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I. Introduction

The inception, development and long run features of industry have long been the subject of much attention. The most basic facts are clear: when an industry is very young, there are few firms in business, but entry is rapid. Subsequently, the number of operating firms levels off and then begins to decline, eventually achieving some "long run" value.

It is not difficult to produce competitive theories of industrial evolution that generate such a time path. For example, a model in which demand increases at a decreasing rate, coupled with increasing optimum firm size due to learning–by–doing does so, as will a suitably selected structure of adjustment costs. Gort and Klepper (1982) provide a good summary of existing approaches along with a substantial amount of information about the data.

The existing theories of entry and exit in competition suffer from two deficiencies. One is that they are not firmly grounded in individual optimization, making it difficult to use what is known about firm behavior in other contexts to restrict the theory, generate predictions that make explicit claims about what should be observed, and develop an integrated theory of all aspects of firm behavior. Second, while the existing theories permit useful classification of entities that tend to induce entry or exit, the abundance of "free parameters" prohibits the development of a body of novel hypotheses; for example, there is little that could not be explained by a demand shift/learning–by–doing model.

This paper presents a preliminary investigation of a model of entry and exit behavior in which individuals optimize and that has few free parameters. It's equilibrium is roughly coincident with the facts described above. Moreover, the structure of equilibrium is sufficiently simple that calculating the equilibrium for given parameter values is not computationally overwhelming, so that structural estimation of the model's parameters—the subject of Part II—is feasible.

The main ingredients of the model are as follows. Inventions—new ideas that might somehow prove useful in production—are random occurrences. But once a new invention exists, there is potential for figuring out exactly how it can be put to use in the production process: innovation. The same applies to subsequent invention of refinements to basic inventions. In the present model there are just two possible inventions: a basic one and a single refinement. Once an invention has occurred firms may attempt to introduce innovations based on it; success is stochastic. Those who succeed may begin production and once some have done so an industry is born. (It is assumed that pre–existing technology is too primitive for exchange based on it to occur in equilibrium.) Subsequently, the basic invention may be refined; the timing of this event is stochastic as well. But once the refinement has come on the scene, innovations based on it are a possibility too. At this point either more new entrants or established firms may innovate, basing their attempts on the refinement. This activity is also stochastic, and, in general, some firms succeed earlier than others. At some point firms that have yet to succeed may find that it is no longer to their advantage to continue trying since others' success has led to a low price for output. Once this is the case (in some instances it will not happen at all) the industry enters an exit phase wherein the total number of firms begins to decline and continues to fall, achieving some long run value as the industry reaches "maturity". In brief, new opportunity produces entry and relative failure to innovate yields exit.

The optimization behavior of firms greatly restricts the possible outcomes. The time paths of the main observables in the model—total number of producing firms and price of output—may (ignoring cases corresponding to extreme parameter values) be described as follows. Each new invention (basic or a refinement) yields immediate entry followed by stability in number of firms. Some time after the refinement has been invented, exit begins and is equal to a constant proportion of the stock of operating firms that have so far failed to generate an innovation based on the refinement. During the period after the basic invention and before refinement, output price is stable. But once refinement has occurred, the price of output begins to fall. However, its decline must cease entirely as exit begins; the exit phase is also one of complete price stability.

The next Section sets out and analyses the model and its equilibrium. Since structural estimation is the goal, comparative dynamics are not presented; all attention is focused on the structure of the model's equilibrium.

II. Theory

In this Section a model of entry and exit in a perfectly competitive industry is set out and analysed. Its elements are very straightforward. New knowledge, or "inventions", emerges constantly in the economy at large, including ideas in both science and industry. Most of this information is of no use whatsoever as far as any given industry is concerned, but some knowledge is applicable. Given some basic invention, firms may try to find a way to put it to use commercially, to "innovate" in the familiar Schumpeterian distinction. Those who succeed in the costly and unpredictable process of innovation begin production and the industry begins. Further inventions yield new innovation opportunities for firms currently operating in the industry as well as others; the latter may find it to their advantage to enter when such an invention occurs. Success at innovation being stochastic, there will, however, be some who lag behind as progress occurs. Because progress lowers costs for competitors, these less successful firms may find it to their advantage to exit. These basic ingredients—new opportunities generating entry and relative lack of success yielding exit—are the key elements in the analysis to follows.

The model imposes three restrictions that should be discussed in advance. First, there are no *direct* costs associated with attempts to implement a new technology. Such activities are costly in that they necessitate foregoing some other activity in the economy and because new techniques cannot be implemented instantaneously once learned. But, if the firm has chosen to participate in the industry in question, learning entails no additional outlay. This assumption will imply that if there are any firms operating in the industry and utilizing any technology other than the most advanced invented so far, such firms will learn a better technology with positive probability—progress must occur if the industry operates at all. It is straightforward to verify that provided such costs are not large, the results to follow are not qualitatively altered by their inclusion.

Second, as indicated earlier, inventions occur *outside* the industry. It is not difficult to replace this assumption with the specification that firms within the industry both invent and innovate, with significant inventions being rare and hard to conceal (although how an invention may be put to use is another matter). The substantive restriction is that the behavior of the industry under consideration does not affect the stochastic process governing arrival of potentially useful ideas. At the cost of additional clutter it is possible to permit various relaxations of this assumption. Moreover it appears that external invention is the rule rather than the exception; see Davies (1979).

Finally, success at innovation is nearly independent of other's success: the probability with which one firm's attempts to innovate succeeds is not affected by others' luck. In other words, it is easier for the firm to sort out implementation internally (via a R & D department, for example) than it is for it to learn from or imitate others. Under this assumption it will follow that the most attractive dates at which to enter the industry are those at which new inventions occur. When innovation depends more strongly on what others know, delaying entry may prove advantageous and the analysis becomes more complex. Provided the interdependency is not large, this assumption may be relaxed.

These three assumptions reduce the difficulty in analysing the model substantially. Moreover, moderate relaxations of them preserve the essential features of equilibrium; the basic ideas that entry is generated by new opportunities and exit by relative failure to exploit them, are also not sensitive. Thus while weakening of these assumptions is not without interest, it will not be pursued here.

The formal analysis follows.

Time, denoted t, is discrete and the horizon is infinite: $t \in \{0,1,\ldots\}$.

The *industry* is defined by the commodity firms might produce and sell to consumers. The *consumer side* of the market is not of immediate interest and is summarized by a time–invariant inverse market demand function $D(Q)$, where Q is industry output; define the product price p by $p \equiv D(Q)$. D is assumed to be continuous, strictly declining and bounded with $\lim_{Q \to \infty} D(Q) = 0$.

The *invention* process is a very simple one. At $t=0$, it is known that there will be at most two inventions of relevance to the industry. The knowledge existing at $t=0$, along with any production techniques based on it, will be referred to as "primitive". The first invention will be interpreted as a "basic" invention and the second as a "refinement". This structure is intended to be a stylization of an industry's development in terms of fundamental breakthrough and subsequent significant improvement; prop planes and jets are an example.

Below, assumptions will be imposed that guarantee no trade will occur prior to implementation of the basic invention. Thus, with respect to observables, there is no loss of generality in letting $t=0$ denote the date at which the basic invention arrives on the scene. (It will be assumed that innovation based on a period t invention may be put to use no sooner that $t+1$. Thus $t=1$ is the earliest date at which production might occur.)

Once the basic invention has emerged, refinement is possible. For any $t \geq 1$, if the refinement has not been discovered earlier, it occurs with fixed probability $\rho \in (0,1)$. Let $T \geq 1$ denote the actual date of refinement.

Altogether then, the basic invention exists at $t=0$ and refinement is possible at any $t \geq 1$, occuring with probability ρ.

The *supply side* of the market comprises a fixed continuum of identical firms $[0,N]$. At any date t, one option for any firm is participation "elsewhere" in the economy. Doing so yields a fixed per period profit of $\pi^0 > 0$. Assuming a perfect capital market and constant interest rate $i > 0$, the option of producing elsewhere has capital value $\pi^0/(1-\gamma)$ where

$\gamma \equiv (1+i) \in (0,1)$.

Participation in the industry requires that π^0 be foregone and makes possible two activities: innovation and production.

Innovation involves attempts to implement inventions. As discussed earlier the innovation process is assumed to entail no direct costs other than foregoing π^0. Firms that have the know–how to implement only technologies based on primitive inventions will be referred to as "knowing θ^0"; all firms are endowed with this information at $t=0$. Any firm knowing how to put at most the basic invention to work (there may be many ways to do so; for simplicity it will be assumed that all yield the same cost) will be referred to as "knowing $\underline{\theta}$", while a firm that can utilize the refinement (once it has been invented) will be said to "know $\bar{\theta}$".

Prior to $t=0$, all firms necessarily know θ^0. At $t=0$, innovation of production techniques using the basic technology is possible. It will be assumed that when the basic invention has occurred (but not the refinement—$0 \leq t < T$) any firm knowing θ^0 succeeds in its efforts to innovate (i.e. learn $\underline{\theta}$) with probability $\beta \in (0,1)$ in any period. If a firm learns $\underline{\theta}$ at t, it may commence production using the innovated technology at $t+1$ or later.

For $t \geq T$, innovation of techniques based on the refinement (i.e. learning $\bar{\theta}$) is also a possibility. The likelihood of success at doing so may depend on the firm's present state of knowledge. Any firm knowing $\underline{\theta}$ learns $\bar{\theta}$ at $t \geq T$ with probability $r \in (0,1)$. Those knowing only θ^0 learn $\underline{\theta}$ with probability $\underline{r}^0 \in (0,1)$ and $\bar{\theta}$ with probability $\bar{r}^0 \in [0,1]$; learning $\underline{\theta}$ is not a prerequisite to learning $\bar{\theta}$ unless $\bar{r}^0 = 0$. Until more of the model has been presented, these probabilities will be left unrestricted.

In brief then, once an invention has occurred implementation is possible and stochastic. When only the basic invention has arrived, participating firms implement it with probability β. Once refinement occurs, firms knowing how to use the basic technology learn how to use the refinement with probability r; others may still learn how to use the basic invention, doing so with probability $\underline{r}^0$, but they may also skip directly to techniques based on the refinement. For them, this latter possibility has probability $\bar{r}^0$.

Turning to *production*, given θ production activities yield one period profits

$$\pi(p;\theta) \equiv \max_{q \geq 0}\{pq - c(q;\theta)\},$$

where $\theta = \theta^0$, $\underline{\theta}$ or $\bar{\theta}$. $c(\cdot\,;\theta)$ gives the factor cost of producing output q using technology based on knowledge θ. Implicit in this specification is that the prices of all factors, including any that might be technology–specific, are constant over time and that there are no direct adjustment costs. One rationalization for this assumption is that any factor specificity is in terms of the underlying inventions rather than in terms of the specific application in this particular industry, and that inventions find applications in numerous industries.

It is assumed that $\pi(\cdot\,;\cdot)$ satisfies

i) $\pi[D(0), \theta^0] = 0$ (i.e. $0 = \operatorname{argmax}_{q \geq 0} \{D(0)q - c(q;\theta^0)\}$); ii) $\pi[D(0), \underline{\theta}] > \pi^0$; and iii) for all $p > 0$, $\infty > \pi(p, \bar{\theta}) > \pi(p, \underline{\theta}) > 0$. (i) requires that the primitive technology is nonviable even if π^0 has been foregone; (ii) states that knowing $\underline{\theta}$ is profitable relative to production elsewhere if p is high enough; and (iii) imposes the condition that if π^0 has been foregone, production given any $\theta \neq \theta^0$ dominates shutting down, but knowing $\bar{\theta}$ is more profitable than knowing $\underline{\theta}$. Also, define $q(p,\theta) \equiv \operatorname{argmax}_{q \geq 0}\{pq - c(q;\theta)\}$. $q(p,\theta^0) = 0$, and for $p > 0$ assume $q(p,\bar{\theta}) > q(p,\underline{\theta})$. (Note that $q(p,\underline{\theta}) > 0$ is implied by $\pi(p,\underline{\theta}) > 0$).

While there is randomness at the individual level, it will be assumed that in the aggregate, given invention of the refinement (a realization of T) the evolution of innovations is deterministic. In general, as is well known, this requirement generates some question as to whether equilibrium exists. Moreover, available existence theorems (e.g. Jovanovic–Rosenthal or Jovanovic–MacDonald) do not apply to this setting. Fortunately, the environment set out above is structured sufficiently that the existence issue may be settled along with the description of equilibrium; that is, the construction given below is itself a direct demonstration of existence.

Given the deterministic aggregate behavior for fixed T, some useful expressions describing industry evolution can be set out. This description depends on the timing of events in each period. The within–period timing convention is as follows: inventions occur first, then firms choose whether to participate in the industry, and finally output is produced and any innovations realized. This specification may be altered with only trivial impact.

At any date t, firms may operate in whatever industry they find advantageous. Let n_t^0 denote the number (strictly, measure) of firms knowing only θ^0 and that participate in the industry at t. n_t^0 evolves according to

$$n_0^0 \in [0,N],$$

$$n_{t+1}^0 = \begin{cases} (1-\beta)n_t^0 - x_{t+1}^0 & 0 \leq t < T \\ (1-\underline{r}^0 - \bar{r}^0)n_t^0 - x_{t+1}^0 & t \geq T, \end{cases}$$

where x_t^0 is net exit (or entry, if $x_t^0 < 0$) by firm's knowing θ^0 at the beginning of period t.[1] n_t^0 is thus the number of firms knowing θ^0 that are "at risk" with respect to learning $\theta \neq \theta^0$ at t. For $0 \leq t < T$, only innovations using basic technology are possible, and these are learned with probability β; for $t \geq T$ innovations using the refinement may also occur, $\underline{r}^0$ and $\bar{r}^0$ being the probabilities of innovation using the basic invention and its refinement respectively.

In an analogous manner, define $\underline{n}_t$ and $\underline{x}_t$ ($\bar{n}_t$ and $\bar{x}_t$) as the number of firms knowing $\underline{\theta}$ ($\bar{\theta}$) that participate or exit at t. The implied evolutions are[2]

$$\underline{n}_0 = 0,$$

$$\underline{n}_{t+1} = \begin{cases} \underline{n}_t + \beta n_t^0 - \underline{x}_{t+1} & 0 \leq t < T \\ (1-r)\underline{n}_t + \underline{r}^0 n_t^0 - \underline{x}_{t+1} & t \geq T, \end{cases}$$

$$\bar{n}_t = 0 \qquad 0 \leq t \leq T,$$

$$\bar{n}_{t+1} = \bar{n}_t + r\underline{n}_t + \bar{r}^0 n_t^0 - \bar{x}_{t+1} \qquad t \geq T.$$

The interpretation of these expressions parallels that given for the evolution of n_t^0. To reduce clutter, assume $\bar{x}_t = 0$ and replace $\underline{x}_t$ by x_t. The restriction on $\bar{x}_t$ will be shown to be nonbinding in equilibrium

Now consider *optimization by an individual firm.* Each takes as given the participation decisions and knowledge of others. In this model, as is familiar from standard competitive analysis, the actions and information of others may be summarized by a price sequence. However, in the present case prices will generally depend on whether the refinement has been invented; i.e. whether $t \gtrless T$, where T, is random. Since the equilibrium price path turns out to be a very simple one, introduction of an elaborate body of notation to describe the price path is not the most straightforward route. Rather, a simpler (and equally correct) route will be followed. Price, knowledge and a variable indexing whether the refinement has arrived are modelled as a joint—Markov process on $[0,D(0)] \times \{\theta^0, \underline{\theta}, \bar{\theta}\} \times \{0,1\}$. The numerous restrictions implied by the structure set out above—for example that prices are deterministic given T—can be left implicit at this point.

Let E_0 be the expectations operator at $t=0$ given $p_0 = D(0)$ (recall there is no trade at $t=0$ irrespective of p), $\theta = \theta^0$ and that the refinement has not been invented. It is assumed that firms are risk neutral and behave so as to maximize

$$E_0\left[\sum_{t=0}^{\infty} \beta^t \tilde{\pi}_t\right],$$

where $\tilde{\pi}_t$ equals π^0 if the firm does not participate, and $\pi(p_t, \theta_t)$ if the firm participates at t when price is p_t and knowledge is $\theta_t \in \{\theta^0, \underline{\theta}, \bar{\theta}\}$.[3]

Given the boundedness of π^0 and $\pi(p; \theta)$, this optimization may be represented by a sequence of pairs of functions $\{U_t(\theta), V_t(\theta)\}_0^{\infty}$ where $U_t(\theta)$ represents the expected present value of profits from t onward given i) an optimal participation policy will be followed; ii) knowledge is θ; and iii) the refinement has not been invented either prior to or at t. $V_t(\theta)$ has the same interpretation except that it takes as given that the refinement has been invented at t or earlier.

The functions $V_t(\cdot)$ and $U_t(\cdot)$ must satisfy the optimality conditions.[4]

$$U_t(\theta^0) = \max_{\iota \in \{0,1\}} \left\{ (1-\iota)\{\pi^0 + \gamma[\rho V_{t+1}(\theta^0) + (1-\rho)U_{t+1}(\theta^0)]\} \right.$$
$$+\iota\gamma\{\beta[\rho V_{t+1}(\underline{\theta}) + (1-\rho)U_{t+1}(\underline{\theta})]$$
$$\left. + (1-\beta)[\rho V_{t+1}(\theta^0) + (1-\rho)U_{t+1}(\theta^0)]\} \right\},$$

$$V_t(\theta^0) = \max_{\iota \in \{0,1\}} \left\{ (1-\iota)[\pi^0 + V_{t+1}(\theta^0)] + \iota\gamma[\underline{r}^0 V_{t+1}(\underline{\theta}) \right.$$
$$\left. + \bar{r}^0 V_{t+1}(\bar{\theta}) + (1-\underline{r}^0-\bar{r}^0)V_{t+1}(\theta^0)] \right\}$$

$$U_t(\underline{\theta}) = \max_{\iota \in \{0,1\}} \left\{ (1-\iota)\{\pi^0 + \gamma[\rho V_{t+1}(\underline{\theta}) + (1-\rho)U_{t+1}(\underline{\theta})]\} \right.$$
$$\left. \iota\gamma\{\pi(p_t, \underline{\theta}) + \gamma[\rho V_{t+1}(\underline{\theta}) + (1-\rho)U_{t+1}(\underline{\theta})]\} \right\}$$
$$= \max_{\iota \in \{0,1\}} [(1-\iota)\pi^0 + \iota\pi(p_t, \underline{\theta})] + \gamma[\rho V_{t+1}(\underline{\theta}) + (1-\rho)U_{t+1}(\underline{\theta})],$$

$$V_t(\underline{\theta}) = \max_{\iota \in \{0,1\}} \left\{ (1-\iota)[\pi^0 + \gamma V_{t+1}(\underline{\theta})] + \iota\{\pi(p_t, \underline{\theta}) + \gamma[r V_{t+1}(\bar{\theta}) \right.$$
$$\left. + (1-r)V_{t+1}(\underline{\theta})]\} \right\},$$

and $$V_t(\bar{\theta}) = \max_{\iota \in (0,1)} \left\{ (1-\iota)[\pi^0 + \gamma V_{t+1}(\bar{\theta}) + \iota[\pi(p_t, \bar{\theta}) + \gamma V_{t+1}(\bar{\theta})] \right\}$$
$$= \max_{\iota \in \{0,1\}} [(1-\iota)\pi^0 + \iota\pi(p_t, \bar{\theta})] + \gamma V_{t+1}(\bar{\theta}).$$

Define $U_t(\bar{\theta}) = V_t(\bar{\theta})$. $U_t(\cdot)$ and $V_t(\cdot)$ may be interpreted straightforwardly. Take $U_t(\theta^0)$, for example. $U_t(\theta^0)$ is the present value of firm profits given knowledge of θ^0 and that the refinement has not been invented by t. If a firm knowing θ^0 does not participate (ι=0) it earns current profit π^0 plus the expected discounted profit associated with knowing θ^0 at $t+1$, where the expectation takes into account that the refinement might be invented at $t+1$. On the other hand, the option of participating ($\iota=1$) yields no current period profit but offers the possibility of knowing $\underline{\theta}$ at $t+1$, the value of which will generally depend on whether the refinement is invented at $t+1$. $U_t(\theta^0)$ is the larger of the values following from the participate/not participate decisions.

Now consider *equilibrium*. At each date t, firm actions (participation (ι), quantity produced (q) and knowledge (θ)) may be summarized by a conditional distribution function $\lambda_t(\iota, q, \theta|\cdot)$ where the conditioning is with respect to whether the refinement occurred at or before t. The sequence of such conditional distribution functions, $\{\lambda_t\}_0^\infty$, is defined to be an equilibrium of the economy if given the (market clearing) price sequence $\{p_t\}$ implied by $\{\lambda_t\}$, the optimizing behavior of all firms coupled with the learning technology implies $\{\lambda_t\}$ itself as the sequence of joint conditional distributions over (ι, q, θ).

The determination of the *structure of an equilibrium* proceeds as follows. First, equilibrium is constructed for $t = T+1,...$; that is, for the periods following that at which the refinement occurred. This construction takes as given fixed values for $n_T^0 \geq 0$, $\underline{n}_T = 0$, and $\bar{n}_T = 0$. These values imply fixed values for the sum $\underline{n}_{T+1} + x_{T+1} = (1-r)\underline{n}_T + \underline{r}^0 n_T^0$ as well as for $\bar{n}_{T+1} = r\underline{n}_T + \bar{r}^0 n_T^0$. It is also assumed that for $t \geq T+1$, $n_t^0 = 0$ and $x_t \geq 0$. Both will be shown to hold in equilibrium.

Next, the structure of period T is determined. Subsequently, behavior in all periods to T but after t=0 is set out. Finally, period 0 is analysed.

The material to follow makes use of the following notation. Define p^* as the unique solution of

$$\frac{\pi^0}{1-\gamma} = \pi(p^*, \underline{\theta}) + \frac{\gamma}{1-\gamma}[r\pi(p^*, \bar{\theta}) + (1-r)\pi^0].$$

p^* has the property that if $t \geq T$, and $p_t = p^*$ in the current and all subsequent periods, a firm knowing $\underline{\theta}$ would be just indifferent about current period participation. Let $\bar{n}^*$ be the number of firms knowing $\bar{\theta}$ which as a group would produce output exactly sufficient to yield price p^* if no other firms produced; that is, $\bar{n}^*$ is the unique solution to

$$p^* = D[\bar{n}^* q(p^*, \bar{\theta})].$$

Next, let $\tilde{p}$ ($> p^*$) be the price at which firms knowing $\underline{\theta}$ would earn exactly π^0 from current production. $\tilde{p}$ solves

$$\pi(\tilde{p}, \underline{\theta}) = \pi^0.$$

Finally, let Q_t denote industry output at t:

$$Q_t \equiv \underline{n}_t q(p_t, \underline{\theta}) + \bar{n}_t q(p_t, \bar{\theta}).$$

In particular, given p^*, define Q^* by $p^* = D(Q^*)$.

At this point, in order to avoid a proliferation of less relevant cases, four parameter restrictions will be imposed. First, it will be assumed that parameters are such that

$$\pi^0 > \gamma[\bar{r}^0\pi(p^*,\bar{\theta}) + (1-\bar{r}^0)\pi^0] \tag{1}$$

is satisfied. The restriction implies that if $p_t = p^*$ prevails for all t beginning next period, firms knowing only θ^0 would not enter during the current period. (Recall that p^* is such that those knowing $\underline{\theta}$ would be indifferent.) Without this restriction it may be that those knowing only θ^0 may learn $\bar{\theta}$ so easily relative to those knowing $\underline{\theta}$, that equilibrium may involve those knowing $\underline{\theta}$ producing at T and exiting at $T+1$, with output sufficient to cause $p_t = p^*$ for all $t \geq T + 1$ being supplied by firms who entered at T knowing only θ^0, and who learned $\bar{\theta}$ at T. Under restriction (1), it will be shown that $\underline{n}_{T+1} > 0$ must hold. It is easy to show that $r \geq \bar{r}^0$ would be sufficient to guarantee the above inequality.[5] That $r \geq \bar{r}^0$ is not necessary is, however, advantageous, for in a richer model allowing innovation effort to be endogenous (for example, Jovanovic and MacDonald) it is possible that the counterpart to $r < \bar{r}^0$ can occur easily enough as a result of the greater incentives to innovate faced by firms whose technological knowledge is presently inferior.

The second restriction is

$$q(p^*, \underline{\theta}) > rq(p^*, \bar{\theta}). \tag{2}$$

This inequality is a relatively mild one, but it plays an important role that will become apparent. The conclusion that it is a mild restriction follows from noting that it is equivalent to the *one period* output of a firm knowing $\bar{\theta}$, when $p_t = p^*$, being smaller than the *total* output a firm presently knowing only $\underline{\theta}$ would expect to produce using technology based on $\underline{\theta}$ if price remained fixed at p^*.

Next, it will be assumed that N is sufficiently large that the expected present value of profits from participation by firms knowing θ^0 may always be driven to $\pi^0/(1-\gamma)$ by suitably many participating, and, that if all N firms participate, so many would learn $\underline{\theta}$ that the supposed participation is not maximal for any firm knowing only θ^0. As stated, this "free entry" restriction depends on the structure of equilibrium. However, it is neither difficult nor very helpful to provide a statement in terms of primitives.

In a similar vein, it will be assumed that

$$\pi^0 < \gamma \min \left\{ \beta\pi[D(0),\underline{\theta}] + (1-\beta)\pi^0,\ \bar{r}^0\pi[D(0),\bar{\theta}] + \underline{r}^0\pi(D(0),\underline{\theta}] + (1-\bar{r}^0-\underline{r}^0)\pi^0 \right\}. \quad (3)$$

Under this restriction, if no firms plan to operate at $t+1$, in which case $p_{t+1} = D(0)$, it would always pay for a firm knowing θ^0 to participate at t. This restriction merely serves to guarantee that the industry is "viable" once the basic invention has arrived on the scene.

Periods following T may now be analysed. Take as given $n^0_T \geq 0$, $\underline{n}_T > 0$ and $\bar{n}_T = 0$, and consider date $t \geq T+1$ assuming $x_t \geq 0$ (no entry of firms knowing $\underline{\theta}$) and $n^0_t = 0$. Recall (1) implies that if $p_t \leq p^*$ for $t \geq T+1$, $n^0_T = 0$ may also be assumed. If $p_t < \tilde{p}$ (defined by $\pi(\tilde{\rho},\underline{\theta}) = \pi^0$) for all $t \geq T+1$, $\underline{n}_T = 0$ may also be imposed.

There are three cases to consider. In the first the maximum number of firms that might operate at $t = T+1$—$\underline{n}_T + (\underline{r}^0 + \bar{r}^0)n^0_T$—is at most $\bar{n}^*$. In this instance the hypothesized equilibrium evolution is[6]

i) $n^0_{T+1} = 0$, $\underline{n}_{T+1} = (1-r)\underline{n}_T + \underline{r}^0 n^0_T$ and $\bar{n}_{T+1} = r\underline{n}_T + \bar{r}^0 n^0_T$;

and ii) $\forall t \geq T+1$ $\quad n^0_{t+1} = 0,$

$$\underline{n}_{t+1} = (1-r)\underline{n}_t,$$

and $\quad \bar{n}_{t+1} = \bar{n}_t + r\underline{n}_t;$

in particular $x_t = 0$ for all $t \geq T+1$. Given $n_T^0 \geq 0$ and $\underline{n}_T > 0$, this evolution is clearly feasible. Moreover, because $q(p,\theta)$ is increasing in p for all θ, and $0 < q(p,\underline{\theta}) < q(p,\bar{\theta})$, it follows that $Q_t \leq Q^*$, and thus that $p_t \geq p^*$. Therefore it is certainly maximal for firms knowing either $\underline{\theta}$ or $\bar{\theta}$ to behave as hypothesized; that $n_t^0 = 0$ is maximal for those knowing θ^0 will be demonstrated below. It follows that the evolution displayed above is equilibrium behavior given $\underline{n}_T + (\bar{r}^0 + \underline{r}^0)\underline{n}_T \leq \bar{n}^*$.

For the second case, suppose instead that $\underline{n}_T + (\bar{r}^0 + \underline{r}^0)n_T^0 > \bar{n}^*$; in particular that $\bar{n}_{T+1}(= r\underline{n}_T + \bar{r}^0 n_T^0) \geq \bar{n}^*$. In this case firms knowing $\bar{\theta}$ at T+1 would by themselves produce output sufficient to cause $p_t \leq p^*$ for all $t \geq T+1$, and should $\underline{n}_t > 0$ for any $t \geq T+1$, $p_t > p^*$ is implied. It follows that the evolution

$$\forall t \geq T+1,\ n_t^0 = 0,\ \underline{n}_t = 0 \text{ and } \bar{n}_t = r\underline{n}_T + \bar{r}^0 n_T^0$$

is equilibrium behavior.[7]

The third and intermediate case—characterized by $\underline{n}_T + (r^0 + \underline{r}^0)n_T^0 > \bar{n}^*$ and $\bar{n}_{T+1}(=r\underline{n}_T + \bar{r}^0 n_T^0) < \bar{n}^*$—generates a slightly more complicated equilibrium evolution. Recall the evolution in the first case: $x_t \equiv 0$. Given the present parameter values, if this "no exit" evolution obtained, eventually $Q_t < Q^*$ would have to occur, implying $p_t < p^*$ for some t. Let $T^* \geq T+1$ denote the first date at which the inequality obtains. The hypothesized evolution is then given by

i) $n_{T+1}^0 = 0,\quad \underline{n}_{T+1} = (1-r)\underline{n}_T + \underline{r}^0 n_T^0 - x_{T+1},\ \bar{n}_{T+1} = r\underline{n}_T + \bar{r}^0 n_T^0;$

ii) $\forall t \geq T+1 \quad n_{t+1}^0 = 0,$

$\underline{n}_{t+1} = (1-r)\underline{n}_t - x_{t+1},$

and $\bar{n}_{t+1} = \bar{n}_t + r\underline{n}_t;$

$$\text{and} \quad \text{iii)} \quad x_t = \begin{cases} 0 & \text{if } T^* > T+1 \text{ and } T+1 \leq t < T^* \\ (1-r)\underline{n}_{T^*-1} + (\bar{n}_{T^*-1} + r\underline{n}_{T^*-1} - \bar{n}^*) \cdot \dfrac{q(p^*,\bar{\theta})}{q(p^*,\underline{\theta})} & \text{if } t = T^* \geq T+1 \\ \left[\dfrac{q(p^*,\bar{\theta})}{q(p^*,\underline{\theta})} - 1\right] r\underline{n}_{t-1} & \text{if } t > T^*. \end{cases}$$

The path of exit (x_t) by firms knowing $\underline{\theta}$ implies that $p_t = D[\underline{n}_t q(p_t, \underline{\theta}) + \bar{n}_t q(p_t, \bar{\theta})] > p^*$ for $T+1 \leq t < T^*$ (assuming $T+1 < T^*$) and $p_t = p^*$ for $t \geq T^*$. If $T+1 = T^*$, $p_t = p^*$ for all $t \geq T+1$. Note also that $n^0_{T^*-1} = 0$ always holds. If $T^* > T+1$, this implied by the assumption $n^0_t = 0$; if $T^* = T+1$, the restriction (1) implies $n^0_T \left[= n^0_{T^*-1}\right] = 0$.

The expressions for $x_t \neq 0$ are obtained as follows. First, x_{T^*} is chosen to yield $p_{T^*} = p^*$, requiring $Q_{T^*} = Q^*$, or

$$\underline{n}_{T^*} q(p^*,\underline{\theta}) + \bar{n}_{T^*} q(p^*,\bar{\theta}) = \bar{n}^* q(p^*,\bar{\theta}).$$

Substitution of $\underline{n}_{T^*} = (1-r)\underline{n}_{T^*-1} - x_{T^*}$ (recall $n^0_{T^*-1} = 0$) and $\bar{n}_{T^*} = \bar{n}_{T^*-1} + r\underline{n}_{T^*-1}$ gives the desired expression. Feasibility requires $0 < x_{T^*} < (1-r)\underline{n}_{T^*-1}$. Should $x_{T^*} = 0$, by definition of T^*, $p_{T^*} < p^*$ violating the requirement $p_{T^*} = p^*$. Similarly, if $x_{T^*} = (1-r)\underline{n}_{T^*-1}$, it follows that

$$\begin{aligned} Q_{T^*} &= (\bar{n}_{T^*-1} + r\underline{n}_{T^*-1}) q(p^*,\bar{\theta}) \\ &< \bar{n}_{T^*-1} q(p^*, \bar{\theta}) + \underline{n}_{T^*-1} q(p^*, \underline{\theta}) \qquad \text{(by (2))} \\ &= Q_{T^*-1}, \end{aligned}$$

implying $p_{T^*-1} < p^*$, again violating the definition of T^*. Thus $0 < x_{T^*} < (1-r)\underline{n}_{T^*-1}$.

For x_t, given $t > T^*$, $p_t = p^*$ and thus $Q_t = Q^*$ must hold for all t, implying

$$\underline{n}_{t-1}q(p^*, \underline{\theta}) + \bar{n}_{t-1}q(p^*, \underline{\theta}) = \underline{n}_t q(p^*, \underline{\theta}) + \bar{n}_t q(p^*, \bar{\theta}).$$

Substitution of $\underline{n}_t = (1-r)\underline{n}_{t-1} - x_t$ and $\bar{n}_t = \bar{n}_{t-1} + r\underline{n}_{t-1}$ gives the desired expression. Once again the feasibility requirement $0 < x_t < (1-r)\underline{n}_{t-1}$ is easily verified under assumption (2).

The role of (2) is clear. Exit by firms knowing $\underline{\theta}$ must maintain $p_t = p^*$. If (2) fails, learning may be so rapid, or firms knowing $\bar{\theta}$ may be so large, that even exit by all firms knowing $\underline{\theta}$ might not prevent $p_t < p^*$.

In terms of optimization by individual firms, that $p_t \geq p^*$ for all t implies all those knowing $\bar{\theta}$ will invariably participate. All firms knowing $\underline{\theta}$ at $t \geq T$ will strictly prefer to participate when $t < T^*$, and will be willing to behave as prescribed by the hypothesized evolution for $t \geq T^*$.[8] Again, that $n_t^0 = 0$ is maximal remains to be demonstrated.

To summarize what has been shown so far, beginning during the period $(T+1)$ following that in which the refinement was invented, and assuming i) fixed values of $n_T^0 \geq 0$ and $\underline{n}_T > 0$, and ii) for all $t > T$, $n_t^0 = 0$ and $x_t \geq 0$, the evolution may take on three forms. If $\underline{n}_T + (\bar{r}^0 + \underline{r}^0)n_T^0 \leq \bar{n}^*$, $p_t > p^*$ and no exit ever occurs. If $\underline{n}_T + (\bar{r}^0 + \underline{r}^0)n_T^0 > \bar{n}^*$ and $r\underline{n}_T + \bar{r}^0 n_T^0 \geq \bar{n}^*$, all firms knowing $\underline{\theta}$ exit at $T+1$ and $\bar{n}_t = r\underline{n}_T + \bar{r}^0 n_T^0$ for all $t \geq T+1$. Should $\underline{n}_T + (\bar{r}^0 + \underline{r}^0)n_T^0 > \bar{n}^*$ and $r\underline{n}_T + \bar{r}^0 n_T^0 < \bar{n}^*$, there is no exit prior to some date $T^* \geq T$ at which time exit by firms knowing $\underline{\theta}$ begins; $p_{t+1} < p_t$ for $t < T^*$ and the level of exit retains $p_t = p^*$ for $t \geq T^*$.

To proceed, consider date T, at which time the refinement is invented. In this part it will be supposed that there is a positive number of firms, $\underline{m}_{T-1} > 0$, that have learned $\underline{\theta}$ prior to T; thus the number participating is constrained by $0 \leq \underline{n}_T \leq \underline{m}_{T-1}$. The number knowing only θ^0 (prior to T) is then $N-\underline{m}_{T-1}$.

The construction for $t > T$ assumed $n_T^0 \geq 0$, $\underline{n}_T > 0$, $\bar{n}_T = 0$, and both $x_t \geq 0$ and $n_t^0 = 0$ for all $t \geq T+1$. These restrictions must be shown to represent optimizing behavior.

First consider $n_t^0 = 0$ for all $t \geq T+1$. Under free entry of firms knowing θ^0, the expected present value of participation at date T given θ^0 must not exceed $\pi^0/(1-\gamma)$. In all three evolutions (for $t \geq T+1$) displayed above, the expected present value of participation given θ^0 is as great (or greater) at $t=T$ than at any $t \geq T+1$. Therefore nonparticipation at $t \geq T+1$ is a maximizing choice for these firms: $n_t^0 = 0$ for $t \geq T+1$.

The conditions $n_T^0 \geq 0$ and $\bar{n}_T = 0$ are merely definitions and need no further consideration. In regard to $x_t \geq 0$ and $\underline{n}_T > 0$, verification requires consideration of periods prior to $t=T$ and is therefore postponed. However, it can be mentioned that those conditions are implied by $\underline{n}_T = \underline{m}_{T-1} > 0$, which is what will be shown below. Thus $\underline{n}_T = \underline{m}_{T-1} > 0$ will be employed here.

Given $\bar{n}_T = 0$ and $\underline{n}_T = \underline{m}_{T-1}$ (exogenous at T), all that needs to be analyzed at $t=T$ is the behavior of those knowing θ^0; in particular, when is $n_T^0 > 0$? Recall that the expected present value of entry at T for such firms cannot, in equilibrium, exceed $\pi^0/(1-\gamma)$. For $\underline{m}_{T-1}$ sufficiently small, given restriction (3), $n_T^0 = 0$ yields any firm knowing only θ^0 expected present value of profits from entry at T in excess of $\pi^0/(1-\gamma)$. Given that N is large and that raising n_T^0 augments both $\underline{n}_{T+1}$ and $\bar{n}_{T+1}$, thereby reducing p_t for $t \geq T+1$, there exists some value of n_T^0 yielding expected present value of profits exactly equal to $\pi^0/(1-\gamma)$. Moreover, this number is nonincreasing and continuous in $\underline{m}_{T-1}$, since increasing $\underline{m}_{T-1}$ raises $\underline{n}_T$ and may lower p_t. Given that N is "large", this value of n_T^0 is in fact declining in $\underline{m}_{T-1}$ for $\underline{m}_{T-1}$ large enough, and takes on its minimum value $n_T^0 = 0$ for some value $\underline{m}_{T-1} < N$. Thus, given any $\underline{m}_{T-1}$, the number of firms knowing θ^0 that participate at t is either that $n_T^0 > 0$ which equates the present value of participation at T to $\pi^0/(1-\gamma)$, or, if no positive n_T^0 will accomplish this, $n_T^0 = 0$.

Now consider any period t such that $0 < t < T$; i.e. a period after the basic invention and before the refinement. (Since $T=1$ may occur, such t need not exist). It will be assumed that for such t, $n_t^0 = 0$. Again, that this behavior is maximal is to be shown. $n_t^0 = 0$ implies the number of firms knowing $\underline{\theta}$ at t, $\underline{m}_t$, is equal to the number that learned $\underline{\theta}$ at $t=0$, βn_0^0, assumed positive.

For firms knowing $\underline{\theta}$, participation at $t < T$ confers no special advantage; in particular, these firms are free to participate as soon as the refinement has been invented. Thus, they will participate at $t < T$ if and only if $p_t \geq \tilde{p}$; i.e. if $\pi(p_t, \underline{\theta}) \geq \pi^0$. The evolution for $1 < t < T$ is given by

$$n_t^0 = 0$$
$$\underline{n}_t = min(\eta, \underline{m}_t)$$
where η solves $\tilde{p} = D[\eta q(\tilde{p}, \underline{\theta})]$.

Notice that p_t is constant for $1 < t < T$, and p_T is not larger than this value, since $\underline{n}_T = \underline{m}_{T-1}$.

Now, consider $t=0$. $\underline{n}_0 = \bar{n}_0 = 0$, by definition. The behavior of firms knowing only θ^0 (i.e. all firms) is to be determined. Given the evolutions for $t \geq 1$ set out above, if n_0^0 is sufficiently small, participation by any one yields expected present value of profits in excess of $\pi^0/(1-\gamma)$. Similarly, for n_0^0 sufficiently large, the expected payoff falls short of $\pi^0/(1-\gamma)$. It is also easy to verify that the expected payoff is continuous and declining in n_0^0, since raising n_0^0 augments $\bar{n}_t$, at least. Thus there exists some feasible $n_0^0 > 0$ such that the expected value of participation at $t=0$ is exactly $\pi^0/(1-\gamma)$, and this value is that hypothesized for the evolution at $t=0$.

It remains to check that three restrictions imposed along the way are nonbinding i) $x_t \geq 0$ for $t \geq T+1$; ii) $\underline{n}_T > 0$; and iii) $n_t^0 \neq 0$ only if $t = 0$ or T.

Since $\underline{m}_{T-1} = \beta n_0^0 > 0$ under the hypothesized evolutions, both (i) and (ii) will obtain if it can be shown that $\underline{n}_T = \underline{m}_{T-1}$. That $\underline{n}_T = \underline{m}_{T-1}$ implies $\underline{n}_T > 0$ is immediate since $\underline{n}_T = 0$ implies $p_T = D(0)$. In regard to the entry condition $x_t \geq 0$ for $t \geq T+1$, there are two situations. In one, as occurs if, for example $x_{T+1} = (1-r)\underline{n}_T$, that no firm knowing $\underline{\theta}$ would strictly prefer to enter at $t \geq T+1$ is immediate because a payoff of at most $\pi^0/(1-\gamma)$ for firms knowing $\underline{\theta}$ defines this situation. $x_t \geq 0$ is then clearly nonbinding. In the other, the payoff to participation exceeds $\pi^0(1-\gamma)$, and so any firm knowing $\underline{\theta}$ and not participating would seek to do so. But if $\underline{n}_T = \underline{m}_{T-1}$, there are no such firms since all possible entry occurred at $t < T+1$. Thus, if $\underline{n}_T = \underline{m}_{T-1}$, (i) and (ii) will follow.

That $\underline{n}_T = \underline{m}_{T-1}$ is easily shown. Suppose $\underline{n}_T < \underline{m}_{T-1}$. Then the expected present value of participation at T given knowledge of $\underline{\theta}$ cannot exceed $\pi^0/(1-\gamma)$. In particular, $p_t \leq \tilde{p}$ must hold, for simply producing at $t=T$ only is an option. Since p_t is constant for $1 < t < T$ and cannot rise at T, the expected present value of profits given knowledge of $\underline{\theta}$ can never exceed $\pi^0/(1-\gamma)$ at any t: both $U_t(\underline{\theta})$ and $V_T(\underline{\theta}) \leq \pi^0/(1-\beta)$. Since free entry implies $U_t(\theta^0) = \pi^0/(1-\beta)$, it follows that the value of participation at $t=0$ given θ^0 is at most

$$\gamma\left\{\beta\frac{\pi^0}{1-\gamma} + (1-\beta)\frac{\pi^0}{1-\gamma}\right\} < \frac{\pi^0}{1-\gamma},$$

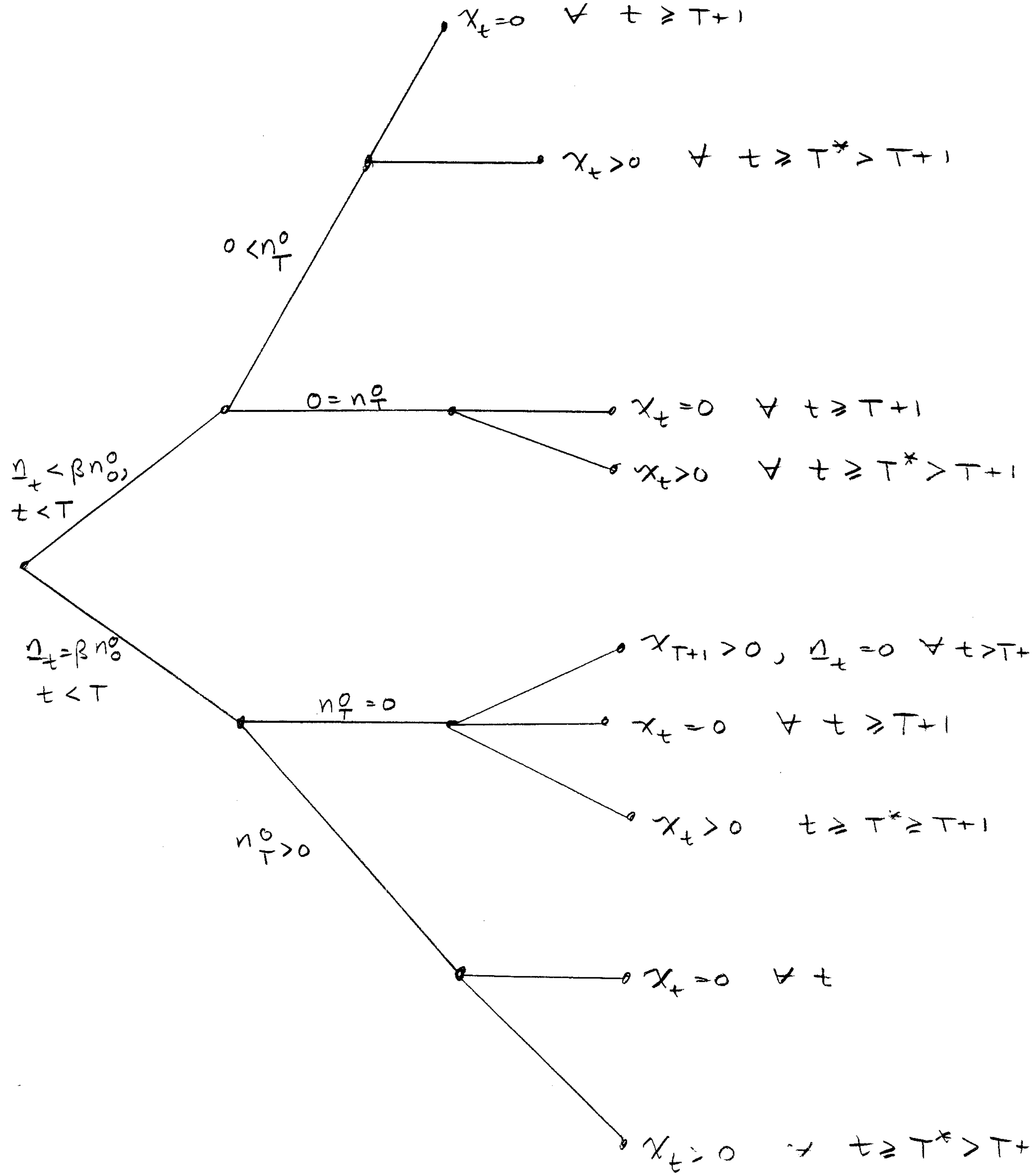
implying $n_0^0 = 0$. But given the assumption that if no firm planned to enter it would pay for some firm to do so, $n_0^0 = 0$ cannot occur in equilibrium, a contradiction. Thus $\underline{n}_T = \underline{m}_{T-1}$.

Observe that this same argument implies that if $\underline{n}_t < \underline{m}_{T-1}$ for $t \leq T-1$, $x_{T+1} = 0$ must occur; $\bar{n}_{T+1} > \bar{n}^*$ (implying $x_{T+1} > 0$) is thus ruled out for that case. Otherwise, the expected present value of profits given $\underline{\theta}$ is $\pi^0/(1-\gamma)$ for all t, in which case no firm knowing only θ^0 would seek to participate at $t=0$. Simiarly, if $\underline{n}_t = \underline{m}_{T-1}$, $x_{T+1} > 0$ implies $n_T^0 = 0$.

Finally, checking $n_T^0 \neq 0$ only if $t=0$ or T is straightforward. In all of the evolutions displayed, the value of participating for firms knowing only θ^0 is exactly $\pi^0/(1-\gamma)$ for $0 \leq t < T$, in which case such firms would be content to behave as hypothesized for those dates. For $t \geq T+1$, participation yields *at most* $\pi^0/(1-\gamma)$, so $n_t^0 = 0$ is maximal for those dates as well.

Figure 1 summarizes the nine possible cases, and is read as follows. $n_0^0 > 0$ is assumed. Along the uppermost branch not all firms knowing $\underline{\theta}$ participate until the refinement is invented; $\underline{n}_t < \beta n_0^0$. When this occurs, at T, all such firms participate and so do some knowing only θ^0; this latter group participates only at T. Subsequently, all firms knowing either $\underline{\theta}$ or $\bar{\theta}$ participate. The other branches are interpreted similarly and make use of the remarks made earlier; for example that $\underline{n}_t < \beta n_0^0$ implies $x_{T+1} = 0$.

<u>**Figure 1**</u>


$x_t = 0 \quad \forall \quad t \geq T+1$
$x_t > 0 \quad \forall \quad t \geq T^* > T+1$
$0 < n^o_T$
$0 = n^o_T$
$x_t = 0 \quad \forall \quad t \geq T+1$
$x_t > 0 \quad \forall \quad t \geq T^* > T+1$
$n_t < \beta n^o_0,$
$t < T$
$n_t = \beta n^o_0$
$t < T$
$x_{T+1} > 0, \; n_t = 0 \quad \forall \; t > T+1$
$n^o_T = 0$
$x_t = 0 \quad \forall \quad t \geq T+1$
$x_t > 0 \quad t \geq T^* \geq T+1$
$n^o_T > 0$
$x_t = 0 \quad \forall \quad t$
$x_t > 0 \quad \forall \quad t \geq T^* > T+1$

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Footnotes

[1]Obviously x_t^0 is constrained both above and below. These constraints are left implicit during the development, but certainly satisfied in the equilibrium.

[2]Constraints on $\underline{x}_t$ and $\bar{x}_t$ are treated in the same manner as those on x_t^0.

[3]Recall $\pi(p;\, \theta^0) = 0$.

[4]In the expressions to follow p_t is price at t and taking into account whether the refinement has occurred.

[5]i.e.

$$\frac{\pi^0}{1-\gamma} \equiv \pi(p^*, \underline{\theta}) + \left[r\frac{\pi(p^*,\ \bar{\theta})}{1-\gamma} + (1-r)\frac{\pi^0}{1-\gamma} \right]$$

$$> \gamma \left[r\frac{\pi(p^*, \bar{\theta})}{1-\gamma} + (1-r)\frac{\pi^0}{1-\gamma} \right]$$

$$\geq \gamma \left[\bar{r}^0 \frac{\pi(p, \bar{\theta})}{1-r} + (1-\bar{r}^0)\frac{\pi^0}{1-\gamma} \right]$$

if $r > \bar{r}^0$.

[6]For brevity, a complete description of $\lambda_t(\iota, q, \theta, |\cdot)$ is omitted. All information can, however, be obtained from the evolution displayed, together with $p_t = q(p_t, \theta)$, $p_t = D(Q_t)$ and $Q_t = \underline{n}_t q(p_t, \underline{\theta}) + \bar{n}_t q(p_t, \bar{\theta})$.

[7]The trivial case in which p_t is such that $\pi(p_t, \bar{\theta}) \leq \pi^0$ is ignored.

[8]If $\underline{n}_T$ is less than the number of firms knowing $\underline{\theta}$ at T, it will be necessary to verify that nonparticipants that know $\underline{\theta}$ would be willing to eschew participation at $t \geq T+1$. It will turn out that consideration of this possibility is not required because $\underline{n}_T$ will take on its maximum possible value; see below.