_{i_0}(\beta, m''_\beta)$, which implies $z'' \notin N(e')$, a contradiction.

By Lemma 1 and Claim 2, $m'' = m'$ and $u'(z'') = u'(z')$, which means $z'' \approx z'$. *Q.E.D.*

Proof of Theorem 1: Let $e \equiv (Q, A, M;\ u_Q) \in \mathscr{E}$ and $z \in N(e)$. Let $e' \equiv (Q', A', M';\ u'_{Q'}) \in \mathscr{E}$ and z' be as described in the statement of Lemma 2. Let $z'' \in \Phi(e')$. Since $\Phi(e') \subseteq N(e')$, $z'' \in N(e')$. By Lemma 2, $z'' \approx z'$. It follows from I.I.P. that $z' \in \Phi(e')$. Then, by CONS, $z'_Q = z \in \Phi(t_Q^{z'}) = \Phi(e)$. Hence, $N(e) \subseteq \Phi(e)$. Thus, we have $\Phi(e) = N(e)$. *Q.E.D.*

If we weaken Consistency to Bilateral Consistency, a weakening that has proved sufficient for characterizations in other contexts,[18] then a characterization of the No-Envy solution cannot be obtained. In fact, many solutions satisfying these properties can be constructed.

Proposition 3: There are proper subcorrespondences of the No-Envy solution

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[18]See Lensberg(1987) and Young(1987).

that satisfy I.I.P. *and* B.CONS.

Proof: Let $\Phi^*(e)$ be an *arbitrary* selection from $N(e)$ for all $e \equiv (Q,I,M; u_Q)$ with $|Q| \geqq 3$. Extend it if needed so that it satisfies I.I.P. Let $\Phi^*(e) = N(e)$ for all $e \equiv (Q,I,M; u_Q)$ with $|Q| \leqq 2$. Then Φ^* is a (typically) proper subcorrespondence of the No-Envy solution that satisfies I.I.P. and B.CONS. *Q.E.D.*

Next we investigate the implications of Converse Consistency. First, we note that the No-Envy solution satisfies the condition. Our next result is that the Pareto solution does not. In fact, it does not even satisfy Weak Converse Consistency.

Proposition 4: The Pareto solution does not satisfy W.C.CONS.

Proof: Let k be *any* positive integer such that $k \geqq 3$. Let $e \equiv (Q,A,M; u_Q) \in \mathcal{E}$ be such that $Q = \{1,2,\cdots,k\}$, $A = \{\alpha_1,\alpha_2,\cdots,\alpha_k\}$, $M = 2k$, and

$$u_1(\alpha_1,2) = u_1(\alpha_2,2k) = u_1(\alpha_3,2k) = \cdots = u_1(\alpha_{k-1},2k) = u_1(\alpha_k,1),$$
$$u_2(\alpha_1,1) = u_2(\alpha_2,2) \quad = u_2(\alpha_3,2k) = \cdots = u_2(\alpha_{k-1},2k) = u_k(\alpha_k,2k),$$
$$\cdots\cdots$$
$$u_k(\alpha_1,2k) = u_k(\alpha_2,2k) = u_k(\alpha_3,2k) = \cdots = u_k(\alpha_{k-1},1) = u_k(\alpha_k,2).$$

Let $z \equiv (\ (\alpha_1,2),\ (\alpha_2,2),\ \cdots,\ (\alpha_k,2)\ )$.

Claim: For all $Q' \subset Q$ with $|Q'| \leqq |Q| - 1$, $z_{Q'} \in P(t^z_{Q'}(e))$. Suppose that there is $z' \equiv (\sigma',m') \in Z(t^z_{Q'}(e))$ such that $u_i(z'_i) \geqq u_i(z_i)$ for all $i \in Q'$ with at least one strict inequality. First, note that $\sigma' \neq \sigma_Q$. (Otherwise z' cannot Pareto-dominate $z_{Q'}$.) Let $j \in Q'$ be such that $\sigma'(j) \neq \sigma(j)$. Since $u_j(z'_j) \geqq u_j(z_j)$, either (1) $j-1 \in Q'$ and $z'_j = (\alpha_{j-1}, m'_{\alpha_{j-1}})$ for some $m'_{\alpha_{j-1}} \geqq 1$ or (2) $z'_j = (\alpha_n, m'_{\alpha_n})$ for some $n \neq j$, $j-1$, and some $m'_{\alpha_n} \geqq 2k$. (If $j = 1$, then define $j - 1 = k$.) If (2) is the case, then there is an agent in Q' that receives a negative amount of money, and he is worse off at z' than at $z_{Q'}$. Hence (1) must hold: $j-1 \in Q'$. Then $\sigma'(j-1) \neq \sigma(j-1) = \alpha_{j-1} = \sigma'(j)$. By the same argument, $j-2 \in Q'$. Repeating this argument, we have that for all $i \in Q'$, $i-1 \in Q'$. But since $|Q'| \leqq |Q| - 1$, there is at least one agent $i_0 \in Q'$ such that $i_0-1 \notin Q'$. Thus the desired contradiction has been obtained, and the claim is proved.

However, z is Pareto-dominated by $z' \equiv (z_k, z_1, z_2, \cdots, z_{k-1})$. Thus, $z \notin P(e)$. *Q.E.D.*

Next we clarify the logical relation between Consistency and Converse Consistency when combined with No-Envy and Independence of Indifferent Permutations.

Corollary 1: If a subcorrespondence of the No-Envy solution satisfies I.I.P. *and* CONS, *then it satisfies* C.CONS.

Proof: The claim immediately follows from Theorem 1 and the fact that N satisfies C.CONS. *Q.E.D.*

The following converse of Corollary 1 is not true, however. Converse Consistency and Independence of Indifferent Permutations together are not strong enough to characterize the No-Envy solution. Again, solutions satisfying these properties can easily be constructed.

Proposition 5: There are proper subcorrespondences of the No-Envy solution that satisfy I.I.P. *and* C.CONS.

Proof: Let $\Phi^*(e)$ be an *arbitrary* selection from $N(e)$ for all $e \equiv (Q, I, M; u_Q)$ with $|Q| = 2$. Extend it if needed so that it satisfies I.I.P. Let $\Phi^*(e) = N(e)$ for all $e \equiv (Q, I, M; u_Q)$ with $|Q| \neq 2$. Then Φ^* is a (typically) proper subcorrespondence of the No-Envy solution that satisfies

I.I.P. and C.CONS. *Q.E.D.*

5. Monotonicity

This section formulates and examines the monotonicity property. Monotonicity says that if z is Φ-optimal for some economy, and preferences are changed in such a way that for each agent, z remains at least as good as any feasible allocation to which it was initially weakly preferred, then z is Φ-optimal for the new economy.

Let $e \equiv (Q, A, M;\ u_Q) \in \mathcal{E}$, $z \in \mathcal{Z}(e)$ and $i \in Q$ be given. Then, u'_i is **obtained from u_i by a monotonic transformation at z** if

$$\{\ z'_i \in \mathcal{Z}_i(e)\ |\ u_i(z'_i) \leqq u_i(z_i)\ \} \subseteq \{\ z'_i \in \mathcal{Z}_i(e)\ |\ u'_i(z'_i) \leqq u'_i(z_i)\ \}.$$ [19]

Monotonicity (MON): $\forall\ e \equiv (Q, A, M;\ u_Q) \in \mathcal{E}$, $\forall\ e' \equiv (Q, A, M;\ u'_Q) \in \mathcal{E}$, $\forall\ z \in \Phi(e)$, if $\forall\ i \in Q$, u'_i is obtained from u_i by a monotonic transformation at z, then $z \in \Phi(e')$.

It is clear that the Pareto solution and the No-Envy solution satisfy Monotonicity.[20] We will also need the following condition.

Complete Indifference (C.I.): $\forall\ e \equiv (Q, A, M;\ u_Q) \in \mathcal{E}$, $\forall\ z \in \mathcal{Z}(e)$, if $\forall\ i,\ j \in Q$, $u_i(z_i) = u_i(z_j)$, then $z \in \Phi(e)$.

If at an allocation, every agent is indifferent between his bundle and any other agent's, then the allocation is Φ-optimal. Note that such an

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[19] $\mathcal{Z}_i(e)$ is the projection of $\mathcal{Z}(e)$ on the i's coordinate: $\mathcal{Z}_i(e) \equiv \{\ z_i \in A \times \mathbb{R}\ |\ \exists\ z' \in \mathcal{Z}(e)$ such that $z'_i = z_i\ \}$.

[20] These solutions satisfy the condition in classical economies as well. Thomson(1987) discusses the monotonicity of the No-Envy solution.

allocation is Pareto-efficient by Proposition 1. If we can attain, at no cost of efficiency, an allocation where all agents are treated equally, we should regard it as desirable.

Obviously, both the Pareto solution and the No-Envy solution satisfy Complete Indifference.

Our second characterization of the No-Envy solution is the counterpart of the following result for classical economies (Thomson(1987)): If Φ satisfies MON, and the condition "if all agents have identical linear preferences, then all allocations that are individually rational from equal division and Pareto-efficient are Φ-optimal," then Φ contains the Walrasian solution from equal division.

Theorem 2: If a subcorrespondence Φ *of the No-Envy solution* N *satisfies* C.I. *and* MON, *then* $\Phi = N$.

Proof: Let $e \equiv (Q, A, M;\ u_Q) \in \mathcal{E}$ and $z \in N(e)$. For each $i \in Q$, let u_i' be such that $u_i'(z_i) = u_i'(z_j)$ for all $j \in Q$. Let $e' \equiv (Q, A, M;\ u_Q')$. By C.I., $z \in \Phi(e')$. Since $z \in N(e)$, $u_i(z_i) \geqq u_i(z_j)$ for all $j \in Q$. Hence, $\{\ z_i' \in \mathcal{Z}_i(e')\ |\ u_i'(z_i') \leqq u_i'(z_i)\ \} \subseteq \{\ z_i' \in \mathcal{Z}_i(e')\ |\ u_i(z_i') \leqq u_i(z_i)\ \}$. Thus, u_i is obtained from u_i' by a monotonic transformation at z. It follows from MON that $z \in \Phi(e)$. Therefore, $N(e) \subseteq \Phi(e)$. *Q.E.D.*

6. Remarks

In this section, we first establish, as corollaries to our theorems, the equivalence with the No-Envy solution of two solutions that have been proposed as solutions to the problem of fair allocation in economies with indivisibilities. Next we examine the robustness of our results to restrictions on the domain.

(a) The **Equal Income Walrasian solution** associates with each economy its set of allocations that are Walrasian for an appropriate choice of a common income. The **Group-No-Envy solution** is designed to ensure that groups are treated fairly in relation to each other. In classical economies, these two solutions are much more restrictive than the No-Envy solution. Yet, on our domain, they are equivalent to it. A direct proof of these equivalences is given by Svensson(1983). They can however be obtained as corollaries of our two characterization theorems. This is because both solutions satisfy all the hypotheses of these theorems.

Let $e \equiv (Q, A, M;\ u_Q)$ be given. Say that $z \in \mathcal{Z}(e)$ is an **equal income Walrasian allocation for** e if there are prices p, one for each of the objects, and a common income M_0 such that for each $i \in Q$, z_i maximizes u_i in $B_i(p, M_0) \equiv \{\ (\alpha, m_0) \in A \times \mathbb{R} \mid p_\alpha + m_0 \leqq M_0\ \}$. Let $\mathbf{W^{ei}}(e)$ be the set of these allocations.

The allocation $z \equiv (\sigma, m) \in \mathcal{Z}(e)$ is a **group-envy-free allocation for** e if for all $Q', Q'' \subseteq Q$ with $|Q'| = |Q''|$, there is no $z' \in \mathcal{Z}(Q', \sigma(Q''), \sum_{j \in Q''} m_j;\ u_{Q'})$ such that $u_i(z'_i) \geqq u_i(z_i)$ for all $i \in Q'$ with at least one strict inequality. Let $\mathbf{N^g}(e)$ be the set of these allocations.[21]

Corollary 2: $W^{ei} = N^g = N$.

Proof: We have two ways to prove the claim. As easily checked, both W^{ei} and N^g are subcorrespondences of N, and satisfy I.I.P. and CONS. Thus, by Theorem 1, $W^{ei} = N^g = N$. It is also clear that W^{ei} and N^g satisfy C.I. and

[21]This concept was introduced for classical economies by Vind(1971) and Varian(1974).

MON.[22] Thus, by Theorem 2, $W^{ei} = N^g = N$. *Q.E.D.*

(b) ADG showed the existence of envy-free allocations for the class of economies studied in this paper. In some cases, it might be more natural to assume that the amounts of money that agents receive are nonnegative. In order for envy-free allocations to exist in this case, there should be a sufficiently large amount of money (Maskin, 1987, and ADG). On the domain so specified, our theorems remain valid.

Our analysis and our results can also be extended to the following more general cases:

(i) the case where the numbers of agents and objects are not equal. For the proof, we add **fictitious agents** or **fictitious objects** so as to obtain an artificial economy with equal numbers of agents and objects (Crawford and Knoer(1979) and ADG). (See the Appendix.)

(ii) the case where each object can be assigned to several agents (e.g., types of houses, types of jobs, or local public goods). For each object α, let n_α be the maximum number of agents to which object α can be assigned. For the proof, we construct an artificial economy where there are n_α identical objects corresponding to each α.

Finally our results also hold true on the following restricted domains:

(iii) the class of economies $(Q, A, M; u_Q)$ where all agents have the same ordering of objects $A \equiv \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ independent of the amount of money they hold, i.e., for all $i \in Q$ and all $m_0 \in \mathbb{R}$, $u_i(\alpha_1, m_0) \geqq u_i(\alpha_2, m_0) \geqq \cdots \geqq u_i(\alpha_n, m_0)$ (up to renaming of the goods).

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[22] Actually, for classical economies, the Equal Income Walrasian solution satisfies MON only in the interior of the feasible set, unless some domain restrictions are imposed. Here, the property holds without further restrictions.

(iv) the class of quasi-linear economies: An economy $e \equiv (Q, A, M; u_Q)$ is **quasi-linear** if for all $i \in Q$, there is a function $K_i : A \to \mathbb{R}$ such that for all $\alpha \in A$ and all $m_0 \in \mathbb{R}$, $u_i(\alpha, m_0) = K_i(\alpha) + m_0$.

Appendix

We describe here a device (Crawford and Knoer(1981) and ADG) that permits us to deal with situations where the numbers of agents and objects are not equal.

The economy $e \equiv (Q, A, M;\ u_Q)$ is **balanced** if $|Q| = |A|$. When $e' \equiv (Q', A', M';\ u'_{Q'})$ is not balanced, we associate with it the balanced economy $e \equiv (Q, A, M\colon\ u_Q)$ as follows: If $|Q'| > |A'|$, then we add $|Q'| - |A'|$ **fictitious objects**, which have no value to any agent: If $\alpha_0 \in A$ is such an object, then for all $i \in Q$, and for all $m_0 \in \mathbb{R}$, set $u_i(\alpha_0, m_0) = m_0$. Similarly, if $|Q'| < |A'|$, then we add $|A'| - |Q'|$ **fictitious agents** who place value only on money: If $i_0 \in Q$ is such an agent, then for all $\alpha \in A$, and for all $m_0 \in \mathbb{R}$, $u_{i_0}(\alpha, m_0) = m_0$.

Let $\mathcal{E}$ be the class of all balanced economies. Let $e \equiv (Q, A, M;\ u_Q) \in \mathcal{E}$. Then let Q_F denote the set of fictitious agents in Q, and A_F the set of fictitious objects. Let $Q_R \equiv Q \backslash Q_F$ be the set of **real agents**, $A_R \equiv A \backslash A_F$ the set of **real objects**. A **feasible allocation for** e is a pair $z \equiv (\sigma, m)$ where $\sigma : Q \to A$ is a bijection and $m \in \mathbb{R}^A$ is such that $\sum_{i \in Q_R} m_{\sigma(i)} = M,$. The feasible allocation $z \equiv (\sigma, m)$ is **envy-free for** e if for all $i \in Q_F$, $m_{\sigma(i)} = \max_{j \in Q_R} \{m_{\sigma(j)}\}$, and for all $i \in Q_R$ and all $\alpha \in A$, $u_i(z_i) \geqq u_i(\alpha, m_\alpha)$.[23]

[23]This definition is due to ADG. They call the allocation a **strongly envy-free allocation for** e.

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