Bargaining Problems with Claims

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1. Introduction

A firm goes bankrupt but its liquidation value is not sufficient to pay back all of its creditors. How should its net worth be divided among them? In this paper we analyze such situations and propose ways in which the claims of the various creditors should be taken into consideration in the computation of the amount attributed to each of them. Our objective is to identify desirable ways of performing this division. We follow the axiomatic approach: We formulate requirements on how the division should be carried out and we search for rules that satisfy all the requirements.

Bankruptcy problems (and taxation problems, which are mathematically equivalent) have been studied by O'Neill (1982), Aumann and Maschler (1985), Moulin, (1987),Young (1987,1988) and Chun (1988). These authors have considered the case when each agent's utility function is a linear function of the monetary amount he is awarded, but here we generalize the model to cover the "non-transferable utility case": Available to the creditors is a set of utility vectors whose upper boundary is not necessarily a hyperplane normal to a vector of equal coordinates. We also specify in our model a "reference point" from which utility gains can be measured. This point is implicitly taken to be the origin in the above-mentioned papers. The incompatibility of the claims implies that the vector of claims is a point outside of the feasible set.

Our formulation can alternatively be seen as a generalization of that adopted by Nash (1950) in his study of bargaining and for that reason we refer to the problems we analyze as "bargaining problems with claims". Consider a labor-management conflict: There are a set of finite alternatives. The two parties have to agree on one of them. If they do not agree, a strike or a lockout may occur. This is the "disagreement point". Labor and management come to the negotiation table with certain expectations. These expectations may have been formed by observing the resolution of similar conflicts in related industries and may not be mutually compatible in the actual situation. They may alternatively represent commitments made in earlier negotiations which, because of changes in the circumstances of bargaining that may have adversely affected the feasible set, may not be jointly realizable any more. According to that interpretation, the disagreement point plays the role of what we referred to earlier by the deliberately neutral term of "reference point" while the vector of expectations of the bargainers was designated by the term of claims point.

The theory of bargaining will serve as a very convenient template for the theory of bargaining with claims that we would like to initiate here. Axiomatic studies of the bargaining problem can be divided into three groups. Starting from Nash (1950) himself, the traditional literature has been mainly based on axioms pertaining to changes in the feasible set. More recently however, the focus has been on the disagreement point (Thomson 1987; Chun and Thomson 1987) and on the number of agents (Thomson 1983). We will show that each of these approaches can be fruitfully adapted to develop a theory of bargaining with claims.

Indeed, the mathematical structure of the class of problems under study here is closely related to that of bargaining theory, and each of the approaches that have proved useful there will prove equally useful here. The overall picture will, however, be quite different. Indeed the developments in bargaining theory have not permitted the identification of a single solution as being best[1] whereas here, our various angles of attack all lead to a same solution. This solution chooses the maximal feasible point on the line connecting the disagreement point to the claims point. Therefore, one can

[1] They are the Nash (1950) solution, the Kalai-Smorodinsky (1975) solution, and the Egalitarian (Kalai 1977) solution.

and c is the *claims point.* Let Σ^n be the class of all n-person problems. The intended interpretation of (S, d, c) is as follows: the agents can achieve any point of S; the point d is a reference point from which they find it natural to measure their utility gains in order to evaluate a proposed compromise; each coordinate of the claims point may represent a promise made to the corresponding agent, or his understanding of what the others had agreed that he would get in some earlier negotiation; or an agreement, perhaps a contract, made before some unfavorable circumstances led to changes in the feasible set that make it impossible. It is assumed that claims are made in good faith, and that although they cannot all be honored, some effort should be made at taking them into account in the determination of the final compromise. What will be a reasonable compromise?

A *solution* is a function $F: \Sigma^n \rightarrow \Re^n$ that associates with each $(S, d, c) \in \Sigma^n$, a unique point of S, $F(S, d, c)$, called the *solution outcome of* (S, d, c).

The following solution, illustrated in Figure 2, will play the central role in our analysis.

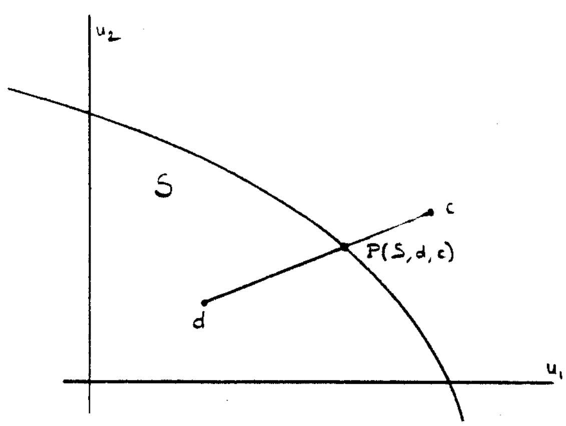

Figure 2.

Definition. The *proportional solution, P*: For all $(S, d, c) \in \Sigma^n$, $P(S, d, c)$ is the maximal point of S on the segment connecting d and c.

much more confidently than would perhaps have been expected, point to one solution as being the most desirable.

2. Preliminaries

An *n-person bargaining problem with claims,* or simply a *problem,* is a triple (S, d, c), where S is a subset of $\Re^n$, d and c are points in $\Re^n$, such that (see Figure 1)

(1) S is convex and closed,

(2) there exist $p \in \Re^n_{++}$[2] and $r \in \Re$ such that for all $x \in S$, $px \leqq r$,

(3) S is *comprehensive,* i.e., for all $x \in S$ and for all $y \in \Re^n$, if $y \leq x$, then $y \in S$,

(4) there exists $x \in S$ with $x > d$,

(5) $c \notin S$, $c \geq d$ and $c \leqq a(S,d)$ where $a_i(S,d) \equiv max\{x_i | x \in S, x \geqq d\}$ for all i.

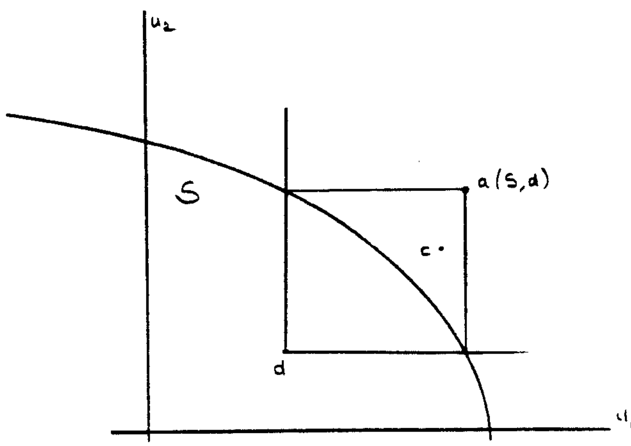

Figure 1.

S is the *feasible set.* Each point x of S is a *feasible alternative.* The coordinates of x are the utility levels, measured in some von Neumann-Morgenstern scales, attained by the agents through the choice of some action. The point d is the *disagreement point*

[2] Vector inequalities: given $x, y \in \Re^n, x \geqq y, x \geq y, x > y$. $\Re^n_{++} \equiv \{x \in \Re^n | x > 0\}$.

We will offer several characterizations of this solution. They will involve several axioms adapted in a straightforward way from the standard theory of bargaining. These axioms are presented first. In addition, a variety of specific additional axioms will be introduced in each of the subsequent sections.

Weak Pareto-Optimality (W.P.O). For all $(S,d,c) \in \Sigma^n$ and for all $x \in \Re^n$, if $x > F(S,d,c)$, then $x \notin S$.

Let $WPO(S) \equiv \{x \in S | \forall x' \in \Re^n, x' > x \text{ implies } x' \notin S\}$ be the set of *weakly Pareto-optimal points of S*. Similarly, let $PO(S) \equiv \{x \in S | \forall x' \in \Re^n, x' \geq x \text{ implies } x' \notin S\}$ be the set of *Pareto-optimal points of S*.

Symmetry (SY). For all $(S,d,c) \in \Sigma^n$, if for all permutations $\pi : \{1,\ldots,n\} \rightarrow \{1,\ldots,n\}$, $S = \pi(S)$, $d = \pi(d)$, and $c = \pi(c)$,[3] then $F_i(S,d,c) = F_j(S,d,c)$ for all $i,j = 1,\ldots,n$.

Boundedness (BDD). For all $(S,d,c) \in \Sigma^n$, $d \leqq F(S,d,c) \leqq c$.

In the following, convergence of a sequence of sets is evaluated in the Hausdorff topology.

Continuity (CONT). For all sequences $\{(S^\nu, d^\nu, c^\nu)\}$ of elements of Σ^n and for all $(S,d,c) \in \Sigma^n$, if $S^\nu \rightarrow S$, $d^\nu \rightarrow d$ and $c^\nu \rightarrow c$, then $F(S^\nu, d^\nu, c^\nu) \rightarrow F(S,d,c)$.

W.P.O requires that there be no feasible alternative that all agents strictly prefer to the solution outcome. *SY* requires that if the problem is invariant under all exchanges of agents, then all agents be treated identically. *BDD* requires that no agent

[3] $\pi(x) \equiv (x_{\pi(i)})_{i=1,\ldots,n}$ and $\pi(S) \equiv \{x' \in \Re^n | \exists x \in \Re^n \text{ such that } x' = \pi(x)\}$.

be worse off at the solution outcome than at the disagreement point, and that no agent be better off at the solution outcome than at the claims point. *CONT* requires that a small change in the problem cause only a small change in the solution outcome.

Let Λ^n be the class of transformations $\lambda : \Re^n \rightarrow \Re^n$ defined as follows: for each $i \in N$, there exist $a_i \in \Re_{++}$ and $b_i \in \Re$ such that for all $x \in \Re^n$, $\lambda_i(x) = a_i x_i + b_i$. Given $S \subset \Re^n$, $\lambda(S) \equiv \{y \in \Re^n | \exists\, x \in S \text{ with } y = \lambda(x)\}$. Note that if $(S, d, c) \in \Sigma^n$, then $(\lambda(S), \lambda(d), \lambda(c)) \in \Sigma^n$ as well.

Scale Invariance (SC.INV). For all $(S, d, c) \in \Sigma^n$ and for all $\lambda \in \Lambda^n$, $F(\lambda(S), \lambda(d), \lambda(c)) = \lambda(F(S, d, c))$.

SC.INV says that subjecting a problem to a positive linear transformation acting independently on each coordinate leads to a new problem that should be solved at the image under this transformation of the solution outcome of the original problem. It is justified by the fact that agents' utilities are of the von Neumann-Morgenstern type. But note that it precludes basing the recommendation for a compromise on interpersonal comparisons of utility.

It is easily verified that all of these conditions are satisfied by the proportional solution. In the two-person case, the proportional solution satisfies Pareto-optimality (which says that $F(S, d, c) \in PO(S)$). The solution also satisfies Pareto-optimality on the domain of n-person "strictly comprehensive"problems (problems (S, d, c) such that for all $x, y \in S$ with $x \geqq y$ there exists $z \in S$ with $z > y$). To obtain Pareto-optimality over the whole of Σ^n, we could follow standard procedures[4] and define a lexicographic extension of the solution. This extension would however not be continuous. Other appealing properties discussed later would also fail to be satisfied.

[4] See, for example, Imai(1983).

The following notation and terminology will be used frequently. Given $S \subset \Re^n$, $int(S)$ is the interior of S, and $r.int(PO(S))$ is the relative interior of $PO(S)$ with respect to $WPO(S)$. Given $x, y \in \Re^n$ such that $x \neq y$, $\ell(x,y)$ is the line passing through x and y. Given $x^1, \ldots, x^k \in \Re^n$, $cch\{x^1, \ldots, x^k\}$ is the smallest convex and comprehensive set containing these k points, and given $S^1, \ldots, S^k \subset \Re^n$, $cch\{S^1, \ldots, S^k\}$ is the smallest convex and comprehensive set containing these k sets.

3. Axioms Concerning Changes in the Feasible Set

In this section, we consider certain changes in the feasible sets and formulate axioms specifying how the solution outcome should respond to these changes. The main one says that an expansion of opportunities, other things being equal, benefits all agents. This requirement, together with a few of the standard conditions, leads to our first characterization of the proportional solution. The axiom and proof are adapted from Kalai (1977).

Strong Monotonicity (ST.MON). For all $(S,d,c), (S',d',c')$ with $(d,c) = (d',c')$ and $S \subseteq S'$, $F(S',d',c') \geq F(S,d,c)$.

Theorem 1. *The proportional solution is the unique solution satisfying W.P.O, SY, SC.INV, and ST.MON.*

Proof. Let $(S,d,c) \in \Sigma^n$ be given. First, suppose that $P(S,d,c) \in PO(S)$. By *S.INV*, we can assume that $d = 0$ and $c = (1, \ldots, 1)$. Then $P(S,d,c) = (\alpha, \ldots, \alpha) \equiv x$ for some α. Let $S' \equiv cch\{x, c_1 e_1, \ldots, c_n e_n\}$, where e_i is the ith unit vector. We have $(S',d,c) \in \Sigma^n$ and since (S',d,c) is invariant under all exchanges of agents and $x \in PO(S')$, we have by *W.P.O* and *SY*, $F(S',d,c) = x$. Now $S' \subset S$ and by *ST.MON*, $F(S',d,c) \geq F(S,d,c)$. Since $x \in PO(S)$, we obtain $F(S,d,c) = x = P(S,d,c)$, as desired. To conclude in the case $P(S,d,c) \in WPO(S) \backslash PO(S)$, we apply a continuity argument involving the earlier conclusion and $ST.MON$. Q.E.D.

Next, we analyze situations where the feasible set may be uncertain. The requirement that we will consider for that case is adapted to our model from an axiom of bargaining theory proposed by Perles and Maschler (1981) and further studied by Peters (1986).

Concavity with Respect to the Feasible Set (S.CAV). For all $(S,d,c),(S',d',c')$ with $(d,c) = (d',c')$ and for all $\alpha \in [0,1]$, $F(\alpha S + (1-\alpha)S', d, c) \geqq \alpha F(S,d,c) + (1-\alpha)F(S',d,c)$. (Note that $(\alpha S + (1-\alpha)S', d, c)$ is a well-defined element of Σ^n.)

$S.CAV$ can be motivated on the basis of timing of decisions. Consider agents *today,* who, *tomorrow,* will face one of two equally likely problems (S,d,c) and (S',d,c), having the same disagreement point and claims point, but different feasible sets. They have two options: they can wait until tomorrow for the uncertainty to be lifted and solve then whatever problem has come up, or they can make contingent contracts. Since the set of feasible utility vectors obtained by such contracts is simply the "average" of S and S', the expected payoff vector associated with the second option is $F(\frac{S+S'}{2}, d, c)$. On the other hand, the expected payoff vector associated with the first option is $\frac{F(S,d,c)+F(S',d,c)}{2}$. This point is typically Pareto-dominated by a point of $\frac{S+S'}{2}$ so that it would be desirable that the agents avail themselves of the possibility of writing contingent contracts. Imposing $S.CAV$ on the solutions guarantees that all agents will benefit from such contracts.

We are now ready to state a characterization of the proportional solution based on $S.CAV$. Since its proof is similar to a proof in Peters (1986), we omit it.

Theorem 2. *The proportional solution is the unique solution satisfying W.P.O, SY, SC.INV, and S.CAV*

4. Axioms Concerning Changes to the Disagreement Point and the Claims Point

In this section, we consider changes in the disagreement point and claims points.

Betweenness with Respect to the Disagreement Point (B.D.P). For all $(S,d,c),(S',d',c') \in \Sigma^n$, for all i and for all $\alpha \in [0,1]$, if $(S,c)=(S',c')$ then $F_i(S,\alpha d+(1-\alpha)d',c) \geqq min\{F_i(S,d,c),F_i(S,d',c)\}$. (Note that $(S,\alpha d+(1-\alpha)d',c)$ is a well-defined element of Σ^n.)

Mid-Point Domination (M.P.D). For all $(S,d,c) \in \Sigma^n$, either $F(S,d,c) \geqq \frac{c+d}{2}$ or $F(S,d,c) \leqq \frac{c+d}{2}$.

B.D.P, adapted from a condition introduced by Chun and Thomson (1988) for bargaining problems under the name of *disagreement point quasi-concavity,* can be motivated on the basis of timing of bargaining, as in the case of *S.CAV,* but this time the uncertainty pertains to the disagreement point. Another difference is that instead of requiring that agents be unanimous in preferring contingent contracts (this would result in the payoff vector $F(S,\frac{d+d'}{2},c)$) we impose the weaker condition that no agent be worse-off under contingent contracts than he would be at the solution outcome of the worse for him of the two constituent problems.

Instead of taking all the features of the problem they face into account, agents may find it natural to evaluate proposed compromises in reference to some summary information. An appealing way of summarizing the important features of a problem certainly seems to be taking that average of the disagreement point and the claims point. A minimal notion of equal treatment is that all agents end up above the average or that they all end up below. This is the content of *M.D.P*, an axiom which generalizes similar conditions of bargaining theory.

We now turn to the results. The proof of Lemma 1 is the same as the proof of Lemma 1 in Chun and Thomson (1988).

Lemma 1. *Let F be a solution satisfying W.P.O, BDD, and B.D.P. Also let $(S,d,c) \in \Sigma^n$ such that $F(S,d,c) \in PO(S)$ be given. Then for all $d' \in [d, F(S,d,c)[$, $F(S,d',c) = F(S,d,c)$.*

Proof. First, note that $(S,d',c) \in \Sigma^n$ for all $d' \in [d, F(S,d,c)[$. Let $d' \in]d, F(S,d,c)[$ be given. Let $\bar{\lambda} \in]0,1[$ be such that $d' = \bar{\lambda} d + (1-\bar{\lambda})F(S,d,c)$, and $\{\lambda^k\}$be a sequence of elements of $]0,1[$ be such that $\lambda^k < \bar{\lambda}$ for all k and $\lambda^k \to \bar{\lambda}$. Also, let $d^k \equiv \frac{d'-\lambda^k d}{1-\lambda^k}$ for all k. Note that $(S,d^k,c) \in \Sigma^n$ for all k. By *B.D.P*, $F_i(S,d',c) \geqq min\{F_i(S,d^k,c), F_i(S,d,c)\}$ for all i and for all k. As $k \to \infty$, $d^k \to F(S,d,c)$, and since $F(S,d,c) \in PO(S)$, it follows from *BDD* that $F(S,d^k,c) \to F(S,d,c)$. Therefore, we obtain $F(S,d',c) \geq F(S,d,c)$. Since $F(S,d,c) \in PO(S)$, we conclude by *W.P.O* that $F(S,d',c) = F(S,d,c)$. Q.E.D.

Theorem 3. *The proportional solution is the unique solution satisfying W.P.O, BDD, CONT, M.P.D, and B.D.P.*

Proof. It is obvious that the proportional solution satisfies the five axioms. Conversely, let F be a solution satisfying the five axioms. Let $(S,d,c) \in \Sigma^n$ be such that $F(S,d,c) \equiv x \in r.int(PO(S))$. From Lemma 1, for all $d' \in [d, F(S,d,c)[$, $F(S,d',c) = F(S,d,c)$. Suppose, by way of contradiction, that $x \neq y \equiv P(S,d,c)$. Pick $d^1 \in [d, F(S,d,c)[$ and $z \in r.int(PO(S))$ such that $z \in \ell(d^1,c)$ and $||c-z|| \geqq ||z-d^1||$. Since $x \in r.int(PO(S))$, such d^1 and z exist. Moreover, since $x \neq y$, it follows that $x \neq z$. Now let $d^2 \equiv 2z - c$. Then $d \in \ell(d^1, c)$ and $||c - z|| = ||z - d^2||$. Since, by *BDD*, $c \geqq d^1$, then $d^1 \geqq d^2$ and therefore, $(S, d^2, c) \in \Sigma^n$. By *M.P.D* and the fact that $z \in PO(S)$, $F(S, d^2, c) = z$. Since $d^1 \in \ell(d^2, z)$, from Lemma 1, $F(S, d^1, c) = z$, a contradiction.

Since an arbitrary problem $(S, d, c) \in \Sigma^n$ can be approximated by a sequence of problems $\{(S^\nu, d^\nu, c^\nu)\}$ of elements of Σ^n such that, for all ν, $d^\nu = d$, $c^\nu = c$ and $F(S^\nu, d^\nu, c^\nu) \in r.int(PO(S^\nu))$, we conclude by *CONT* that $F(S, d, c) = P(S, d, c)$ for all $(S, d, c) \in \Sigma^n$, as desired

Q.E.D.

In the above theorem, W.P.O cannot be strengthened to Pareto-optimality as shown in the following:

Theorem 4. *There is no solution satisfying P.O and BDD if $n \geq 3$.*

Proof. Let $n = 3$, $S = cch\{(1,1,0),(0,1,1)\}$, $d = (0,0,0)$, and $c = (2/3, 2/3, 2/3)$. It is easily verified that $(S, d, c) \in \Sigma^n$ and that $PO(S) = [(1,1,0),(0,1,1)]$. However, for all $x \in PO(S)$, $x_2 > c_2$, in contradiction with *BDD*. Q.E.D.

Also, we should note that in Theorem 3, BDP cannot be strengthened to the following condition which is the counterpart for our domain of a condition introduced by Chun and Thomson (1987) for bargaining problems.

Disagreement Point Concavity (D.CAV). For all $(S, d, c), (S', d', c') \in \Sigma^n$ and for all $\alpha \in [0, 1]$, if $S = S'$ and $c = c'$, then $F(S, \alpha d + (1-\alpha)d', c) \geqq \alpha F(S, d, c) + (1 - \alpha)F(S, d', c)$.

The proof of the next result is obtained by a simple adaptation of an argument found in Chun and Thomson(1987).

Theorem 5. *There is no solution satisfying W.P.O, BDD, and D.CAV.*

5. Axioms Concerning Changes in the Populations

In this section, we focus on the behavior of solutions across cardinalities. To accommodate changes in the number of agents, we generalize the model along the lines of a similar generalization of the bargaining problem (Thomson 1983). There is an infinite universe $I = \{1, 2, \ldots\}$ of ***potential agents***, only a finite number of whom are present at a given time. Let M be the class of finite subsets of I. Given $M \in \mathcal{M}$, $\Re^M$ is the Cartesian product of $|M|$ copies of $\Re$ indexed by the members of M. Let Σ^M be the class of subsets of $\Re^M$ satisfying all the assumptions previously imposed on the members of Σ^n. Let $\Sigma \equiv \cup_{M \in \mathcal{M}} \Sigma^M$. A *solution* is a function defined on Σ that associates with every $M \in \mathcal{M}$ and every $(S, d, c) \in \Sigma^M$ a unique member of S interpreted as the compromise recommended for that problem.

We will present an alternative characterization of the proportional solution using axioms concerning changes in the set of agents. All of the axioms introduced earlier for the fixed population case can of course be extended to the variable population case by insisting that they hold *for all M*. For instance, *W.P.O* can be written as:

Weak Pareto-Optimality (W.P.O). For all $M \in \mathcal{M}$ and for all $(S, d, c) \in \Sigma^M$, $F(S, d, c) \in WPO(S)$.

We omit the straightforward reformulation of those of our conditions that we will also need. We will also use the following.

Independence of Unclaimed Alternatives (I.U.A). For all $M \in \mathcal{M}$ and for all (S, d, c), $(S', d', c') \in \Sigma^M$, if $(d, c) = (d', c')$, and $S' = \{x \in S | x \leqq c\}$, then $F(S', d', c') = F(S, d, c)$.

Anonymity (AN). For all $M, M' \in \mathcal{M}$ with $|M| = |M'|$, for all $S \in \Sigma^M$, $S' \in \Sigma^{M'}$ and for all one-to-one functions $\gamma : M \to M'$, if $S' = \{x' \in \Re^{M'} | \exists\, x \in S$ with $x'_{\gamma(i)} = x_i$ for all $i \in M\}$, $d'_{\gamma(i)} = d_i$ and $c'_{\gamma(i)} = c_i$ for all $i \in M$, then $F_{\gamma(i)}(S') = F_i(S)$ for all $i \in M$.

Sequential Invariance (SEQ.INV). For all $M, N \in \mathcal{M}$, for all $(S, d, c) \in \Sigma^M$ and for all $(S', d', c') \in \Sigma^N$, if $M \subseteq N$, $(S'_M, d'_M) = (S, d)$, $c'_M = c$ and $F_M(S', d', c') \in int(S)$, then $F(S, d, c) = F(S, F_M(S', d', c'), c)$.

Population Monotonicity (POP.MON). For all $M, N \in \mathcal{M}$, for all $(S, d, c) \in \Sigma^M$ and for all $(S', d', c') \in \Sigma^N$, if $M \subseteq N$, $(S'_M, d'_M, c'_M) = (S, d, c)$, then $F_M(S', d', c')$ $\leq F(S, d, c)$

I.U.A requires that the part of the feasible set that is not dominated by the claims point be ignored. Since each coordinate of the claims point can be regarded as the maximal utility level to which each agent is entitled, feasible alternatives that are not dominated by the claims point can be thought of as being irrelevant.[5] *AN* says that the solution should be invariant under exchanges of the names of agents. It is a straightforward generalization of SY.

The main axiom of this section, *SEQ.INV*, is motivated by considering the possibility of a varying population. Let N be a population of agents facing the problem (S', d', c'). Assuming the solution F is being used, this results in the compromise $F(S', d', c')$. Then, it is found that the claims of the agents in some subgroup $N \setminus M$ are in fact, invalid. Is the original agreement moot, or should the remaining agents take it as a starting point. An agent who is hurt by ignoring it might feel that the

[5] A related axiom was introduced for bargaining problems by Peters (1986) under the name of *independence of nonindividually rational alternatives.*

community is reneging on an agreement, whereas agents who benefit might argue that any earlier agreement is irrelevant. Imposing $SEQ.INV$ permits a reconciliation of there two viewpoints. A related axiom, introduced for bargaining theory by Thomson (1988), was itself inspired by a condition proposed by Kalai (1977) for bargaining problems with a fixed population under the name of ***step-by-step negotiation.***

$POP.MON$, which generalizes a condition used by Thomson(1983) says that the arrival of new claimants, with resources remaining fixed, penalizes all the agents initially present.

We will need additional pieces of notation. For $M \in \mathcal{M}$, let e_M be the $|M|$-dimensional vector with all coordinates equal to 1. Given $M, N \in \mathcal{M}$ such that $M \subset N$, and $x \in \Re^N$, let x_M be the projection of x onto $\Re^M$.

Theorem 6. *The proportional solution is the unique solution satisfying W.P.O, BDD, CONT, AN, SC.INV, I.U.A, and SEQ.INV.*

Proof. It is obvious that the proportional solution satisfies the seven axioms. Conversely, let F be a solution satisfying all of the axioms.

(i) We show that $F = P$ on Σ^M for $|M| = 2$. Without loss of generality, we take $M = \{1,2\}$. Let $(S,d,c) \in \Sigma^M$ be given. Recall that since $|M| = 2$, $P(S,d,c) \in PO(S)$. By $SC.INV$, we may assume that $d = 0$ and $c = e_M$. Let $x \equiv P(S,d,c)$ and $S^3 \equiv \{x' \in S | x' \leqq c\}$. Note that x is the only point of $WPO(S)$ with equal coordinates: $x \equiv ae_M$ for some a"in"Re_{++}.

Now the proof consists in constructing a sequence $\{T^\nu\}$ of anonymous problems involving the original members of M as well as a new agent, who, without loss of generality, we take to be agent 3.[6] Let $N \equiv \{1,2,3\}$. The first step of the construction

[6] This construction is the same as that of Thomson (1983, Theorem 3).

of T^ν consists in replicating S^3 in $\Re^{\{2,3\}}$ and $\Re^{\{3,1\}}$ respectively. Formally, let

$$S^1 \equiv \{(y_2, y_3) \in \Re^{\{2,3\}} | \exists (x_1, x_2) \in S^3 \text{ with } y_2 = x_1 \text{ and } y_3 = x_2\} \text{ and}$$

$$S^2 \equiv \{(y_3, y_1) \in \Re^{\{3,1\}} | \exists (x_1, x_2) \in S^3 \text{ with } y_3 = x_1 \text{ and } y_1 = x_2\}.$$

Now pick a sequence $\{\epsilon^\nu\}$ in $\Re_{++}$ such that $\epsilon^\nu \to 0$. Let $x^\nu \equiv (a - \epsilon^\nu)e_N$, $d' = 0e_N$, $c' = e_N$, and $T^\nu \equiv \text{cch}\{S^1, S^2, S^3, x^\nu\}$. It is clear that, for ϵ^ν small enough, $(T^\nu, d', c') \in \Sigma^N$, $S^1 = T^\nu_{\{2,3\}}$ and $S^2 = T^\nu_{\{3,1\}}$.

For ν large enough, by *W.P.O* and *AN*, $F(T^\nu, d', c') = x^\nu$, and by *SEQ.INV*, $F(S^3, d, c) = F(S^3, x^\nu_M, c)$. Since $\epsilon^\nu \to 0$ as $\nu \to \infty$, $x^\nu_M \to x$. Therefore, by *BDD*, $F(S^3, d, c) = x$. Finally, by *I.U.A*, $F(S, d, c) = F(S^3, d, c) = x \equiv P(S, d, c)$, as desired.

(ii) We show that $F = P$ on Σ. Let $N \in \mathcal{M}$ with $|N| > 2$ and $(T, d', c') \in \Sigma^N$ be given. Let $\bar{T} \equiv \{x \in T | x \leqq c'\}$. By *SC.INV*, we may assume that $d' = \epsilon e_N$ for some $\epsilon \in \Re_{++}$. Now suppose that $(\bar{T}, d', c')$ satisfies the following condition (A):

Condition (A): $WPO(S) \cap \{x \in \bar{T} | x > 0\} = PO(S) \cap \{x \in \bar{T} | x > 0\}$ and, for all $M \subset N$ with $|M| = 2$, $(\bar{T}_M, d'_M, c'_M) \in \Sigma^M$.

Note that by *BDD*, $F_M(\bar{T}, d', c') \in int(T_M)$ for all $M \subset N$. By step (i), $F(\bar{T}_M, d'_M, c'_M) = P(\bar{T}_M, d'_M, c'_M)$ for all $M \subset N$ with $|M| = 2$. Therefore, by *SEQ.INV*, $F(\bar{T}, d', c') = \beta d' + (1 - \beta)c'$ for some $\beta > 0$. By *W.P.O*, $F(\bar{T}, d', c') = P(\bar{T}, d', c')$.

Since an arbitrary $(\bar{T}, d', c') \in \Sigma^N$ can be approximated by a sequence of problems $\{(\bar{T}^\nu, d'^\nu, c'^\nu)\}$ of elements of Σ^N such that, for all ν, $d'^\nu = d'$, $c'^\nu = c'$ and $(\bar{T}^\nu, d'^\nu, c'^\nu)$ satisfies condition (A), we conclude by *CONT* that $F(\bar{T}, d', c') = P(\bar{T}, d', c')$.

Finally, by *I.U.A*, we obtain the desired conclusion. Q.E.D.

The proof of the next result which is closely related, is omitted.

Theorem. *The proportional solution is the unique solution satisfying W.P.O, AN,* *SC.INV, C.I.I.A, POP.MON, and CONT.*

6. Concluding comment.

In some recent work, Gupta and Livne (1988) considered bargaining problems enriched by specifying a reference point inside of the feasible set, and characterized a solution to this class of problems. They provided a variety of interpretations for this reference point, including that of a starting point for the negotiations. Their structure is sort of dual to the claims problems analyzed here.

We should also note that, when specialized to the class of bargaining problems with claims that exhibit the transferable utility property, our solution does reduce to one of the solutions discussed in the literature quoted in the introduction. Axiomatic characterizations of that solution in that context can be found in Chun (1988). The solution is also a special case of a larger class characterized by Young (1987).

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