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Abstract

This paper presents a simple, competitive equilibrium model of exhaustible resource extraction in which the price can remain constant or decline monotonically for all time. It is driven by technological change that results from the accumulation of knowledge by forward looking, cost minimizing firms. Because of the characteristics of knowledge, the technology exhibits both externalities and increasing returns. A new existence result and a feasible procedure for calculating sub-optimal dynamic equilibria are established. Results from the computation of a sample equillbrium are also presented.

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1. INTRODUCTION

This paper presents a simple, perfect foresight, exhaustible resource model in which the spot price of the resource can remain constant or decline monotonically for all time. It is motivated by the observation that in long time series, prices for most natural resources fluctuate about a constant or a declining trend. The model departs from from existing models by allowing for the possibility of endogenous technological change. The observation that technological change has acted to mitigate resource scarcity is old. The substantive contribution of this paper is to contruct a fully specified competitive equilibrium in which technological change is the result of intentional actions taken by forward looking agents. By assumption, the production of new knowledge by any individual firm has positive external effects on the production of all other firms in the economy. Despite the crucial assumption that the social marginal product of knowledge is increasing, the model has a well defined competitive equilibrium with externalities. To prove that this equilibrium exists and that it can be approximated by a sequence of computable, finite horizon equilibria, the paper develops a new approach for the analysis of a suboptimal dynamic equilibrium.

Standard resource pricing models generate two logically distinct sets of implications. The most immediate results are the implications for the behavior of a firm, taking as given a path for the future price of the resource. These results can be derived immediately from a specification of the extraction technology of the firm. If a specification of the demand for the resource at all future dates is added, the model can also yield the price path itself. Under a constant marginal cost extraction technology as assumed in the original paper by Hotelling [14], it is difficult to distinguish the implications for the firm from those for the industry. Unless the path for future prices is increasing at the rate of interest, the optimal behavior for any firm is either to sell all its stock or to buy as much as possible. Under virtually any specification of demand, prices must therefore go up at the rate of interest. The exact specification of demand is need only to determine the level of the price at some point.

In more general models that allow the marginal cost of extraction to vary with the rate of extraction or cumulative extraction, the distinction between the implications for the firm and the industry is more obvious. In this case, a firm facing a given path for the price of the extracted resource must choose quantities so that the difference between the price and the marginal cost of extraction increases at the rate of interest. By itself, this puts no restrictions on the trend in prices. Implications for the behavior of prices alone cannot be determined until demand for the resource is specified and an intertemporal form of "supply equals demand" is imposed.[1]

The simplest form of demand in a resource model is to assume that there is a given interest rate and a stationary market inverse demand curve, $p = D(q)$. Following the strategy of Lucas and Prescott [17], it is then a simple matter to solve for the equilibrium quantities and prices. Let $u(q)$ be defined as the integral under the demand curve up to q. Then the equilibrium quantities and prices follow immediately from the solution to the problem of maximizing the infinte sum (or intergral) of $u(q(t))$, discounted at the interest rate, subject to the extraction technology.

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[1] Solow [26] gives a simple presentation of the intuition behind these results. The book by Dasgupta and Heal [6] gives a comprehensive introduction to resource models. Devarajan and Fisher [8] review the contribution of Hotelling and part of the large literature, much of it recent, that has developed on natural resource pricing.

Once this kind of demand is assumed, it is clear that prices must ultimately be increasing. Without calculating intertemporal optimization conditions, finiteness of the initial resource stock implies that the quantity of the resource extracted or consumed in any period must go to zero as time goes to infinity. In any model with a stationary demand for a flow of the resource, the price ultimately increases as the quantity supplied goes to zero. Changes in the extraction costs faced by firms can alter the pattern of intertemporal supply, but they can not avoid this implication. Pindyck [21] shows that given aggregate extraction costs for the industry that increase as reserves fall, the price of a resource can fall for some initial period as exploration takes place and new discoveries are made. The logical foundation for this form of aggregate extraction costs is challenged in Swierzbinski and Mendelsohn [29]. Their analysis suggests that in models where the aggregate is derived from a specification of the extraction costs and exploration strategy of individual firms, the price should not fall even for an initial interval. In any case, since the demand for the resource is stationary, the prices must eventually rise. Backstop models, as introduced by Nordhaus [20], and Dasgupta and Heal [5], modify the demand for the resource by assuming the existence of a perfect substitute available in infinite supply at a constant cost.[2] Assuming that the backstop is currently available, demand for the exhaustible resource will have an intercept at the cost of the substitute and resource prices will be bounded; but effective demand for the resource is still stationary and the price must ultimately be increasing. If the demand is assumed to be a function of the stock of an extracted resource that does not depreciate, e.g. gold, the analysis in terms of a flow demand clearly does

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[2] See Dasgupta and Heal [6] for more recent papers which rely on the presence of some kind of backstop technology.

not apply. The stock of the resource in use increases as extraction takes place, and as demonstrated in Levehari and Pindyck [16], the price for the resource can fall forever. However, once depreciation is allowed, the stock must eventually be falling and the prices will must once again increase in the limit. Except in the very special case of a demand for a stock of a resource that does not depreciate, the only way to generate prices that fall forever is for the demand curve for the resource to shift to the left over time.

In the model proposed here, the demand for the resource is derived from a stationary production process which uses the resource and other inputs to produce a consumption good. The perceived demand for the resource at a point in time is its marginal productivity schedule. Shifts in this schedule arise from the accumulation of knowledge, an intangible capital input in the production process. The marginal product of knowledge is assumed to be globally increasing in the stock of knowledge. This departure from the usual concavity assumption on production is natural in this context. Without it, one would be faced with the implausible prospect of an optimal steady state level of knowledge. In a world with a stationary labor force, the marginal product of additional knowledge at the steady state would be so low that it would no longer be worth the trouble it takes acquire new knowledge. Research would stop. The other distinguishing assumption about knowledge is that it cannot be perfectly patented or completely kept secret. Consequently, research by any individual firm has positive external effects on the production of all other firms in the economy. In related work, Romer [23] develops the implications of these assumptions in simple growth model with no exhaustible resources.

As in conventional models of physical capital, firms can choose to convert output into additional units of knowledge by means of a deterministic investment technology; that is, they can forego output and do research. The assumption that the technological change assciated with the accumulation of knowledge is "resource saving" is captured by the assumption that knowledge and the resource are substitutes in production. For example, it is possible to produce telecommunications services with lots of copper wire and little knowledge or with little copper and lots of knowledge (i.e. fiber optic technology.) Stated in this context, the essence of the model here is to assert that the price of copper is falling because of the development of this kind of knowledge; and simultaneously to assert that the research responsable for these inovations is undertaken by profit maximizing agents who seek to economize on resource expenditures. In the last section, detailed attention is devoted to a rigorous proof that these assertions are consistent in part because they sound suspiciously like newspaper explanations that "the price went up, so the demand went down, so the price went down."

The formal analysis of the suboptimal equilibrium in this model follows a new approach. It is now common theoretical practice to characterize Pareto optimal dynamic equilibria by studying a suitably chosen maximization problem. The equilibrium here is not Pareto optimal and cannot be characterized by solving a maximization problem. As in other models of sub-optimal dynamic equilibria, for example, perfect foresight Sidrauski models of inflation (Brock [2]) or dynamic models with more conventional kinds of externalities (Brock [3], Hochman and Hochman [13]), it is relatively easy to write down a set of difference (or differential) equations which would characterize the behavior of an equilibrium if one were known to exist. Except in the restrictive cases where these equations can be explicitly solved, it is much more difficult to show that an equilibrium does exist. For a broad class of dynamic models, the argument here establishes a new method for proving the existence of a suboptimal dynamic equilibrium and justifies a feasible method for calculating the equilibrium quantities and prices. The existence proof and the justification for the numerical method rely on the observation that although equilibria cannot be derived by solving a maximization problem, they are fixed points of a mapping defined by a maximization problem. They follow by an application of the Maximum Principle.[3]

The next section begins with a stripped down, two period version of the model which illustrates the equilibrium analysis in a familiar, finite dimensional context. Section 3 presents the full infinite horizon model and derives the difference equations which characterize the equilibrium. Section 4 presents the results of a numerical solution of these equations for a specific economy. Because they are more technical, the formal results are deferred until the Section 5. To provide the required consistency or "diagnostic" check on the logical structure of the model, we prove first that an equilbrium exists. As a corollary to that proof, we establish the approximation result that shows that an infinite horizon equilibrium can be approximated by taking limits of finite horizon equilibria. A concluding section discusses some limitations and possible extensions of the model.

2. TWO PERIOD MODEL

The analytical difficulty in the model outlined above arises entirely from the externality and non-convexity associated with knowledge. To illustrate the equilibrium analysis in the simplest possible context, consider a simple two period model. Let each of S identical firms have a technology

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[3] For a statement of the Maximum Principle in the form in which we will use it, see Hildenbrand [12].

for producing goods in period two from knowledge produced in period one and a set of other inputs. Assume provisionally that the research technology for producing knowledge is linear, so one unit of knowledge k is produced for each unit of foregone consumption in period one. As indicated in the introduction, each firm is assumed to contribute unavoidably to the aggregate stock of knowledge and to benefit from it because secrecy is only partial and property rights for knowledge are not completely defined. Let $F(k,K,x)$ denote the (differentiable) production function of a representative firm, where $K = \sum_{j=1}^{S} k_j$ is the aggregate stock of knowledge and x is a vector of all other inputs. Assuming the usual form of competitve behavior, each firm takes both prices and the aggreagate stock of knowledge as given in its optimization problem. For the existence of a competitive equilibrium with externalities, F must be concave in the arguments k and x that are controlled by each firm. Without loss of generality, we can also assume that F is homogeneous of degree one in k and x. If it is not, we can add an additional factor to the vector x to make it so. (See Rockafellar [22], p. 67.) Because the scale and number of firms in equilibrium is indeterminate, we can simplify the notation by assuming that the number of firms equals the number of (identical) consumers, so per firm and per capita quantities are equivalent. To focus solely on the choice of the level of knowledge, assume that the factors x, other than knowledge, cannot be augmented or consumed directly and are available in fixed supply at a per capital level $\bar{x}$. Let $\bar{e}$ denote the intial endowment of consumption goods in period one and assume that the initial stock of knowledge is zero. Since $F(k,K,x)$ is assumed to be homogeneous in k and x and increasing in K, it will exhibit increasing returns to scale. Suppose further that the true (per capita) social production function, $\mathcal{F}(k) = F(k,Sk,\bar{x})$, is globally convex in k. For example, let F take the form $F(k,K,x) = k^{\alpha}K^{\nu}x^{1-\alpha}$ with $\alpha < 1$ and $\alpha+\nu > 1$.

Let $U(c_1,c_2)$ denote a concave, differentiable, strictly increasing utility function for each agent. Consider the problem of proving that an equilibrium for this economy exists and of characterizing its qualitative properties. First, define a family of constrained maximization problems, $P(K)$, by maximizing representative utility subject to the technology, taking as given a level K for the aggregate level of knowledge. Formally this gives

$$P(K) \qquad \max U(c_1,c_2)$$

$$\text{subject to } c_1 \le \bar{e} - k,$$
$$c_2 \le F(k,K,x),$$
$$x \le \bar{x}.$$

For arbitrary choices of K, the solution to this problem will have no economic meaning since the optimal value of $\hat{k}$ from this problem may not satisfy the aggregate consistency condition, $K = S\hat{k}$.

Now define a function $W(k,K)$ by substituting the constraints for the maximization problem into the objective: $W(k,K) = U(\bar{e}-k,F(k,K,\bar{x}))$. By the assumed form for F, W is a concave function of k for each fixed K. Assuming enough steepness on the boundary to ensure the existence of an interior solution, the optimal choice of k for each value of K is given as the solution to the equation $D_1W(k,K) = 0$. Equilibrium quantities can now be derived by inserting the condition $K = Sk$ into the first order condition for the choice of k:

$$(2.1) \qquad D_1W(k,Sk) = 0.$$

Let k^* be a root of this equation, hence also a solution to the problem $P(Sk^*)$. By the standard necessary conditions for the existence of a constrained optimum, each of the three constraints in the explicit statement of the problem $P(Sk^*)$ will have associated to it a Lagrangian or Kuhn-Tucker multiplier. Since firms take the aggregate variable K as given, both consumers and producers will face concave maximization problems in equilibrium. By a simple computation, it follows that the multipliers can be used to decentralize the quantities implied by k^* in a competitive equilibrium. Using the sufficient conditions for the concave maximization problem of the firm, the choices k^* and $\bar{x}$ are optimal for a firm facing prices equal to the values of the multipliers and a given value Sk^* for the aggregate level of knowledge. Similarly, in the concave utility maximization problem of a consumer faced with the multipliers as prices and with an income determined by the value of the endowment $(\bar{e},\bar{x})$, standard sufficient conditions show that $c_1 = \bar{e} - k^*$, $c_2 = F(k^*,Sk^*,\bar{x})$ will be optimal values for consumption.

This kind of argument demonstrates that any solution to equation 2.1 can be interpreted as an equilibrium for this simple economy. In fact, in this model and its generalization to an infinite horizon model, one can show that k represents the quantities in a competitive equilibrium if and only if it is a solution to the appropriate version of equation 2.1 (Romer [23]). As the example here suggests, it is generally a simple matter to derive such an equation from a specification of the preferences and technology for an economy. The qualitative properties of any possible equilibrium follow directly from an examination of its properties.

Proving the existence of an equilibrium for this kind of model is therefore equivalent to proving the existence of a root of the equation 2.1 or its generalization. In a finite dimensional problem, it may be simple to show directly that a root must exist for some specified class of utility and production functions, but in an infinite dimensional problem it is likely to be difficult. Given a path for a variable $K(t)$ in an infinte horizon dynamic model, the first order condition for the representative agent maximization problem analogous to the problem $P(K)$ is an Euler equation together with an initial condition and a transversality condition at infinity. This Euler equation will depend on the given path $K(t)$, but just as in the finite dimensional case, one can substitute $Sk(t)$ for $K(t)$ into these equations. The analog of equation 2.1 is then an autonomous system of difference equations with an initial condition and a terminal condition "at infinity". The difficulty arises in showing that there exists a solution that satisfies the boundry conditions. Except in specially chosen models, these equations and boundary conditions cannot be derived from any maximization problem. (This is easy to check using conditions given in Dechert [7] that any difference equation system arising from a maximization problem must satisfy.) If they were the necessary conditions for some problem $\tilde{P}$, a theorem ensuring the existence of a maximum for $\tilde{P}$ would imply the existence of a solution to the equations satisfying the boundary conditions. In a continuous time model with a single state variable, Romer [23] shows directly that a solution satifying the boundary conditions exists by using the geometry of the two-dimensional phase plane. For economies with more than one state variable, no comparable geometrical approach is available.

An alternative way to demonstrate the existence of an equilibrium is to exploit the structure imposed by the individual maximization problem in a fixed point argument. In the two period example considered above, consider a correspondence M which sends K into the aggregate result of the solutions to the individual maximization problem:

$$M(K) = S[\underset{k}{\text{argmax}}\ W(k,K)].$$

If W is continuous (and in general if any constraint set depending on K varies continuously) the Maximum Principle guarantees that M will be upper-hemi-continuous (u.h.c.). If W is concave as a function of its first argument, M will be convex valued. If M maps a compact, convex set into itself, a version of the Kakutani fixed point theorem will imply the existence of a fixed point, and therefore of an equilibrium.

3. INFINITE HORIZON RESOURCE MODEL

Let $c_t \in \mathbb{R}$ represent the consumption of the single consumption good in period t, let $u:\mathbb{R}_+ \longrightarrow \mathbb{R}$ denote a momentary utility function. ($\mathbb{R}^n_+$ will denote the non-negative orthant in $\mathbb{R}^n$.) Let $\beta < 1$ be a discount factor. The objective function for this model will be of the form $\sum_{t=0}^{\infty} \beta^t u(c_t)$. In the discussion that follows, it is convenient to speak as though the good c_t is a composite commodity and the objective function represents the usual preferences of each of an infinte family of individuals linked by intergenerational altruism. Alternatively, following Lucas and Prescott [17], we can treat the model as a partial equilibrium model of the market for a particular good, here a consumption good, not the resource itself. In this case, interpret $u(c_t)$ as the integral up to c_t of the area under a stationary demand curve for the this consumption good, and β as one divided by one plus the exogenously given interest rate.

To minimize extraneous complications and to highlight the interaction between the accumulation of knowledge and resource pricing, we will assume that the stock of knowledge and the stock of the resource are the only state variables necessary to summarize the dynamics in this model. Thus, assume that all other factors of production like physical capital, labor, etc. are available in a fixed per capita supply $\bar{x}$, and have no use other than as inputs in this production process. This kind of assumption can be relaxed at the cost of increasing the dimensionality of the difference equation system which results. In the first approach to this problem, we did not feel that adding the familiar features of physical capital accumulation, a growing population, etc. would add enough insight to justify the additional complexity in the exposition and the numerical computation of an equilibrium. Nonetheless, all of the theoretical results in Section 5 apply to the general case with multiple state variables. Letting r_t denote resource inputs, output in period t can be written as $F(k_t,K_t,r_t,x_t)$. As above, F will be assumed to be concave and homogeneous of degree one in all other inputs when the aggregate stock of knowledge, K_t, is held constant. Homogeneity implies that the scale and number of firms is indeterminate, so that we are free to set the number of firms equal to the number, S, of consumers. Henceforth, all quantities are measured in per capita (equivalently, per firm) units. Having made this observation, we can drop the other factors x_t. Because each of the identical firms will choose $x_t = \bar{x}$ for all t, we can suppress this argument and neglect it in the subsequent discussion. To futher simplify the exposition, we assume that private knowledge k and public knowledge K enter F as a kind of composite good. Thus, let $\psi\colon \mathbb{R}^2_+ \to \mathbb{R}$ be the aggregating function for knowledge, and let $f\colon \mathbb{R}^2_+ \to \mathbb{R}$ describe output as a function of composite knowledge and the resource inputs. Period t output can therefore be written $f(\Psi(k_t,K_t),r_t) = F(k_t,K_t,r_t,\bar{x})$. In keeping with the assumptions on F, f will be assumed to be concave jointly in its two arguments and Ψ will be concave as a function of k when K is held constant. Increasing marginal productivity of knowledge at the social level is captured by the assumtion that $\Psi(k_t,Sk_t)$ is a convex function of k_t, and that $f(\Psi(k_t,Sk_t),r_t)$ is therefore convex in k_t for any fixed r_t.

To describe the resource extraction technology, let A_t denote the stock of the resource remaining in the ground, and define a function $h:\mathbb{R}_+^2 \longrightarrow \mathbb{R}$ such that $h(r_t,A_t)$ measures the cost, in units of the output good, of current period extraction r_t. As expected of a cost function, h is increasing in r_t and is convex as a function of both units. For technical reasons, it is convenient to assume that the marginal cost of extraction in any period is strictly increasing in the rate of extraction; hence we assume that $h(r_t,A_t)$ is strictly convex as a function of r_t for any fixed A_t. Because extraction costs increase with cumulative extraction, h is assumed to be decreasing in A_t.[4]

Finally, let $G:\mathbb{R}_+^2 \longrightarrow \mathbb{R}$ describe the research technology; if I_t is the output devoted by the firm to research, $G(I_t,k_t)$ denotes the resulting increment in the stock of knowledge. As is standard for a production function, G is assumed to be concave. Note that the production functions f and G, and the cost function h, all have the standard concavity or convexity properties. The model departs from the norm only through the increasing returns present in Ψ.

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[4]Because there is no discovery in this model, the objection raised by Swierzbinski and Mendelsohn to the specification of extraction costs used by Pindyck and others does not apply to the model here.

We can now describe the constraints for this economy. Recalling that all quantities are measured in per firm and per capita magnitudes, the first constraint is that current output must be at least as large as the sum of current consumption, current output invested in research and current extraction costs:

$$f(\varphi(k_t,K_t),r_t) - c_t - I_t - h(r_t,A_t) \geq 0. \tag{3.1}$$

Since G gives the increment to the firm's stock of knowledge, and since knowledge is assumed not to depreciate, the evolution of k_t is given by

$$g(I_t,k_t) + k_t - k_{t+1} \geq 0. \tag{3.2}$$

Finally, A_t decreases one for one with r_t so

$$A_t - r_t - A_{t+1} \geq 0. \tag{3.3}$$

Note that each of these constraints can be written in the form $C_t(y) \geq 0$ for a concave function C_t and a vector $y = (k_t,k_{t+1},A_t,A_{t+1},c_t,r_t,I_t)$. These functions may differ with the date t because of the dependence of the first constraint on K_t.

Given an arbitrary non-negative sequence $K = \{K_t\}_{t=0}^{\infty}$, we can specify the representatitve agent utility maximization problem for this economy analogous to the problem $P(K)$ in the two period economy. In the infinite horizon case, define the problem $P_\infty(K)$ as follows:

$$P_\infty(K) \qquad \text{maximize} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

over the set of non-negative sequences $\{k_t\}$, $\{A_t\}$, $\{c_t\}$, $\{I_t\}$, and $\{r_t\}$ satisfying the constraints 3.1, 3.2 and 3.3. Because of the assumptions described above, $P_\infty(K)$ is a concave maximization problem for any arbitrary path K; i.e. the problem involves maximizing a concave objective over a convex set of sequences. The following assumption collects the convexity assumptions noted above on the functions f, φ, G, and h. To simplify the statement of the Euler equations for this problem, we also assume that these functions are differentiable. Since the domain of a function need not be an open set, throughout the paper "differentiable" will mean that a function is differentiable on the interior of its domain and continuous on the entire domain.[5]

ASSUMPTION 1: The functions u , f and g are concave, increasing and twice continuously differentiable; h is convex, strictly convex and increasing in the first argument, decreasing in the second argument, and twice continuously differentiable; Ψ is increasing, concave in its first argument, and twice continuously differentiable.

Increasing marginal productivity of knowledge is assumed to ensure that the accumulation of new knowledge never stops. In an infinite horizon maximization problem, a production function that is globally convex with respect to an augmentable capital good raises the possibility that the supremum for the problem may be unbounded and that the problem may fail to have an optimum in any sense, overtaking or otherwise. In the model here, the finiteness of the objective function, the existence of a solution to the problem $P_{\infty}(K)$, and the existence of a social planning optimum all follow from an additional assumption on the research technology. Because of strongly diminishing returns to investment in research in any given period, there is a technologically determined upper bound on the rate of growth of the stock of knowledge. Combined with a bound on the degree of increasing returns in Ψ,

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[5]More precisely, the functions are continuous in the relative topology on the domain as a subset of $\mathbb{R}^n$. This is needed only to rule out jumps on the boundary of the domain.

this will ensure that the maximum feasible rate of growth of consumption is not "too big." The intuitive basis for diminishing returns in research is clear.[6] For example, even though it may be possible to develop the knowledge necessary to produce electricity from controlled nuclear fusion by spending a small fraction of total GNP on the research effort over the next twenty or thiry years, it would most likely be impossible to develop it by next year even if unlimited resources were devoted to the effort. The first part of Assumption 2 states the bound on the degree of increasing returns in production; the second part determines the bound on the rate of growth of knowledge implied by the decreasing returns in research.

ASSUMPTION 2:

i) There exist numbers $b_1, b_2, \sigma \in \mathbb{R}$, with $\sigma > 1$, such that $f(\Psi(k,Sk),A_0) \leq b_1+b_2k^{\sigma}$.

ii) There exists $\gamma > 1$, with $\beta\gamma^{\sigma} < 1$, such that for all $I_t \in \mathbb{R}$ and all $k_t \in \mathbb{R}_+$, $G(I_t,k_t) < (\gamma-1)k_t$.

The first part of this assumption ensures that for any amount of resource usage up to the maximum amount available, output is bounded by a function which grows as k to the power σ. The bound on G implies that for all t, $k_{t+1}-k_t < (\gamma-1)k_t$. Hence, $k_t \leq k_0\gamma^t$ for any initial value k_o. In equilibrium, this implies K_t can grow no faster than γ^t as well. Then output can grow no faster than $\gamma^{\sigma t}$. Since $\beta\gamma^{\sigma}$ is assumed to be less than one, any feasible consumption path will be summable with respect to β^t. Since the utility function is concave, $u(c_t)$ will also be summable with

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[6] Especially to anyone who does research for a living.

respect to β^t. Consequently, as demonstrated in lemmas 1 and 2 in Section 5, solutions to the problem $P_\infty(K)$ and to the social planning problem for this economy will exist.

Finally, we need specific restrictions on the derivatives of the functions f and G. In the notation used throughout, the symbol "Du" denotes the derivative of u, "D_1f" denotes the partial derivative of f with respect to its first argument, etc. In the usual abuse of notation, the symbol Ψ will be used to denote both the aggregating function for the two types of knowledge and a specific argument of the function f. Which use is intended is always clear from the context.

ASSUMPTION 3: Let Ψ, r, and k be positive.

i) Normalization: $D_1g(0,k) = 1$,

ii) Substitution: $D_{12}f(\Psi,r) < 0$.

The normalization in 3.i defines the units of knowledge; one unit of knowledge is that amount which could be produced in the limit as one unit of consumption good is invested at an arbitrarily slow rate. From the discussion in the introduction, it is clear that the substitution assumption 3.ii is crucial for the results which follow. This assumption characterizes the effect knowledge has on production. With prices held constant, increases in knowledge lead producers to economize on resource utilization. There is no theoretical presumption in favor of this kind of interaction. It is a statement about the technology which in principle can be verified by an engineer. The justification here is purely empirical; without this assumption, resource prices in this model must ultimately be increasing. Recall that f is not homogeneous of degree one in Ψ and r because other arguments x are being held constant; if it were, $D_{12}f < 0$ would not be possible.

As an example of the kind of function used in the numerical example, let f take the form $f(\Psi,r) = (\Psi^{\alpha} + r^{\rho})^{\nu}$, where α, ρ, and ν lie between 0 and 1 and ν is strictly less than one. This kind of function allows the resource to be important in production in the sense that starting from zero units of resource usage, the marginal product of the resource is infinite. In an equilibrium for an economy with this technology, some amount of the resource will always be used in production. But, the assumptions that f is increasing in both arguments and that it satisfies the substitution condition together imply that the resource can not be necessary for positive production. For $\Psi_1 > \Psi_2$, the graph of $f(\Psi_1,r)$ as a function of r must lie above that for $f(\Psi_2,r)$. By the substitutionn assumption, the slope of the first function (the higher one) must be less than the slope of the second function at any point $r > 0$. These two requirements are consistent only if the intercepts for the two functions are different, hence not both equal to zero.

Authors concerned with exhaustible resources that are necessary for production are generally concerned with the possibility of maintaining a specified level of consumption as resouce stocks decline.[7] In this model, this is trivially possible. In fact, per capita consumption will generally

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[7] Dasgupta and Heal [5], Solow [27], Stiglitz [28], Ingham and Simmons [15] are early references in this area. Dasgupta and Heal [6] gives a simple presentation of some of the basic results. Mitra [19] examines the role of exogenously specified population growth. Cass and Mitra [4] give a complete characterization of the technological requirements for consumption to be bounded away from zero in a quite general model. They weaken the assumptions on the technology to allow for the possibility that technological progress may be the result of a form of capital accumulation. They also suggest that this kind of capital may not exhibit conventional decreasing returns. However, they do not consider the existence of competitive equilibria in this context or the behavior of resource prices over time.

grow without bound. For us, the canonical resource is a specific good like copper, not an aggregate like energy. We are not concerned here with more speculative questions concerning possible "limits to growth" because it seems appropriate first to try to explain observed empirical regularities in variables like resource prices. Resources here are not assumed to be necessary in production because falling resource prices are possible only in the presence of opportunities for substitution which could not be present if they were. Moreover, for specific resources like copper, oil, etc., it seems quite unlikely that production would go to zero in their absence.

Returning to the characterization of a competitive equilibrium for this economy (discussion of the social optimum being deferred until the end of this section), a set of Hamiltonian-like equations can easily be derived for the maximization problem $P_\infty(K)$. Define a Lagrangian $\mathcal{L}$ as follow:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda_t[k_t - k_{t+1} + g(f(\Psi(k_t,K_t),r_t) - c_t - h(r_t,A_t) \ , \ k_t)]$$
$$+ \mu_t[A_t - A_{t+1} - r_t] \ .$$

As a function, $\mathcal{L}$ depends on the sequences for the choice variables c_t and r_t, the exogenous state variable K_t, the endogenous state variables k_t and A_t, and the shadow prices for these endogenous state variables, λ_t and μ_t. As usual, one first maximizes out the variables c_t and r_t, solving for them as functions of the other variables. Then the partial derivatives $\frac{\partial\mathcal{L}}{\partial k_t}$ and $\frac{\partial\mathcal{L}}{\partial A_t}$ define first order difference equations for the shadow prices λ_t and μ_t respectively. The constraint equations 3.2 and 3.3 define the corresponding difference equations for the state variables. Each of these four equations depends on the values of k_t, A_t, λ_t, μ_t, and K_t. For any given path K, the necessary conditions for specific paths $\hat{k}_t$ and $\hat{A}_t$ to be a solution to $P_\infty(K)$ are that there exist paths $\hat{\lambda}_t$ and $\hat{\mu}_t$ such that $\hat{k}_t$, $\hat{A}_t$, $\hat{\lambda}_t$, $\hat{\mu}_t$, and K_t satisfy this system of equations and satisfy a set of four boundary conditions. Two of the boundary conditions are given by the initial values for the state variables k_0 and A_0; the other two are given by the transversality conditions $\lim_{t\to\infty} \hat{\lambda}_t \hat{k}_t = \lim_{t\to\infty} \hat{\mu}_t \hat{A}_t = 0$. (For a proof of the necessity of these tranversality conditions in a problem like $P_\infty(K)$, see Ekeland and Scheinkman [10].)

Proceeding as in the two period model, we can substitue out for K to find the equations that will characterize a competitive equilibrium with externalities. Recalling that the economy consists of S identical firms, we can substitute in the equilibrium condition $K_t = Sk_t$ to get a coupled, autonomous, first order system of difference equations in k_t, A_t, λ_t, and μ_t, with two initial conditions and two terminal conditions. As before, any solution $(k_t^*, A_t^*, \lambda_t^*, \mu_t^*)$ to these equations that satisfies the boundary conditions can be supported as a competitive equilibrium with externalities using the shadow prices λ_t^* and μ_t^* as equilibrium present value (i.e. time zero) prices for private knowledge and the resource. In period t, the spot price, p_t, of the resource in terms of the consumption good will be equal to the marginal rate of transformation betweem resources and consumption goods, $p_t^* = D_2 f(\Psi(k_t^*, Sk_t^*), r_t^*)$. The present value price for the consumption good at time t can be calculated by dividing μ_t^* by this spot price. Because of the symmetry in the problem, these results do not require that there exist a market for private knowledge. The discussion up to this point has suggested that integrated firms engage in both resource extraction and the production of the output good, but these activities can also be decentralized so they are carried out by separate competitive firms. As in the two period model, the proof of these results follows directly from a version of the Kuhn-Tucker Theorem. For a detailed example of this kind of argument in an infinte dimensional space, see Romer [23].

Existence of a competitive equilibrium therefore follows immediately from the existence of a solution to these equations satisfying the boundary conditions. Showing directly that such a solution exists does not appear to be feasible in general; but the non-constructive argument in Section 5 shows that under assumptions 1, 2, and 3, one must exist. Given that one exists, it is still a non-trivial matter to characterize its behavior. Because the analysis relies on the fact that k_t^* grows without bound and that A_t^* approaches a point on the boundary of the feasible set of values, one cannot simply linearize this system around a steady state for the dynamical system. Moreover, there is no hope in general of solving the complete non-linear system. Nonetheless, useful information can be extracted by examining the individual equations. First, observe that the equation $\frac{\partial \mathcal{L}}{\partial A_t} = 0$ gives

$$\mu_t^* = \mu_{t-1}^* + \lambda_t^*(D_I g)(D_A h).^8$$

Because $D_A h$ is negative, the present value shadow price for in-ground resources decreases. (If $D_A h = 0$, this reproduces a form of Hotelling's rule; the present value shadow price of the in-ground resources must be constant.) Recall that the spot price p_t^* of the extracted resource in terms of the consumption good is given by the marginal rate of transformation, $p_t^* = D_r f$ ($= D_2 f(\Psi(k_t^*, Sk_t^*), r_t^*)$). Recall also that finiteness of the initial stock of the resource implies that r_t^* must eventually be decreasing towards a limit of zero. If f is strictly concave, the second derivative $D_{rr} f$ is

[8]When no ambiguity can arise, expressions like $D_A h$ will be used in place of the more cumbersome (but more explicit) form $D_2 h(r_t^*, A_t^*)$.

negative. If k_t^* were constant, this would imply that p_t^* would ultimately be increasing as r_t^* goes to zero; but if k_t^* increases fast enough (so $\Psi(k_t^*, Sk_t^*)$ increases fast enough), and if $D_{12}f(\Psi, r)$ is negative, $D_r f$ may decrease.

The behavior of $D_r f$ over time is constrained not just by feasiblity but also by the requirement that the input r_t^* be optimally chosen. (That is r_t^* must be optimally chosen by firms that take the aggregate path for Sk_t^* as given. None of the discussion so far pertains to the social optimization problem of a planner who can take account of the externality in this economy. See below for a discussion of this problem.) The equation for maximizing out c_t, $\frac{\partial \mathcal{L}}{\partial c_t} = 0$, gives

$$\beta^t Du(c_t^*) = \lambda_t^* D_I g.^{9}$$

In equilibrium, if k_t^* is increasing without bound, c_t^* must be bounded from below; constant consumption at the level implied by no resource inputs and no new investment in research would always be feasible. Then $Du(c_t^*)$ is bounded from above and $\lambda_t^* D_I g$ must go to zero with β^t. The equation for r_t^*, $\frac{\partial \mathcal{L}}{\partial r_t} = 0$, implies

$$\lambda_t^*(D_I g)(D_r f - D_r h) = \mu_t^*.$$

Since μ_t^* and $\lambda_t^*(D_I g)$ are both decreasing, optimal resource usage by competitive firms does not rule out a path for $p_t^* = D_r f$ that is monotonically decreasing, provided that k_t^* increases without bound. Note that if $D_A h$ were equal to zero, μ_t^* would be constant. Since $D_r h$ must be

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[9] In this equation and the subsequent discussion, we neglect the non-negativity constraints on c_t, I_t, and r_t. In the actual calculation of an equilibrium, we will need to verify that they are satisfied. If not, these equations will hold as inequalities.

bounded from below by zero, $D_r f$ would then have to be increasing in the limit.

To see that unbounded growth in k_t^* is a possible equilibrium outcome for this system, consider the functional form described above for f, $f(\psi, r) = (\psi^{\alpha} + r^{\rho})^{\nu}$. Suppose that ψ takes the form $\psi(k,K) = k^{\xi}K^{\eta}$. If ξ is less than one but $(\xi+\eta)\alpha\nu > 1$, knowledge will have the required decreasing private marginal productivity and increasing social productivity. If r_t^* is forced to be identically zero, output takes the form

$$c_t + I_t \leq (k_t^{\xi} K_t^{\eta})^{\alpha\nu} .$$

For a model of this form, Romer [23], [24] proves that there exists an equilibrium in which k_t and c_t grow without bound.[10] In fact, the growth rate for capital can approach the asymptotic maximal growth rate γ^t. This does not constitute a proof that in the economy with resources, k_t will exhibit similar growth. Since the equilibria here are all second-best equilibria, it is not necessarily the case that the equilibrium in the economy with resources will yield higher utility than that achieved in the economy without resources. Nonetheless, the natural presumption is that in the economy with the resources, an equilibrium with unbounded growth in k_t will exist. The same increasing returns in k which drive growth in the economy without the resource are present in the economy with the resource. Moreover, as the stock A_t of the resource approaches zero, this economy resembles more and more closely the economy with no resource.

None of this constitutes a proof that for some specification of this model, the price of the resource will fall. We do not attempt to establish an analytical result of this kind. Simulations described in the next section

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[10] In fact the model there is in continuous time, but for the discrete version these results will hold.

demonstrate that the trend in prices, both initially and asymptotically, depends not only on the functional forms chosen to describe the technology, but also on the relative size of the initial endowments of knowledge and the resource. It would not be hard to specify conditions like those in Romer [23], [24] that guarantee that k_t^* grows without bound. The difficulty is that the behavior of $D_r f(\Psi, r)$ over time depends not just on whether or not k_t^* (hence also Ψ) grows over time, but rather on the on the rate of growth of k_t^* relative to the rate of decrease of r_t^*. Growing k_t^* causes $D_r f$ to increase because of the substitution assumption, but falling r_t^* causes it to fall because of concavity. To determine the magnitudes of these rates starting from specific initial conditions, one must solve the equation system forward, using the transversality conditions to determine the initial values for the shadow prices. For this model, this can be done numerically, but not analytically.

This leaves four important steps in the analysis to be completed. First, some proof must be given that under the assumptions above, this economy has a competitive equilibrium. Second, further exploration of the qualitative properties of the competitive equilibrium must demonstrate that in at least some cases, prices can indeed fall. As suggested above, this can only be done numerically, but this raises an additional complication. Any numerical approach to this problem must rely on a limiting argument using solutions to truncated T-period economies. It is not literally possible to evaluate conditions at infinity. Thus, the third task is to establish an approximation result which states that equilibria for T-period economies converge to an equilibrium for the infinite horizon economy as T goes to infinity. This is also an important check on the robustness of the model. Any equilibrium in a perfect foresight model which depended too strongly on the infinite future would be highly suspect. Note that since the equilibria for these economies are not the solutions to maximization problems, this problem is distinct from the much easier problem of showing that solutions to trucated maximization problems converge to the the solution of an infinte horizon maximization problem.

The numerical analysis of a specific example is contained in the next section and the existence and approximation results follow in the section 5. Before going on to these, we conclude this section with the fourth of the steps alluded to above, a comparison of the competitive equilibrium with the social optimum for this economy. In most models of equilibrium with externalities, this forms the bulk of the analysis, but the discussion here will be brief. In part, this is becuase we are interested primarily in explaining historical price movements and feel that an equilibrium without government intervention is a better historical model that one with intervention. Over the time horizon for which we have data on resource prices, substantial government tax and subsidy schemes are a quite recent innovation. In addition, the analysis of the social optimum in this context would add little to what is already known; the presence of an exhaustible resource has little bearing on the welfare anaysis of this model. Exactly as one would guess from a simple static model with a positive externality, the sub-optimality in this economy arises because competitive agents accumulate too little knowledge. In the optimum, k_t will grow more quickly than in the competitive equilibrium. The social optimum can be supported as a competitive equilibrium with taxes under a variety of tax schemes that directly or indirectly subsidize the production of knowledge. Romer [24] discusses these results in the context of a model without an exhaustible resource. They carry over essentially without modification. The only implication worth noting here is that since k_t grows more rapidly in the social optimum, resource prices in this model will fall more rapidly (or rise less rapidly) than they do in the competitve equilibrium.

In the formal analysis, the existence of the social optimum follows immediately from Lemmas 1 and 2 in Section 5. Again, the key feature is that because of the form of the research technology, the maximum rates of growth of knowledge and consumption are not "too" fast. The equations describing the social optimum can be derived as above by substitutuing the expression Sk_t in for K_t before taking the derivative $\frac{\partial \mathcal{L}}{\partial k_t}$ instead of after doing so. This change in the equation for the evolution of λ_t constitutes the only difference between the equations for the competitive equilibrium and the social optimum. Choosing taxes that will support the optimum is then simply a matter of getting the equations for λ_t for the two models to agree.[11]

Finally, all the usual cautions about second best analysis apply. Roughly speaking, any policy that stimulates research will be welfare improving. Thus, if the consumption good is transportation services from automobiles and the resource is petroleum, regulations like mandated minimum fuel efficiency standards can be welfare improving if they stimulate research that has a common property element. In contrast, speed limit restrictions should lower the demand, hence the price, for petroleum, and discourage research. Taking this view to the extreme, it is conceivable that a reduction in the inital stock of the resource could be welfare improving. The increase in the price of the resource may cause an increase in the amount of research suffcient to outweigh the direct welfare loss from the reduction in the stock

[11]Brock [2] or Hochman and Hochman [13] present exactly this kind of analysis for dynamic models with externalities.

of the resource. For example, Great Britain may have benefited from its relatively meager endowment of wood if it contributed to an earlier transition to coal and an earlier development of steam power.

4. NUMERICAL EXAMPLE

As suggested above, calculating the competitive equilibrium prices and quantities for a given specification of this economy is equivalent to solving a two point boundary value problem for a four dimensional difference equation system. As noted above, it is conceptually impossible to evaluate any of the variables for this system at the "boundary" defined by $t = \infty$. Any numerical procedure must rely on calculating the values at some large but finite value T. The difficulty in implementing this procedure lies in the choice of the appropriate termainal values for the state variables and the shadow prices at T+1. Rather than rely on some ad hoc procedure, the approach taken here is to define and calculate equilibria for truncated, T-period finite horizon economies. The proof in the next section shows that any sequence of equilibria for these economies has a sub-sequence which converges to an equilibrium for the infinite horizon economy.

To define these economies, truncate the preferences at $t = T$, but leave the technology unchanged. For any given sequence $\{K_t\}_{t=0}^{\infty}$, the truncated problem $P_T(K)$ is to maximize $\sum_{t=0}^{T} \beta^t u(c_t)$ over all feasible infinite sequences of capital and resource stocks. This is equivalent to the finite dimensional problem of maximizing these preferences subject to the technology up to T, treating the values of k_{T+1} and A_{T+1} as freely chosen terminal values. (For $t \geq T+1$, letting $c_t = r_t = 0$ and letting k_t and A_t be constant is feasible.) For this finite dimensional problem, a simple application of the Kuhn-Tucker theorem shows that the second set of boundary conditions is just $\lambda_T k_{T+1} = \mu_T A_{T+1} = 0$. Since the technology for $t > T$ is irrelevant, the values of K_t for $t > T$ are irrelevant; to specify the problem $P_T(K)$, it is sufficient to specify the first $T+1$ components of K. An equilibrium for this truncated economy is therefore a $T+1$ vector K such that the solution $\{k\}_{t=0}^{T+1}$ to $P_T(K)$ satifies $K = Sk$. As before, substituting $K_t = Sk_t$ into the difference equations which describe the first order conditions for the problem $P_T(K)$ gives a set of autonomous difference equations in k_t, A_t, λ_t and μ_t. In fact, they are identical to the equations for the infinite horizon economy. The only difference is that the boundary conditions at infinity have been replaced by the boundary conditions $\lambda_T k_{T+1} = \mu_T A_{T+1} = 0$ which can be evaluated directly.

For simplicity and without loss of generality, the number of firms S was set equal to one. The functional forms chosen were as follows:

$$f(\psi, r) = F\cdot(\psi + r^{\rho})^{\nu} \quad \rho = 0.8, \ \nu = 0.6$$

$$\psi(k, K) = P\cdot k^{\alpha} K^{\eta} \quad \alpha = 0.8, \ \eta = 0.9$$

$$G(I, k) = (\gamma - 1)[1 - \exp(-\frac{I}{k(\gamma - 1)})] \quad \gamma = 1.03$$

$$U(c) = c$$

$$h(r, A) = H\cdot\frac{r^2}{A}$$

The parameters F, P and H are multiplicative constants chosen to scale the functions. The choice of functional form was determined primarily by analytical convenience. Despite the numerical approach used here, this was a concern because of questions of convergence. In principal it is straightforward to start with values for $z = (k, A, \lambda, \mu) \in \mathbb{R}^4$ at either $t = 0$ or at $t = T$ and use the difference equations to work to the other boundary, iterating until the conditions at each boundary are met. In practice, the equations relating values at t to values at $t+1$ are given in implicit form by the first order conditions. The functions here were chosen so that a simple Newton method would exhibit global convergence to a value for the variables at t given values at $t+1$. In fact, it can be shown that for a given value of z_t, there may exist more than one value of z_{t+1} which satisfies the implicitly defined difference equation; but for each z_{t+1}, there is a unique value of z_t. This does not necessarily imply that there are multiple equilibria for this economy. Starting from a specific value of z_0, many of the possible paths satisfying the implicit difference equations may fail to satisfy the necessary terminal conditions. Nonetheless, this suggests a numerical strategy of choosing a value for z_{T+1} and working backwards. For any given values of k_0 and A_0, trial and error lead to the value of z_{T+1}, satisfying the terminal conditions, such that solving the difference equations backwards from this point lead to a value of z_0 with k_0 and A_0 as its first two components.

The most troubling of the specified functional forms is the linear utility function. Under the Lucas-Prescott interpretation of this function, this corresponds to a market demand curve for the output good which is infinitely elastic at a price of one, so it may be possible to justify this form for a small country. In principle, it is straightforward to extend the analysis to more general, strictly concave functions. In practice, the required numerical analysis may require methods which are more sophisticated than the elementary ones we used.

The discount factor β was set at 0.95 so the real interest rate was roughly 5%. Accordingly, one time period is taken to be approximately one year. Values for the truncation point T ranging from 100 to 400 years in the future were tried. For the specific initial conditions used in the results reported below, the value function (equal to the present discounted value of consumption) for the 200 year truncated economy differed by less than one-half of one percent from the value function for the 300 year truncated economy, which in turn differed from the value function for the 400 year economy by less than 0.05% The 200 year results were therefore taken as having essentially converged to the ∞ horizon results and are the only ones reported. Figure 1 graphs the behavior over time of the real spot price of the exhaustible resource for one choice of the initial values for A_0 and k_0. Subsequent graphs give the paths for the resource stock, resource usage, the rate of growth of the capital stock and the level of consumption for this example. Out of many simulations, these results are presented because the behavior of the resource price differs so starkly from that expected in more conventional models. Here the price initially rises, then falls monotonically.

In general, the behavior of the price is determined by the offsetting effects of growing knowledge and shrinking resource usage. Not surprisingly, by adjusting the relative magnitude of the initial stocks and the scaling factors in the definitions of the functions, it was possible to generate price paths which fell monotonically or increased monotonically. Monotonically falling prices could be generated in cases where the resource was exhausted by the terminal date T and in cases where it was not. In no case were the non-negativity conditions for c_t, r_t, or investment $I_t = f - c_t - h$ violated. As one would expect from the behavior of the model with no resource, in all cases the capital stock eventually grew at a rate close to the maximum feasible rate of 3% per year; consumption grew accordingly.

5. EXISTENCE AND APPROXIMATION

5.1 An Existence Result

The first step in the formal analysis of this model is to be precise about the sequence space on which the objective functional is defined. The model requires a space which can accommodate sequences for k_t and c_t which may grow at the rate γ^{σ}. For obvious reasons it will be useful if the feasible sequences are contained in a compact set in this space. By assumption 2, $\beta\gamma^{\sigma} < 1$. Let δ be some constant satisfying $1 > \delta > \beta$ and $\delta\gamma^{\sigma} < 1$. Let $\mathscr{L}_1(Y,m,\mathscr{B})$ denote the usual Banach space of integrable functions defined on a set Y which is made into a measure space by a measure m on a sigma algebra $\mathscr{B}$. For this application, let Y be the set of non-negative integers, let $\mathscr{B}$ denote the set of all subsets of Y and let m be the measure which assigns mass δ^t to the element t. The norm will then take the form

$$\|k\|_{\delta} = \sum_{t=0}^{\infty} \delta^t |k_t| .$$

Let $\ell_1(\delta)$ denote this space for these choices. This norm is extended to the n-fold cartesian product of $\ell_1(\delta)$, denoted $(\ell_1(\delta))^n$, by replacing the absolute value by any norm on $\mathbb{R}^n$ equivalent to the usual norm.

As has been argued elsewhere (Romer [25]), $\mathscr{L}_1$ spaces are convenient for maximization problems of the type considered here because of the availability of simple characterizations for weak upper-semi-continuity and weak compactness. It is well known that the usual discounted objective functionals are not continuous in the weak topology, but upper-semi-continuity and compactness are sufficient for finding solutions to maximization problems. These functionals are weakly upper-semi-continuous. As opposed, for example, to compactness in the Mackey topology, a simple characterization of weak $\mathcal{L}_1$ compactness is available. The choice of an $\mathcal{L}_1$ topology is particularly compelling in a discrete time model. Since the underlying measure space is purely atomic, the weak and norm topology coincide. (Dunford and Schwartz [9, IV.8].) Thus we can exploit the advantages of sets which are weakly compact and functions which exhibit a form of norm continuity.

Let k_0 and A_0 be positive initial values for the state variables in this economy; they will be constant throughout the discussion which follows. Define a subset $\Delta \subset \ell_1(\delta)$ by

$$\Delta = \{ \{K_t\}_{t=1}^{\infty} \in \ell_1(\delta): K_1 \le \gamma S k_0 \text{ and } K_{t+1} \le \gamma K_t \}.$$

Holding constant the initial values, define the feasibility correspondence $\Gamma:\Delta \subset \ell_1(\delta) \longrightarrow (\ell_1(\delta)_+)^4$ as the map which sends the path K into the set of non-negative sequences $\{k_{t+1}, A_{t+1}, c_t, r_t\}_{t=0}^{\infty}$ which satisfy the constraints 3.1, 3.2, and 3.3 in the definition of the problem $P(K)$. As noted in the discussion of assumption 2, k_t is bounded by $k_0\gamma^t$; for any $K \in \Delta$, K_t is bounded by $Sk_0\gamma^t$. Then by assumption 2, output and consumption are bounded by an affine function of $\gamma^{\sigma t}$. Since δ was chosen so that $\delta\gamma^{\sigma} < 1$, the correspondence Γ does indeed map Δ into $(\ell_1(\delta)_+)^4$.

Given an element $z = \{k_{t+1}, A_{t+1}, c_t, r_t\}_{t=0}^{\infty}$ in $(\ell_1(\delta))^4$, we can trivially extend the preference functional to a functional $V:(\ell_1(\delta)_+)^4 \longrightarrow \mathbb{R}$ by setting $V(z) = \sum_{t=0}^{\infty} \beta^t u(c_t)$. Then we can rewrite the problem $P_\infty(K)$ as

$$P_\infty(K) \qquad \max \{ V(z) : z \in \Gamma(K) \} .$$

The following lemmas verify the conditions needed to apply the Maximum Principle to this problem.

LEMMA 1: For given positive k_0 and A_0, there is a compact set $\Omega \subset (\ell_1(\delta)_+)^4$ such that for any $K \in \Delta$, $\Gamma(K) \subset \Omega$. Moreover, $\Gamma(K)$ is itself compact for each $K \in \Delta$.

PROOF: By the continuity of the functions in the constraints 3.1, 3.2 and 3.3 and the fact that convergence in $\ell_1(\delta)$ implies pointwise convergence, $\Gamma(K)$ is closed. Define Ω as the set of non-negative sequences $(\{k_{t+1}\},\{A_{t+1}\},\{c_t\},\{r_t\})$ such that $k_t \le k_0\gamma^t$, $A_t \le A_0$, $c_t \le b_1 + b_2k_0\gamma^{\sigma t}$ and $r_t \le A_0$. (Here b_1, b_2, γ, and σ are as specified in assumption 2.) Since the bounds in this definition are all summable with respect to δ^t, it follows that Ω is compact in both the weak and the norm topology by Dunford and Schwartz [9, IV.13.3]. By the discussion following the definition of Γ, it follows that for all $K \in \Delta$, $\Gamma(K)$ is contained in Ω and hence is itself compact.

The proof of compactness is made simple by the choice of δ satisfying $\delta\gamma^\sigma < 1$ in the definition of the norm for the sequence space. Note that it also implies that neither Ω nor $\Gamma(K)$ has an interior point.

LEMMA 2: The function $V:(\ell_1(\delta)_+)^4 \longrightarrow \mathbb{R}$ is continuous in the norm topology.[12]

PROOF: Let $\{z^n\} \subset (\ell_1(\delta)_+)^4$ be a sequence converging to a point z^*. Then z^n converges pointwise to z^* and is bounded in the $(\ell_1(\delta))^4$ norm; that is, there exists $B \in \mathbb{R}$ such that $\sum_{t=0}^{\infty} \delta^t \|z_t^n\| \le B$ for all n. Then for all n and all t, $\|z_t^n\| \le \delta^{-t}B$. Since the utility function u is continuous, the sequence u^n defined by $u_t^n = u(c_t^n)$ converges pointwise to $u^* = u(c_t^*)$, where c^n (c^*) is the third component sequence of z^n (z^*). Since the function u is concave, there exist non-negative constants a_o and a_1 such that $u(0) \le u_t^n \le a_o + a_1 B\delta^{-t}$ for all n and all t. Since $\beta < \delta$, these bounds are summable with respect to β^t. By the Dominated Convergence Theorem, $\sum_{t=0}^{\infty} \beta^t u(c_t^n)$ converges to $\sum_{t=0}^{\infty} \beta^t u(c_t^*)$ and V is continuous.

To apply the Maximum Principle to this problem, it remains to establish that the feasibility correspondence $\Gamma:\Delta \subset \ell_1(\delta)_+ \longrightarrow (\ell_1(\delta)_+)^4$ is continuous. Upper-hemi-continuity follows easily from the continuity of the functions defining the constraints and the fact that convergence in the norm implies pointwise convergence. The difficulty, as always, is to show that the

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[12] As noted above, integral functionals of the general form of V are weakly upper-semi-continuous in an $\mathcal{L}_1$ topology. Also as noted, since the measure space used here is purely atomic, the weak and norm topologies coincide. Hence, it follows that V is u.s.c. in the norm topology. We are able to prove that V here is actually continuous in the norm topology because we have assumed that the utility function u is bounded from below. This was done purely for simplicity. If u is not bounded from below -- for example if $u(c)$ took the form $u(c) = \ln(c)$ -- then we would need to modify the proof and treat both u and V as upper-semi-continuous, extended-real valued, concave functions.

corrspondence is lower-hemi-continuous. Given a sequence of paths K^n converging to K^* in $\ell_1(\delta)_+$ and an element $z^* \in \Gamma(K^*)$, one must construct a sequence of paths z^n converging to z^* such that $z^n \in \Gamma(K^n)$ for all n . The key observation in the proof is as follows. The correspondence in finite dimensional space which specifies the feasible values $(c_t, r_t, k_{t+1}, A_{t+1})$ given (K_t, k_t, A_t) is lower-hemi-continuous. Suppose that for all $s \leq t$, we are given sequences $\{z_s^n\}_{n=0}^{\infty}$, such that z_s^n converges to z_s^* as n goes to infinity and such that z_s^n is feasible given z_{s-1}^n. Then we can use the lower-hemi-continuity of this finite dimensional correspondence to construct a new sequence in n, z_{t+1}^n, that satisfies the constraints and converges to z_{t+1}^*. Proceeding inductively, this gives a sequence z^n (of sequences) with $z^n \in \Gamma(K^n)$, such that z^n converges <u>pointwise</u> to z^*; but by the bound γ^{σ} on feasible growth rates and the choice of δ less than $\gamma^{-\sigma}$, $\{z^n\}$ is uniformly bounded by a sequence summable with respect to δ^t. Then the Dominated Convergence Theorem implies that $\{z^n\}$ actually converges to z^* in the $(\ell_1(\delta))^4$ norm. This argument once again relies heavily on the summable upper bound implied by the choice of $\delta < \gamma^{-\sigma}$.

To make this argument precise, define a correspondence Ψ from $\mathbb{R}_+^3$ into $\mathbb{R}_+^4$ as follows: $\Psi(K_t, k_t, A_t)$ is the set of elements $(k_{t+1}, A_{t+1}, c_t, r_t) \in \mathbb{R}_+^4$ such that

$$k_t - k_{t+1} + G(f(\Psi(k_t, K_t), r_t) - c_t - h(r_t, A_t) \ , \ k_t) \geq 0$$

and

$$A_t - A_{t+1} - r_t \geq 0 \ .$$

LEMMA 3: Ψ is lower-hemi-continuous.

PROOF: The proof is a tedious but straightforward exercise in finite dimensional space. It is contained in an appendix available on request.

LEMMA 4: Γ is continuous.

PROOF: We use the definitions and results from Hildenbrand [12, Part 1]. Since the image of $\Delta \subset \ell_1(\delta)_+$ under Γ is contained in the compact set Ω, Γ is u.h.c if and only if it is closed. By the continuity of the constraint functions, Γ is closed. As above, let $\{K^n\}$ be a sequence in Δ converging to K^*, and let $z^* \in \Gamma(K^*)$. Since $z^* \in \Gamma(K^*)$, we know that $z_1^* \in \Psi(K_0^*, k_0, A_0)$. Using the l.h.-continuity of Ψ, construct a sequence $z_1^n \in \Psi(K_0^n, k_0, A_0)$, $n \geq 0$, converging to z_1^*. Suppose we have defined z_s^n for all n and for all $s \leq t$. Since $z_{t+1}^* \in \Psi(K_t^*, k_t^*, A_t^*)$, use the l.h.-continuity of Ψ again to construct a sequence z_{t+1}^n, $n \geq 0$, converging to z_{t+1}^*. By construction, the sequence $z^n = \{z_t^n\}_{t=0}^{\infty}$ is an element of $\Gamma(K^n)$ for each n, and z^n converges pointwise to z^*. Since z^n is contained in Ω for all n, the set $\{z^n\}_{n=0}^{\infty}$ is uniformly bounded by a sequence that is summable with respect to δ^t. By the Dominated Convergence Theorem, z^n actually converges to z^* in the $(\ell_1(\delta))^4$ norm. Hence, we can conclude that Γ is l.h.c. By the remarks just after the proof of lemma 2, it is upper-hemi-continuous. Hence it is continuous.

Lemmas 1 through 4 are sufficient to apply the maximum principle to the problem $P_\infty(K)$ and conclude that the correspondence defined by the maximizing values is u.h.c. As currently defined, this correspondence, the argmax correspondence, is a mapping from $\Delta \subset \ell_1(\delta)_+$ into $(\ell_1(\delta)_+)^4$. For the fixed point argument which follows, it is convenient to modify the definition of the constraint correspondence slightly so that both it and the argmax correspondence map a set into itself. Recall that Ω, as defined in the proof of lemma 1, is the set of feasible sequences k, A, c, and r in $(\ell_1(\delta)_+)^4$.

Refering to the definitions of Δ and Ω if necessary, note also that S times the projection of Ω with respect to its first argument is contained in Δ. Thus, define $\tilde{\Gamma}:\Omega \longrightarrow (\ell_1(\delta)_+)^4$ as the correspondence which sends the quadruple of sequences $z = (\{k_{t+1}\},\{A_{t+1}\},\{c_t\},\{r_t\})_{t=0}^{\infty}$ into the set $\Gamma(S\{k_t\}_{t=1}^{\infty})$. That is, the action of $\tilde{\Gamma}$ on z is equivalent to the action of Γ on S times the first component sequence of z. $\tilde{\Gamma}$ is continuous since Γ is. By Lemma 1 its image lies in Ω. Since the constraints 3.1, 3.2 and 3.3 define a convex set of feasible values for any fixed path $K(t)$, the correspondences Γ and $\tilde{\Gamma}$ are convex valued. For the problem $P_\infty(K)$, define the argmax correspondence M from Ω into itself,

$$M(z) = \underset{w}{\operatorname{argmax}} \{ V(w) : w \in \tilde{\Gamma}(z) \subset (\ell_1(\delta)_+)^4\}.$$

Since V is concave and $\tilde{\Gamma}$ is convex valued, $M(z)$ is convex valued. By the Maximum Principle, it is upper-hemi-continuous. By lemma 1, it maps a compact set into itself. By the Kakutani extension of the Schauder fixed point theorem, M has a fixed-point in Ω. As a result, we can conclude,

THEOREM 1: Under assumptions A.1 to A.3 and given positive initial values for k_0 and A_0, there exists a sequence K^* and a solution $(\{k^*_{t+1}\},\{A^*_{t+1}\},\{c^*_t\},\{r^*_t\})_{t=0}^{\infty}$ to the problem $P_\infty(K^*)$, such that $K^* = Sk^*$.

Given this fixed point, K^*, constructing a competitive equilibrium is straightforward. The necessary conditions for a solution to the concave maximization problem $P_\infty(K^*)$ consist of the difference equations referred to in Section 3 plus the two tranversality conditions at infinity. The multipliers λ^*_t, and μ^*_t will be present value shadow prices for the knowledge and the in-ground resource at the future date t. Even if there is no market for knowledge, these can be used to derive prices for any goods that are traded. In particular, the spot price for the resource will be given by the marginal rate of transformation as given in Section 3. The proof that this defines a competitive equilibrium consists of showing that, at these prices, price-taking firms and consumers will choose the quantities associated with the fixed point K^*. Given the information from the necessary conditions, this is a simple application of the well known sufficient conditions for a dynamic maximization problem to the problem of the firm and of the consumer. For a detailed treatment of this general approach to decentralizing a possibly time-dependent dynamic maximization problem as a competitive equilibrium, see Romer [23].

Superficially, it may appear that this argument generates prices for an infinite dimensional equilibrium without applying the Hahn-Banach theorem and without some kind of interiority condition, but this is not the case. The proof of the necessary conditions for the kind of problem considered here is essentially a version of the Kuhn-Tucker Theorem in an infinite dimensional space. That theorem requires an interiority assumption typically referred to as a Slater condition. The problem $P_\infty(K^*)$ will satisfy a Slater condition and the usual necessary conditions. What is clear is that this condition and the duality theory used in the proof of the necessary conditions must rely on a different topology than the one used here; as observed in the remark after Lemma 1, in the topology used here, all the sets of interest have empty interior. One of the useful features of the approach outlined here is that it allows this kind of separation between the arguments needed to prove the existence of a fixed point and those used to generate prices.

5.2 An Approximation Result

If the fixed point K^* were known, it is easy to show that the solution to the infinite horizon problem $P_\infty(K^*)$ can be found as the limit of the solutions to the finite horizon truncated versions of $P_\infty(K^*)$. By itself, this is not helpful in calculating K^*, precisely because K^* must be found simultaneously with the solution to $P_\infty(K^*)$. The generalization needed to justify the procedure used in Section 4 is a result which shows that the limit of a sequence of fixed points for finite horizon economies converges to a fixed point for the infinite horizon economy.

Consider a sequence of T-period truncated versions of the infinite economy. Preferences are defined as before except that the summation is truncated at T. The technology can be described exactly as before. The operative constraints are those on quantities up to T and the non-negativity constraints up to T+1 , but there is no harm in using the full constraint set in $(\ell_1(\delta)_+)^4$. Let V^T denote the truncated preference functional, $\sum_{t=0}^{T} \beta^t u(c_t)$. By appending a sequence of constants (k,A,c,r) with $c = r = 0$ to the solution of the truncated problem, we can view this solution as an element in $(\ell_1(\delta)_+)^4$. Suppose we are given a sequence $\{z^T\}$ of equilibria for for the finite economies; i.e. a sequence $\{z^T\}$ such that

$$z^T \in \operatorname{argmax} \{ V^T(\hat{z}) : \hat{z} \in \tilde{\Gamma}(z^T) \} ,$$

where $\tilde{\Gamma}$ is as defined above. Since $z^T \in \Omega$ for all T and since Ω is compact, this sequence must have a convergent sub-sequence. Because of the dependence of V^T on T, the results above are not enough to conclude anything about any limit z^* of the sequence z^T, but a slight generalization will suffice. Let $\bar{N}$ denote the set of integers union the point ∞ , and make $\bar{N}$ a metric space by defining $d(n,m) = | \frac{1}{n} - \frac{1}{m} |$ with the convention

$\frac{1}{\infty} = 0$. Then we can define a single function $\tilde{V}$ over $\bar{N} \times (\ell_1(\delta)_+)^4$ by setting $\tilde{V}(T,z) = V^T(z)$. (Recall that z is formally a quadruple of sequences, but V and V^T depend trivially on all but the sequence corresponding to consumption.) To prove that the extended function $\tilde{V}$ is continuous, note that convergent sequences in $\bar{N} \times (\ell_1(\delta)_+)^4$ have values for T which are eventually constant or which tend to ∞. For the first type of sequence, lemma 2 above will apply. Thus, it remains to consider convergent sequences such that T goes to infinity. But since $\beta\gamma^\sigma$ is less than one and since momentary utility can grow no faster than $\gamma^{\sigma t}$, we can put a uniform bound on the difference between V^T and V, and this bound can be made arbitrarily small as T goes to infinity. Thus we have

LEMMA 5: $\tilde{V}:\bar{N} \times (\ell_1(\delta)_+)^4 \longrightarrow \mathbb{R}$ is continuous.

Now we can use the Maximum Theorem to conclude as before that the correspondence which sends (T,z^T) into $\operatorname{argmax} \{ \tilde{V}(T,z) : z \in \tilde{\Gamma}(z^T) \}$ is u.h.c. Then if z^T converges to z^* as T goes to ∞, we can conclude that z^* is an element of $\operatorname{argmax} \{ \tilde{V}(\infty,z) : z \in \tilde{\Gamma}(z^*) \}$ and thus that z^* is an equilibrium for the infinite horizon economy. In fact we know more. Since any sequence z^T of finite horizon equilibria will be contained in the compact set Ω , it must have a convergent subsequence. This establishes

THEOREM 2: Any sequence of equilibria for T-period finite horizon economies has a convergent subsequence which converges to an equilibrium for the infinite horizon economy.

Note that this gives an alternative proof of the existence of an infinite horizon equilibrium, but it does not offer any real savings in effort. In either case, the key step is to establish the upper-hemi-continuity of the argmax correspondence by checking that the conditions for the Maximum Theorem are met.

6. DISCUSSION

This model is both too simple and too complicated. The ways in which it is too simple are obvious. The numerical analysis needs to be refined and extended to more complicated functions. The entire framework needs to be extended to allow for other goods. Evidence such as that provided by Barnett and Morse [1] suggests that not only have resource prices been falling, extraction costs have also been falling. As the model stands, extraction costs fall only with decreases in the current rate of extraction r_t. Generating falling prices, falling extraction costs and increasing resource output over some interval will require some addition to the model like exploration and discovery or technological change in the extraction technology as well as the production technology for output. The intent here was to suggest a possibility in a simple context. If it is judged to be of interest, complicating elements designed to make it more realistic can easily be added.

The sense in which the model is too complicated is perhaps more worrisome. This is not a tightly parameterized model. There is wide latitude for generating different kinds of behavior using different specifications for the initial conditions and the functional forms. The extensions described in the last paragraph will only increase this freedom. Given the computational difficulties and the current state of the available data, there is little hope of directly estimating some set of fundamental parameters.

This does not imply that the model had no scientific content. In the analysis of exhaustible resources, it makes a great deal of difference whether this model or a more conventioanl extension of the Hotelling model offers a better explanation of the behavior of prices over the last several hundred years. Any number of possible interventions like price controls, research subsidies, fuel efficiency standards for automobiles, etc. will have very different positive and normative effects in the two models. Observing that it will not be easy to distinguish between them is not the same as claiming it does not matter which (if either) is right. Moreover, the model does have implications for in empirical research. For example, it suggests that some attempt should be made to quantify the effect (if any) of technological change on the derived demands for resources. Until more evidence on this point is available, future tests of resource models may best be conducted along the lines used in Miller and Upton [18] or Farrow [11], taking as given the path for the market price of the resource and testing only the efficiency conditions for the firm.

We emphasize the model here suggests only a possibility. It should be absolutely clear that the past behavior of resource prices, technological change, and growth offers no guarantee about future trends. The assumption in this model that knowledge exhibits global increasing marginal productivity could simply be wrong. We could be on the verge of the end of a long period of explosive growth in knowledge, resource productivity, and per capita consumption. Nonetheless, there is nothing in the data to suggest that we have already reached such a point; and there is no compelling theoretical reason to believe that we must reach such a point anytime soon.

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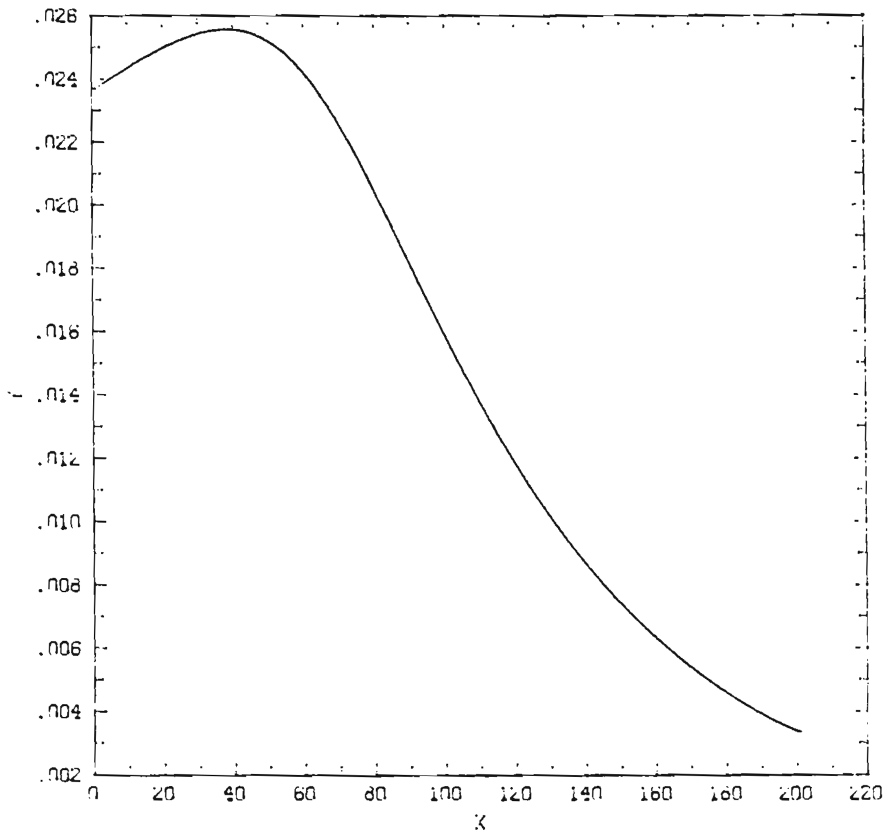

FIGURE 1: LEVEL OF THE RESOURCE PRICE AS A FUNCTION OF TIME

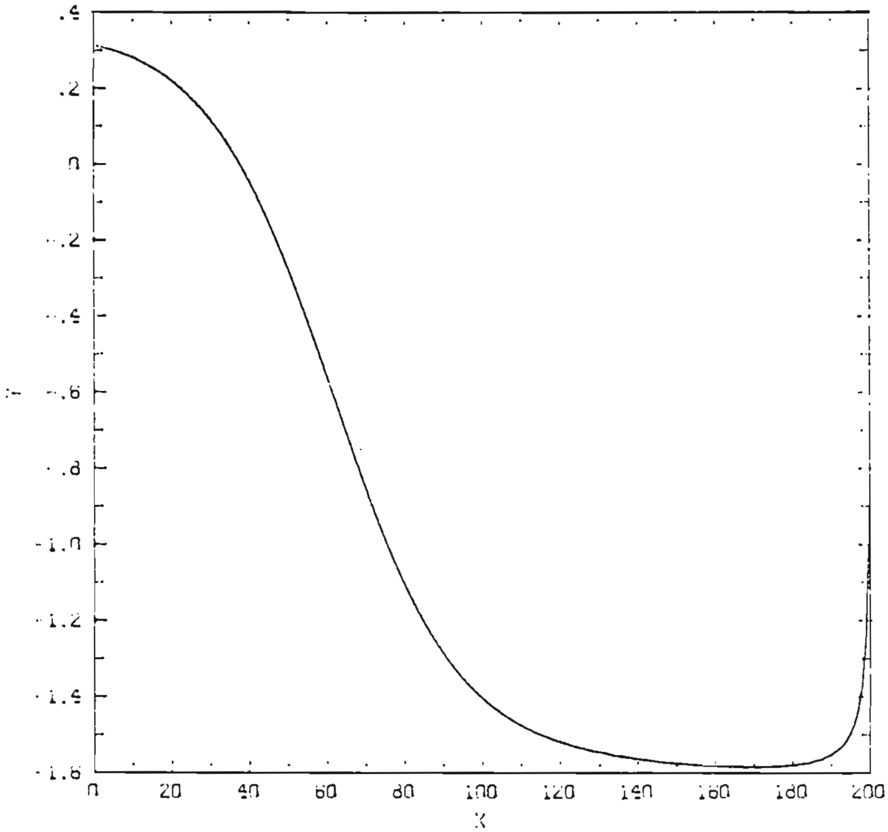

FIGURE 2: PERCENTAGE RATE OF CHANGE OF THE RESOURCE PRICE AS FUNCTION OF TIME

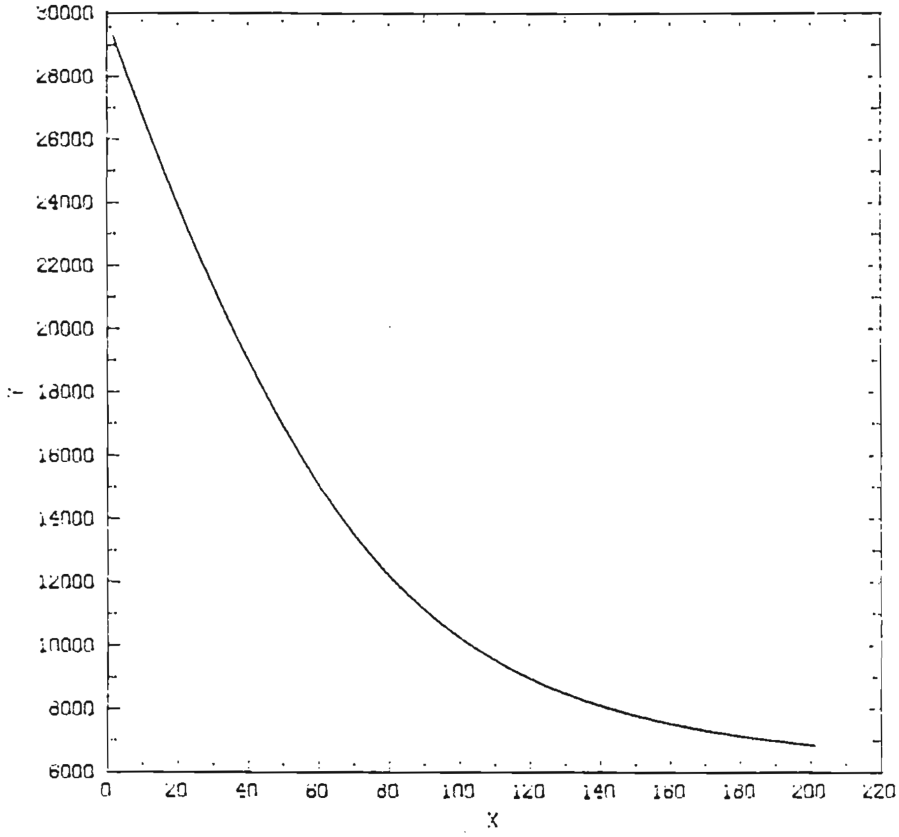

FIGURE 3: LEVEL OF THE RESOURCE STOCK AS A FUNCTION OF TIME

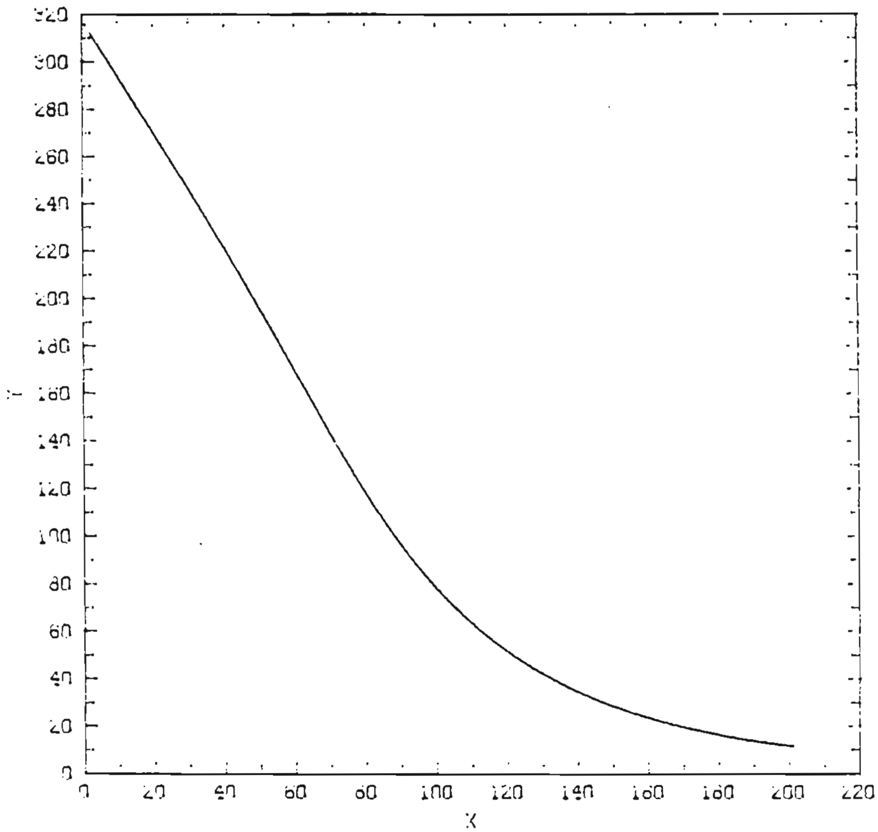

FIGURE 4: RATE OF DEPLETION OF THE RESOURCE STOCK AS A FUNCTION OF TIME

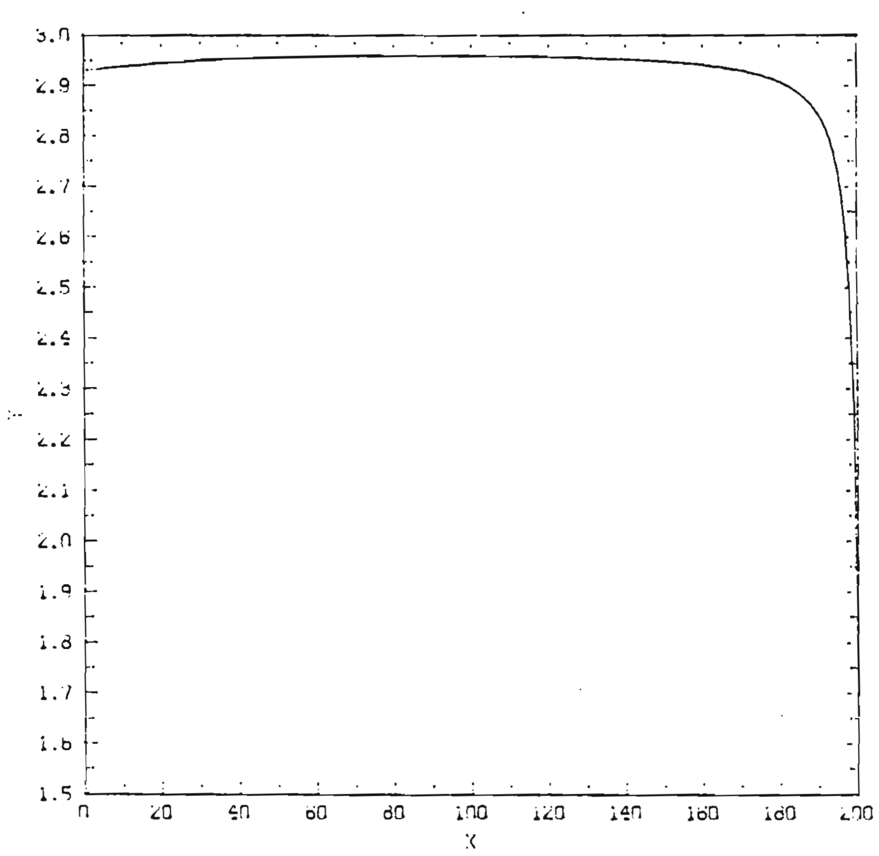

FIGURE 5: PERCENTAGE RATE OF GROWTH OF CAPITAL AS A FUNCTION OF TIME

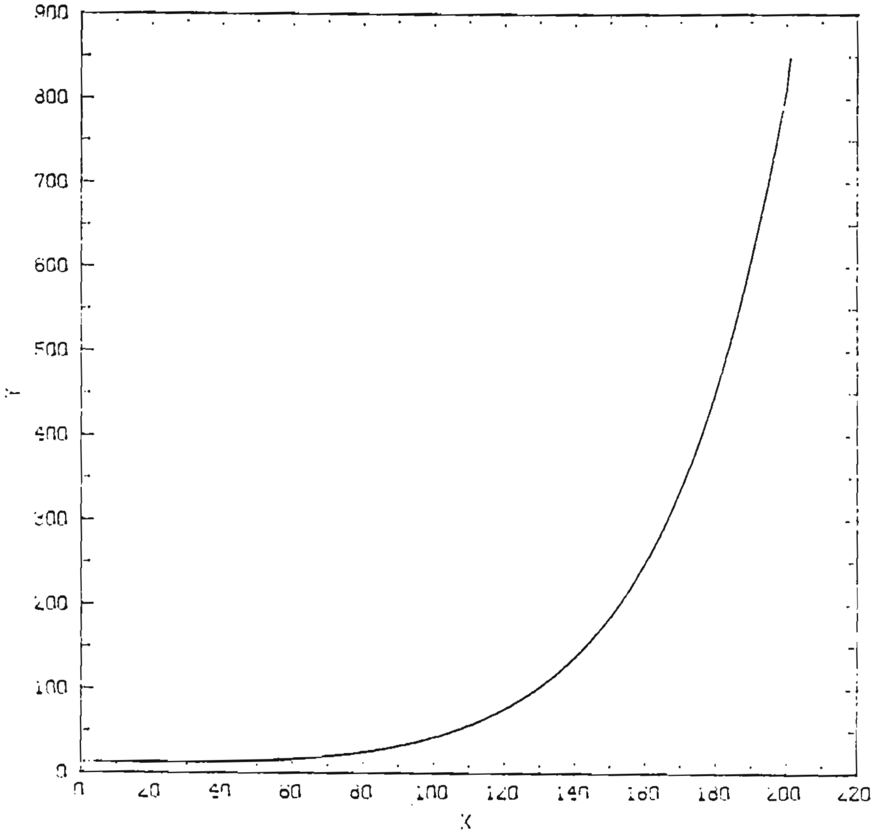

FIGURE 6: LEVEL OF CONSUMPTION AS A FUNCTION OF TIME