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λ–CONSTANT COMPARATIVE STATICS

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Intertemporal Substitution and λ-Constant Comparative Statics*

Whether a particular specification of preferences admits "sufficient" intertemporal substitutability is a question for consumption theory, both applied and pure. What is sufficient or sizable is likely to be an empirical question, but characterizing the degree of intertemporal substitutability admitted by a specification of preferences falls within the scope of pure theory. This paper contributes to the pure theory of consumption by characterizing intertemporal substitution along intertemporal consumption profiles.

The problem of allocating wealth to consumption over time is central to the theory of consumption and interest (Fisher [1930] 1977; Friedman 1957; Modigliani and Brumberg 1954); indeed, it is the cornerstone of the intertemporal substitution models in macroeconomics. Applications outside macroeconomics include labor supply over the life cycle (e.g., Weiss 1972; Heckman 1974; Ghez and Becker 1975) and the welfare cost of intertemporal taxation (e.g., Judd 1987).

The approach I take to characterize intertemporal substitution is to relate the intertemporal variation in consumption to intertemporal variation in prices; for instance, $x^*_{t+1} - x^*_t \approx a\cdot[p_{t+1} - p_t]$. I establish the condition under which the factor of proportionality a is a particular type of compensated own-price effect, one in which compensation preserves the value of the marginal utility of wealth. If the condition holds, one may perform a relatively simple comparative statics exercise to characterize intertemporal

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*I thank Eric Bond and workshop participants at the University of Rochester for helpful comments. I have also benfitted from discussions with Eric Bond, James Heckman, Robert King, and William Thomson.

substitution. The exercise is called marginal-utility-of-wealth- (or λ-) constant comparative statics. Transformed into an elasticity, the factor of proportionality is a unit-free scalar measure of intertemporal substitutability.

The basic idea of linking intertemporal substitution and λ-constant comparative statics is not new. Heckman (1974) employs λ-constant comparative statics in the context of life-cycle consumption demand and labor supply. However, the link is only implicit in that work as Heckman does not justify the applicability of the technique. MaCurdy (1981, 1069-74) is the first to discuss the relationship between a movement along a life-cycle profile (intertemporal substitution) and λ-constant comparative statics (a compensated shift in the profile). Although many labor economists since Heckman and MaCurdy have asserted the analytical equivalence of intertemporal substitution and λ-constant comparative statics, none has established such a result formally. In addition, interest in λ-constant comparative statics transcends labor economics: King (1988) places λ-constant effects at center stage in evaluating a general equilibrium business cycle model.

Is it obvious that α is a comparative statics effect? Withdraw for the moment from the intertemporal context to consider a schoolboy's consumption of baseball cards and water pistols. Is the difference in consumptions of cards and pistols approximately proportional to the difference in their prices, with the factor of proportionality a λ-constant own-price effect? It should surprise no one that the answer is generally no. But in no way are cards and pistols fundamentally different from x^*_{t+1} and x^*_t. What is more startling is that the answer is in particular cases yes.[1]

The analysis of this paper formally establishes the counter-intuitive link between intertemporal substitution and λ-constant comparative statics. Section 1 is set in the simplest intertemporal setting to clarify a key concept: the intertemporal elasticity of substitution. A precise definition relating intertemporal consumption variation to intertemporal price variation is presented and compared to the intertemporal analogs of the Hicks-Allen and direct elasticities of substitution, and the λ-constant own-price elasticity of demand.

A richer intertemporal setting is the foundation for the analysis of section 2. I employ a discrete time, finite horizon, perfect foresight model. With access to a perfect capital market, the consumer trades from his endowment both intertemporally and across goods within each period. By focusing on λ-constant demands and comparative statics, this section develops the concepts used in section 3.

The link between intertemporal substitution and λ-constant comparative statics is established in section 3 and Appendix B. I prove that λ-constant comparative statics is valid in characterizing the intertemporal variation of consumption if and only if lifetime utility is intertemporally additive in consumption.

Two extensions are investigated in section 4: addictive behavior and

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[1]The theorem established in section 3, below, implies that: if the schoolboy's utility function is block additive with baseball cards and water pistols in distinct blocks, then the factor of proportionality is the λ-constant own-price effect in the demand for water pistols. Of course, the quality of the approximation depends on the units in which the goods are measured. If the units are chosen to keep the prices of baseball cards and water pistols close, the quality of the approximation is high.

uncertainty. The analysis of addiction indicates that λ-constant comparative statics may be used to characterize intertemporal substitution in only a restrictive class of addiction models. Although resolution of uncertainty induces variation in the marginal utility of wealth, nevertheless, the λ-constant own-price elasticity of demand is shown to be an important element characterizing intertemporal substitution in the presence of an uncertain future.

The paper closes with a brief summary of the results and a conclusion.

1. Intertemporal Elasticity of Substitution

In this section, a few basic ideas are highlighted in the simplest intertemporal setting. The goal is to present a scalar measure of intertemporal substitutability. A precise definition of the intertemporal elasticity of substitution is presented and compared to more traditional elasticities of substitution.

Assume that a consumer chooses an intertemporal consumption vector $\mathbf{x} = (x_0, \ldots, x_{T-1})$ to maximize lifetime utility $u = \Sigma_{t=0}^{T-1} U(x_t)$ subject to the wealth constraint, $W = \Sigma_{t=0}^{T-1} p_t x_t$. The restrictions--a single consumption good, zero rates of time preference and interest, time-invariant momentary utility functions, and perfect foresight--are relaxed in sections 2-4.

Central to the analysis of intertemporal substitution is intertemporal variation of consumption based on intertemporal variation in prices (or interest rates). Let η_t denote the intertemporal elasticity of substitution such that, for $p_{t'}$ close to p_t,

$$(1)\qquad \frac{x^*_{t'} - x^*_t}{x^*_t} \equiv \eta_t \cdot \frac{p_{t'} - p_t}{p_t} \qquad \text{or} \qquad \hat{x}(t) \equiv \eta_t \cdot \hat{p}(t),$$

with asterisks denoting optimal values, and t and t' denoting two time periods. This definition is natural: the intertemporal elasticity of substitution η_t carries a clear interpretation in terms of intertemporal choice.

An explicit expression for η_t is easily derived. Expressing the period t and period t' necessary conditions logarithmically generates

$$(2)\qquad \log U'(x^*_{t'}) - \log U'(x^*_t) = \log p_{t'} - \log p_t.$$

First-order expansions around the scalars x^*_t and p_t yield

$$(3.1)\qquad \frac{U''(x^*_t)}{U'(x^*_t)} \cdot [x^*_{t'} - x^*_t] \approx [p_{t'} - p_t]/p_t$$

or

$$(3.2)\qquad \hat{x}(t) \approx \frac{U'(x^*_t)}{U''(x^*_t)x^*_t} \cdot \hat{p}(t).$$

For small intertemporal price variation, the relationship is exact. Consequently, under the conditions of the simple model, the intertemporal elasticity of substitution η_t equals $U'(x^*_t)/U''(x^*_t)x^*_t$.

The intertemporal elasticity of substitution η_t is generally distinct from standard elasticities of substitution--the Hicks-Allen (cross-price) elasticity of substitution and the direct elasticity of substitution--applied to the intertemporal context. In the intertemporal context, the Hicks-Allen elasticity of substitution $\sigma_{t,t'}$ is a measure of the responsiveness of demand in period t to a change in the price in period t', holding lifetime utility constant. Hence η_t and $\sigma_{t,t'}$ are conceptually distinct. Analytically, $\sigma_{t,t'} \equiv \epsilon^s_{t,t'}/s_{t'}$, with $\epsilon^s_{t,t'}$ denoting the compensated cross-price elasticity of demand for x_t, and $s_{t'}$ the lifetime budget share of expenses in period t'. With additive lifetime utility, the Hicks-Allen elasticity of substitution $\sigma_{t,t'}$ is related to the two periods' intertemporal elasticities of substitution.

$$(4) \qquad \sigma_{t,t'} = -\psi \cdot \eta_t \cdot \eta_{t'},$$

with $\psi \equiv \left[\Sigma_{\tau=0}^{T-1} U'(x_\tau^*)x_\tau^*\right] \Big/ \left[\Sigma_{\tau=0}^{T-1} U'(x_\tau^*)^2/U''(x_\tau^*)\right]$.[2] Therefore, although $\sigma_{t,t'}$ is related to the intertemporal elasticity of substitution η_t, the two are conceptually and analytically distinct.

The intertemporal elasticity of substitution η_t is also distinct from the direct elasticity of substitution, a measure of curvature of an indifference surface. Evaluated at some intertemporal consumption vector $\mathbf{x} = (x_0, \ldots, x_{T-1})$, the direct elasticity of substitution between x_t and $x_{t'}$ is

$$(5.1) \qquad S_{t,t'} \equiv \frac{\partial \log(x_{t'}/x_t)}{\partial \log[U'(x_t)/U'(x_{t'})]},$$

with $du = dx_\tau = 0$ for all $\tau \neq t$ or t'. With additive lifetime utility, the

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[2] The derivation of this result is available from the author on request.

restriction that consumption in other periods is held fixed is irrelevant. From McFadden (1963),

$$(5.2) \qquad S_{t,t'} = -\frac{\dfrac{1}{U'(x_t)\cdot x_t} + \dfrac{1}{U'(x_{t'})\cdot x_{t'}}}{\dfrac{U''(x_t)}{U'(x_t)^2} + \dfrac{U''(x_{t'})}{U'(x_{t'})^2}} > 0,$$

since $U'' < 0$. $S_{t,t'}$ is independent of x_τ for all $\tau \neq t$ or t'.[3]

The intertemporal elasticity of substitution η_t does result from a particular comparative statics exercise. Differentiating period t's necessary condition holding the marginal utility of wealth constant at λ^* yields $\partial x_t^*/\partial p_t = \lambda^*/U''(x_t^*)$. Converting this expression to an elasticity results in

$$(6) \qquad \frac{p_t}{x_t^*}\frac{\partial x_t^*}{\partial p_t} = \frac{U'(x_t^*)}{U''(x_t^*)x_t^*} = \eta_t.$$

Consequently, η_t is the marginal-utility-of-wealth-constant <u>own</u>-price elasticity of demand for x_t.

Although the conceptual distinction between the three elasticities--the Hicks-Allen and direct elasticities of substitution, and the λ-constant own-price elasticity--remains, the analytical distinction vanishes if the lifetime utility function is CES.

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[3] If $S_{t,t'}$ were evaluated at the optimal consumption pair $(x_t^*, x_{t'}^*)$, and $T = 2$, then $S_{t,t'} = \sigma_{t,t'}$.

$$(7) \qquad u = \frac{1}{\gamma} \sum_{\tau=0}^{T-1} \delta_t \, x_t^{\gamma},$$

with $\gamma < 1$. Direct computation reveals that $-\eta_t = \sigma_{t,t'} = S_{t,t'} = 1/(1-\gamma)$. Since the CES specification is common in applied work, this equality has facilitated the link between macroeconomic research, which usually uses the direct elasticity of substitution,[4] and the empirical research on life-cycle labor supply, which emphasizes the λ-constant own-price elasticity.[5]

2. λ-Constant Comparative Statics

In this section, I set the foundation for linking intertemporal variation in consumption to the result of a marginal-utility-of-wealth- or λ-constant comparative statics exercise. Focusing on the technique of λ-constant comparative statics, I begin the analysis with the presentation of a standard, T period model of a consumer's intertemporal choice.

An Intertemporal Model

Preferences, which are intertemporally strongly separable, are represented by a block-additive lifetime utility function.

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[4]Frequently, equilibrium business cycle models refer to the magnitude of intertemporal substitution, but only rarely is a precise measure defined apart from the expression $1/(1-\gamma)$. Mankiw, Rotemberg, and Summers (1985) and Hall (1988) are explicit in adopting the direct elasticity of substitution evaluated at the optimum.

[5]Contributions to the literature on life-cycle labor supply which link the λ-constant own-price elasticity (or effect) to intertemporal substitution include: MaCurdy (1981; 1985), Killingsworth (1983), Browning, Deaton, and Irish (1985), Altonji (1986), Pencavel (1986), and Abowd and Card (1987). The link is implicit in Heckman (1974).

$$(8) \qquad u = \sum_{t=0}^{T-1} U(\mathbf{x}_t)\beta^t = \sum_{t=0}^{T-1} U(x_{1t}, \ldots, x_{Nt})/(1+\rho)^t,$$

with $\mathbf{x}_t = (x_{1t}, \ldots, x_{Nt})$ denoting the N-dimensional consumption vector for time t, β the discount factor, and ρ the rate of time preference. The momentary utility function U is strictly concave in its N arguments.

The consumer faces perfectly foreseen, exogenous sequences of income $\{y_t\}_{t=0}^{T-1}$, endowments $\{\bar{\mathbf{x}}_t\}_{t=0}^{T-1}$, prices $\{\mathbf{p}_t\}_{t=0}^{T-1}$, and interest rates $\{r_t\}_{t=0}^{T-1}$. With lifetime borrowing and lending opportunities, the consumer's consumption choice must satisfy

$$(9) \qquad \sum_{t=0}^{T-1} R_t(y_t + \mathbf{p}_t \cdot \bar{\mathbf{x}}_t) = \sum_{t=0}^{T-1} R_t \mathbf{p}_t \cdot \mathbf{x}_t.$$

The discounting term is $R_t \equiv \prod_{\tau=0}^{t-1} \left[\frac{1}{1+r_\tau}\right]$ for $t > 0$, and $R_0 = 1$. Let $W \equiv \sum_{t=0}^{T-1} R_t(y_t + \mathbf{p}_t \cdot \bar{\mathbf{x}}_t)$ denote full-wealth at time zero and $R_t \mathbf{p}_t$ denote the discounted price vector; then the full-wealth constraint is

$$(10) \qquad W = \sum_{t=0}^{T-1} R_t \mathbf{p}_t \cdot \mathbf{x}_t \equiv \sum_{t=0}^{T-1} \sum_{i=1}^{N} R_t p_{it} x_{it},$$

which is the discounted sum of NT expenditure terms.

The consumer's choice is the solution to a (singly) constrained utility maximization problem in NT choice variables. With λ denoting the Lagrange multiplier associated with the full-wealth constraint, the necessary conditions are

$$(11.1)\qquad \beta^t U_i(x_t) - \lambda R_t p_{it} = 0, \qquad i = 1, \ldots, N, \quad t = 0, \ldots, T-1,$$

$$(11.2)\qquad W - \sum_{t=0}^{T-1} \sum_{i=1}^{N} R_t p_{it} x_{it} = 0,$$

which are NT + 1 equations in NT + 1 unknowns. (Strict concavity of the momentary utility function is sufficient to satisfy the second-order conditions.) Therefore, the uncompensated (or Marshallian) demands which solve this system are

$$(12)\qquad x^*_{it} = g_{it}(R_0 p_0, \ldots, R_{T-1} p_{T-1}, W), \qquad i = 1, \ldots, N, \quad t = 0, \ldots, T-1.$$

These demands satisfy all the textbook properties—homogeneity, negativity, symmetry, and adding up. A fundamental decomposition of the consumer's problem (Frisch 1959; Barten 1964) implies that the total effect of a <u>change</u> in the ℓth price on the consumption of the kth good is the sum of <u>three</u> terms: the specific substitution effect, the general substitution effect, and the wealth effect.

$$(13)\qquad \frac{\partial x^*_k}{\partial p_\ell} = \lambda^* H^{k\ell} - \frac{\lambda^*}{\partial \lambda^*/\partial W} \frac{\partial x^*_k}{\partial W} \frac{\partial x^*_\ell}{\partial W} + (\bar{x}_\ell - x^*_\ell) \frac{\partial x^*_k}{\partial W},$$

with k indexing some pair (i, t) and ℓ some pair (j, t'). λ^* is the marginal utility of wealth; $H^{k\ell}$ is the (k, ℓ)th element of the inverse of the Hessian matrix of the lifetime utility function.

$$(14) \qquad \mathbf{H} \equiv \left[\frac{\partial^2 u}{\partial x_k \partial x_\ell} \right]^{-1} .$$

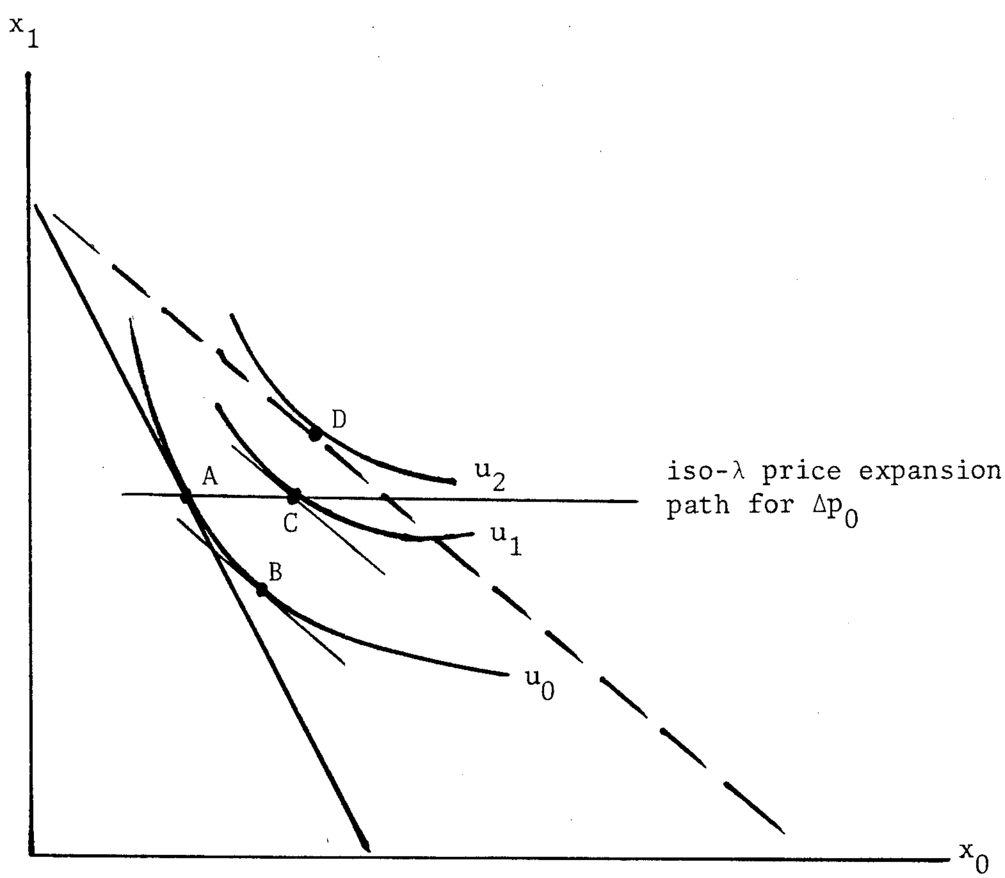
Intertemporal additivity of lifetime utility implies that the NT-by-NT Hessian matrix is block diagonal. Therefore, **H** is also block diagonal, implying that all "cross-period" specific substitution effects equal zero.

The first two terms in equation (13) combine to form the pure substitution effect: if in response to the change in p_ℓ income is compensated to keep utility constant, the pure substitution effect measures the response of consumption. The first term, the specific substitution effect also corresponds to a compensated price effect: if in response to the price change income is compensated to keep the marginal utility of wealth λ^* constant, then the specific substitution effect measures the response of consumption (Frisch 1959). In the current analysis, I refer to the term $\lambda^* H^{k\ell}$ as a λ-constant comparative statics effect.

For the case of one (superior) consumption good in each of two periods and no endowment, the comparative statics effects of a decrease in the price of x_0 are illustrated in Figure 1. The movement from A to D is the uncompensated effect. The traditional Hicksian decomposition to substitution and wealth effects is: A to B is the substitution effect (i.e., compensation to preserve utility at u_0); B to D is the wealth effect. Alternatively, the movement from A to C preserves the value of the marginal utility of wealth at λ^*--the specific substitution effect; consequently, the iso-λ^* price expansion path for variation in p_0 is a horizontal line.[6] The movement from C to D combines

[6]This iso-λ^* price expansion path is horizontal if and only if utility is additive. The sign of the cross-partial derivitive U_{01} determines whether the

Figure 1. λ-Constant Decomposition



the general substitution effect and the traditional wealth effect. This implies that C to B is the general substitution effect: with superior consumption goods, the general substitution effect of a decrease in p_0 reduces consumption in each period.

λ-Constant Demands

Since the analysis to follow relies on λ-constant comparative statics, it is useful to replace the uncompensated demands with λ-constant (or Frisch) demands. Return to the necessary conditions represented by equation (11.1). For each time period t, an N-equation system can be solved for the λ-constant demands,

$$(15) \qquad x^*_{it} = x_i(\lambda^* \beta^{-t} R_t \mathbf{p}_t), \qquad i = 1, \ldots, N, \quad t = 0, \ldots, T-1,$$

with $x_i(\cdot)$ a time-invariant function of the scaled price vector. (λ-constant demands can be derived for any value of λ; in writing equation (15), the optimal value λ^* is employed.) The λ-constant demands exhibit the following properties: (a) homogeneity of degree zero in period t's prices $\mathbf{p}_t$ and the scalar $\phi_t^{-1} \equiv \lambda^* \beta^{-t} R_t$; (b) symmetry of cross-price effects; and (c) negativity. (See Browning, Deaton, and Irish (1985) for derivations of these results as well as the dual representation of λ-constant demands.)

The λ-constant comparative statics results are derived as follows. One takes the total differential of equation (11.1) yielding for each period t

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iso-λ^* price expansion slopes up or down.

$$(16)\quad \begin{bmatrix} U_{11}(t) & \cdots & U_{1N}(t) \\ \vdots & \ddots & \\ U_{N1}(t) & & U_{NN}(t) \end{bmatrix} \begin{bmatrix} dx^*_{1t} \\ \vdots \\ dx^*_{Nt} \end{bmatrix} = \lambda^* \beta^{-t} R_t \begin{bmatrix} dp_{1t} \\ \vdots \\ dp_{Nt} \end{bmatrix}.$$

More compactly,

$$(17)\quad \mathbf{U}(t)d\mathbf{x}^*_t = \phi_t d\mathbf{p}_t.$$

Pre-multiplying by the inverse of the N-by-N Hessian matrix,

$$(18)\quad d\mathbf{x}^*_t = \phi_t \mathbf{U}^{-1}(t)d\mathbf{p}_t.$$

Consequently, the λ-constant cross-price effects (and own-price effects for $j = i$ in equation (19.1)) are

$$(19.1)\quad \frac{\partial x^*_{it}}{\partial p_{jt}} = \lambda^* \beta^{-t} R_t \cdot \frac{U^{ij}(t)}{|\mathbf{U}(t)|}$$

$$(19.2)\quad \frac{\partial x^*_{it}}{\partial p_{jt'}} = 0, \qquad \text{for all } t' \neq t,$$

with $U^{ij}(t)$ denoting the (i, j)th co-factor of $\mathbf{U}(t)$.

For concreteness consider an example with two goods—consumption c_t and leisure ℓ_t—in each period and in which the rate of time preference equals the time-invariant rate of interest. The necessary conditions include

$$(20.1) \quad U_c(c_t, \ell_t) = \lambda p_t$$

$$t = 0, \ldots, T-1,$$

$$(20.2) \quad U_\ell(c_t, \ell_t) = \lambda w_t,$$

with w_t denoting the price of leisure, the wage rate.[7] Totally differentiating the two-equation system (for each t) around the optimum, one derives the following λ-constant comparative statics results:

$$(21) \quad \begin{bmatrix} \frac{\partial c_t^*}{\partial p_t} & \frac{\partial c_t^*}{\partial w_t} \\ \frac{\partial \ell_t^*}{\partial p_t} & \frac{\partial \ell_t^*}{\partial w_t} \end{bmatrix} = \frac{\lambda^*}{U_{cc}U_{\ell\ell} - U_{c\ell}^2} \begin{bmatrix} U_{\ell\ell} & -U_{\ell c} \\ -U_{c\ell} & U_{cc} \end{bmatrix}.$$

Therefore, the own-price effects are both negative and the cross-price effects depend on the sign of the cross-partial derivative $U_{c\ell}$. In particular, with income compensation to hold λ constant at $\lambda^* > 0$, consumption is increasing in the wage rate if and only if $U_{c\ell}$ is less than zero (Weiss 1972; Heckman 1974).

Interpretation

Beginning with Heckman (1974), λ-constant comparative statics has been employed to characterize intertemporal variation in consumption based on intertemporal variation in prices. This is striking: comparative statics exercises, such as λ-constant comparative statics, perturb the optimum; that

[7] Appendix A contains the λ-constant demands for consumption and leisure for three commonly employed specifications of momentary utility.

is, some exogenous variable changes. However, with perfect foresight nothing changes in the intertemporal choice problem. Since prices in different periods are associated with different goods, there is intertemporal price variation but each particular price remains unchanged.

What is the link between λ-constant comparative statics and intertemporal substitution? That λ-constant demands fully characterize intertemporal choice is not sufficient to establish the link. Marginal-utility-of-wealth- constant (Frisch), utility-constant (Hicksian), and wealth-constant (Marshallian) demands each fully characterizes intertemporal consumption choice. Furthermore, that the marginal utility of wealth is constant over the life cycle is not sufficient to justify λ-constant comparative statics; utility and wealth are also constant over the life cycle. The link between λ-constant comparative statics and intertemporal substitution must be established formally.

3. Characterizing Intertemporal Consumption Variation

In this section, I establish formally that λ-constant comparative statics fully characterizes the intertemporal variation of consumption if lifetime utility is intertemporally block-additive as in equation (8). However, the validity of λ-constant comparative statics is not general. Indeed, the class of lifetime utility functions over which λ-constant comparative statics is valid is shown to be only slightly more general than equation (8).

Sufficiency

Does the lifetime utility function in equation (8) guarantee the validity of λ-constant comparative statics in characterizing intertemporal variation in consumption? Yes. To establish this result, it is useful to define the λ-constant demands using logarithms. Taking logs of each of the NT necessary conditions results in

$$(22) \quad t \log \beta + \log U_i(\mathbf{x}_t) = \log \lambda^* + \log R_t + \log p_{it}, \quad \begin{matrix} i = 1, \ldots, N, \\ t = 0, \ldots, T-1. \end{matrix}$$

Since $\log \beta$ equals $-\log(1+\rho)$ and $\log R_t$ equals $-\Sigma_{\tau=0}^{t-1} \log(1+r_\tau)$, the λ-constant demand functions in N arguments are

$$(23) \quad x^*_{it} = x_i\left[\log \lambda^* + t \log(1+\rho) - \sum_{\tau=0}^{t-1} \log(1+r_\tau) + \log \mathbf{p}_t\right], \quad \begin{matrix} i = 1, \ldots, N, \\ t = 0, \ldots, T-1. \end{matrix}$$

For any good i in time period t', a first-order expansion of $x^*_{it'}$ around x^*_{it} yields

$$(24) \quad x^*_{it'} \approx x^*_{it} + \sum_{n=1}^{N} p_{nt} \cdot \frac{\partial x_i}{\partial p_{nt}} \cdot$$

$$\left\{\log(1+\rho)\cdot[t' - t] - \sum_{\tau=t}^{t'-1} \log(1+r_\tau) + [\log p_{nt'} - \log p_{nt}]\right\},$$

with $\partial x_i/\partial p_{nt}$ denoting the partial derivatives computed from equation (23). Approximating $\log(1+\rho)$ by ρ, $\log(1+r_\tau)$ by r_τ, and $\log p_{nt'} - \log p_{nt}$ by $[p_{nt'} - p_{nt}]/p_{nt}$ yields

$$(25) \quad x^*_{it'} - x^*_{it} \approx \sum_{n=1}^{N} \frac{\partial x_i}{\partial p_{nt}} \cdot \left\{[p_{nt'} - p_{nt}] + p_{nt}\rho \cdot [t' - t] - p_{nt} \sum_{\tau=t}^{t'-1} r_\tau\right\},$$

for all i, t, and t'. Therefore, intertemporal variation in consumption depends on intertemporal variation in prices, and terms in the rates of time preference and interest. The magnitude of the intertemporal consumption response is in each case determined by a λ-constant comparative statics effect.

Expressing this result in percentage variations generates the principal result.

$$(26) \qquad \hat{x}_i(t) \approx \sum_{n=1}^{N} \eta_{in}(t)\cdot\left\{\hat{p}_n(t) + \rho\cdot[t' - t] - \sum_{\tau=t}^{t'-1} r_\tau\right\},$$

with $\eta_{in}(t)$ denoting the λ-constant elasticity of demand for good i with respect to the price of good n at time t.

Equation (26) extends the definition of the intertemporal elasticity of substitution to the N good case with $r_t \neq \rho$. Thus the $\eta_{in}(t)$ are intertemporal (cross-price) elasticities of substitution in consumption of good i in period t. Hence the validity of λ-constant comparative statics.

If the price of only the jth good varies intertemporally, then

$$(27) \qquad \hat{x}_i(t) \approx \eta_{ij}(t)\cdot\hat{p}_j(t) + \eta_{i\lambda}(t)\cdot\left\{\rho\cdot[t' - t] - \sum_{\tau=t}^{t'-1} r_\tau\right\},$$

where $\eta_{i\lambda}$ is the elasticity of x_i with respect to λ, which by the homogeneity of λ-constant demands equals $\Sigma_{n=1}^{N} \eta_{in}(t)$.

A common simplification is to let r_t be constant intertemporally, then $\Sigma_{\tau=t}^{t'-1} r_\tau = r\cdot[t' - t]$. Therefore, equation (27) reduces to

$$(28) \qquad \hat{x}_i(t) \approx \eta_{ij}(t)\cdot\hat{p}_j(t) + \eta_{i\lambda}(t)\cdot(\rho - r)[t' - t].$$

Since concavity implies that $\eta_{i\lambda}$ is less than zero, consumption of any good i trends down or up through time as $\rho \gtrless r$. The steepness of the consumption profiles depends on the magnitudes of all the period t intertemporal elasticities of substitution, as well as the difference $\rho - r$.

Generality?

λ-constant comparative statics is not generally valid in characterizing intertemporal variation of consumption. This proposition is established by a counter-example to generality. Let N = 1 and T = 2, and let preferences be represented by the lifetime utility function $u = x_0x_1$. The necessary conditions directly yield the λ-constant demands.

$$(29) \qquad x_0^* = \lambda^* p_1 \quad \text{and} \quad x_1^* = \lambda^* p_0,$$

with $\lambda^* = W/2p_0p_1$. Note that $\partial x_0^*/\partial p_0 = 0$. Consequently, if λ-constant comparative statics were valid in this example, then the difference $x_1^* - x_0^*$ must be unrelated to the price difference $p_1 - p_0$. However,

$$(30) \qquad x_1^* - x_0^* = -\lambda^*[p_1 - p_0] \neq 0.$$

Therefore, λ-constant comparative statics is not generally valid.

That λ-constant comparative statics is not generally valid might not be surprising. However, the preferences in this example are strongly separable:

$u = \exp[\log x_0 + \log x_1] = x_0 x_1$. Although the optimal consumptions are invariant to increasing, monotonic transformations of lifetime utility, this result indicates that the validity of λ-constant comparative statics is not.

To consider this proposition more generally, let $u = G[U_0(x_0) + \ldots + U_{T-1}(x_{T-1})]$ with $G'[\cdot] > 0$, so preferences are intertemporally, strongly separable. The necessary conditions include

$$(31) \qquad G'[U(x_0) + \ldots + U(X_{T-1})]\cdot U'(x_t) = \lambda p_t, \qquad t = 0, \ldots, T - 1.$$

The λ-constant exercise yields for each period t

$$(32.1) \qquad U_t''(x_t^*)dx_t^* + [G''(\cdot)/G'(\cdot)]\cdot U_t'(x_t^*) \sum_{\tau=0}^{T-1} U_\tau'(x_\tau^*)dx_\tau^* = \tilde{\lambda}^* dp_t,$$

with $\tilde{\lambda}^* \equiv \lambda^*/G'(\cdot)$. Solving the system of T - 1 equations simultaneously results in

$$(32.2) \qquad \frac{\partial x_t^*}{\partial p_t}\cdot\frac{p_t}{x_t^*} = \frac{U_t'(x_t^*)}{\{U_t''(x_t^*) + [G''(\cdot)/G'(\cdot)]Q\}x_t^*},$$

where Q is a complicated expression in the marginal utilities of the T - 1 consumption goods. In general, $Q \neq 0$. From equation (3.2), $\hat{x}(t) \approx \left[(U_t'(x_t^*)/U_t''(x_t^*)x_t^*\right]\cdot\hat{p}(t)$. Consequently, within the class of strongly separable preferences, the validity of λ-constant comparative statics requires that G" equal zero. Increasing affine transformations preserve the usefulness of λ-constant comparative statics.[8]

Necessity

Within the context of a discrete time, perfect-foresight model with a single linear constraint in exogenous variables such as wealth, prices and interest rates: What is the most general utility function which supports the validity of λ-constant comparative statics?

With $u = F(\mathbf{x}_0, \ldots, \mathbf{x}_{T-1})$, the necessary conditions include

$$(33) \qquad \frac{\partial F}{\partial x_{it}}(\mathbf{x}_0, \ldots, \mathbf{x}_{T-1}) = \lambda R_t p_{it}, \qquad \begin{array}{l} i = 1, \ldots, N, \\ t = 0, \ldots, T-1. \end{array}$$

Take the total differential (around the optimum) of this system of NT equations; the resulting linear system is $\mathbf{F}d\mathbf{x}^* = \lambda^* \mathbf{R}d\mathbf{p}$, or

$$(34) \qquad \begin{bmatrix} \mathbf{F}(0,0) & \ldots & \mathbf{F}(0,T-1) \\ \vdots & \ddots & \\ \mathbf{F}(T-1,0) & & \mathbf{F}(T-1,T-1) \end{bmatrix} \begin{bmatrix} d\mathbf{x}_0 \\ \vdots \\ d\mathbf{x}_{T-1} \end{bmatrix} = \lambda^* \begin{bmatrix} \mathbf{R}_0 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{R}_{T-1} \end{bmatrix} \begin{bmatrix} d\mathbf{p}_0 \\ \vdots \\ d\mathbf{p}_{T-1} \end{bmatrix}$$

with $\mathbf{F}(t,t') = \left[\dfrac{\partial^2 F}{\partial x_{it}\partial x_{jt'}}\right]$ denoting the N-by-N Hessian matrix in period t; $d\mathbf{x}_t = (dx_{1t}, \ldots, dx_{Nt})'$ and $d\mathbf{p}_t = (dp_{1t}, \ldots, dp_{Nt})'$ denoting the N dimensional consumption and price vectors in period t; and $\mathbf{R}_t = R_t I$ denoting for each t an N dimensional scalar matrix.

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[8]Normalizations with $G'' \neq 0$ can be used if $\tilde{\lambda} \equiv \lambda / G'(\cdot)$ rather than λ is held constant in the comparative statics exercise. MaCurdy (1981, 1061) employs this technique.

The first step of the derivation links intertemporal consumption variation and λ-constant comparative statics to restrict the form of the λ-constant demands function. The second step uses the restrictions on λ-constant demands to restrict the Hessian matrix $\mathbf{F}$. The final step retrieves the lifetime utility function from the restrictions imposed on $\mathbf{F}$. The three steps of the derivation are relegated to Appendix B. The result is that lifetime utility must be strictly linear.

$$u = \sum_{t=0}^{T-1} f(\mathbf{x}_t, \mathbf{z}_t). \tag{35}$$

with $\mathbf{z}_t$ denoting an M-vector of time-varying preference parameters.

Sufficiency was established using equation (8), a particular case of equation (35). However, it is a straightforward exercise to establish sufficiency of the lifetime utility function in equation (35). Therefore, the results of this section and Appendix B establish the following theorem:

THEOREM: In the presence of (a) exogenous sequences of prices, income, and endowments, and (b) a perfect capital market, λ-constant comparative statics is valid in characterizing the intertemporal variation of consumption of an agent with perfect foresight if and only if the lifetime utility function is $u = \Sigma_{t=0}^{T-1} f(\mathbf{x}_t, \mathbf{z}_t)$.

Of course, if the consumer cannot commit to a sequence of consumption, the lifetime utility function must be further restricted to be time consistent.

4. Extending the Model

In this section, I investigate two extensions: addictive behavior and uncertainty. Addiction, in allowing the utility of current consumption to depend on previous consumption, might render intertemporal additivity too restrictive. With uncertainty, realized opportunities deviate from expected opportunities, inducing variation in λ over the life cycle. Nevertheless, λ-constant comparative statics can be a useful technique in characterizing intertemporal substitution models with addiction or uncertainty.

Addiction

Beginning with Marshall (1920, 666), economists have been intrigued with the behavior summarized by the Shakespearean dictum: "Use doth breed a habit." Economic models of addiction, both myopic and rational, have relied on intertemporally nonseparable preferences (Pollak 1970; Pollak 1976; Becker and Murphy 1988).

Let preferences be represented by a lifetime utility function

$$(36) \qquad u = \sum_{t=0}^{T-1} U(x_t, S_t),$$

where S_t is the stock of consumption capital which depends on consumption behavior in periods zero through t-1. Although lifetime utility is intertemporally additive in x_t and S_t jointly, intertemporal additivity in the x_t alone does not follow. Consequently, as a general proposition, λ-constant comparative statics is not appropriate in analyzing intertemporal substitution properties of such an addiction model.

The technique is valid in some cases. Consider one example. Let the momentary utility function U be additive in x_t and S_t.

$$(37) \qquad U(x_t, S_t) = \log x_t + a_t S_t;$$

and let the equation of motion governing the stock of consumption capital be

$$(38) \qquad S_t = (1 - \delta)S_{t-1} + \log x_{t-1},$$

with δ denoting the depreciation rate. In this example, past consumption affects current utility but not the marginal utility of current consumption. By repeated substitution, S_t can be expressed

$$(39) \qquad S_t = \sum_{\tau=0}^{t-1} (1-\delta)^{t-\tau-1} \log x_\tau .$$

Substituting S_t into equation (37) produces a lifetime utility which is intertemporally additive in consumption.

$$(40) \qquad u = \sum_{t=0}^{T-1} b_t \log x_t, \qquad b_t \equiv 1 + \sum_{\tau=t}^{T-1} (1 - \delta)^{\tau-t} a_{\tau+1} .$$

Consequently, λ-constant comparative statics can be a valid technique in characterizing intertemporal substitution even though momentary utility in period t depends on previous consumptions.[9]

This example highlights a subtle but important feature of the theorem:

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[9]A similar result applies outside the context of consumption. The analysis of production in the presence of a learning curve (e.g., Spence 1981) is very similar to the addiction model of consumption: output is increasing in and marginal cost is decreasing in the stock of production capital (accumulated output). The technique of λ-constant comparative statics would be valid only if total output could be represented in a reduced form as the intertemporal sum of functions of inputs and parameters dated at time t.

lifetime utility must be expressible by an intertemporally additive function of consumption, but in equation (35) $f(\mathbf{x}_t, \dot{\mathbf{z}}_t)$ need not be period t's momentary utility function.

One can use λ-constant comparative statics to establish that the model employed in this example can exhibit important features of addiction such as reinforcement or tolerance. However, it is incapable of capturing such features as withdrawal or a bi-modal distribution of consumption. (See Becker and Murphy (1988) for a discussion of the essential characteristics of addictive behavior.) Becker and Murphy employ "adjacent complementarity", which requires $U_{xs} > 0$, to generate a bi-modal distribution of consumption through multiple steady states. Adjacent complementarity is inconsistent with the requirements of the theorem, hence λ-constant comparative statics is not appropriate for analyzing the intertemporal substitution aspects of such an addiction model. Although Becker and Murphy use λ-constant comparative statics to characterize particular features of their model, they do avoid using the technique to characterize intertemporal variation in consumption induced by intertemporal variation in prices.

It might be possible to retain the validity of λ-constant comparative statics more generally. In equation (36), lifetime utility is intertemporally additive in x_t and S_t jointly. This suggests that the usefulness of the technique would be preserved if the wealth constraint could be expressed linearly in x_t and S_t. If so, a shadow price could be assigned to each period's stock of consumption capital, and the solution to the transformed problem could be analyzed in the usual way to characterize intertemporal substitution (King 1988, 29).

Uncertainty

Is the link between λ-constant comparative statics and intertemporal substitution broken if the assumption of perfect foresight is dropped? A key result is that with sequential resolution of uncertainty the marginal utility of wealth is not constant: optimizing behavior implies that λ_t^* follows a random walk with drift, or more formally a martingale (Hall 1978; MaCurdy 1985). MaCurdy (1985) and Browning, Deaton, and Irish (1985) use λ-constant demands to analyze the intertemporal problem under uncertainty. They find that λ-constant demands provide a tractable and conceptually clear framework for analyzing the effects of new information. Browning, Deaton, and Irish (1985, 515-16) write, "The great advantage of Frisch [λ-constant] demands is that they separate out the effects of movements along the path (which occur with or without perfect foresight) from those movements of the path itself caused by new information."

The decomposition into movements along versus shifts in life-cycle consumption profiles can be illustrated succinctly as follows. For this example assume that lifetime utility is intertemporally additive, sub-utilities are time-invariant, the rate of interest equals the rate of time preference, only the jth price varies intertemporally and the price variation is transitory. Aside from intertemporal additivity, the assumptions can be relaxed as a simple extension. The actual consumption choice in period t' deviates from expected consumption by a term which is proportional to the deviation of price from its expectation.

$$(41) \qquad x^*_{it'} - x^e_{it'} \approx \frac{\partial g_{it'}}{\partial p_{jt'}} [p_{jt'} - p^e_{jt'}],$$

where the factor of proportionality is the uncompensated cross-price effect. (Recall from equation (12) that $g_{it}(\cdot)$ denotes the Marshallian demand for good i at time t.) Combining this parametric change with the anticipated intertemporal variation results in

$$(42.1) \quad x^*_{it'} - x^*_{it} \approx \frac{\partial x_i}{\partial p_{jt}} [p^e_{jt'} - p_{jt}] + \frac{\partial g_{it'}}{\partial p_{jt'}} [p_{jt'} - p^e_{jt'}]$$

or

$$(42.2) \quad \hat{x}_i(t) \approx \eta_{ij}(t) \cdot \hat{p}^e_j(t) + \epsilon_{ij}(t') \cdot \hat{p}^u_j(t'),$$

with $\hat{p}^e_j(t)$ denoting the expected intertemporal (percentage) variation in the price of good j, $\hat{p}^u_j(t')$ the percentage change in the price of good $x_{jt'}$, and $\epsilon_{ij}(t')$ the uncompensated cross-price elasticity of demand for good $x_{it'}$ with respect to the price of good $x_{jt'}$. The total intertemporal variation is the sum of a movement along the intertemporal profile (the λ-constant effect), and the change induced by a parametric shift in the profile (the uncompensated effect).

The total intertemporal variation can be expressed via an alternative decomposition. Combine the general substitution effect and the income effect to form the λ-varying effects; then the uncompensated effect is the sum of λ-constant and λ-varying effects. Therefore,

$$(43.1) \quad x^*_{it'} - x^*_{it} \approx \frac{\partial x_i}{\partial p_{jt}} [p_{jt'} - p_{jt}] + \frac{\partial x_i}{\partial \lambda} \frac{\partial \lambda}{\partial p_{jt}} [p_{jt'} - p^e_{jt'}]$$

$$(43.2) \quad \hat{x}_i(t) \approx \eta_{ij}(t)\cdot\hat{p}_j(t) + \xi_{ij}(t')\cdot\hat{p}^u_j(t'),$$

with $\xi_{ij}(t')$ denoting the period t sum of the general substitution effect and the income effect expressed as an elasticity. The total intertemporal variation is the sum of the λ-constant effect of the realized intertemporal price variation plus the λ-varying effect induced by the deviation of $p_{jt'}$ from its expected value.

Both decompositions establish that λ-constant comparative statics is central to the characterization of intertemporal variation in consumption even in the presence of sequentially resolved uncertainty. Of course, the power of the decomposition rests on the magnitude of the wealth effects associated with surprises.[10]

5. Conclusion

Marginal-utility-of-wealth-constant comparative statics is a useful technique for analyzing intertemporal substitution. However, the results do not always correspond to intertemporal substitution. The theorem derived in section 3 establishes that intertemporal additivity is both necessary and sufficient for λ-constant comparative statics to characterize intertemporal substitution.

Since intertemporal additivity is commonly employed, the potential

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[10]The longer the horizon and the smaller the rate of time preference, the smaller are the wealth effects likely to be. Bewley (1977, Theorem 3.2) establishes that if income follows an arbitrary stationary process, then as the horizon approaches infinity and the rate of time preference approaches zero, the marginal utility of wealth converges to a constant. This result does not require perfect capital markets: Bewley establishes the result under the restriction that the consumer cannot borrow.

applicability of the technique is widespread. Furthermore, λ-constant comparative statics terms are easy to compute. The cross term $\partial x^*_{it}/\partial p_{jt}$ is proportional to the (i,j)th co-factor of period t's Hessian matrix. Relative to wealth-constant or utility-constant comparative statics, the λ-constant comparative statics exercise reduces the number of equations and endogenous variables by $(T-1)N + 1$, a reduction at least as large as the number of time periods T.

The results apply outside the intertemporal context. Perhaps the most important application is to insuring against uncertainty. With some minor changes in notation, state-contingent consumption demand with full insurance satisfies the conditions of the theorem. Replace periods t with states of nature s and the discounting terms β^t and R_t with probabilities π_s. Since expected utility is additive across states, the condition of the theorem is satisfied without ruling out state-dependent preferences or actuarially unfair insurance.[11]

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[11]Rosen (1985, 1157) uses λ-constant comparative statics to analyze implicit contracts.

APPENDIX A

Preference Specifications and λ-Constant Demands

In this appendix, I present λ-constant demands for several tractable specifications of momentary utility. For illustration, consumption and leisure are the only two goods in each period. The interest rate is assumed to be constant, the preference parameters are restricted to be time invariant, and subsistence levels are set to zero. Each of these restrictions can be relaxed, but at the cost of a more complex notation.

Constant Relative Risk Aversion

The first specification of momentary utility is additive in consumption and leisure.

$$\text{(A1)} \qquad U(c_t, \ell_t) = \phi \left[\frac{c_t^{\gamma_1} - 1}{\gamma_1}\right] + (1 - \phi)\left[\frac{\ell_t^{\gamma_2} - 1}{\gamma_2}\right],$$

with $0 \leq \phi \leq 1$ and $\gamma_1, \gamma_2 < 1$. (See, e.g., Heckman and McCurdy (1982).) The limiting case of $\gamma_1 \to 0$ and $\gamma_2 \to 0$ is logarithmic utility.

$$\text{(A2)} \qquad U(c_t, \ell_t) = \phi \log c_t + (1 - \phi) \log \ell_t.$$

For specification (A1), the two necessary conditions at time t imply the following λ-constant consumption and leisure demands:

$$\text{(A3)} \qquad \log c_t^* = \frac{1}{1 - \gamma_1} \left[\log \phi - \log p_t - \log \lambda^* + (r - \rho)t\right]$$

with α_c and α_ℓ constants related to ϕ and γ. Therefore, the λ-constant own-price elasticities of demand are $\eta_{cc} = - [1 - (1-\phi)\gamma]/(1-\gamma) < 0$ and $\eta_{\ell\ell} = - (1 - \phi\gamma)/(1-\gamma) < 0$. The signs of the cross-price elasticities of demand depend on whether γ is positive of negative: $\eta_{c\ell} = - (1-\phi)\gamma/(1-\gamma) \gtrless 0$ and $\eta_{\ell c} = - \phi\gamma/(1-\gamma) \gtrless 0$ as $0 \gtrless \gamma$.

Kydland and Prescott in their study of fluctuations generated by a real business cycle model employed this generalized Cobb-Douglas specification (as well as an intertemporally nonseparable specification). In calibrating their model, Kydland and Prescott set $\phi = 1/3$ and $\gamma = - 1/2$. These values generate λ-constant demand elasticities of $\eta_{cc} = -2$, $\eta_{\ell\ell} = -7/9$, $\eta_{c\ell} = 2$, and $\eta_{\ell c} = 1$. These are sizable. For instance, perfectly foreseen 10 percent intertemporal growth in the wage produces 20 percent intertemporal growth in consumption. Transforming the leisure demand elasticity to an hours supply elasticity yields $\eta_{hh} = 14/9 \approx 1.56$. Thus Kydland and Prescott's intertemporally additive specification allows for considerable intertemporal substitution.

Quasi-Linear

The third specification of momentary utility treats leisure as a borderline inferior good.

$$\text{(A8)} \qquad U(c_t, \ell_t) = \frac{1}{\gamma}\left[\left[c_t + \frac{\ell_t^\theta}{\theta}\right]^\gamma - 1\right]$$

with $\theta < 1$ and $\gamma < 1$. (See, e.g., Greenwood, Hercowitz, and Huffman (1988).) With the quasi-linear specification, $U_{c\ell} < 0$. The limiting case of $\gamma \to 0$ yields $U(c_t, \ell_t) = \log(c_t + \ell_t^\theta/\theta)$.

(A4) $$\log \ell_t^* = \frac{1}{1-\gamma_2}\left[\log (1-\phi) - \log w_t - \log \lambda^* + (r-\rho)t\right].$$

(As an approximation, the difference $r - \rho$ replaces $\log[(1+r)/(1+\rho)]$.) Equations (A3) and (A4) imply that the λ-constant own-price elasticities of demand are $\eta_{cc} = -1/(1-\gamma_1) < 0$ and $\eta_{\ell\ell} = -1/(1-\gamma_2) < 0$; the λ-constant cross-price elasticities are both zero because $U_{c\ell} = 0$ for this specification.

Generalized Cobb-Douglas

The second specification of momentary utility is not necessarily additive in consumption and leisure.

(A5) $$U(c_t, \ell_t) = \frac{1}{\gamma}\left[c_t^{\phi}\, \ell_t^{1-\phi}\right]^{\gamma},$$

with $0 \leq \phi \leq 1$ and $\gamma < 1$. (See, e.g., Kydland and Prescott (1982) and King (1988).) The cross derivative $U_{c\ell} \gtrless 0$ as $\gamma \gtrless 0$. Therefore, this specification is more flexible than the first, but it also contains logarithmic utility as a special case (as $\gamma \to 0$).

The necessary conditions imply that

(A6) $$\log c_t^* = \alpha_c - \left(\frac{1-(1-\phi)\gamma}{1-\gamma}\right) \log p_t - \frac{(1-\phi)\gamma}{1-\gamma} \log w_t - \frac{1}{1-\gamma} \log \lambda^* + \frac{1}{1-\gamma}(r-\rho)t$$

(A7) $$\log \ell_t^* = \alpha_\ell - \frac{\phi\gamma}{1-\gamma} \log p_t - \left(\frac{1-\phi\gamma}{1-\gamma}\right) \log w_t + \frac{1}{1-\gamma} \log \lambda^* + \frac{1}{1-\gamma}(r-\rho)t,$$

The necessary conditions imply a very simple expression for leisure demand.

$$\text{(A9)} \qquad \log \ell_t^* = \frac{1}{1-\theta}(\log p_t - \log w_t)$$

without a time trend. This expression triples as the Marshallian, Hicksian, and Frisch (λ-constant) demands for leisure. The own- and cross-price elasticities of demand for leisure are $\eta_{\ell\ell} = -\frac{1}{1-\theta} < 0$ and $\eta_{\ell c} = \frac{1}{1-\theta} > 0$.

The expression for λ-constant consumption demand is more complicated.

$$\text{(A10)} \qquad c_t^* = \left[\lambda^* p_t\right]^{\frac{-1}{1-\gamma}} \left(\frac{1+r}{1+\rho}\right)^{\frac{t}{1-\gamma}} - \frac{1}{\theta}\left(\frac{p_t}{w_t}\right)^{\frac{\theta}{1-\theta}} .$$

Given λ^*, consumption c_t^* is decreasing in price p_t and increasing in the wage rate w_t.

APPENDIX B

Proof of the Theorem

THEOREM: In the presence of (a) exogenous sequences of prices, income, and endowments, and (b) a perfect capital market, λ-constant comparative statics is valid in characterizing the intertemporal variation of consumption of an agent with perfect foresight if and only if the lifetime utility function is $u = \Sigma_{t=0}^{T-1} f(x_t, z_t)$.

PROOF:

Sufficiency

Establishing sufficiency is a simple exercise: follow the steps in equations (22) - (28) for the more general specification of momentary utility.

Necessity

In general, the vector of λ-constant demands in period t is

$$\text{(B1)} \qquad x_t^* = x\left[\lambda^* R_0 p_0, \ \ldots, \ \lambda^* R_{T-1} p_{T-1}, \ z_t\right],$$

with z_t denoting the M-vector of time-varying preference parameters. To be valid in characterizing intertemporal substitution, λ-constant comparative statics must satisfy

$$\text{(B2)} \quad x_{it'}^* - x_{it}^* \approx \sum_{n=1}^{N} \frac{\partial x_i}{\partial p_{nt}} \cdot \left\{ [p_{nt'} - p_{nt}] - p_{nt} \sum_{\tau=t}^{t'-1} r_\tau \right\} + \sum_{m=1}^{M} \frac{\partial x_i}{\partial z_{mt}} \cdot [z_{mt'} - z_{mt}]$$

for all i, t, t', λ^*, and prices. Therefore,

$$\text{(B3)}\qquad x_i\left[\lambda^* R_0 \mathbf{p}_0,\ \ldots,\ \lambda^* R_{T-1}\mathbf{p}_{T-1},\ \mathbf{z}_{t'}\right] \approx x_i\left[\lambda^* R_0 \mathbf{p}_0,\ \ldots,\ \lambda^* R_{T-1}\mathbf{p}_{T-1},\ \mathbf{z}_t\right]$$

$$+ \sum_{n=1}^{N} \frac{\partial x_i}{\partial p_{nt}} \cdot \left\{ [p_{nt'} - p_{nt}] - p_{nt} \sum_{\tau=t}^{t'-1} r_\tau \right\} + \sum_{m=1}^{M} \frac{\partial x_i}{\partial z_{mt}} \cdot [z_{mt'} + z_{mt}],$$

which is approximately equal to period t's λ-constant demand function evaluated at $R_t\mathbf{p}_t = R_{t'}\mathbf{p}_{t'}$ and $\mathbf{z}_t = \mathbf{z}_{t'}$. Since this final expression is not a function of $\mathbf{p}_t$,

$$\text{(B4)}\qquad \frac{\partial x^*_{it'}}{\partial p_{jt}} = 0,$$

which must hold for all i, j, t, and $t' \neq t$. Therefore, the λ-constant demand functions are restricted to the form

$$\text{(B5)}\qquad \mathbf{x}^*_t = \mathbf{x}(\lambda^* R_t \mathbf{p}_t,\ \mathbf{z}_t), \qquad t = 0,\ \ldots,\ T-1.$$

Let $\mathbf{\Delta}(t,t)$ denote period t's N-by-N matrix of normalized price derivatives: $\left[\dfrac{\partial x_{it}}{\partial (p_{jt}/\phi_t)}\right]$. The system of NT λ-constant demands given in equation (B5) implies the following matrix equation:

$$\text{(B6)}\qquad \begin{bmatrix} d\mathbf{x}_0 \\ \vdots \\ d\mathbf{x}_{T-1} \end{bmatrix} = \lambda^* \begin{bmatrix} \mathbf{\Delta}(0,0) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{\Delta}(T-1,T-1) \end{bmatrix} \begin{bmatrix} R_0 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & R_{T-1} \end{bmatrix} \begin{bmatrix} d\mathbf{p}_0 \\ \vdots \\ d\mathbf{p}_{T-1} \end{bmatrix},$$

or $d\mathbf{x}^* = \lambda^* \mathbf{\Delta R} d\mathbf{p}$. From equation (34), $d\mathbf{x}^* = \lambda^* \mathbf{F}^{-1}\mathbf{R}d\mathbf{p}$. Therefore, $\mathbf{F\Delta} = \mathbf{I}$.

$$(B7) \qquad \begin{bmatrix} \mathbf{F}(0,0)\boldsymbol{\Delta}(0,0) & \cdots & \mathbf{F}(0,T-1)\boldsymbol{\Delta}(T-1,T-1) \\ \vdots & \ddots & \\ \mathbf{F}(T-1,0)\boldsymbol{\Delta}(0,0) & & \mathbf{F}(T-1,T-1)\boldsymbol{\Delta}(T-1,T-1) \end{bmatrix} = \underset{(NT \times NT)}{\mathbf{I}} .$$

Consequently, **F** must be block diagonal and **Δ** must satisfy

$$(B8) \qquad \begin{bmatrix} \boldsymbol{\Delta}(0,0) \\ \vdots \\ \boldsymbol{\Delta}(T-1,T-1) \end{bmatrix} = \begin{bmatrix} \mathbf{F}^{-1}(0,0) \\ \vdots \\ \mathbf{F}^{-1}(T-1,T-1) \end{bmatrix}.$$

The final step follows from the block diagonal property of **F**: the lifetime utility function must be strictly additive.

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