Rochester Center for

Economic Research

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Working Paper No. 192
August 1989

University of

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*University of Rochester. I would like to thank NSF for its support under grants SES 85-11136 and SES 88-09822, Peyton Young for his comments, and Koichi Tadenuma for his careful reading of the paper and for his calculation of the example of Section III-6. All errors are my own.

The Consistency Principle

I. *General introduction*

The objective of this paper is to describe the role played recently in the comparative study of solutions by a fundamental principle, which we will call the ***consistency principle.*** This principle unifies important developments in diverse areas ranging from abstract game-theoretic models to concrete taxation and apportionment problems.

Although most of the literature reviewed here appeared in the last five years, the principle itself is a very old one. In fact, it is likely at the root of a solution proposed in the Talmud more than 2000 years ago for the adjudication of conflicting claims.

A ***decision problem*** is given by a ***list of agents*** together with a ***set of alternatives*** available to them, and their ***preferences*** defined over this set. These preferences conflict. A ***solution*** is a rule that associates with every decision problem D ***in some admissible class*** $\mathscr{D}$ an outcome x in the feasible set of that problem; x is the ***solution outcome of D.*** Depending upon the context, the rule may be seen normatively, as a recommendation that an impartial arbitrator could make on how the problem should be solved, or, it may be intended as a description or as a prediction of the way the problem would be solved by the agents on their own.

Bargaining problems and ***bargaining solutions*** on the one hand, ***resource allocation problems*** and ***allocation rules*** on the other, are canonical examples illustrating these general notions.

Two main methodologies have been adopted in the study of solutions. One is ***axiomatic.*** Appealing properties of solutions, or ***axioms***, are formulated, and the existence of solutions satisfying all the axioms is investigated. Such studies often result in characterization theorems, that is, theorems identifying a particular solution as being the only one to satisfy a given list of axioms. According to this methodology, the primary concepts are the properties of solutions. These properties are used as building blocks in the construction of desirable solutions. Bargaining has been studied in this way to a very large extent.

The other methodology proceeds in the opposite direction, starting with solutions, which are taken as primary concepts. Solutions are chosen on the basis of their intuitive appeal, sometimes as formal representations of schemes actually used in practice, and it is asked whether each particular solution satisfies properties of interest. The study of resource allocation rules has as a whole been conducted in this manner.

However, the last few years have seen a considerable expansion of the axiomatic methodology. Significant progress has been made in the study of problems to which it had been traditionally applied, and a variety of new classes of models for which this methodology was discovered to be equally powerful have been identified. Our purpose here is to review the role played in these developments by the consistency principle.

Contrarily to most studies, in which a specific class of problems is first chosen and the restrictions forced on solutions by various combinations of axioms are determined, we will examine here a wide range of models and present results unified by the common application of a specific principle. Of course, other properties will be involved but consistency will always be the central one.

In Part II, we give a general statement of the consistency principle and we discuss several natural variants. In Part III, we examine a sequence of models: bargaining problems, games in coalitional form, bankruptcy and taxation problems, quasi–linear cost allocation problems, resource allocation problems in private good economies, and apportionment problems. We show how the consistency principle has been adapted for each of these models, how it has permitted the characterization of some existing solutions, and how it has led to the discovery and the characterization of new solutions. Part IV contains some concluding comments.

II. *The Consistency Principle*

We start with a very general statement of the consistency principle, and we discuss several useful variants.

1. *General concepts.*

Let $\mathcal{I} = \mathbb{N}$ be an infinite list of ***potential agents.***[1] Let $\mathcal{P}$ be the collection of all finite subsets of $\mathcal{I}$ with generic elements P, Q,... . For each group Q ϵ $\mathcal{P}$, there is an ***outcome space relative to Q***, X^Q, that is, a space from which the alternatives available to the group Q are taken. $\mathcal{D}^Q$ is the class of ***problems*** that the members of Q could conceivably face. Each element of $\mathcal{D}^Q$ is given by a ***feasible set***, a certain subset of X^Q satisfying some regularity assumptions, together with the preferences of the members of Q over this feasible set. How rich is the class of admissible problems is a modelling choice whose importance should be quite clear by the end of this article.

Given a group Q ϵ $\mathcal{P}$ and a problem D ϵ $\mathcal{D}^Q$, we would like to identify which alternative of D will be the compromise reached on their own by the agents in Q, or the recommendation made to them by some impartial arbitrator. However, instead of considering each problem separately, we will be

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[1]$\mathbb{N}$ is the set of natural numbers.

more ambitious and look for a general rule that would be applicable to all the problems that any admissible group could face.

Therefore, let $\mathscr{D} \equiv \bigcup_{Q \in \mathscr{P}} \mathscr{D}^Q$ and $X \equiv \bigcup_{Q \in \mathscr{P}} X^Q$.

Definition. A ***solution on*** $\mathscr{D}$ is a function $F: \mathscr{D} \to X$ that associates with every $Q \in \mathscr{P}$ and with every $D \in \mathscr{D}^Q$ an alternative $F(D)$ in the feasible set of $D \subset X^Q$. This alternative is called the ***solution outcome of D.***

2. *The Fundamental Definition.*

A solution satisfies consistency if whenever it recommends x as solution outcome for some admissible problem involving some group Q, then it recommends the restriction of x to any subgroup P as solution outcome of the subproblem, faced by this subgroup, obtained from the original problem by attributing to the members of the complementary subgroup $Q \backslash P$ their components of x.

To formally state the principle, we need first to clarify what is meant by a subproblem.

Definition. Given two groups $P, Q \in \mathscr{P}$ with $P \subset Q$, a problem $D \in \mathscr{D}^Q$, and finally an alternative x in the feasible set of D, the ***subproblem of D with respect to P and x*** is the problem comprising all the alternatives of D at which the members of the complementary subgroup $Q \backslash P$ achieve their components of x. We denote it $t_P^x(D)$.

Note that $t_P^x(D)$ may or may not satisfy all the properties that are required of the members of $\mathscr{D}^P$.

We are now ready to state the fundamental definition.

Fundamental Definition: A solution $F: \mathscr{D} \to X$ satisfies ***Consistency*** if for all groups $P, Q \in \mathscr{P}$ with $P \subset Q$ and for all problems $D \in \mathscr{D}^Q$, if x is the solution outcome of D, then the restriction of x to the subgroup P is the solution outcome of the subproblem of D with respect to P and x, provided this subproblem belongs to $\mathscr{D}^P$: for all P, Q ϵ $\mathscr{P}$ with P $\subset$ Q, for all D ϵ $\mathscr{D}^Q$, if $x = F(D)$ and $t_P^x(D) \in \mathscr{D}^P$, then $x_P = F(t_P^x(D))$.

Once the problem D ϵ $\mathscr{D}^Q$ has been solved at some point x by applying the solution F, how does the situation appear to the subgroup P? Assume that the members of the complementary subgroup Q\P have accepted the payoffs specified for them by F. From their viewpoint, all alternatives in D yielding them the payoffs $x_{Q\backslash P}$ are equivalent. Therefore, from the viewpoint of the members of P, this set of alternatives, if it constitutes a well–defined problem, really is the problem that has to be solved. Will solving it produce the payoffs x_P initially assigned to them? A solution for which the answer is always yes, as stated in the Fundamental Definition, has a sort of internal consistency that might greatly help in ensuring that agents respect agreements.

Consider also the not uncommon situation when the alternatives to be selected from are the result of contributions made by agents in a natural temporal sequence. Then, the agents who are done first might want to receive their payoffs and leave the scene. A consistent solution would prevent some of the remaining agents to want to renegotiate among themselves on the basis that now they really face a different problem.[2]

Note that in the above discussion we considered groups of "named" individuals. Two groups P and P′ may be composed of the same number of individuals with the same characteristics, and yet be treated differently by a solution. Although the assumption is made in most of the models reviewed here that all agents are fundamentally equal and, in particular, that identical

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[2]This motivation is due to Lensberg (1985).

agents should be treated identically, our formalism is chosen so as to accomodate the possibility of treating agents differently simply on the basis of who they are. This will provide us with useful flexibility. For example, in voting bodies, some voters may have more power than others (e.g. the Security Council of the United Nations). Similarly, in bankruptcy court, some claims may have higher priority than others.

3. *Variants of the Fundamental Definition.*

At various points in the preceding definitions other choices could have been made. We discuss next the nature of these choices.

(a) ***Single–valuedness of solutions.*** We require solutions to associate with every admissible problem a ***unique*** outcome ("***the*** solution outcome of D..."). Whether a solution is meant to offer predictions or recommendations, uniqueness of the solution outcome is of course greatly desirable. Fortunately, there are interesting classes of problems for which a large number of appealing single–valued solutions can be defined; then, it is natural to limit one's search to such solutions. The axiomatic theory of bargaining has developed with the almost universal requirement of single–valuedness for that reason. However, in many branches of economics and game theory, single–valuedness is virtually impossible to obtain or comes at a very high price. (For instance, most of the solutions discussed in economic theory, such as the Walrasian solution and the core, are multi–valued.) Domain restrictions occassionally exist that guarantee single–valuedness (gross substitutability guarantees single–valuedness of the Walrasian solution,) but they are often too strong to be of much use.

To permit multi–valued solutions, replace the statements "if $x = F(D)$" and "then $x_P = F(t_P^x(D))$" of the Fundamental Definition by "if $x \in F(D)$" and "then $x_P \in F(t_P^x(D))$" respectively.

(b) ***Number of potential agents.*** We have assumed the set of potential agents $\mathcal{I}$ to be countable infinite. In a number of applications, it is more natural to draw agents from a finite list. Alternatively, modelling the set of potential agents as a continuum may have mathematical advantages. On occasions, these alternative choices for $\mathcal{I}$ have significant implications for the theorems. Some of the results that we will present require that indeed there be a fair amount of flexibility in the specification of the class of admissible groups Q; for instance, one may have to have access to Q's of arbitrarily large cardinality. For others, a limited class of Q's suffices; in some cases, it in fact suffices to have access to Q's of cardinality 3.

To deal with these cases, write "$\mathcal{I} \subset \mathbb{N}$" or "$\mathcal{I} = \mathbb{R}$"[3] instead of "$\mathcal{I} = \mathbb{N}$".

(c) ***Restrictions on the subgroup.*** Starting from some group $Q \in \mathcal{P}$, and having solved at x some problem D that it faces, the Fundamental Definition asks us to investigate how the subproblems $t_p^x(D)$ faced by ***each of the subgroups P of Q*** would be solved. However, in some situations, it may be natural to limit one's attention to subgroups of small cardinality. In particular, when we are concerned with modelling non–cooperative behavior and the principle is meant to express the stability of a compromise under challenges by subgroups, it makes sense to require that only small subgroups can form since coordinated action may be difficult for large groups. In fact, the consistency principle is sometimes written with the restriction that only subgroups of cardinality 2 can form. Usually, but not always, excluding subgroups of cardinality greater than 2 weakens the axiom in some minor way, and many characterization proofs still go through, although with some extra work. It is

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[3]$\mathbb{R}$ is the set of real numbers.

also natural to exclude degenerate subgroups of cardinality 1. This weakening has the same sort of consequence.

The size of the subgroups is not the only relevant consideration however. The set of agents may be endowed with some additional structure, and the admissibility of a subgroup may be decided so as to reflect this structure. For instance, if the situation under study is intertemporal allocation, agents can be indexed by time; then, allowing only subgroups consisting of successive generations, perhaps with overlapping lifespans, is quite natural.

Alternatively, the set of agents may have a graph structure, representing channels of communication or some other relevant aspect of social organization, such as hierarchies or family structures, and only connected components of the graph may be considered admissible subgroups.

To allow for these various possibilities, let $\alpha: \mathscr{P} \to \mathscr{P}$ be a correspondence associating with every $Q \in \mathscr{P}$ a list $\alpha(Q)$ of admissible subgroups of Q. Then, adjust the Fundamental Definition by replacing "for all P, $Q \in \mathscr{P}$ with $P \subset Q$" by "for all P, $Q \in \mathscr{P}$ with $P \in \alpha(Q)$".

(d) ***Definition of the subproblem.*** Describing how the original problem appears to be subgroup P after the members of the complementary group Q\P have received their components of the outcome is not always straightforward. In some cases (the class of games in coalitional form is the most prominent example), several specifications make sense. The crucial point however is the dependence of the subproblem on the original problem ***and*** the compromise that is being evaluated. The notion of a subproblem discussed here should therefore be contrasted with notions that depend only on the original problem. (For instance, a subgame of a game in coalitional form is simply the restriction of the vector giving the worths of all the coalitions in the original game to the coordinate subspace corresponding to the list of coalitions that are subsets of a particular subgroup of the set of players.)

The subproblem could also be made to depend on the solution itself. Hart and Mas–Colell (1989) propose such a notion and Suematsu (1988) explores a general formulation in that spirit. This specification has proved useful too, and in fact has permitted the characterization of other solutions. But, in order to preserve the unity of this review, we will not pursue it here.

(e) ***Relation between the solution outcomes of the original problem and of its subproblems.*** The Fundamental Definition requires ***coincidence*** of the restriction to the subgroup of the initial compromise with the solution outcome of the subproblem faced by this subgroup. More generally, we could request that a certain relation between the two outcomes holds. The weaker requirement of Pareto–domination of one outcome by the other has been convenient in bargaining theory.

To formalize this choice, replace "then $x_P = F(t_P^x(D))$" of the Fundamental Definition by "then $x_P \leqq F(t_P^x(D))$".[4]

(f) ***Admissibility of the subproblem.*** The Fundamental Definition applies only if the subproblem is in the admissible domain. A stronger version can be obtained by ***requiring*** the subproblem to be in the admissible domain. Although in some applications, the subproblem is automatically admissible, for some others this is far from being the case.

To so strengthen the Fundamental Definition, replace "if $x = F(D)$ and $t_P^x(D) \in D^P$, then $x_P = F(t_P^x(D))$" by "if $x = F(D)$ then $t_P^x(D) \in D^P$ and $x_P = F(t_P^x(D))$".

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[4]Vector inequalities $x \geqq y$, $x \geq y$, $x > y$.

III. *Applications.*

Together, the models described below cover a very broad range of problems commonly studied. At one extreme is our first canonical example, the class of bargaining problems; no information about the physical features of the alternatives among which a choice has to be made is retained in the specification of a bargaining problem. At the other extreme is our second canonical example, the class of resource allocation problems. There, the set of alternatives is endowed with vector space and topological structures.

It is useful to distinguish between models on the basis of their informational content. Indeed, how much information is available is relevant to the way conflicts are resolved in practice and has clear normative implications. The first point is strongly supported by experimental work. Yaari and Bar–Hillel (1984) confronted a group of subjects with several problems involving different sets of physical alternatives having the same representation in utility space and found systematic differences in the way the problems were solved, depending on the interpretation given to these alternatives. The position that only utility information is relevant is termed "Welfarism" by Sen (1979). Welfarism, intended as a descriptive theory, is in clear violation of these results. The second point, that from a normative perspective, welfarism is inadequate as well, is forcefully made by Roemer (1986a), who argues in favor of a "resourcist" position, calling for a precise description of the concrete features of the alternatives available.

Of course, the acceptance of these views does not imply rejection of all abstract formulations. Indeed, the advantage of such formulations is their wide applicability. For instance, the Shapley value, a solution concept originally developed for coalitional form games, has been very successfully applied to a variety of concrete problems, from the computation of power indices in voting bodies to cost allocation and distribution of goods in economics. On the other hand, when extra information is available, it can be used to enrich the class of admissible solutions, as illustrated by the example of exchange economies; there, the set of available physical choices has a very special structure (it is a convex, compact subset of a vector space); preferences can be meaningfully required to satisfy properties that would not be well defined otherwise (such as monotonicity, smoothness, convexity); finally, allocation rules can be constructed that make use of this special structure (an example is the Walrasian solution).

We will consider the following classes of problems:

(1) Bargaining problems,

(2) Games in coalitional form,

(3) Bankruptcy and taxation problems,

(4) Quasi–linear cost allocation problems,

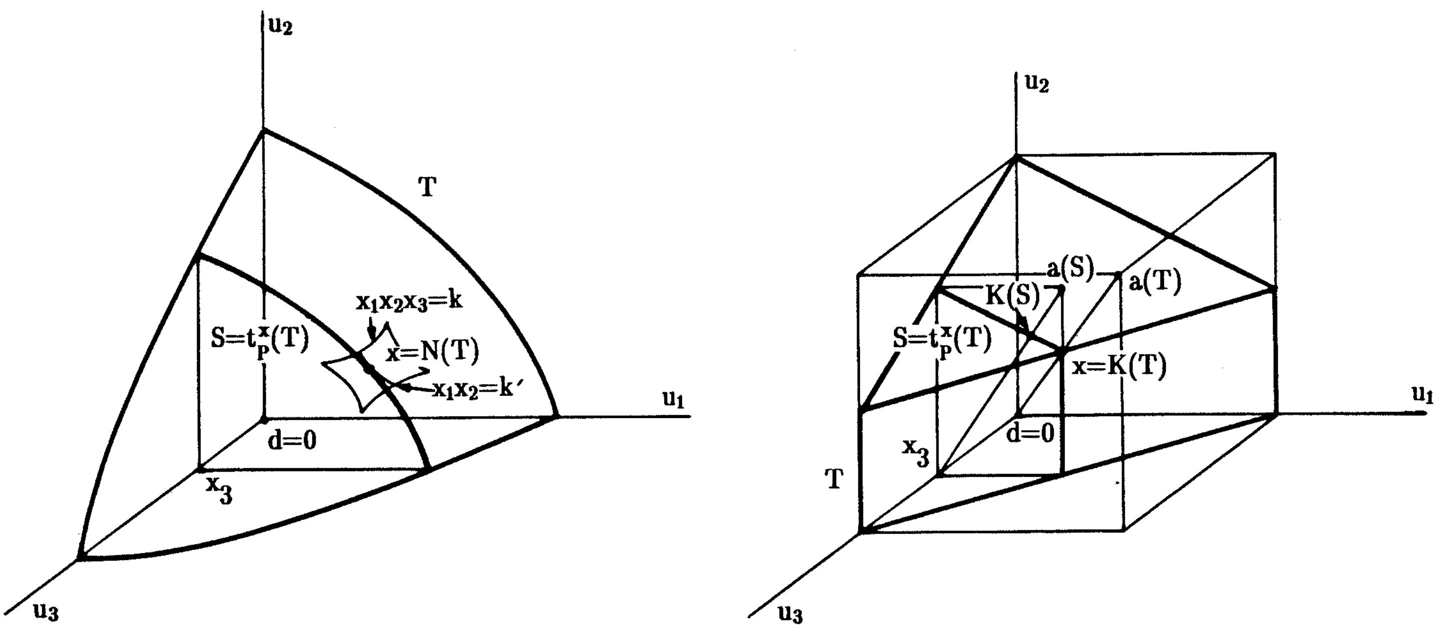
(5) Resource allocation problems in exchange economies, and

(6) Apportionment problems.

For each of these classes, we restate the Fundamental Definition when applied to the class and we present the main results that have been based on the principle.

1. ***Bargaining problems.***

Examples. A typical bargaining problem, involving a group of three agents Q = {1,2,3}, is represented in Figure 1a: there is a feasible set T, which is a subset of the three–dimensional utility space, and a disagreement point d = 0. The points of T represent choices available to the agents. What compromise will they reach? Nash (1950) predicted the point x maximizing over T the product of utility gains from d. Assuming agent 3 to be content with x_3, let us consider the subset of T consisting of all points where his utility is x_3. S represents the options open to agents 1 and 2 once agent 3 has received x_3. Now, we note that (x_1,x_2) is the maximizer of the product of their utility gains in S, an observation illustrating the fact that the Nash solution satisfies consistency.



The Nash solution satisfies consistency

(a)

The Kalai–Smorodinsky solution does not satisfy consistency

(b)

Consistency in bargaining theory

Figure 1

Another popular solution was introduced by Kalai and Smorodinsky (1975): these authors proposed to select the maximal feasible point proportional to the ***ideal point***, the point whose i^{th} coordinate is equal to the maximal feasible utility for agent i. Figure 1b shows that this solution does not satisfy consistency. Indeed, calling x the Kalai–Smorodinsky solution outcome of the three–person problem T, we note that the Kalai–Smorodinsky solution outcome of the subproblem of T relative to $P = \{1,2\}$ and x is not x_P!

General definitions. A ***bargaining problem*** is a pair $(S,d) \in 2^{\mathbb{R}^Q} \times \mathbb{R}^Q$: there is a group of agents Q who can attain any of the points of the ***feasible set*** S, a subset of their utility space $\mathbb{R}^Q$, by unanimously agreeing on it. If they fail to reach an agreement, they get d, the ***disagreement point.***. We assume, as is standard, that S is convex and compact, and that there exists at least one point of S that strictly dominates d. We also require S to be d–comprehensive, (if a point is feasible, then any point that it dominates and dominates d is also feasible). This is to guarantee that the solutions that we will want to consider always select outcomes that are at least weakly Pareto–optimal. Finally, and to simplify the exposition, we assume $d = 0$ and we write S instead of (S,0). Let Σ^Q be the class of problems satisfying all of the above assumptions, $\Sigma \equiv \bigcup_{Q \in \mathscr{P}} \Sigma^Q$, and $X \equiv \bigcup_{Q \in \mathscr{P}} \mathbb{R}^Q$. A ***solution*** is a function that associates with every $Q \in \mathscr{P}$ and every $S \in \Sigma^Q$ a unique point of S.

Definition. The solution $F\colon \Sigma \to X$ satisfies ***Consistency*** if for all $P, Q \in \mathscr{P}$ with $P \subset Q$, for all $S \in \Sigma^P$ and $T \in \Sigma^Q$, if $S = t_P^x(T) \equiv \{x' \in \mathbb{R}^P \mid \exists y \in T$ with $y_{Q\backslash P} = x_{Q\backslash P}$ and $y_P = x'\}$ where $x = F(T)$, then $x_P = F(S)$.

Geometrically, $t_P^x(T)$ is simply the section of T through x parallel to $\mathbb{R}^P$.

Among the bargaining solutions commonly studied, only two satisfy Consistency. They are the ***Nash solution*** N (N(S) is the maximizer of Πx_i for $x \in S$), and the ***Lexicographic Egalitarian solution*** L (L(S) is the point of S that is maximal in the lexicographic order)[5]. Now consider a list $\{f_i | i \in \mathbb{N}\}$, where f_i: $\mathbb{R}_+ \to \mathbb{R}$ is strictly monotone, continuous, and such that for each Q, the function f^Q: $\mathbb{R}_+^Q \to \mathbb{R}$ defined by $f^Q(x) \equiv \sum_Q f_i(x_i)$ is strictly quasi-concave. Then, given $Q \in \mathscr{P}$ and $S \in \Sigma^Q$, let $F(S) \equiv \operatorname{argmax}\{f^Q(x) | x \in S\}$: any such ***separable additive*** solution F also satisfies Consistency.[6] The ***Egalitarian solution*** E (Kalai, 1977: E(S) is the maximal point of S of equal coordinates) does not satisfy Consistency but it satisfies the slightly weaker condition obtained by requiring "$x_P \leqq F(t_P^x(T))$" instead of "$x_P = F(t_P^x(T))$". This requirement will be called ***Weak Consistency.***

In bargaining theory, Consistency was first used by Harsanyi (1959).[7] Harsanyi felt that the Nash solution was the appropriate solution for two–person problems and he asked whether one could deduce in some natural way how n–person problems should be solved. He showed that if a solution is consistent and coincides with the Nash solution for two–person problems, then it coincides with the Nash solution for all cardinalities. Lensberg (1985), who recently rediscovered the condition, is the author of the most general theorems involving it. In particular, he showed that Harsanyi's hypothesis that

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[5]Given $x, x' \in \mathbb{R}^Q$, let $\tilde{x}, \tilde{x}'$ be obtained from x and x′ by rewriting their coordinates in increasing order. We say that x is lexicographically greater than x′ if either $\tilde{x}_1 > \tilde{x}'_1$ or $[\tilde{x}_1 = \tilde{x}'_1$ and $\tilde{x}_2 > \tilde{x}'_2]$ ..., or for some k, $[\tilde{x}_i = \tilde{x}'_i$ for all $i \leq k$ and $\tilde{x}_{k+1} > \tilde{x}'_{k+1}]$.

[6]The Nash solution is the member of this family obtained by choosing $f_i(x_i) = \log x_i$ for all i and for all $x_i \in \mathbb{R}_+$.

[7]Under the name of the Bilateral Equilibrium condition.

two–person problems be solved according to the Nash solution could be replaced by elementary axioms, as detailed below.

Results. To present the results we need to formulate a few other requirements on solutions. ***Pareto–optimality***: the solution does not select a semi–strictly dominated outcome; ***Weak Pareto–optimality***: the solution does not select a strictly dominated outcome; ***Anonymity***: the solution is invariant under exchanges of the names of the agents; ***Scale invariance***: the solution is invariant under positive linear rescaling, independent agent by agent, of the utilities; ***Individual Monotonicity***: an expansion of the feasible set along the ith axis benefits agent i; ***Continuity***: small changes of problems do not produce large changes in solution outcomes; ***Population Monotonicity***: the arrival of additional agents unaccompanied by an expansion of opportunities is costly to all agents initially present.[8]

Theorem 1 (Lensberg 1988): The Nash solution is the only solution satisfying Pareto–optimality, Anonymity, Scale Invariance, and Consistency.

Theorem 2 (Lensberg 1985): The Lexicographic Egalitarian solution is the only solution satisfying Pareto–optimality, Anonymity, Individual Monotonicity, and Consistency.

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[8]***Pareto–optimality***: if $x \geq F(S)$, then $x \notin S$; ***Weak Pareto–optimality***: if $x > F(S)$, then $x \notin S$; ***Anonymity***: $F(\pi(S)) = \pi(F(S))$, where $\pi: Q \to Q'$ with $|Q| = |Q'|$ is any one–to–one function exchanging the names of the agents; ***Scale invariance***: $F(\lambda(S)) = \lambda(F(S))$, where $\lambda: \mathbb{R}^Q \to \mathbb{R}^Q$ is any positive linear rescaling, independent agent by agent, of the utilities; ***Individual Monotonicity***: if $S' \supset S$ and $S'_{Q\backslash i} = S_{Q\backslash i}$, then $F_i(S') \geqq F_i(S)$ (notation: S_P is the projection of S on $\mathbb{R}^P$); ***Continuity***: if $\{S^\nu\}$ is a sequence of elements of Σ^Q such that $S^\nu \to S \in \Sigma^Q$, then $F(S^\nu) \to F(S)$; ***Population Monotonicity***: if $P \subset Q$, $S \in \Sigma^P$, $T \in \Sigma^Q$ and $T_P = S$, then $F_P(T) \leqq F(S)$.

Theorem 3 (Thomson 1984): The Egalitarian solution is the only solution satisfying Weak Pareto–optimality, Anonymity, Continuity, Population Monotonicity, and Weak Consistency.

Theorem 4 (Lensberg 1987): The separable additive solutions are the only solutions satisfying Pareto–optimality, Consistency, and Continuity.[9]

2. *Games in coalitional form.*

Example. Consider a group of three differently skilled agents. The productivity of each subgroup S depends on the complementarities between the skills of the agents composing it, and is measured by a single number v_S. Let v be the list of all these v_S given in Figure 2. We would like to reward agents as a function of what they can contribute to the various subgroups. A well–known method is the core: pick a payoff vector $x = (x_1,x_2,x_3)$ that cannot be "improved upon" by any subgroup: for all S, $v_S \leqq \sum_S x_i$. The vector $x = (10,10,30)$ is in the core. Now, assuming agent 3 to have accepted $x_3 = 30$, how does the situation appear to the remaining agents $P = \{1,2\}$. Agent 1 on his own can achieve 0; by cooperating with agent 3 and paying him x_3, he can achieve $0 = v_{\{1,3\}} - x_3 = 30 - 30$. In either case, he obtains 0. A similar computation for agent 2 involves comparing 0 and 10 ($= v_{\{2,3\}} - x_3 = 40 - 30$), for a maximum of 10. Finally, together, agents 1 and 2 can achieve $v_Q - x_3 = 50 - 30 = 20$. Is (x_1,x_2) in the core of the 2–person game so defined? The answer is yes. This illustrates that the core is consistent.

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[9]Refinements of Theorem 1 appear in Lensberg and Thomson (1988) and Thomson (1985b). A version pertaining to a domain of non–convex problems is given by Foster and Vohra (1988). A result related to Theorem 4 is given by Young (1988b).

On the other hand, the equally well-known solution due to Shapley (1953)[10] is not consistent, as also illustrated in Figure 2. Shapley recommends the payoff vector (70/6, 100/6, 130/6) for v, and (55/6, 115/6) for the resulting subgame.

Game	The core of v contains x =	Subgame of v with respect to {1,2} and x	The core of sub–game contains	The Shapley value of v is y =	Subgame of v with respect to {1,2} and y	The Shapley value of subproblem is game
$v_1=0$	10	0	10	70/6	50/6	55/6
$v_2=0$	10	10	10	100/6	110/6	115/6
$v_3=0$	30			130/6		
$v_{12}=20$		20			170/6	
$v_{13}=30$						
$v_{23}=40$						
$v_{123}=50$						

Consistency for coalitional form games
Figure 2[11]

General definitions. A ***game in coalitional form*** is a vector $v \in \mathbb{R}^{2^{|Q|}-1}$: there is a group Q of agents whose members can gather in ***coalitions***.[12] What each coalition can achieve on its own is measured by a number, its ***worth***, which is given as one of the coordinates of v. Restrictions may be imposed on v making, for instance, the game monotonic (if $S \supset T$, then $v_S \geqq v_T$), or

[10]The solution is defined below. As we noted earlier, this solution does satisfy consistency for a definition of the subgame involving the solution itself (Hart and MasColell, 1989).

[11]In this Figure, we write v_1 for $v_{\{1\}}$, v_{12} for $v_{\{1,2\}}$ and so on.

[12]A coalition is a ***non–empty*** subset of Q.

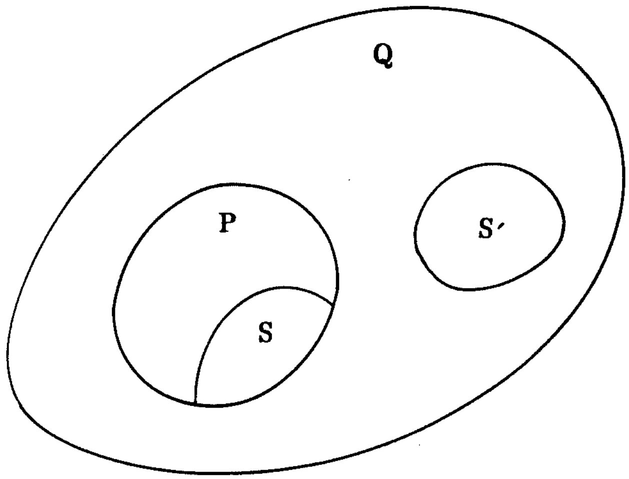
super–additive (the worth of a coalition is greater than the sum of the worths of the coalitions comprising a partition, no matter what that partition is). Let $\mathcal{G}^Q$ be a class of admissible games for the group Q, $\mathcal{G} \equiv \bigcup_{Q \in \mathcal{P}} \mathcal{G}^Q$, and $X \equiv \bigcup_{Q \in \mathcal{P}} \mathbb{R}^Q$. A ***solution*** is a correspondence that associates with every Q and every $v \in \mathcal{G}^Q$ a non–empty set of vectors x in $\mathbb{R}^Q$ such that $\sum_Q x_i \leqq v_Q$. The ith coordinate of such a vector represents one of the possible payments to agent i for being involved in the game.

Definition: The solution $F\colon \mathcal{G} \to X$ satisfies ***Consistency if*** for all $P, Q \in \mathcal{P}$ with $P \subset Q$, for all $v \in \mathcal{G}^Q$, and for all $x \in F(v)$, $t_P^x(v) \in \mathcal{G}^P$ and $x_P \in F(t_P^x(v))$ where $t_P^x(v)$ is the game $w \in \mathcal{G}^P$ defined by

$$w_P = v_Q - \sum_{Q \setminus P} x_i$$

$$w_S = \max\{v_{S \cup S'} - \sum_{i \in S'} x_i \mid S' \subset Q \setminus P\} \text{ otherwise.}$$

The definition of the subgame is illustrated in Figure 3. Given the payoff vector $x \in \mathbb{R}^Q$, the worth of the coalition S in the subgame of v relative to P and x is computed under the assumption that S can obtain the cooperation of any subgroup S' of Q not overlapping with P, provided each of the members of S' receives his payoff of the proposed compromise x. After these payments are made, what remains for S is the difference $v_{S \cup S'} - \sum_{S'} x_i$. Maximizing behavior on the part of S involves finding $S' \subset Q \setminus P$ for which this difference is maximal. (Note that the worths of two distinct coalitions S_1 and S_2 may be established by assuming cooperation with two overlapping subgroups S_1' and S_2' of $Q \setminus P$.)



Defining the subgame of $v \in \mathcal{G}^Q$ relative to $P \subset Q$ and $x \in \mathbb{R}^Q$.

Figure 3

Apart from the core, another well–known solution is the ***prenucleolus***, which associates with every $v \in \mathcal{G}^Q$ the vector $x \in \mathbb{R}^Q$ with $\sum_Q x_i = v_Q$ whose associated vector of "excesses" $e(x) \in \mathbb{R}^{2^{|Q|}-1}$, where for each $S \subset Q$, $e_S(x) \equiv v_S - \sum_S x_i$ is lexicographically minimal among all such vectors; for the ***Shapley–value***, $x_i = \sum_{\substack{S \subset Q \\ S \ni i}} k_S(v_S - v_{S \setminus i})$, where $k_S \equiv \frac{(|S|-1)!(|Q|-|S|)!}{|Q|!}$; for the ***prekernel***, x is such that for all $i,j \in Q$, $\max\{e_S(x) \mid i \in S, j \notin S\} = \max\{e_S(x) \mid j \in S, i \notin S\}$.

Consistency first appears in the context of games in coalitional form in Davis and Maschler (1965), Maschler and Peleg (1967), Maschler, Peleg and

Shapley (1972) and Aumann and Dréze (1974). It is satisfied by the core, the prenucleolus and the prekernel.[13]

Results. We will need the following properties of solutions. ***Pareto–Optimality:*** payoffs add up to the worth of the grand coalition; ***Individual Rationality:*** each agent is awarded at least what he can achieve on his own; ***Super–Additivity:*** the set of payoff vectors of the sum of two games contains all the sums of payoff vectors of each of the games; ***Invariance:*** the multiplication by a common positive constant of all the utilities and the addition of arbitrary constants to each of the utilities affect the payoffs in the same way; ***Symmetry:*** if two agents contribute the same amount to all coalitions, they are awarded the same payoff; ***Anonymity:*** the solution is invariant under exchanges of the names of the agents; ***Converse Consistency:*** if a Pareto–optimal payoff vector is such that its restriction to any two person subset of the set of agents is a solution outcome of the corresponding subgame, then it is a solution outcome of the original game.[14]

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[13]Other solution concepts satisfy Consistency: the pseudokernel, the pseudonucleolus, the pseudobargaining set.

[14]***Pareto–Optimality:*** $\sum_Q F_i(v) = v_Q$; ***Individual Rationality:*** for all $i \in Q$, $F_i(v) \geqq v_i$; ***Super–Additivity:*** $F(v_1+v_2) \supset F(v_1) + F(v_2)$; ***Invariance:*** if there exists $\alpha \in \mathbb{R}_{++}$ and $\beta \in \mathbb{R}^Q$ such that for all $S \subset Q$, $w_S = \alpha v_S + \sum_S \beta_i$, then $F(w) = \alpha F(v) + \beta$; ***Symmetry:*** $v_{S\cup\{i\}} = v_{S\cup\{j\}}$ for all $S \subset Q\backslash\{i,j\}$ implies $F_i(v) = F_j(v)$; ***Anonymity:*** for all $v \in \mathcal{G}^Q$, $v' \in \mathcal{G}^{Q'}$ with $|Q| = |Q'|$, if there is a one–to–one function π: $Q \to Q'$ such that $v_S = v'_{\{\pi(i)|i\epsilon S\}}$ for all $S \subset Q$, then $F_i(v) = F_{\pi(i)}(v')$; ***Converse Consistency:*** if $\sum_Q x_i = v_Q$ and for all $P \subset Q$ with $|P| = 2$, $x_P \in F(t_P^x(v))$, then $x \in F(v)$.

***Theorem* 5** (Sobolev 1975). The prenucleolus is the only single–valued solution satisfying Anonymity, Invariance, and Consistency.

***Theorem* 6** (Peleg 1986a). Let $\mathscr{G}_0$ be the class of games for which the core is non–empty. The core is the only solution on $\mathscr{G}_0$ satisfying Individual Rationality, Super–Additivity, and Consistency.

***Theorem* 7** (Peleg 1986a). The prekernel is the only solution satisfying Pareto–Optimality, Invariance, Symmetry, Consistency, and Converse Consistency.[15]

We noted earlier that the Shapley–value does not satisfy Consistency. However, as shown by Hart and Mas–Colell (1988, 1989), it satisfies Consistency with respect to a subgame that depends on the solution itself. The Shapley–value can in fact be characterized with the help of this condition.

Next, we consider a richer formulation where what each coalition can achieve is given as a ***subset*** v_S of the utility space $\mathbb{R}^S$ pertaining to that coalition. Each v_S, which is interpreted as the set of utility vectors achievable by the coalition S on its own, is required to satisfy certain properties which we will not list. These games will be called NTU (non–transferable utility) games, as opposed to the TU (transferable utility) games described earlier. Let $\mathscr{H}^Q$ be the class of admissible games involving the group Q, $\mathscr{H} \equiv \bigcup_{Q \in \mathscr{P}} \mathscr{H}^Q$, and $X \equiv \bigcup_{Q \in \mathscr{P}} \mathbb{R}^Q$. A solution here associates with every $Q \in \mathscr{P}$ and every $v \in \mathscr{H}^Q$ a non–empty subset of v_Q.

[15]Results related to Theorem 5 appear in Suematsu (1988) and to Theorem 6 in Tadenuma (1989). A recent result involving consistency is Dutta (1988). A characterization of the core of market games along the lines of Theorem 7 appears in Peleg (1986b).

Definition. The solution $F: \mathscr{H} \to X$ satisfies ***Consistency*** if for all P, Q $\epsilon\ \mathscr{P}$ with $P \subset Q$, for all $v \in \mathscr{H}^Q$, and for all $x \in F(v)$, $t_P^x(v) \in \mathscr{H}^P$ and $x_P \in F(t_P^x(v))$, where $t_P^x(v)$ is the game $w \in \mathscr{H}^P$ defined by

$$w_P = \{y \in \mathbb{R}^P \mid (y, x_{Q\setminus P}) \in v_Q\}$$
$$w_S = \bigcup_{S' \subset Q\setminus P} \{y \in \mathbb{R}^S \mid (y, x_{S'}) \in v_{S\cup S'}\} \text{ if } S \subset P,\ S \neq P.$$

Theorem 8 (Peleg 1985). Let $\mathscr{H}_0$ be the class of NTU games with a non–empty core. The core is the only solution on $\mathscr{H}_0$ satisfying Individual Rationality, Consistency and Converse Consistency.[16]

Extensions of these results to the case of games with coalition structures appear in Peleg (1986a) and Tadenuma (1989).

3. *Bankruptcy and taxation problems.*

Examples. We start with two problems discussed in the Talmud.

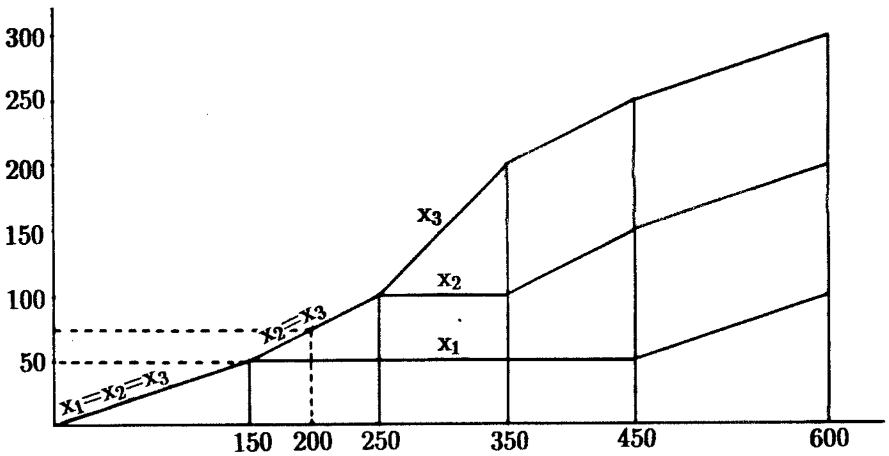
The ***contested garment*** problem: two men disagree over the ownership of a garment, worth 100. The first man claims half of it (50) and the other claims it all (100). Assuming both claims to be made in good faith, how should the worth of the garment be divided among the two men? The Talmud recommends 25 to the first one and 75 to the second.

The ***estate problem***: a man has three wives whose marriage contracts specify that in case of his death they should receive 100, 200 and 300 respectively. The man dies and his estate is found to be worth only 100. How should that amount be divided among the wives? The Talmud recommends equal division. If the estate is worth 300, the Talmud recommends proportional division, but if it is worth 200, it recommends (50, 75, 75)!

[16]We omit the precise formulation of Converse Consistency, which is patterned after the condition of that name that we stated earlier for TU games.

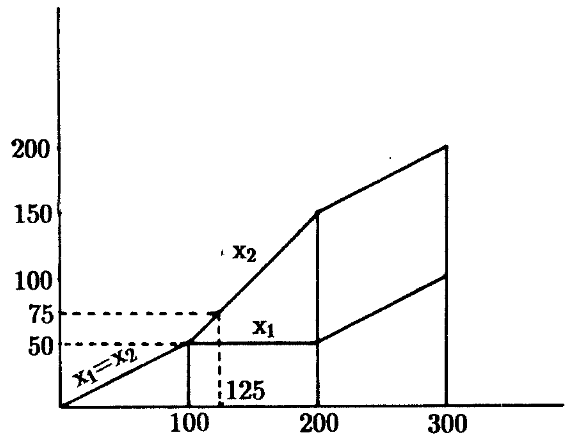
To clarify the mystery posed by the numbers proposed as solutions to these two problems we should first of all find a natural formula that would generate them. Consider the following method, described for the n–person case as a function of the amount available for distribution, and illustrated in Figure 4 for the two Talmudic problems: The first units are divided equally until each claimant has received an amount equal to half of the smallest claim; then, the claimant with the smallest claim does not receive anything for a while; instead, equal division of any additional unit is applied among all others until each of them has received an amount equal to half of the second smallest claim ... the algorithm proceeds in this way until a value of the estate equal to $\Sigma c_i/2$; then, each claimant has received half of his claim; for values of the estate greater than $\Sigma c_i/2$, payments are computed in a symmetric way by successively equating incremental losses instead of gains. The reader can check that this method applied to the two Talmudic problems does yield the numbers given in the Talmud.

Now, for an estate of 200 in the 3–person problem, the amounts awarded to claimants 1 and 2 are 50 and 75 respectively, for a total of 125. Applying the 2–person method to an estate of 125 claimed by the first two claimants returns the same numbers 50 and 75! In fact, given any value of the estate, if x denotes the solution to the 3–person problem, applying the 2–person method to any pair $\{i,j\}$ for an estate of $x_i + x_j$ yields the settlement (x_i,x_j). The Talmudic solution is consistent!



Claims are $(c_1,c_2,c_3) = (100,200,300)$

(a)



Claims are $(c_1,c_2) = (100,200)$

(b)

The Talmudic solution to the estate problem.
The value of the estate is measured horizontally.
The payments to the claimants are measured vertically.

Figure 4

General Definitions. A ***bankruptcy problem*** is a pair $(c,E) \in \mathbb{R}^Q_+ \times \mathbb{R}_+$ with $\sum_Q c_i \geqq E$: Q is a group of claimants on the net worth E of a bankrupt firm, c_i being the claim of claimant i. Bankruptcy problems have been considered by O'Neill (1982), Aumann and Maschler (1985), and Chun (1988).

A different interpretation of pairs in $\mathbb{R}^Q_+ \times \mathbb{R}_+$ gives the class of tax collection problems: a ***tax collection problem*** is a pair $(w,T) \in \mathbb{R}^Q_+ \times \mathbb{R}_+$ with $T \leqq \sum_Q w_i$: Q is a group of ***taxpayers*** with ***incomes*** given by the coordinates of w, and who among themselves must cover the ***cost*** T of a certain project. Let $\mathscr{T}^Q$ be the class of these problems, $\mathscr{T} \equiv \bigcup_{Q \in \mathscr{P}} \mathscr{T}^Q$, and $X \equiv \bigcup_{Q \in \mathscr{P}} \mathbb{R}^Q$. A ***solution*** is a function associating with every $Q \in \mathscr{P}$ and every $(w,T) \in \mathscr{T}^Q$, a vector in $\mathbb{R}^Q$ with coordinates adding up to T. Taxation problems have been extensively investigated by Young (1986, 1987a, 1987b, 1988a) and we will focus on that model.

Interesting examples of solutions are: the ***proportional solution*** which gives the vector of taxes x as λw, λ being adjusted, as in the next three examples, so that $\Sigma x_i = T$; the ***leveling tax***, where $x_i = \max\{w_i - 1/\lambda, 0\}$; ***Stuart's solution***, where $x_i = \max\{0, w_i - w_i^{1-\lambda}\}$; ***Cassel's solution***, where $x_i = w_i^2/(w_i + 1/\lambda)$.

Taxpayers' incomes	Proportional taxation applied to		Rank taxation applied to	
w	(w,30)	$(w_1,w_2,15)$	(w,30)	$(w_1,w_3,20)$
10	5	5	5	10/3
20	10	10	10	
30	15		15	40/3

Proportional taxation is consistent but rank taxation is not

Figure 5

Figure 5 illustrates the fact that proportional taxation is consistent whereas the following ***rank taxation*** method[17] is not: order the taxpayers in increasing order of incomes. Then assess them proportionately to their positions (for example, agent of rank 5 is assessed 5/3 times what agent of rank 3 is assessed).

[17] I owe this example to P. Young.

Consider now the following class of solutions. Let $f:\mathbb{R}_+ \times [a,b] \to \mathbb{R}_+$, where $[a,b] \subset [-\infty,+\infty]$, be continuous, weakly monotonic in its second argument and such that $f(w_i,a) = 0$ and $f(w_i,b) = w_i$ for all w_i. Then, given $(w,T) \in \mathcal{T}$, let $x = F(w,T)$ if for some λ, $x_i = f(w_i,\lambda)$ for all i and $\Sigma x_i = T$. Young (1986) calls these methods ***parametric***. It is straightforward to check that they are all consistent:

Definition. The solution $F:\mathcal{T} \to X$ satisfies ***Consistency*** if for all P, Q $\in \mathcal{P}$ with $P \subset Q$, and for all $(w,T) \in \mathcal{T}^Q$, if $x = F(w,T)$, then $x_P = F(t_P^x(w,T))$, where $t_P^x(w,T) = (w_P, \underset{P}{\Sigma} x_i)$.

Results. We will consider the following additional requirements: ***Symmetry***: taxpayers with identical incomes are assessed the same taxes; ***Continuity***: small changes in the parameters of the problem do not produce large changes in taxes; ***Homogeneity***: if incomes and aggregate tax are multiplied by the same positive number, so is the vector of taxes; ***Progressivity***: taxpayers with greater incomes pay relatively greater taxes; and ***Decomposability***: taxes can be assessed indifferently at one time or in installments.[18]

Theorem 9 (Young 1987b): The parametric solutions are the only solutions satisfying Symmetry, Continuity and Consistency.

Within the class of parametric solutions, a narrow subclass of great interest can be identified.

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[18]***Symmetry***: $w_i = w_j$ implies $F_i(w,T) = F_j(w,T)$; ***Continuity***: $(w^\nu,T^\nu) \to (w,T)$ implies $F(w^\nu,T^\nu) \to F(w,T)$; ***Homogeneity***: $F(\alpha w,\alpha T) = \alpha F(T,w)$ for all $\alpha > 0$; ***Progressivity***: $w_i > w_j > 0$ implies $F_i(w,T)/w_i > F_j(w,T)/w_j$; ***Decomposability***: $F(w,T+T') = F(w,T) + F(w-F(w,T),T')$.

Theorem 10 (Young 1986): A parametric solution satisfies Progressivity, Homogeneity, and Decomposability if and only if it can be represented in one of the following ways

$$f(w_i,\lambda) = \lambda w_i \qquad 0 \leqq \lambda \leqq 1 \qquad \text{or}$$

$$f_p(w_i,\lambda) = w_i - w_i/(1+\lambda w_i^p)^{1/p} \quad 0 \leqq \lambda \leqq \infty, \qquad p > 0 \qquad \text{or}$$

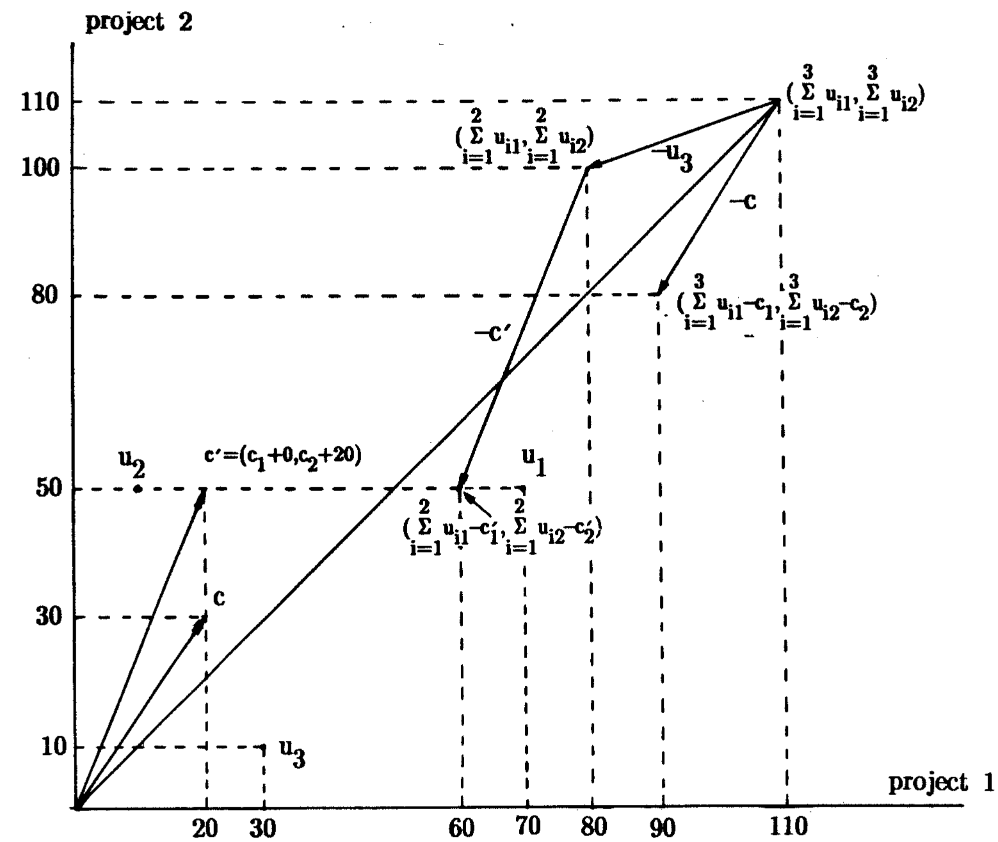
$$f_\infty(w_i,\lambda) = \max\{0,w_i-1/\lambda\} \qquad 0 \leqq \lambda \leqq \infty.$$

Other interesting subclasses of the class of consistent solutions are identified by Young (1987a, 1988a).

A family of problems closely related to taxation problems is the class of surplus–sharing problems studied by Moulin (1985a). Such a problem is a pair $(w,s) \in \mathbb{R}_+^Q \times \mathbb{R}_+$ where w_i is the investment in a joint venture made by agent $i \in Q$ and $s > 0$ is the surplus generated by this venture. Moulin uses Consistency together with some other natural conditions to characterize one–parameter families of surplus–sharing methods that generalize both equal sharing and proportional sharing.

(4) *Quasi–linear cost allocation problems.*

Example. Three agents have the choice between two projects, a_1 and a_2, costing $(c_1,c_2) = (20,30)$. The benefits they derive from these projects are $u_1 = (70,50)$, $u_2 = (10,50)$, and $u_3 = (30,10)$. Which project should be selected and how should its cost be allocated? Consider the method consisting in first selecting the project generating the highest surplus and then choosing contributions so that all agents receive an equal share of this surplus. For the example, the best project is a_1 since $u_{11} + u_{21} + u_{31} - c_1 = 70 + 10 + 30 - 20 = 90 > u_{12} + u_{22} + u_{32} - c_2 = 50 + 50 + 10 - 30 = 80$, and the utility levels are $(90/3,90/3,90/3) = (30,30,30)$. To check whether the method is



Consistency for quasi–linear cost–allocation problems

Figure 6

consistent, we note that in order to guarantee that agent 3 receives a utility of 30, agents 1 and 2 should pay him $30 - 30 = 0$ if they choose a_1 and $30 - 10 = 20$ if they choose a_2, leading to an "adjusted cost vector" $c' = (20+0, 30+20) = (20,50)$. The project that produces the highest surplus is of course a_1 since $u_{11} + u_{21} - c_1' = 70 + 10 - 20 = 60 > u_{12} + u_{22} - c_2' = 50 + 50 - 50 = 50$. Equal sharing of that surplus yields the utilities $(60/2,60/2) = (30,30)$, as initially determined. This is because the method is consistent.

General Definitions. Given a finite set A of ***public projects***, a ***quasi–linear cost allocation problem*** is a pair $(u,C) \in \mathbb{R}^{|A|Q} \times \mathbb{R}^{|A|}$. C is the ***cost vector.*** Each coordinate of C is the cost of the corresponding project. In addition, there is a private good called "money". The preferences of agent i, defined over the product $A \times \mathbb{R}$, admit a quasi-linear utility representation: given the project a and given agent i's holdings of money m_i, his utility is $u_{ia} + m_i$. Let $\mathcal{M}^Q$ be the class of these problems, $\mathcal{M} \equiv \bigcup_{Q \in \mathcal{P}} \mathcal{M}^Q$, and $X \equiv \bigcup_{Q \in \mathcal{P}} \mathbb{R}^Q$. A ***solution*** is a function that associates with every $Q \in \mathcal{P}$ and every $(u,C) \in \mathcal{M}^Q$ a vector $x \in \mathbb{R}^Q$ such that $\sum_Q x_i \leqq \max_{a \in A}(\sum_Q u_{ia} - C_a)$.

Moulin (1985a, 1985b) carried out an extensive analysis of this class of problems, which generalizes the class of bankruptcy and taxation problems as it is given in a space of higher dimensionality.

Moulin describes a rich class of solutions. They can be interpreted as variants of the Egalitarian solution since they are based on equating utility gains. They differ from each other in the specification of the reference point from which utility gains are measured.

Definition. The solution $F: \mathcal{M} \to X$ satisfies ***Consistency***[19] if for all P, $Q \in \mathcal{P}$ with $P \subset Q$, for all $(u,C) \in \mathcal{M}^Q$, if $x = F(u,C)$, then $x_P = F(t_P^x(u,C))$, where $t_P^x(u,C) = (u_P, C')$ with $C'_a = C_a + \sum_{Q \setminus P} (x_i - u_{ia})$ for all $a \in A$.

Results. We will consider solutions satisfying the following requirements: ***Pareto–Optimality***: the decision maximizes the net aggregate benefit; ***Anonymity***: the solution is invariant under exchanges of the names of agents; ***Independence of the Zero of the Utility Functions***: the solution is invariant under the addition of an arbitrary constant to the agents' utilities; and ***Independence***

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[19]Moulin uses the term "separability".

of the Zero of the Cost Function: any increase in the cost function uniform across all alternatives is distributed evenly among the agents.[20]

***Theorem* 11** (Moulin 1985a): A solution $F:\mathcal{M} \to X$ satisfies Pareto–Optimality, Anonymity, the two Independence axioms and Consistency if and only if there is $g:[\mathbb{R}^A]^2 \to \mathbb{R}$ satisfying

(i) $g(x+\alpha(1,\ldots,1),z) = g(x,z) + \alpha$ for all $x, z \in \mathbb{R}^A$, $\alpha \in \mathbb{R}$; $g(0,z) = 0$ for all $z \in \mathbb{R}^A$,

(ii) $g(x,z+\alpha(1,\ldots,1)) = g(x,z)$ for all $x, z \in \mathbb{R}^A$, $\alpha \in \mathbb{R}$,

and such that for all $Q \in \mathcal{P}$ and for all $(u,C) \in \mathcal{M}^Q$ and for all $i \in Q$,

$$F_i(u,C) = (1/|Q|)\max_{a\in A}(\sum_Q u_{ia}-C_a) + (1/|Q|)\{(|Q|-1)g(u_i,\sum_Q u_i-C) - \sum_{Q\setminus i} g(u_j,\sum_Q u_i-C)\}.$$

The class identified by this theorem is quite large. By imposing further conditions it is possible to characterize an interesting subclass:

We will assume that F is ***Cost Monotonic***: an increase in the cost function is borne by all agents; and is ***Immune to Manipulation by Disposal of Utility***: no agent can benefit by pretending his utility to be smaller than what it really is.[21]

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[20]***Pareto–Optimality***: $\sum_Q x_i = \max_{a\in A}(\sum_Q u_{ia} - C_a)$; ***Anonymity***: $F(\pi(u),C) = \pi(F(u,C))$ for any one–to–one function $\pi:Q \to Q'$ where $|Q| = |Q'|$; ***Independence of the Zero of the Utility Functions***: if for some α, $v_i = u_i + \alpha(1,\ldots,1)$, and $v_j = u_j$ for all $j \neq i$, then $F_i(v,C) = F_i(u,C) + \alpha$ and $F_j(v,C) = F_j(u,C)$ for all $j \neq i$; ***Independence of the Zero of the Cost Function***: if $C' = C + \alpha(1,\ldots,1)$, then $F_i(u,C') = F_i(u,C) - \alpha/|Q|$ for all i.

[21]***Cost Monotonicity***: if $C \leqq C'$, then $F(u,C') \leqq F(u,C)$; ***Immunity to Manipulation by Disposal of Utility***: if $u_i < v_i$ and $u_j = v_j$ for all $j \neq i$, then

Theorem 12 (Moulin 1985a): A solution of the form identified in Theorem 11 satisfies Cost Monotonicity and is Immune to Manipulation through Disposal of Utility if and only if $g(x,z) = \tilde{g}(x)$ for some monotonic $\tilde{g}:\mathbb{R}^A \to \mathbb{R}$.

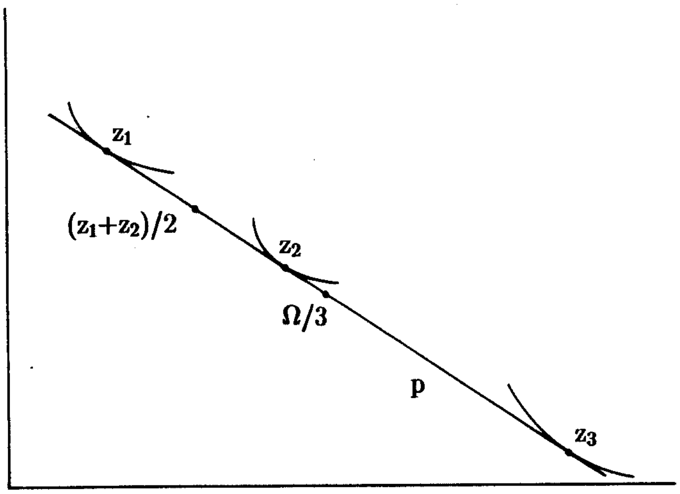
(5) ***Resource allocation problems in exchange economies.***

Examples. Suppose that you had to determine a fair allocation of the resources available on the planet earth. After attributing to each individual what you think he deserves, you focus on a particular continent and add up what the inhabitants of that continent have received. If you had to fairly allocate that amount among them, would you give back to each of them exactly what that person had initially received? If the answer is yes and if the answer would be yes for all continents, all countries within continents, towns within countries,..., independently of preferences and independently of what is to be distributed, then the method of fair division that you are using is consistent.

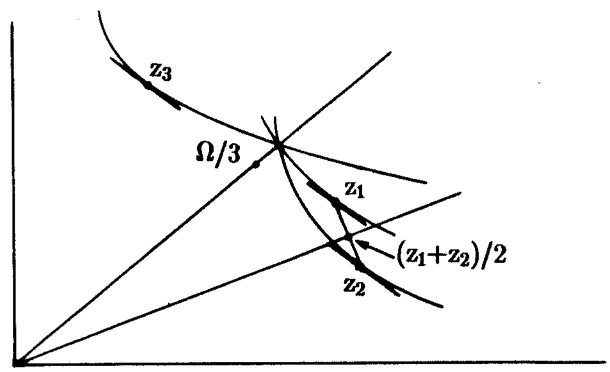
For instance, suppose that you recommend that resources be allocated by operating the Walrasian mechanism from equal division, as illustrated in the two–commodity example of Figure 7a. If the aggregate bundle Ω is available, the mechanism leads to the allocation $z = (z_1,z_2,z_3)$ with associated equilibrium price p. Considering now the amount $z_1 + z_2$ to be distributed to the subgroup made up of the first two agents, and applying the same mechanism (now, each agent starts out with $(z_1+z_2)/2$), takes us back to the allocation (z_1,z_2), the same price serving as equilibrium price. Each of the two agents ends up with exactly the same bundle. This simple example illustrates the fact that the Walrasian solution from equal division is consistent.

$F_i(u,C) \leqq F_i(v,C)$.

On the other hand, Figure 7b shows that the solution that selects the efficient allocations at which utilities are equal, using the utility representations obtained by calibrating along the ray through the aggregate bundle, is not consistent. Indeed, although in the three person economy $z = (z_1, z_2, z_3)$ satisfies this criterion, agents 1 and 2's indifference curves through z_1 and z_2 respectively, do not intersect on the ray passing through $z_1 + z_2$.



The Walrasian solution from equal division is consistent
(a)



The Egalitarian solution is not consistent
(b)

Consistency in pure exchange economies

Figure 7

General definitions. A ***problem of fair division*** is a pair $(u,\Omega) \in U^Q \times \mathbb{R}^\ell_+$: there are ℓ commodities and a group Q of agents; $u_i: \mathbb{R}^\ell_+ \to \mathbb{R}$ is agent i's continuous, quasi-concave and monotone ***utility function***; $\Omega \in \mathbb{R}^\ell_+$ is the ***aggregate endowment***. This formulation is to be distinguished from the usual set-up in which each agent is entitled to a particular share of Ω, his initial endowment; here, we assume instead that agents are ***collectively*** entitled to Ω. Let $\mathscr{E}^Q$ be a class of admissible economies, $\mathscr{E} \equiv \bigcup_{Q \in \mathscr{P}} \mathscr{E}^Q$, and $X \equiv \bigcup_{Q \in \mathscr{P}} \mathbb{R}^{\ell|Q|}$. A ***solution*** is a correspondence that associates with every $Q \in \mathscr{P}$ and every $(u,\Omega) \in \mathscr{E}^Q$ a non–empty subset of $\{z \in \mathbb{R}^{\ell|Q|} \mid \sum_Q z_i \leqq \Omega\}$, the set of feasible allocations for (u,Ω).

Definition. The solution $F: \mathscr{E} \to X$ satisfies ***Consistency***[22] if for all $P, Q \in \mathscr{P}$ with $P \subset Q$, and for all $(u,\Omega) \in \mathscr{E}^Q$, if $z \in F(u,\Omega)$, then $z_P \in F(t_P^z(u,\Omega))$, where $t_P^z(u,\Omega) = (u_P, \sum_P z_i)$.

We stated the condition for correspondences. This is probably the most natural formulation since resource allocation rules are rarely single–valued; one would not want to eliminate the Walrasian solution from equal division W_{ed} from consideration, for instance, just because it does not usually select a single allocation.

We noted earlier that this solution satisfies consistency. However, there are other important solutions that do too. The ***Pareto correspondence*** P is among them; so are its intersections with the ***no–envy*** correspondence F (Foley 1967; $F(u,\Omega)$ is the set of feasible allocations z such that for no pair $\{i,j\}$, $u_i(z_j) > u_i(z_i)$; at such an allocation no agent would want to exchange bundles with anyone else,) or with the ***egalitarian–equivalent*** correspondence E^* (Pazner and Schmeidler 1978; $E^*(u,\Omega)$ is the set of feasible allocations z such that for some $z_0 \in \mathbb{R}_+^\ell$ and for all i, $u_i(z_0) = u_i(z_i)$). However, neither the intersection of P with the individually rational correspondence from equal division I_{ed} ($I_{ed}(u,\Omega)$ is the set of feasible allocations z Pareto–dominating equal division) nor the Core correspondence from equal division satisfies the condition.

[22]Thomson (1988) uses the phrase "stability under arbitrary formation of subgroups".

Results. Consistency can be used to characterize the Walrasian solution from equal division when imposed in conjunction with other related requirements expressing a certain form of invariance of the solution under deletion, as well as addition, of agents. A representative example is the following result, which involves the condition of ***Replication Invariance***: if an allocation is chosen by the solution for some economy, then its k–times replica is chosen by the solution for the k–times replica of the economy.[23]

Theorem **13** (Thomson 1988). Consider a domain of economies where preferences have differentiable numerical representations. If a solution that selects Individual rational from equal division and Pareto optimal allocations satisfies Consistency and Replication Invariance, then it is a subcorrespondence of the Walrasian solution from equal division.

Other results in the same spirit can be found in Thomson (1988).

Consider now the following class of economies in which indivisible goods are present. For instance, there are n jobs that have to be assigned to n agents. Salary adjustments can be made to compensate agents for being assigned less desirable jobs. The notion of an envy–free allocation applies to this situation just as well. Envy–free allocations do not always exist, but when they do, there often is a continuum of them and the natural question there is how to make selections from this continuum. This question was addressed by Tadenuma and Thomson (1989). To state their results we need the following very mild condition: ***Independence of Irrelevant Permutations***. If an allocation obtained by exchanges of bundles from one that is chosen by the solution leaves

[23] ***Replication invariance***: $z \in F(u,\Omega)$ implies $k_*z \in F(k_*u,k\Omega)$, where k_*z is the k–replica of z and k_*u is the k–replica of u.

unaffected the welfares of all agents, then it should also be chosen by the solution.

Theorem **14** (Tadenuma and Thomson, 1989). If a subcorrespondence of the no–envy solution satisfies Consistency and Independence of Irrelevant Permutations, then it coincides with the no–envy solution.

We will omit the formal statement of the two additional conditions needed for the next result: ***Bilateral Consistency*** is simply Consistency applied to only two persons subgroups, and ***Converse consistency*** is patterned after the condition of the same name used in the section on coalitional form games.

Theorem **15** (Tadenuma and Thomson, 1989). There is an infinity of subsolutions of the no–envy solution satisfying Bilateral Consistency and Independence of Irrelevant Permutations, or Converse Consistency and Independence of Irrelevant Permutations. However, if a subsolution of the no–envy solution satisfies Bilateral Consistency, Independence of Irrelevant Permutations and Converse Consistency, then it coincides with the no–envy solution.

(6) *Apportionment problem.*

Example. One of the oldest problems in political science is that of attributing seats to states in order to achieve proportional representation. The problem arises because rounding is necessary and it is important because which rounding method is used may dramatically affect the representation of small states. Consider the three–state apportionment problem described in Figure 8, and let us solve it according to two well–known methods, respectively advocated by Jefferson and Hamilton, and defined as follows.

For ***Jefferson's method***, choose a divisor of the populations of states so that the whole numbers contained in the quotients sum to the total number of seats. Then, give to each state its whole number. For ***Hamilton's method***, define the "quota" of each state to be the ratio of its population to the aggregate population times the total number of seats. Give to each state the whole number contained in its quota. Assign the remaining seats to those states having the largest fractions.

Note that states S_1 and S_3 have been allocated a total of $6 = 1 + 5$ seats under Jefferson's method. Applying this method to the problem of allocating 6 seats to these states produces exactly the same apportionment (1,5). Jefferson's method is consistent. However, Hamilton's method is not since when there are three states, S_1 and S_3 together receive $6 = 1 + 5$ seats , but applying the method to the allocation of 6 seats among them produces the allocation (2,4).

States' Populations S	Jefferson's method applied to (S,10)	Jefferson's method applied to $(S_1,S_3,6)$	Hamilton's method applied to (S,10)	Hamilton's method applied to $(S_1,S_3,6)$
200	1	1	1	2
500	4		4	
590	5	5	5	4

Jefferson's method is consistent but Hamilton's method is not.

$S = (200,\ 500,\ 590)$

Figure 8

General definitions. An apportionment problem is a pair $(s,H) \in \mathbb{N}^Q \times \mathbb{N}$: the members of Q are states with ***populations*** given by the coordinates of s; H is the number of seats in the ***parliament.*** The objective is to allocate the seats to the states proportionately to their populations, or as "close to proportionately as possible".[24] Let $\mathscr{A}^Q$ be the class of these problems, $\mathscr{A} \equiv \bigcup_{Q \in \mathscr{P}} \mathscr{A}^Q$, and $X \equiv \bigcup_{Q \in \mathscr{P}} (\mathbb{N} \cup \{0\})^Q$. A solution is a correspondence that associates with every $Q \in \mathscr{P}$ and every $(s,H) \in \mathscr{A}^Q$ a vector in $(\mathbb{N} \cup \{0\})^Q$ with coordinates adding up to H.

Balinski and Young (1982) carry out an extensive analysis of apportionment. One of the important axioms they consider is Consistency.[25]

Definition. The solution $F: \mathscr{A} \to X$ satisfies ***Consistency*** if for all $P, Q \in \mathscr{P}$ with $P \subset Q$, and for all $(s,H) \in \mathscr{A}^Q$, if $x \in F(s,H)$, then $x_P \in F(t_P^x(s,H))$, where $t_P^x(s,H) \equiv (s_P, \sum_P x_i)$. Also, if $y \in F(t_P^x(s,H))$, then $(y, x_{Q \setminus P}) \in F(s,H)$.

The first part of the condition corresponds directly to what we have already seen a number of times. The second part is new. It says that if the subproblem relative to P and $x \in F(s,H)$ admits a solution outcome y different from x_P, then the original problem admits as solution outcome the juxtaposition of y with $x_{Q \setminus P}$.

Results. In addition to consistency, we will impose the following properties on solutions. ***Balancedness:*** whenever two states have equal populations, their apportionments do not differ by more than one seat; ***Anonymity:*** the solution is invariant under exchanges of names of the states; ***Homogeneity:*** if the populations of the states change by the same proportions, the apportionment

[24]The more general problem of allocating seats "proportionately to both population and parties" has been examined by Balinski and Demange (1986).

[25]Under the name of "uniformity".

should be unchanged; ***Weak Proportionality***: if there is a feasible apportionment proportional to the populations, it is the only one recommended by the solution; ***Rank–preserving***: if a state has a greater population than another one, it should receive at least as many seats at any apportionment recommended by the solution.[26]

Let $r:\mathbb{C}\times(\mathbb{N}\cup\{0\}) \to \mathbb{R}$ be a monotone decreasing function of its second argument ($\mathbb{C}$ is the set of rational numbers) and let $\mathcal{F}$ be the class of all functions f: $\mathbb{C}^Q \times (\mathbb{N}\cup\{0\}) \to (\mathbb{N}\cup\{0\})^Q$ defined recursively as follows:

(i) for $H = 0$, $f(s,H) = (0,...,0)$

(ii) if $f(s,H) = x$, then $f(s,H+1)$ is found by giving $x_i + 1$ seats to some state i such that $r(s_i,x_i) \geqq r(s_j,x_j)$ for all $j \neq i$, and x_j seats to each $j \neq i$.

Finally, the ***rank index solution relative to r*** is defined by $F(s,H) \equiv \{x \mid x = f(s,H) \text{ for some } f \in \mathcal{F}\}$.[27]

Theorem **16** (Balinski and Young 1982). The rank index solutions are the only solutions satisfying Balancedness, Anonymity, and Consistency.

Given a monotone increasing function $d = \mathbb{N}\cup\{0\} \to \mathbb{N}\cup\{0\}$ such that $a \leqq d(a) \leqq a+1$, the ***divisor solution based on d*** is defined by $F(s,H) = \{x \mid d(x_i-1) \leqq s_i/\lambda \leqq d(x_i) \text{ and } \Sigma x_i = H \text{ for some } \lambda\}$.

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[26]***Balancedness***: $s_i = s_j$ implies $|F_i(s,H) - F_j(s,H)| \leqq 1$; ***Anonymity***: $F(\pi(s),H) = \pi(F(s,H))$, where $\pi:Q \to Q'$ with $|Q| = |Q'|$ is an arbitrary permutation; ***Homogeneity***: $F(\lambda x,H) = F(s,H)$ for all $\lambda \in \mathbb{C}_{++}$, the set of positive rational numbers; ***Weak Proportionality***: if $x = \lambda s$ for some λ and $\Sigma x_i = H$, then $F(s,H) = \{x\}$; ***Rank–Preserving***: if $s_i > s_j$, and $x \in F(s,H)$, then $x_i \geqq x_j$.

[27]Jefferson's method is the member of the family obtained for r defined by $r(s_i,x_i) = s_i/(x_i+1)$.

By imposing some of the other conditions, Theorem 14 can be refined to give

Theorem **17** (Balinski and Young, 1982): The ***divisor solutions*** are the only ones to satisfy Homogeneity, Anonymity, Weak Proportionality, Rank–Preservingness and Consistency.

IV. *Concluding comments*

We hope to have convinced the reader that the consistency principle is powerful and versatile. In fact, there are other models in which consistency has been used. We will simply mention that Epstein (1986) has used the principle in a characterization of certain allocation rules in intergenerational problems, Toda (1988) and Sasaki (1988) have used it to provide a characterization of the core of matching games, and Tadenuma and Thomson (1987) have explored its implication for the study of resource allocation in public good economes.

It is also worth noting that in all of the examples we examined, subproblems are obtained by varying the number of agents. But there are other ways of affecting the dimensionality of problems. Roemer (1986a,b, 1988) considered exchange economies and allowed for variations in the number of commodities instead. He used the resulting consistency condition to characterize egalitarian type solutions. See also Donaldson and Roemer (1987) for a characterization of "welfarist" solutions.

While the consistency principle has been an important unifying theme in a variety of areas, it should be pointed out however that the techniques of proof of the various characterizations that have been based on it have little in common, due to the very different mathematical structures of these problems.

Bibliography

Aumann, R., and J. Drèze, "Cooperative games with coalition structures," *International Journal of Game Theory* 3 (1974), 217–237.

_______________ and M. Maschler, "Game theoretic analysis of a bankruptcy problem from the Talmud," *Journal of Economic Theory* 36 (1985), 195–213.

Balinski, M. and P. Young, *Fair Representation*, Yale University Press, 1982.

_______________ and G. Demange, "Generalizing proportionality; or how to simultaneously apportion seats to region and to parties," Ecole Polytechnique DP (January 1986).

Chun, Y., "The proportional solution for rights problems," *Mathematical Social Sciences* 15 (1988), 231–246.

Davis, M. and M. Maschler, "The kernel of a cooperative game," *Naval Research Logistics Quarterly* 12 (1965), 223–259.

Donaldson, D. and J. Roemer, "Social choice in economic environments with dimensional variation," *Social Choice and Welfare* 4 (1987), 253–276.

Dutta, B., "The Egalitarian solution and reduced game properties," Technical Report No. 8906 (March 1989).

Epstein, L., "Intergenerational consumption rules: an axiomatization of utilitarianism and egalitarianism," *Journal of Economic Theory* 38 (1986), 280–297.

Foley, D., "Resource allocation and the public sector," *Yale Economic Essays* 7 (1987), 45–98.

Foster, D. and R. Vohra, "Consistency and the Nash bargaining solution," Ohio State University working paper WPS 88–22 (1988).

Harsanyi, J.C., "A bargaining model for the cooperative N–person games," in Tucker, A.W. and R.D. Luce (eds.), *Contributions to the theory of games*, Princeton University Press (1959), 324–358.

Hart, S. and A. Mas–Colell, "The potential of the Shapley–value," Chapter 9 in *The Shapley–Value: Essays in Honor of Lloyd S. Shapley* (A.E. Roth, ed.), Cambridge University Press, (1988) 127–137.

_______________ and _______________ "Value and potential," *Econometrica*, (1989), 589–614.

Kalai, E., "Proportional solution to bargaining problems: interpersonal utility comparisons," *Econometrica* 45 (1977), 1023–1030.

_______________ and M. Smorodinsky, "Other solutions to Nash's bargaining problem," ***Econometrica*** 43 (1975), 513–518.

Lensberg, T., "Stability, collective choice and separable welfare," Ph.D. Dissertation, Norwegian School of Economics and Business Administration, Bergen, Norway (1985).

_______________, "Stability and collective rationality," ***Econometrica*** 55 (1987), 935–961.

_______________, "Stability and the Nash solution," ***Journal of Economic Theory*** 45 (1988), 330–341.

_______________ and W. Thomson, "Characterizing the Nash bargaining solution without Pareto optimality," ***Social Choice and Welfare*** 5 (1988), 237–254.

Maschler, M. and B. Peleg, "The structure of the kernel of a cooperative game," ***Siam Journal of Applied Mathematics*** 15 (1967), 569–604.

_______________, B. Peleg and L. Shapley, "The kernel and the bargaining set for convex games," ***International Journal of Game Theory*** 1 (1972), 73–93.

Moulin, H., "The separability axiom and equal sharing methods," ***Journal of Economic Theory*** 36 (1985a), 120–148.

_______________, "Egalitarianism and utilitarianism in quasi–linear bargaining," ***Econometrica*** 53 (1985b), 49–67.

_______________, "Equal or proportional division of a surplus, and other methods," ***International Journal of Game Theory*** 16 (1987), 161–186.

Nash, J.F., "The bargaining problem," ***Econometrica*** 18 (1950), 155–162.

O'Neill, B., "A problem of rights arbitration from the Talmud," ***Mathematical Social Sciences*** 2 (1982), 345–371.

Pazner, E. and D. Schmeidler, "Egalitarian equivalent allocations: A new concept of economic equity," ***Quarterly Journal of Economics*** 92 (1978), 671–687.

Peleg, B., "An axiomatization of the core of cooperative games without side–payments," ***Journal of Mathematical Economics*** 14 (1985), 203–214.

_______________, "On the reduced game property and its converse," ***International Journal of Game Theory*** 15 (1986a), 187–200.

_______________, "An axiomatization of the core of market games," Hebrew University of Jerusalem DP (1986b), ***Mathematics of Operations Research***, forthcoming.

Roemer, J., "The mismarriage of bargaining theory and distributive justice," ***Ethics*** 97 (1986a), 88–110.

______________, "Equality of resources implies equality of welfare," ***Quarterly Journal of Economics*** 101 (1986b), 751–784.

______________, "Axiomatic bargaining theory on economic environments," ***Journal of Economic Theory*** 45 (1988), 1–31.

Sasaki, H., "Axiomatization of the core for two–sided matching problems," Nagoya City University DP (March 1988).

Sen, A., "Utilitarianism and welfarism," ***Journal of Philosophy*** 76 (1979), 463–489.

Shapley, L., "A value for n–person games," in ***Contributions to the Theory of Games II*** (Annals of Mathematics Studies 28), ed. by H.W. Kuhn and A.W. Tucker, Princeton: Princeton University Press (1953), 307–317.

Sobolev, A.I., "The characterization of optimality principles in cooperative games by functional equations," ***Mathematical Methods in the Social Sciences*** 6 (1975), 150–165 (in Russian).

Suematsu, T., "Consistency and solutions to cooperative games," University of Rochester mimeo, 1988.

Tadenuma, K., "Consistency of solutions for social decision problems," University of Rochester mimeo, 1989.

______________ and W. Thomson, "Consistency in economies with public goods," University of Rochester mimeo, forthcoming.

______________ and ______________, "No–envy, consistency, and monotonicity in economies with indivisibilities," University of Rochester DP No. 178, March 1989.

Thomson, W., "Monotonicity, stability and egalitarianism," ***Mathematical Social Sciences*** 4 (1984), 15–28.

______________, "On the Nash solution," University of Rochester mimeo (1985).

______________, "A study of choice correspondences in economies with a variable number of agents," ***Journal of Economic Theory*** 46 (1988), 247–259.

Toda, M., "The consistency of solutions for marriage problems," University of Rochester mimeo, February 1988.

Yaari, M. and M. Bar–Hillel, "On dividing justly," ***Social Choice and Welfare*** 1 (1984), 1–24.

Young, P., "Taxation and bankruptcy," University of Maryland Mimeo (July 1986).

________________, "When is a tax increase fairly distributed?," University of Maryland mimeo (1987a).

________________, "On dividing an amount according to individual claims or liabilities," ***Mathematics of Operations Research*** 12 (1987b), 398–414.

________________, "Distributive justice in taxation," ***Journal of Economic Theory*** (1988a).

________________, "Consistent solutions to the bargaining problem," University of Maryland, School of Public Affairs mimeo (1988b).