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REPEATED GAMES, FINITE AUTOMATA, AND COMPLEXITY*

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Abstract

We study the structure of Nash equilibria in 2-player repeated games played with finite automata, when complexity considerations enter into the players' decision calculus. We employ a more general measure of complexity than that used by previous authors, viz., the complexity of a machine depends not only on the number of states but also on the complexity of the transition mechanism in the machine. We show that, in this set-up, the resulting Nash equilibria are trivial: the machines recommend actions each period which are stage game best responses to one another.

*This is an updated version of an earlier paper with the same title. In particular, we consider more general measures of complexity. We are grateful to Professor Ariel Rubinstein for comments and suggestions. We are also grateful to seminar participants at the Universities of Rochester and Arizona, particularly to John Boyd and Mahmoud El-Gamal.

Repeated Games, Finite Automata, and Complexity

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1. Introduction

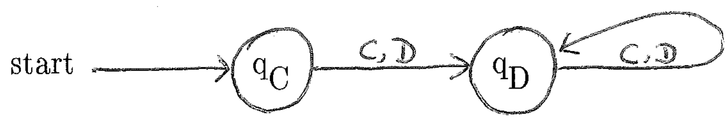
In this paper we study the structure of Nash equilibria in 2–player repeated games played with finite automata. Such an approach was first suggested by Aumann (1981) for the study of decision making with bounded rationality; recent applications include Neyman (1985) in the context of a finitely repeated prisoners' dilemma game using machines of fixed sizes (see also Zemel (1986)); Kalai and Stanford (1988), who show that every strategy of an infinitely repeated game can be uniquely described by an automaton of minimal size; and Ben–Porath and Peleg (1987), who show that any payoff of an infinitely repeated game can be approximated using automata with a finite set of states (for a more complete list of references, the reader should consult the excellent survey in Kalai (1987)). In the current paper we are concerned with the effects of complexity and implementation costs on the set of equilibria in an infinitely repeated game, issues that are analyzed in Rubinstein (1986) and Abreu and Rubinstein (1988). In particular, we examine the set of equilibria when a more general measure of complexity is employed by the players.

We build on the model of Abreu and Rubinstein (1988) where the strategic interaction between the two players occurs over the simultaneous selection of finite automata, or (Moore) machines, with which to implement their repeated game strategies. Such machines consist of (i) a finite set of states, one of which is identified as the initial state, (ii) an output function determining the action taken at the t^{th} stage of the game as a function of the t^{th} state, and (iii) a transition function describing the $(t+1)^{st}$ state as a function of the t^{th} state as well as the action by the other player at the t^{th} stage of the game. Any pair of machines for the two players then determines a sequence of action pairs and ultimately a repeated game payoff.

In Abreu and Rubinstein (1988) preferences for player i defined over machine pairs (M_i, M_j) are increasing in the repeated game payoffs induced by the machines, and decreasing in the complexity of M_i. Complexity is measured by the number of states a machine possesses; thus the lower the number of states, the less costly is the implementation of the strategy. Within this framework, Abreu and Rubinstein (1988) show that the set of Nash equilibrium payoffs in the repeated game is dramatically reduced from the standard Folk Theorem result (cf. Fudenberg and Maskin (1985)). In particular, since the player's preferences are allowed to be lexicographic in repeated game payoffs and complexity, this result demonstrates a non–robustness of the Folk Theorem to "infinitesimal" considerations of decision costs.

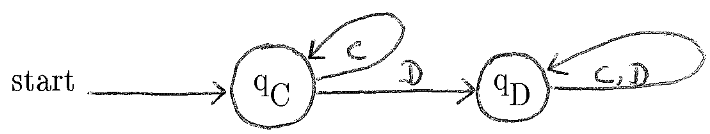
Abreu and Rubinstein (1988) note that their measure of implementation cost captures the following complexity issues: (i) if monitoring an opponent's behavior is costly, players will attempt to economize on states held to keep track of the other's actions, and (ii) if "punishment states", that is, states which lead (through the output function) to actions designed to punish deviant behavior by the opponent, are costly to maintain, players will eliminate such states if they are never used. In particular, this shows that "empty" threats to take certain actions cannot exist in equilibrium. However they also mention that their measure is quite special, and ignores other possible dimensions with which to measure a machine's complexity such as the complexity of the transition mechanism. (This point is also raised in Rubinstein (1986).) Indeed, we believe that their measure does not allow for a sufficiently fine distinction between machines that appear quite different from the perspective of implementation costs.

A simple example will serve to illustrate this point. Consider the repeated prisoner's dilemma, where in each period a player has two options: Cooperate (C) or Defect (D). One possible strategy for player 1 is to employ a machine that, regardless of 2's choice of action, plays action C initially and then switches to D forever. Diagrammatically, this machine can be represented as

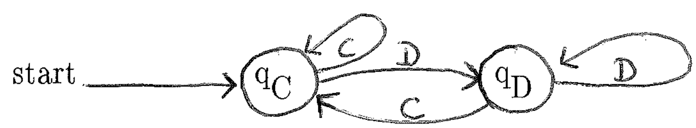


(Here, as in the next two figures, player 1's machine recommends C (resp. D) when the state is q_C (resp. q_D). The letters on the arrows represent 2's choice of action, and the arrows themselves represent the transitions.)

A second possible machine for player 1 that appears slightly more complicated is one which adopts the "grim" trigger strategy (play C until 2 plays D, then play D forever). This also involves two states:



Finally, a tit–for–tat strategy can also be implemented using only two states:



All three of these machines employ two states, and hence are of equal complexity as defined by Abreu and Rubinstein (1988). Yet we would argue that the first machine is inherently less "complex" than the second, and the second less "complex" than the third, where clearly this augmented definition of complexity is a function of the number of transitions or distinct routes between states. The rationale for this measure is related to those noted by Abreu and Rubinstein (1988): if monitoring the opponents' behavior is costly, then at each state players will attempt to economize on the number of events which require distinct transitions. Thus the grim strategy is less complex under this measure than tit–for–tat, since at state q_D player 1 no longer needs to monitor player 2's behavior; similarly the first machine is less complex than the second. In general, if two machines differ only in that one has more transitions emanating from each state, then such a machine requires a higher degree of precision in the monitoring of the opponent's behavior. Second, it may simply be that the transitions, like states, are costly to maintain; the lower the number of contingencies the machine has to identify to proceed, the less costly is the implementaion of the strategy.

In what follows, the preference structure of Abreu–Rubinstein (1988) is extended to capture this notion of transitional complexity as one aspect of the complexity of a machine. In section 2, we present 3 different complexity measures, of varying degrees of restrictiveness. Given a machine M_i for player i, a machine M_j is at least as good as another machine M_j' if (i) M_j yields at least as high a repeated game payoff as M_j', and (ii) M_j is at most as complex as M_j'. Such preferences as we consider include the case where transitional complexity enters lexicographically and last, thus giving an "infintesimal" change over Abreu–Rubinstein (1988), but also includes numerous other cases. In particular, we allow for the case where players use different complexity measures from each other, reflecting possible differential advantages.

The main result of the paper (Theorem 2 below) is simply stated: under <u>any</u> of these measures, any Nash equilibrium of the machine game is such that at each stage of the repeated game the players adopt one–shot Nash equilibrium actions. In payoff space this implies that, if preferences are lexicographic with repeated game payoffs evaluated according to the limit of means, the only equilibrium payoffs to the players are those that are (rational) convex combinations of stage–game Nash equilibrium payoffs. So, for example, when finite automata play the repeated prisoners' dilemma, the <u>only</u> Nash equilibrium payoff is that resulting when both players play the suboptimal strategy "defect" forever. Interestingly, for some other games–such as the "Battle of the Sexes," the set of equilibrium payoffs resulting from our formulation of preferences, is the same as that in Abreu–Rubinstein (1988). A comparison of our results with theirs, together with an intuitive explaination of our result is provided after the proof of our main result. In the course of proving our main theorem, we also prove that in the limit–average payoffs case, the machines in a N.E. must start cycling between states right from the beginning. This is in contrast to the result in Abreu–Rubinstein (1988), where the machines eventually settle down into a cycle. We also show (Theorem 1) that all the qualitative results of Abreu–Ruvinstein (1988) regarding the lengths of cycles, etc., continue to hold in our framework.

The paper is organized as follows: Section 2 gives the description of the model; Section 3 contains the statement and proof of the main theorem; Section 4 includes a discussion of possible extensions of the model; and the Appendix shows that the relevant results in Abreu and Rubinstein (1988) continue to hold in the current model.

2. The model

2.1 Notation and Definitions

Let $G = (S_1,S_2,u_1,u_2)$ denote a two–person game in normal form, where S_i is a finite set of actions for player i and $u_i:S_1\times S_2 \rightarrow \mathbb{R}$ is i's payoff function. Let N(G) denote the set of Nash equilibria of the game G; i.e. $(s_1^*,s_2^*) \in N(G)$ iff $\forall\ (s_1,s_2) \in S_1\times S_2$, $u_1(s_1^*,s_2^*) \geq u_1(s_1,s_2^*)$ and $u_2(s_1^*,s_2^*) \geq u_2(s_1^*,s_2)$.

The supergame of G, denoted G^∞, consists of an infinite sequence of repetitions of G at times t = 1,2,3,... . At period t the players simultaneously select an element of S_i, denoted s_i^t, which become common knowledge.

The players determine a sequence of actions in the supergame G^∞ by simultaneously selecting finite automata, or (Moore) machines [see Hopcroft and Ullman (1979)]. A machine for player i, denoted M_i, is a four–tuple $(Q_i,q_i^1,\lambda_i,\mu_i)$, where Q_i is a finite set of states, $q_i^1 \in Q_i$ is the initial state, $\lambda_i:Q_i\rightarrow S_i$ is the output function, and $\mu_i:Q_i\times S_j\rightarrow Q_i$ is the transition function. A pair of machines (M_1,M_2) induces a sequence of state pairs (q^t) and action pairs (s^t) in the obvious manner:

$$q^1 = (q_1^1, q_2^1),$$
$$s^t = (\lambda_1(q_1^t), \lambda_2(q_2^t)),$$
$$q^{t+1} = (\mu_1(q_1^t, s_2^t), \mu_2(q_2^t, s_1^t)).$$

Since the machines are finite both q_i^t and q^t eventually cycle. Let T_i denote the length of the cycle of q_i^t, and T that of q^t; further let the first cycle in q^t begin and end at t_1 and t_2, respectively. We refer to the time period $t < t_1$ as the pre–cycle phase. By the stationarity of the output and transition functions the continuation of q^t after t_2+1 is just like after t_1.

$\pi_i(M_1,M_2)$ denotes the repeated game payoff induced by the machines (M_1,M_2), where this is evaluated either according to the limit of means or with discounting:

(i) limit of means

$$\pi_i(M_1,M_2) = \lim_{T'\to\infty} \frac{1}{T'} \sum_{t=1}^{T'} u_i(s^t) = \frac{1}{t_2-t_1+1} \sum_{t=t_1}^{t_2} u_i(s^t) \ ;$$

(ii) discounting

$$\pi_i(M_1,M_2) = \frac{1-\delta}{\delta} \sum_{t=1}^{\infty} \delta^t u_i(s^t) \ .$$

2.2 Measures of Complexity

As mentioned in the Introduction, we consider 3 different measures of complexity designed to capture the notion of monitoring costs. The following notation will be used: $|Q_i|$ will denote the number of elements in Q_i. $R(q_i)$ will denote the number of distinct transitions from $q_i \in Q_i$. That is, $R(q_i)$ is the number of equivalence classes at q_i, where s_j and $\hat{s}_j$ are defined to be equivalent at q_i if $\mu_i(q_i, s_j) = \mu_i(q_i, \hat{s}_j)$. Finally, $R(M_i)$ will denote the number of distinct transitions in

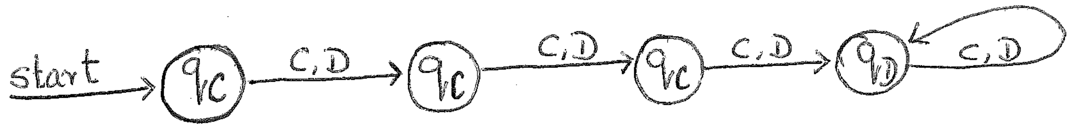
M_i, i.e., $R(M_i) = \Sigma_{q_i \in Q_i} R(q_i)$. Where M_i is understood we shall use the more compact notation R_i for $R(M_i)$.

Measure 1 The simplest measure of state–transtional complexity that we consider is obtained by counting the number of state–action pairs (q_i, s_j) that require distinct transitions. This is easily seen to be the same as $R(M_i)$. Thus, under this measure of complexity, M_i is more complex than M_i' if and only if $R(M_i) > R(M_i')$. Note that: (i) this measure completely orders all machines with respect to their complexity, and (ii) under this measure the examples of the previous section are increasing in complexity, although they all employ only 2 states.

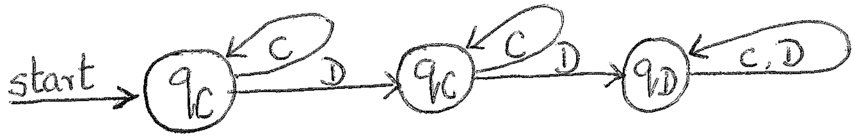
Measure 2 Our second complexity measure extends the first. Rather than just count the number of transitions in a machine, we define a 2–dimensional measure of complexity $(|Q_i|, R(M_i))$ for a machine M_i. Under this measure, M_i is more complex than M_i' if $|Q_i| \geq |Q_i'|$ and $R(M_i) \geq R(M_i')$ with at least one strict inequality. As defined, this measure does not completely order machines according to their degree of complexity. We note, however, that any extension making it a complete ordering would not change our results. In particular, the measure could be completed lexicographically in either way: M_i is more complex than M_i' iff $|Q_i| > |Q_i'|$ or $|Q_i| = |Q_i'|$ and $R(M_i) > R(M_i')$; or M_i is more complex than M_i' iff $R(M_i) > R(M_i')$ or $R(M_i) = R(M_i')$ and $|Q_i| > |Q_i'|$. As a special case, therefore, is the one where the number of transitions are appended on to the Abreu–Rubinstein (1988) preferences lexicographically and last.

We also note that measures 1 and 2 could yield different rankings of complexity, i.e., neither need nest the other. Consider for example the machines described below which implement strategies in the repeated–prisoner's dilemma:

Machine 1



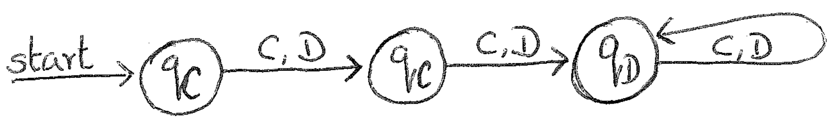
Machine 2



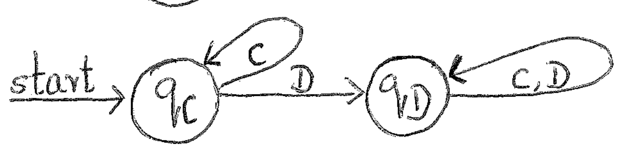
Then, the lexicographic completion of measure 2 with states first would pick machine 1 as more complex, while measure 1 would always pick machine 2 as the more complex.

Measure 3[1] By considering a 2–dimensional measure of complexity, measure 2 does capture both potential size and maintenence costs (number of states), and monitoring costs (number of transitions). However, measure 2 alsodoes not permit a sufficiently fine distinction between machines. To illustrate, consider the following 2 machines implementing strategies in the Prisoner's dilemma:

Machine 3



Machine 4



Both use 3 transitions; machine 3 uses 3 states while machine 4 uses only 2. Measure 2 would therefore unambiguously select machine 4 as the less complex. This, however, does not always seem reasonable, especially when states are "cheap" but transitions are

[1]John Boyd independently suggested this as a measure of complexity. Mahmoud El–Gamal suggested to us the possibility of incorporating differential transition costs such that not all transitions in the machine have the same cost. For example, if more transitions emanate from one state than another, transitions from the first state would have a greater weight than those from the second. We believe this would have no impact on our results.

not: the transitions in machine 3 involve no monitoring at all, while in machine 4 this is not true for those emanating from q_C. The measure appears in this case to pay insufficient attention to monitoring costs.

The dilemma is easily resolved when we note that every state must have at least one transition emanating from it. Monitoring costs at a state exist only if there is more than one transition leading from it. Thus, we define M_i to be more complex than M_i' if $(R(M_i) - |Q_i|) \geq (R(M_i') - |Q_i'|)$ and $|Q_i| \geq |Q_i'|$ with at least one strict inequality.

As with measure 2, this measure is not complete, but this is irrelevant for our purpose. The reader is free to complete these measures in any way he or she finds desirable. Some aspects of completing these measures are however interesting. It was observed earlier that measures 1 and 2 could yield different rankings of complexity. The same is true of measures 1 and 3, and of measures 2 and 3. Consider, for example, the case where measure 3 is completed lexicographically with net transitions $(R(M_i) - |Q_i|)$ first. (We write M1 for machine 1, etc.) Then, between M3 and M4, measure 3 picks M3 as the less complex,[2] while M3 would be the more complex under any completion of measure 2. Similarly, between M1 and M2, measure 3 under lexicographic completion with states first, would pick M2 as the less complex, while measure 1 would pick M1 for this spot. It is interesting to observe, on the other hand, that measure 1 is a special completion of the 2–dimensional measure 3, obtained by summing the coordinates $(R(M_i) - |Q_i|, |Q_i|)$ of the latter. In general, however, none of the measures need nest any of the others.[3]

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[2]Measure 3 is very flexible. If made lexicographic with states first it would rank M3 as more complex than M4.

[3]We are currently engaged in studying a variation on measures 2 and 3, where the only transitions that have a cost associated with them are those between distinct states. Preliminary results show that actions other than stage game Nash equilibrium actions can form part of a machine game equilibrium.

2.3 Equilibrium in the machine game

It is irrelevant for the results of this paper which of the meausres of complexity is considered in the sequel. Therefore, in what follows, we will use the terms "more complex," or "at least as complex as." Any of the 3 measures may be used to make the definitions precise. In particular, we allow players to use different measures from each other to reflect possible differential advantages: states may matter more to player 1 than transitions, while for player 2 the situation may be reversed, with indeed, states being irrelevant.

Player 1's preferences over machine pairs (M_1, M_2), denoted by $\gtrsim_1$ are assumed to satisfy the following: if (a) M_1' is at most as complex as M_1 and (b) $\pi_1(M_1', M_2) \geq \pi_1(M_1, M_2)$, then $(M_1', M_2) \gtrsim_1 (M_1, M_2)$. If (a) holds, and (b) holds strictly, $(M_1', M_2) >_1 (M_1, M_2)$. Similarly, if (b) holds while M_1' is strictly less complex than M_1.

Player 2's preferences over pairs (M_1, M_2), denoted by $\gtrsim_2$, are defined analagously.

<u>Definition</u>. A Nash equilibrium of the machine game is a pair (M_1^*,M_2^*) such that for all M_1,M_2, the following relations hold:

$$(M_1^*,M_2^*) \gtrsim_1 (M_1,M_2^*),$$

$$(M_1^*,M_2^*) \gtrsim_2 (M_1^*,M_2).$$

Let $N(G_m)$ denote the set of Nash equilibria of the machine game.

3. Results

We begin this section by first proving that all the qualitative results of Abreu–Rubinstein (1988) regarding the structure of the machines and equilibrium play continue to hold in our framework. These results will be used below to prove our main result that all Nash equilibria of the machine game are now trivial.

A point to be noted in the sequel is the following: when a machine is physically unaltered except for the dropping of one or more transitions, it becomes unambiguously less complex under all 3 measures. The same thing happens when a state is dropped, for now at least one transition – leading from the dropped state – also gets eliminated.

Theorem 1: For all $(M_1, M_2) \in N(G_m)$,

(i) $T_1 = T_2 = T$;

(ii) no state is repeated within a cycle by either player;

(iii) no non–cycle state is repeated within the pre–cycle phase by either player;

(iv) the set of pre–cycle states and cycle states are distinct.

Proof. Abreu and Rubinstein (1988) prove (i)–(iv) for discounting when only the number of states matters; as is shown in the Appendix their proof goes through in the current model as well. They further note that with the limit of means, while (i)–(iii) continue to hold, there may exist a "Phase II" in which cycle states are employed, yet not in the correct (according to the cycle) order; thus (iv) is not necessarily satisfied. If Phase II is empty then the proof for the discounting case goes through with the limit of means; hence what remains to be shown is that, when transitions matter, Phase II is always empty.

Suppose to the contrary that q_i were both a pre–cycle and cycle state for player i.

Let t,t' denote respectively the first time q_i occurs and the first time it occurs in the cycle phase, and let $q_j = q_j^t$ and $q_j' = q_j^{t'}$; clearly $q_j \neq q_j'$. Let $\tilde{Q}_1,\tilde{Q}_2$ be the set of cycle states for 1 and 2, respectively. It is clear that $\forall\ \hat{t} \in [t,t']\ q_j^{\hat{t}} \in \tilde{Q}_j$, else player j could save a state and at least 1 transition by replacing q_j with q_j' at t (recall that the cycle has begun at (q_i,q_j')). It follows that $\forall\ \hat{t} \in [t,t']\ q_i^{\hat{t}} \in Q_i$ as well.

We claim that for at least one player, say j, $\exists\ \tilde{q}_j \in \tilde{Q}_j$ and $s_i,s_i' \in S_i$ such that

(*) $\qquad \mu_j(\tilde{q}_j,s_i) \neq \mu_j(\tilde{q}_j,s_i').$

If (*) were not true, then there would exist a unique path through the sets $\tilde{Q}_i$ and $\tilde{Q}_j$, implying that the cycle would have begun at t, contradicting our hypothesis that (q_i,q_j) is not part of the cycle.

Now in the cycle the states in $\tilde{Q}_j$ occur in a particular order, say $q_j^1,\ldots,q_j^C$. But then if (*) holds, player j can adopt the following strategy and save on the number of transitions employed: play q_j' at t, and define

$$\mu_j'(q_j^k,s_i) = q_j^{k+1} \qquad \forall\ s_i \in S_i,\ k < C,$$
$$\mu_j'(q_j^C,s_i) = q_j^1 \qquad \forall\ s_i \in S_i.$$

Of course, $q_j' = q_j^k$ for some k, so j employs strictly less transitions than previously, with the same repeated game payoffs and the same number of states.

Similarly, if it were i for whom (*) held, i could adopt a different machine and do strictly better. QED

Abreu and Rubinstein (1988) show that if the players' preferences are weakly monotonic in repeated game payoffs and states, the set of potential equilibrium payoffs is quite limited relative to the Folk Theorem. The following result shows that if in addition players' preferences take transitional complexity into account, the potential equilibrium paths themselves become trivial.

Theorem. For all $(M_1,M_2) \in N(G_m)$,

(i) $(\lambda_1(q_1^t),\lambda_2(q_2^t)) \in N(G)\ \forall\ t$; i.e. any Nash equilibrium of the machine game induces one–shot Nash equilibrium actions at each stage of the repeated game.

(ii) In the limit–average case the cycle begins in the first period, i.e., the set of pre–cycle states is empty.

Proof: By Theorem 1, we claim that $\forall\ q_i \in Q_i$ and $s_j,s_j' \in S_j$,

$$(^{**}) \qquad \mu_i(q_i,s_j) = \mu_i(q_i,s_j').$$

To see this suppose that q_i is such that (**) does not hold, where $q_i = q_i^t$; let $q_j = q_j^t$. Then there exists $\bar{s}_j \in S_j$ such that

$$\mu_i(q_i,\bar{s}_j) = \bar{q}_i \neq \mu_i(q_i,\lambda_j(q_j)),$$

where the state $\mu_i(q_i,\lambda_j(q_j))$ is the unique (by Theorem 1) successor to q_i along the path of states induced by the machines M_1,M_2. But then by defining a new machine M_i' which differs from M_i only in that $\forall\ s_j \in S_j$, $\mu_i'(q_i,s_j) = \mu_i(q_i,\lambda_j(q_j))$ player i receives the same repeated game payoff with the same number of states but with at least one less transition. This contradicts the assumption that $(M_1,M_2) \in N(G_m)$.

Suppose now that, per absurdem, the set of pre–cycle states in the limit–average case were not empty. Let N_1, N_2 denote respectively the number of pre–cycle and cycle states in M_1. We will construct an alternative machine $M_1{}^*$ for player 1 such that (i) $M_1{}^*$ makes use of only the N_2 cycle states in both the pre–cycle and cycle phases, and (ii) $(M_1{}^*, M_2) >_1 (M_1, M_2)$. Note that the existence of such an $M_1{}^*$ does not contradict (iv) in Theorem 1 above since $(M_1{}^*, M_2)$ may not be in $N(G_m)$, but $(M_1, M_2) \in N(G_m)$ is contradicted.

Since neither players repeats pre–cycle states in the pre–cycle phase, the pre–cycle phase has length N_1 periods. Let n be the remainder from dividing N_1 by N_2, and let $\{q_1^1, ..., q_1^{N_2}\}$ be 1's cycle states in M_1 in the order in which they occur in the cycle. Define M_1^* by $M_1^* = \{Q_1^*, q_1^{1*}, \lambda_1^*, \mu_1^*\}$ where

$$\begin{aligned} Q_1^* &= \{q_1^1, ..., q_1^{N_2}\} \\ q_1^{1*} &= q_1^{N_2-n+1} \\ \lambda_1^*(q_1^k) &= \lambda_1(q_1^k) \\ \mu_1^*(q_1^k, s_j) &= \mu_1(q_1^k, s_j). \end{aligned}$$

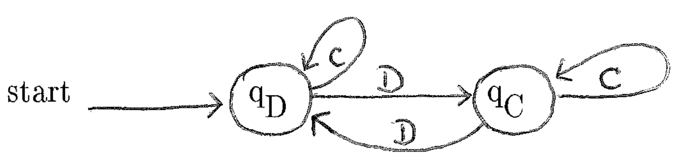
That is, M_1^* starts at $q_1^{N_2-n+1}$ and uses the same output and transition functions as M_1. By (**), this has no impact on player 2's transitions, and by construction the cycle begins at $(N_1 + 1)$, and yields player 1 exactly the same repeated game payoff. but M_1^* uses strictly fewer states and transitions than M_1, so certainly $(M_1^*, M_2) >_1 (M_1, M_2)$, or $(M_1, M_2) \notin N(G_m)$. a contradiction. This proves the second part of the Theorem.

Since pre–cycle states are empty in the limit–average case, equation (**) of course proves the first part of the Theorem: if i's transitions are independent of what j does then j will prefer to play stage–game best response strategies at each point in time.

QED

Why is this result true? The intuition underlying the formal proof is as straighforward as the following verbal argument suggests: if transitions have a cost associated with them, players' machines in equilibrium must use all the available transitions. Thus, whereas in Abreu and Rubinstein (1988) a <u>state</u> will be discarded if it is never employed, (so that punishment states must occur at least once), in addition a <u>transition</u> will now be discarded if it is never employed (by making the transition equivalent to some transition which is employed). This implies that "punishment contingencies", i.e. those state–action pairs which lead (through the transition function) to a punishment state, must occur at least once. But then these contingencies are now part of the equilibrium path of play, implying that there cannot be <u>any</u> threats in equilibrium, empty or otherwise.

To illustrate the difference between the Abreu and Rubinstein (1988) model and the current model, consider again the repeated prisoners' dilemma, where preferences are lexicographic and repeated game payoffs are evaluated according to the limit of means. If both players adopt the machine implementing the "grim" strategy, the resulting equilibrium path is (C,C) forever. Yet this is not an equilibrium when the number of states matters, since both have an incentive to drop the "punishment state" q_D (clearly this conclusion holds in the current model as well). Alternatively, Rubinstein (1986) shows that (C,C) can be sustained for all but the initial period if the players both adopt the following "pretty grim" strategy:



This constitutes equilibrium behavior in the model of Abreu and Rubinstein (1988) since now the punishment state q_D occurs along the equilibrium path, if only once. However when transitions matter this will no longer be an equilibrium, since in particular the transition from q_C to q_D, or rather the "punishment contingency" (q_C,D), is never pursued. In fact, an implication of the above result is that the <u>only</u> equilibrium in the repeated prisoners' dilemma is where both players play D forever. On the other hand, Abreu and Rubinstein (1988) show that in the repeated battle of the sexes game all Nash equilibria are such that players at each stage are at one of the one–shot Nash equilibria. Hence in this game the additional consideration of transition costs will have no effect on the potential equilibrium payoffs.

4. Conclusion

In this paper we have explored the sensitivity of the results of Abreu and Rubinstein (1988) to their measure of a machine's complexity. We have shown that if, in addition to the number of states, the complexity of the transition mechanism enters into this measure the set of possible Nash equilibrium payoffs is dramatically reduced in certain games (e.g. prisoners' dilemma) and is untouched in others (e.g. battle of the sexes). However we concur with Abreu and Rubinstein (1988) that the story on implementation costs in repeated games is nowhere near complete, and hope to explore further such issues in future research.

In general the motivation of Rubinstein (1986), Abreu and Rubinstein (1988) and ourselves concerns limiting the extent of the Folk Theorem in infinitely repeated games by explicitly including complexity considerations in the players' preferences, where complexity relates to the cost of implementing a repeated game strategy. These considerations are facilitated by the use of finite automata, which provide simple measures of such complexity. Another use of finite automata, as noted in the Introduction, is in the study of bounded rationality in finitely repeated games, in particular the finitely repeated prisoners' dilemma (Neyman (1985), Zemel (1986)), where now the maximum number of states a player can used is fixed at the outset. In these papers the motivation is to generate cooperation as an equilibrium, in part to justify the empirical regularity of such behavior (cf. Axelrod (1980)). Given the results of the current paper, then, it appears that these two strands of the literature have actually passed each other by, in that Neyman (1985) gets cooperation in finite time while we get only non–cooperation in infinite time! An implication of this fact appears to be the necessity of developing a more general theory of bounded rationality and decision costs in repeated games which would address the issue of complexity both in terms of the computation of optimal strategies as well as the actual implementation of such strategies.

Appendix

Abreu and Rubinstein (1988) prove Theorem 1 above for discounting using the following three results. We modify their proofs to show that the results hold in the current model.

Let $(M_1,M_2) \in N(G_m)$. Define $A_i(k_1,k_2)$ as the average (discounted or undiscounted) payoff to i between the periods k_1 and k_2 inclusive, and let $\pi_i^*(k)$ denote the average discounted payoff to i from period k on.

<u>Lemma A1</u>. Suppose $q_1^{k_1} = q_1^{k_2+1}$, $k_2 \geq k_1$. Then $A_2(k_1,k_2) = \pi_2^*(k_1)$.

<u>Proof</u>. Suppose it were the case that $A_2(k_1, k_2) > \pi_2{}^*(k_1)$. Note that this implies that $q_2^{k_1} \neq q_2^{k_2+1}$. Define a new machine $M_2{}^*$ for player 2 identical to M_2 but with the following change:

$$\mu_2{}^*(q_2^{k_2}, s) = q_2^{k_1} \qquad \forall \text{ s such that } \mu_2(q_2^{k_2}, s) = q_2^{k_2+1}.$$

Clearly, $M_2{}^*$ is no more complex than M_2 but $\pi_2(M_1, M_2{}^*) > \pi_2(M_1, M_2)$, contradicting $(M_1, M_2) \in N(G_m)$.

Similarly, if $A_2(k_1, k_2) < \pi_2{}^*(k_1)$, player 2 could do better with a machine $M_2{}^*$ identical to M_2 but with:

$$\mu_2{}^*(q_2^{k_1-1}, s) = q_2^{k_2+1} \qquad \forall \text{ s suc that } \mu_2(q_2^{k_1-1}, s) = q_2^{k_1}.$$

Q.E.D.

Let m_i denote the minimal t such that q_i^t is repeated.

<u>Lemma A2</u>. $m_1 = m_2$.

<u>Proof</u>. Suppose $m_2 > m_1$; let $\overline{m}_1 > m_1$ satisfy $q_1^{\overline{m}_1} = q_1^{m_1}$. Define M_2' by

$$Q_2' = Q_2 \backslash \{q_2^{m_1}\},$$
$$\lambda_2'(q_2) = \lambda_2(q_2) \ \forall \ q_2 \in Q_2' ,$$
$$\mu_2'(q_2,\cdot) = \mu_2(q_2,\cdot) \ \forall \ q_2 \neq q_2^{m_1-1},$$
$$\mu_2'(q_2^{m_1-1},s) = q_2^{\overline{m}_1} \ \forall \ s \in S_1.$$

Since $q_2^{m_1-1}$ and $q_2^{m_1}$ are not repeated in (M_1,M_2), the sequence of states in (M_1,M_2') is $q^1,\ldots,q^{m_1-1},q^{\overline{m}_1},q^{\overline{m}_1+1},\ldots$ By Lemma A1 $\pi_2(M_1,M_2') = \pi_2(M_1,M_2)$, but M_2' is less complex than M_2 (since $q_2^{m_1}$ and the transitions leading to and from it are absent in M_2', whereas only 1 transition has been added); a contradition.

Q.E.D.

Recall that t_1 is the beginning of the initial cycle.

<u>Lemma A3</u>. $t_1 = m_1$.

<u>Proof</u>. Let $\overline{m}_i$ be the minimal $t > m_i$ such that $q_i^t = q_i^{m_i}$, and suppose that $\overline{m}_1 > \overline{m}_2$. If $q_2^{\overline{m}_1} \neq q_2^t$ for any $t < \overline{m}_1$, define M_2' by

$$Q_2' = Q_2 \backslash \{q_2^{\overline{m}_2}\},$$
$$\lambda_2'(q_2) = \lambda(q_2) \ \forall \ q_2 \in Q_2',$$

$$\mu_2'(\cdot,\cdot) = \mu_2(\cdot,\cdot) \;\forall\; (q_2,s) \neq (q_2^{\overline{m}_1-1}, s_1^{\overline{m}_1-1}),$$

$$\mu_2'(q_2^{\overline{m}_1-1}, s_1^{\overline{m}_1-1}) = q_2^{m_1}.$$

Then by Lemma A1 $\pi_2(M_1,M_2') = \pi_2(M_1,M_2)$, but (for the same reason as in lemma A2), M_2' is less complex than M_2, a contradiction. If $q_2^{\overline{m}_1} = q_2^k$ (where $k \neq m_1$), define M_1' by

$$Q_1' = Q_1 \backslash \{q_1^{m_1}\},$$

$$\lambda_1'(\cdot) = \lambda_1(\cdot),$$

$$\mu_1'(q_1^t, s_2^t) = \mu_1(q_1^t, s_2^t) \text{ if } t \neq m_1-1, \overline{m}1-1,$$

$$\mu_1'(q_1^t, s) = q_1^{\overline{m}_2} \quad \forall\; s \in S_2,\; t = m_1-1,$$

$$\mu_1'(q_1^{\overline{m}_1-1}, s_2^{\overline{m}_1-1}) = q_1^k,$$

$$\mu_1'(q_1^{\overline{m}_1-1}, s) = \mu_1(q_1^{\overline{m}_1-1}, s) \;\forall\; s \neq s_2^{\overline{m}_1-1}.$$

Then the sequence of states is $q^1,\ldots,q^{m_1-1}, q^{\overline{m}_2}, q^{\overline{m}_2+1},\ldots,q^{\overline{m}_1-1}$ followed by the cycle $(q^k, q^{k+1},\ldots,q^{\overline{m}_1-1})$. By Lemma A1, $\pi_1(M_1,M_2) = \pi_1(M_1',M_2)$, but now M_1' is less complex than M_1. Contradiction. QED

Lemma A3 shows that all pre–cycle states are distinct and Lemma A2 shows that the cycles in q_1^t and q_2^t begin at the same time. To see that all cycle states are distinct, and therefore that $T_1 = T_2 = T$, a construction similar to that in Lemma A2 is used (see Abreu and Rubinstein (1988), p. 1270).

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