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Low Frequency Filtering and Real Business Cycles

Abstract

It is common practice in business cycle analysis for researchers to filter out low frequency components, so as to focus their efforts on business cycles rather than economic growth. In recent real business cycle research, this "trend elimination" has most frequently been undertaken with the HP filter due to Robert Hodrick and Edward Prescott [1980].

In this paper, we provide a detailed discussion of the HP filter from time and frequency domain perspectives, motivating it as a generalization of the well known exponential smoothing filter. Further, we show that the HP filter—in parameterizations applicable in large samples—contains a centered fourth difference, so that it is capable of rendering stationary time series that are "difference stationary" and, indeed, integrated of higher order.

However, our application of the HP filter to U.S. time series and to the simulated outcomes of real business cycle models leads us to question its widespread use as a unique method of trend elimination, prior to computation of moments summarizing model and actual time series. We provide examples of how standard HP practice produces major departures from alternative, more traditional views of cyclical fluctuations. Notably, HP filtering dramatically alters measures of persistence, variability and comovement. Thus, we recommend some alterations in existing practice of communicating research results on business cycle models.

I. Introduction

A hallmark of modern equilibrium business cycle theory is the view that growth, business cycles and seasonal variations are to be studied within a unified framework. Even though rational economic agents are presumed to respond differently to shocks of varying duration, dynamic economic theory generally imposes concrete and extensive restrictions across frequencies. For one example, the manner in which agents respond to varying seasonal opportunities provides information about how they will respond to temporary opportunities during the course of business cycles. For another, the manner in which labor supply responds to the permanent wage and wealth changes occurring during economic growth restricts responses at business cycle frequencies.

Yet, beginning with Kydland and Prescott [1982], many studies in the real business cycle research area apply the HP filter—due to Hodrick and Prescott [1980]—to both time series generated from an artificial economy and actual data before making conclusions about the properties of the model and its congruence with observations. This practice corresponds to an implicit definition of "the business cycle frequencies" and a decision to downplay the implications of the model at other frequencies, generally on the grounds that these represent growth rather than business cycles.

Reading the literature on real business cycles, one can easily come to the conclusion that the practice of "low frequency filtering" has relatively minor consequences for how one thinks about economic models and their consistency with observed time series. This paper demonstrates that the opposite conclusion is true: the practice of low frequency filtering has major consequences for both the "stylized" facts of business cycles and the perceived operation of dynamic economic models. In fact, the components of time series removed by mechanical application of the HP filter are sufficiently important that one may feel that the baby is being thrown out with the bath water in terms of business cycle research. Consequently, we provide some specific recommendations for alterations in practice, i.e., in the reporting of research results on business cycles.

Our motivation for this investigation derived from two sources, which we provide to the reader as a puzzle to be investigated in the remainder of the paper.

Implications for Simulated Time Series

Our first exposure to the potential importance of HP filtering involved an attempt to replicate some results obtained in Gary Hansen's [1985] analysis of the effects of varying the labor supply elasticity in a basic real business cycle model.[1]

Table I–1 provides a few population moments, basically those reported by Hansen [1985], with and without HP filtering.[2] The model economy is one in which the single source of uncertainty is a trend stationary technology shock to total factor productivity; it is detailed in King, Plosser and Rebelo [1988a] and reviewed in section III below. From this table, it is clear that HP filtering alters the moment implications of the model in a quantitatively important manner, but that this influence is not constant across series.

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[1]We thank Gary Hansen for providing some simulation results for unfiltered versions of his model [1985] that confirmed our conjectures that filtering, not model solution methods or model parameter values, lay at the heart of major differences in moments.

[2]Reporting of these selected moments is common in discussions of implications of real business cycle models, as–for example—in McCallum's [1989] survey, Tables 1 and 2.

First, HP filtering, which extracts a component from the original series, lowers volatility as measured by the standard deviation columns in table 1. Second, HP filtering alters the relative volatilities of different series (the standard deviation of a variable divided by that of output). In particular, it increases the relative volatility of investment and hours while lowering that of consumption, the real wage and the capital stock. Third, the correlations between individual series and output—a measure of cyclical sensitivity—are substantially altered by HP filtering. Notably, the cyclical variation in capital and labor input is dramatically altered by filtering. In the unfiltered economy, capital's correlation with output is .73 and that of labor with output is .79. With filtering, capital's correlation drops to .07 and that of labor rises to .98.

<u>HP Filtering of Some U.S. Post War Time Series</u>

Our second indication of the potential importance of HP filtering came from Marianne Baxter's empirical work (Baxter [1988] and Baxter and Stockman [1988]) on stylized facts of economic fluctuations in the United States and other countries.[3] To provide some empirical background to our subsequent analysis, we begin by displaying an application of the HP filter to a measure of aggregate economic activity and a measure of labor input. These are the logarithm of U.S. real gross national product, which we denote y_t, and the logarithm of per capita average hours worked, which we denote N_t.

[3]We thank Marianne Baxter for suggesting the revealing examples contained in this section and for technical assistance in producing these results.

Like other low frequency filters, the HP filter can be viewed as extracting growth and cyclical components from the data.[4] To start, let us focus on the y_t series and begin by dividing y_t into a linear trend (γt) and a residual deviation from a linear trend (so that the residual $y_t^r = y_t - \gamma t$). If the growth component is assumed to be a deterministic trend, then the business cycle component is y_t^r. Under the HP filtering procedure, by contrast, the time series is permitted to have a stochastic growth component. In addition to extracting a linear trend—if one exists in the series under study—HP filtering also removes some additional variation whose properties depend on the series in ways detailed in section II below. The HP cyclical component is then defined as the difference between the original time series and the HP growth component.

Implications for Real GNP: Figure I–1 plots the HP growth component versus the linear trend component of real gross national product. A relatively common reaction to this figure is that these ways of removing growth are not too different. But it turns out that there are major consequences for business cycle components.

In order to study the practical implications of alternative detrending methods, we construct the HP *stochastic* growth component by subtracting a linear trend from the HP growth component (taking the vertical difference between the series in Figure 1). We call this component $HP^g(y_t)$.

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[4]In part, our discussion in this section and below involves the issue of how best to define business cycles. One possibility—which is sometimes discussed in evaluation of mechanical procedures such as the HP filter—would be to select some mechanical method that broadly replicated the stylized facts reported by NBER researchers following Mitchell [1927,1951] and Burns and Mitchell [1941]. However, preliminary work by King and Plosser [1989] leads us to believe that the NBER methods should be subject to some scrutiny as well.

Summarizing our definitions, the alternative decompositions are

$$y_t = \gamma t + y_t^r = \gamma t + HP^c(y_t) + HP^g(y_t),$$

i.e., the HP cyclical and stochastic growth components sum to the residual from the deterministic trend.

Figure I–2 makes clear that the HP stochastic growth component constitutes a major portion of the departure of output from a linear trend, so that the implied cyclical components arising from these two methods of trend elimination are very different both in terms of magnitude and persistence. Notably, the autocorrelation correlation coefficient of y_t^r at the annual lag is .72. Figure I–3 plots the HP cyclical component ($HP^c(y_t)$) of real GNP, which is a rapidly fluctuating series, as may be judged from its autocorrelation structure, which is presented in Table I–2, panel A. The autocorrelation at a year lag (four quarters), for example, is only .09, which is an order of magnitude smaller than the autocorrelation coefficient for $HP^r(y_t)$ at the same lag.

Implications for GNP and Labor Input: Figure I–4 plots y_t^r versus the departures of our labor input measure from its mean (i.e., $N_t - \overline{N}$). There is not a strong relationship: the contemporaneous correlation is only .06. However, when one compares the HP cyclical components of output and hours in Figure I–5, there is a striking coincidence: the contemporaneous correlation is .86. Further information on the implications of filtering for correlations among variables is contained in Table I–2.

Outline of Our Analysis

To this point, we have shown that low frequency filtering—using the Hodrick and Prescott [1980] filter—has important implications on moments of

U.S. time series and a simulated real business cycle model. In developing an explanation of the origin of these results and their practical consequences for business cycle research we proceed as follows. In section II, we first discuss what linear filtering is and then review some facts about linear filters. We next derive the HP growth and cyclical filters as a direct generalization of the well-known exponential smoothing filter of Brown [1962]. In section III, we apply the HP filter to some basic real business cycle models to further develop the sorts of implications suggested by section III. Building on these results, in section IV, we provide a concrete set of recommendations about how to best report results of investigations into business cycles. Section V is a brief summary and conclusion.

II. Low Frequency Filtering

There is a lengthy history in macroeconomics of filtering time series. For example, there is extensive use of moving averages of time series by Mitchell [1930] and Kuznets [1930] in their analyses of business cycles and economic growth. Further, applications of moving averages and other linear filters can sometimes lead to important statistical artifacts in time series. For example, Fishman [1969] summarizes research that points out how the apparent long swings in economic activity suggested by Kuznets [1961] might potentially have arisen simply from his application of moving averages rather than as a property of underlying economic time series.

In this section, our objectives are twofold. First, in section II.1, we provide an overview of analytical tools for studying the implications of linear filters. This section should be skipped by readers who are comfortable with introductory treatments of frequency domain methods (e.g., Harvey [1981, Chapter 3]). Second, in section II.1, we motivate the HP filter as a generalization of the familiar class of exponential smoothing (ES) procedures studied by Brown [1962]. Throughout our discussion in this section, we will focus our attention on a representative time series y_t, which we treat as the logarithm of an original series so that its first difference is a growth rate.

In filtering y_t, a researcher is motivated by one of several objectives: (i) extraction of a component such as a growth, cyclical or seasonal component; (ii) transformation to induce stationarity; or (iii) mitigation of measurement error that is assumed to be particularly important at specific frequencies. We concentrate on the first two motivations, since a detailed treatment of measurement error would require grappling with details of a specific application.

To focus our discussion, then, consider the idea that a particular economic model makes predictions about a "business cycle" component of a time series and that the researcher views the series as containing both growth and business cycle components,

$$y_t = y_t^g + y_t^c ,$$

where y_t^g is the growth component and y_t^c is the business cycle component. Representing y_t^g as a moving average (possibly two sided) of observed y_t permits extraction of the growth component (y_t^g) and the cyclical component (y_t^c). That is, suppose that we assume that

$$y_t^g = \sum_{j=-\infty}^{\infty} g_j \, y_{t-j} = G(B) \, y_t ,$$

where B is the backshift operator with $B^n x_t = x_{t-n}$ for $n \gtrless 0$. Then, since

$y_t^c = y_t - y_t^g$, it follows that y_t^c is also a moving average of y_t,

$$y_t^c = [1 - G(B)]\ y_t \equiv C(B)\ y_t .$$

In the language of filtering theory, G(B) and C(B) = [1 – G(B)] are linear filters and we now explore some of their properties.

II.1 Some Facts About Linear Filters

In order to discuss why a specific linear filter may be described as a low frequency filter, we are led to consideration of the Fourier transform of a linear filter (also called the frequency response function of the filter). For example, the frequency response of the growth filter is

$$\tilde{G}(\omega) = \sum_{j=-\infty}^{\infty} g_j \exp\ (-ij\omega)$$

where i denotes the imaginary number $\sqrt{(-1)}$ and where ω is frequency measured in radians, i.e., $-\pi \leq \omega \leq \pi$.

Gain and Phase Decomposition: At a given frequency ω, the frequency response $\tilde{G}(\omega)$ is simply a complex number, so that it may be written in polar form as $\tilde{G}(\omega)=\Gamma(\omega)\exp(-i\Psi(\omega))$, where $\Gamma(\omega)=|\tilde{G}(\omega)|$ and $\Psi(\omega)$ are real numbers for fixed ω. In these expressions and below, $|x|$ denotes the modulus of x (the square root of the product of x and its conjugate). The gain of the linear filter, $\Gamma(\omega)$, yields a measure—at the specified frequency ω—of the increase in the amplitude of the filtered series over the original series. The phase, $\Psi(\omega)$, yields a measure of the time displacement attributable to the linear filter, again at the specified frequency ω. The frequency response function can be decomposed into the gain function $\Gamma(\omega)$ and phase function $\Psi(\omega)$ by replicating the preceding decomposition at each value of ω.

To take a concrete example, suppose that a time series is strictly periodic with a period of $2\pi/\omega^*$. Then application of the linear filter G would simply alter the range of this periodic function by $\Gamma(\omega^*)=|G(\omega^*)|$, as illustrated in Figure II-1a. Further, Figure II-1b illustrates the hypothetical phase shift effect of a linear filter.

Symmetric Filters: In our analysis, we will focus on filters that possess a symmetry property in that $g_j = g_{-j}$. For any such filter, it is possible to show that

$$\tilde{G}(\omega) = g_o + 2 \sum_{j=1}^{\infty} g_j \cos (j\omega)$$

using the trigonometric identity $2\cos(x) = \{\exp(ix) + \exp(-ix)\}$. Symmetric filters have the important property that they do not induce a phase shift, i.e., $\Psi(\omega) = 0$ for all ω, since the Fourier transform $\tilde{G}(\omega)$ is real for a symmetric filter. Thus, the gain function is equal to the frequency response, so that we use these terms interchangeably below.

Further, in the class of symmetric filters, it is easy to see that

$$\tilde{G}(0) = \sum_{j=-\infty}^{\infty} g_j = 1$$

is a necessary and sufficient condition for a filter to have the property that it has unit gain at zero frequency.[5] Thus, by extension, the associated cyclical filter $C(B) = [1 - G(B)]$ will place zero weight on zero frequency whenever $\tilde{G}(0) = 1$.

[5]This property obtains for symmetric filters since $\cos(0)=1$ and it follows directly that $\tilde{G}(0)=1$. Moreover, this property holds as well for nonsymmetric filters since $\exp(0) = 1$ implies that $\tilde{G}(0)=1$ under the condition that the filter weights sum to unity.

By restricting attention to symmetric filters, then, we can simply express their implications by plotting the gain for various values of ω. Figure II-2a depicts the gain function of an idealized growth filter that emphasizes only frequencies up to some maximum ω^*. It has unit gain for frequencies $0 \leq \omega \leq \omega^*$ and zero gain for $\omega^* \leq \omega \leq \pi$. Figure II-2b shows the gain of its cyclical counterpart, C(B) = [1 - G(B)]. However, as discussed by Koopmans [1974, pages 176–185], it is not possible to actually apply such an ideal filter to a finite length data set, since its construction requires an infinite number of weights. Adaptation of the ideal filter to a finite weight context—including procedures such as truncation of the filter weights—generates problems that make the resulting filter imperfect. For example, Koopmans [1974, figure 6.7, page 185], demonstrates that the squared gain of the filter obtained via truncation is not flat over either $0 \leq \omega \leq \omega^*$ or $\omega^* \leq \omega \leq \pi$. Rather, there is "leakage" from those frequencies for which the ideal filter's gain is zero to those for which it is unity. For this reason, we consider the linear filters arising from some commonly analyzed minimization problems, which take explicitly into account the length of the data set.

II.2 Analysis of Some Common Linear Filters

In many practical contexts, one frequently approaches the task of extracting unobserved components by solving a minimization problem. Two noted examples are the problem (ES) which leads to the exponential smoothing filters for growth and cyclical components and the problem (HP) which leads to the Hodrick-Prescott [1980] filters for growth and cyclical components.[6]

[6] Our formulation of the HP problem is slightly different from that originally presented in Hodrick and Prescott [1980], in terms of treatment of endpoints.

$$\text{(ES)} \qquad \min_{\{y_t^g\}_{t=1}^{T}} \sum_{t=1}^{T} \left[(y_t - y_t^g)^2 + \lambda\, (y_t^g - y_{t-1}^g)^2 \right]$$

$$\text{(HP)} \qquad \min_{\{y_t^g\}_{t=0}^{T+1}} \sum_{t=1}^{T} \left[(y_t - y_t^g)^2 + \lambda\, [(y_{t+1}^g - y_t^g) - (y_t^g - y_{t-1}^g)]^2 \right].$$

In practice, the ES program would contain an additional parameter—a constant mean of the growth rate—to permit the minimal extraction of a deterministic linear trend for each chosen value of λ. The HP program automatically involves minimal extraction of a linear trend component, since this specification involves no *change* in the growth rate. Thus, throughout our discussion, we proceed as though a linear trend had already been removed from data.

Each of these minimization problems contains a parameter λ that "penalizes" changes in the growth component (in problem (ES)) or in the acceleration of the growth component (in problem (HP)). Below, we will use the first order conditions from these problems to characterize the associated linear filters. For the minimization problem (ES), the first order condition takes the form

$$0 = -2\ [y_t - y_t^g] + 2\lambda\ [y_t^g - y_{t-1}^g] - 2\lambda\ [y_{t+1}^g - y_t^g].$$

For the minimization problem (HP), the first order condition takes the form

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However, this difference is unimportant given our focus on the "infinite sample" version of the filter.

$$0 = -2(y_t - y_t^g) + 2\lambda[(y_t^g - y_{t-1}^g) - (y_{t-1}^g - y_{t-2}^g)]$$
$$- 4\lambda[(y_{t+1}^g - y_t^g) - (y_t^g - y_{t-1}^g)]$$
$$+ 2\lambda[(y_{t+2}^g - y_{t+1}^g) - (y_{t+1}^g - y_t^g)].$$

In each case, then, the first order condition links $y_t^c = y_t - y_t^g$ to changes in the growth component in adjacent periods. Below, this shared characteristic will play an important role in analysis of the growth and cyclical filters associated with these minimization problems.

In studying the optimal linear filters that solve these first-order conditions, we will consider the limiting version that obtains as the historical record length (T) is driven to infinity. This results in relatively simple formulae describing the filters and provides the maximum opportunity for these to match the perfect low frequency filter described earlier. In this case, each of the first order conditions can be written in the form $F(B)y_t^g = y_t$. The F(B) polynomials associated with the two problems are:

$$F_{ES}(B) = -\lambda B^{-1} + (1 + 2\lambda) - \lambda B = [\lambda(1-B)(1-B^{-1}) + 1]$$

$$F_{HP}(B) = [\lambda B^{-2} - 4\lambda B^{-1} + (6\lambda+1) - 4\lambda B + \lambda B^2] = [\lambda(1-B)^2 (1-B^{-1})^2 + 1].$$

In order to find the growth and cyclical filter, we need to invert F(B) since $G(B)=[F(B)]^{-1}$ and $C(B)=1-G(B)=[F(B)-1][F(B)]^{-1}$. The details of this process are relatively easy for the ES filter; Appendix A records the more tedious calculations for the HP filter.

<u>Growth and Cyclical Components Via Exponential Smoothing</u>

The extraction of low frequency components via exponential smoothing has a long tradition in economics, having been employed—to cite only one example—in Friedman's [1957] research on the permanent income hypothesis. In contrast to that application, however, the problem (ES) leads to a two sided exponential smoothing filter since we do not constrain y_t^g to be a function solely of past history. Manipulating the relevant first order condition for the ES filter, we find that

$$C(B) = [F(B)-1][F(B)]^{-1} = \frac{\lambda\ [1-B][1-B^{-1}]}{1 + \lambda[1-B][1-B^{-1}]}.$$

Thus, we find that the ES cyclical filter contains forward and backward differences. A key implication of this finding is that the ES filter would render stationary Nelson and Plosser's [1982] differenced stationary stochastic processes and also integrated processes of order two, whose growth rates are not stationary.

Our convenient expression for the cyclical filter's Fourier transform is

$$\tilde{C}(\omega) = [F(\exp(-i\omega)) - 1]/F(\exp(-i\omega))$$

$$= \frac{\lambda\ [1 - \exp(-i\omega)]\ [1 - \exp(i\omega)]}{1 + \lambda\ [1 - \exp(-i\omega)]\ [1 - \exp(i\omega)]} = \frac{2\lambda\ [1 - \cos(\omega)]}{1 + 2\lambda\ [1 - \cos(\omega)]},$$

where the third equality makes use of the trigonometric identity discussed earlier. Thus, the cyclical filter has zero weight at the zero frequency (since $\cos(0) = 1$) and assigns a weight close to unity at high frequencies (since $\cos(\pi) = -1$, $\tilde{C}(\pi) = 4\lambda/(1 + 4\lambda)$, which is close to one for large λ). Figure II-3 plots the gain of this cyclical filter for some alternative values of the smoothing parameter λ. Higher values of λ shift the gain function upward, raising the gain closer to unity for each fixed frequency.

Analysis of the cyclical filter in the time domain is slightly messier. To undertake this analysis, we define θ to be the smallest root of F, i.e., $F(\theta)=F(\theta^{-1})=0$. This parameter is related to λ by the equation $\theta = \{(1+2\lambda) - [(1+2\lambda)^2 - 4\lambda^2]^{1/2}\}/(2\lambda)$, so that it is real and less than 1 for any $\lambda > 0$. The growth filter can then be expressed as $G(B) = F(B)^{-1} = (\theta/\lambda)[1-\theta B]^{-1}[1-\theta B^{-1}]^{-1}$. From a straightforward expansion,

$$y_t^g = \left[\frac{(\theta/\lambda)}{1-\theta^2} \right] \left[\sum_{s=0}^{\infty} \theta^s \, y_{t-s} + \sum_{s=0}^{\infty} \theta^s \, y_{t+s} \right],$$

i.e., the growth component is a two sided exponentially weighted moving average of the original series. Similarly, the cyclical filter can be expressed as:

$$C(B) = \frac{\theta \;\; [1-B][1-B^{-1}]}{\lambda \;\; [1-\theta B][1-\theta B^{-1}]},$$

which also makes clear that the effects of second differencing $[1-B][1-B^{-1}]$ in the numerator are partly undone by the presence of $[1-\theta B][1-\theta B^{-1}]$ in the denominator. In fact, if θ were unity (which is true in the limit as $\lambda \to \infty$), then numerator and denominator terms would cancel. In practical applications θ is closer to .9, so that while this filter will render stationary an integrated time series, it will generally preserve more low frequency content than the first difference filter.

Filter weights for the cyclical filter C(B) are shown in Figure II-4 for a sample value of the smoothing parameter (λ=60). Larger values of λ—which penalize changes in the growth component—lead to smoother growth components. Thus, they lead to values of θ closer to unity (in the limit as $\lambda \to \infty$, $\theta \to 1$

so that $C(B) = 1$, i.e., $y_t^c = y_t$). The values of θ for some alternative values of λ are given in Table II-1.

Growth and Cyclical Filters via the Hodrick-Prescott Method

It turns out that the HP filters are closely related to those derived above. Manipulating the relevant first order condition, the HP cyclical filter C(B) may be written as

$$C(B) = [F(B)-1][F(B)^{-1}] = \frac{\lambda\ [1-B]^2\ [1-B^{-1}]^2}{1 + \lambda[1-B]^2\ [1-B^{-1}]^2}$$

Hence, the HP cyclical filter is also capable of rendering stationary any integrated process up to fourth order, since there are four differences in the numerator.

As with the exponential smoothing filter explored earlier, it turns out that the Fourier transform of the cyclical component filter has a particularly simple form:

$$\tilde{C}(\omega) = \frac{4\lambda[1-\cos(\omega)]^2}{1+ 4\lambda[1-\cos(\omega)]^2} .$$

Thus, the cyclical component filter places zero weight on the zero frequency $[\tilde{C}(0) = 0]$ and close to unit weight on high frequencies $[\tilde{C}(\pi) = 16\ \lambda/(1 + 16\ \lambda)]$. These features are reflected in Figure II-5, which plots the gain for various values of λ. Increasing λ shifts the gain function upward, moving a given frequency's gain closer to unity.

Developing time domain representations of the filter is once again more involved (see Appendix A). The first order condition F(B) may be factored into $(\lambda/\theta_1\theta_2)\ [(1 - \theta_1 B)(1 - \theta_2 B)(1 - \theta_1 B^{-1})(1 - \theta_2 B^{-1})]$, where θ_1 and θ_2 are complex conjugates whose value depends on λ. (These parameters are the zeros of F that satisfy $|\theta_i| < 1$). With this factorization, we can develop a two sided moving average expression for the growth component

$$y_t^g = \left[\frac{\theta_1 \theta_2}{\lambda}\right] \left[\sum_{j=0}^{\infty} \left[A_1 \theta_1^j + A_2 \theta_2^j\right] y_{t-j} + \sum_{j=0}^{\infty} \left[A_1 \theta_1^j + A_2 \theta_2^j\right] y_{t+j} \right]$$

where the parameters A_1 and A_2 depend on θ_1 and θ_2 in a manner spelled out in Appendix A. It may be shown that the coefficient $[A_1 \theta_1^j + A_2 \theta_2^j]$ is a real number for each j and that A_1 and A_2 are complex conjugates. Hence, the growth component is a two sided moving averages involving a kind of "double exponential smoothing." Table II-2 indicates the values of the θ's and A's that arise with different values of the smoothing parameter λ. Figure II-6 plots the filter weights of the cyclical filter for the λ=1600 value that has most frequently been employed, following Hodrick and Prescott [1980].[7]

Combining the results of this section, we conclude that the HP filter will render stationary series that are integrated (up to fourth order), but that it also removes substantial low frequency variation. On the other hand, the HP filter will preserve more low frequency content than the first difference which is commonly employed for the purpose of achieving stationarity. As in the case of the ES filter, this property derives from the fact that the (fourth) differences in the numerator are partly undone by the $[(1 - \theta_1 B)(1 - \theta_2 B)(1 - \theta_1 B^{-1})(1 - \theta_2 B^{-1})]$ terms which appear in the denominator, since the modulus of θ_i is about .9 with the smoothing parameter λ set equal to 1600. Another way to reach this conclusion is to examine Singleton's [1988, figure 2] comparison of the squared gain of the HP and first difference filter.

[7]See Appendix A for a discussion of the rationalization of λ = 1600 from the standpoint of an unobserved components model.

Comparisons of ES and HP Filters

There is a single parameter on which the gain of the ES and HP cyclical filters depends, the smoothing parameter. To compare the filters, we chose $\lambda = 1600$ for the HP cyclical filter and required that the gain of the HP and ES cyclical filters be equal at the frequency $\pi/16$, which corresponds to a period of 8 years (32 quarters).

The results of this comparison are given in Figure II–7: the HP filter looks more like the ideal filter presented in Figure 2, since its gain function is more nearly zero for frequencies below $\pi/16$ and more nearly unity for frequencies above it.

Inverse Optimal Linear Filtering

Given the form of the HP filter, one can ask "for what set of statistical structures is the HP filter an optimal linear filter in the sense of minimizing the mean square error as in Wiener [1949] and Whittle [1963]?"[8] We treat this question in Appendix B and summarize here the results of our investigation.

If innovations to the growth and cyclical components are uncorrelated, we find that a necessary condition for the HP filtering procedure to be optimal is that the stochastic growth component have a random walk growth rate, i.e., that it be second difference stationary in an extension of the Nelson and Plosser [1982] terminology. However, this condition is not sufficient. For the HP filter to be optimal, we must further require either that the cycle consist of uncorrelated events or that there be an identical dynamic

[8]This question was first posed to us by Mark Watson, who also provided useful hints about how to proceed with answering the question. However, Watson should not be held responsible for any potential errors in following these leads or for our interpretation of the results.

mechanism that propagates changes in the growth rate and innovations to the business cycle component.

In real business cycle models growth and business cycles do not arise as separate phenomena, so that these models provide no theoretical justification for decomposition into growth and cycles. The simplest way to introduce growth into a real business cycle model is to assume that the level of Harrod–neutral technical progress expands at a constant rate. This induces common deterministic trends into time series and economic fluctuations are stationary stochastic processes about this common trend. In this case there is a clear–cut separation between growth and cycles; growth is responsible for the common deterministic trend while cycles are the fluctuations around that trend. If we make exogenous technical progress stochasitc and assume that it follows an integrated process (a kind of "stochastic growth"), then these will generally set in motion complex responses that may resemble economic fluctuations (see King, Plosser and Rebelo [1988b, Section II]). Thus, it is difficult to separate growth and fluctuations in this context. The dividing lines virtually disappear in models of endogenous economic growth, in which transient displacements to the dynamic system have permanent consequences for the paths of economic quantities (King and Rebelo [1986]). However, given that there are a variety of motivations for filtering—some which do not hinge on an interest in precise growth versus cycle decompositions—we next explore the consequences of low frequency filtering in standard real business cycle models.

III. Filtering A Real Business Cycle Model

Our next objective is to investigate how application of a low frequency filter influences the time series generated by an artificial economy. The specific economy that we study is one that we have explored in detail elsewhere (King, Plosser, and Rebelo [1988a]), so that our presentation is deliberately brief. For reference purposes, the economy is close to that studied by Hansen [1985] and Prescott [1986], which contain examples of the application of HP filtering to model and actual time series.

The Basic Neoclassical Model

The deep structure of the model economy—preferences, technology and resource constraints—is specified as follows:

Preferences: The representative agent values sequences of consumption (C_t) and leisure (L_t) according to

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t u(C_t, L_t) \right],$$

where for simplicity we work with the loglinear momentary utility function $u(C_t, L_t) = \log(C_t) + \eta\log(L_t)$. In this expression E_0 is the expectation conditioned on information available at time zero.

Technology: The production and accumulation technologies are

$$Y_t = A_t \left[K_t^{1-\alpha}(N_t X_t)^{\alpha}\right] \qquad \text{and} \qquad K_{t+1} = (1-\delta)K_t + I_t,$$

where Y_t is output, N_t is labor input, K_t is capital, I_t is investment and δ is the rate of depreciation. The production function is constant returns-to-scale with $0 < \alpha < 1$. The exogenous variables are X_t, which is a labor augmenting technological shift that satisfies $X_{t+1}/X_t = \gamma_x > 1$, and A_t, which is a stationary total factor productivity shift that satisfies

$\log(A_t/\underline{A}) = \rho \log(A_{t-1}/\underline{A}) + \epsilon_{At}$ with $\underline{A} > 0$, $\rho > 0$, and ϵ_t is an iid random variable with $E(\epsilon_t) = 0$ and $E(\epsilon_t) = s_\epsilon^2$.

Resource Constraints. The resource constraints for goods and leisure are

$$C_t + I_t = Y_t \qquad \text{and} \qquad N_t + L_t = 1.$$

Values for technology and preference parameters are given in Table III-1.

Approximate Dynamics

The equilibrium quantities for consumption, investment, output, capital and real wages will fluctuate stochastically around a common deterministic trend induced by X_t. On the other hand, hours are stationary random variables. Approximating this system, we can develop a state space system for the logarithms of variables so that each variable can be written in the form $\log(Y_t) = \log(Y) + \log(X_t) + \hat{y}_t$, where $\hat{y}_t$ is interpretable as the deviation from trend.[9] The state space system which describes the vector $z_t = [\hat{y}_t\ \hat{c}_t\ \hat{i}_t\ \hat{k}_t\ \hat{w}_t\ \hat{N}_t]'$ then is:

$$z_t = \Pi\, s_t$$

with state evolution governed by

$$s_{t+1} = M\, s_t + \epsilon_{t+1} \qquad \text{and} \qquad M = \begin{bmatrix} \mu_1 & \pi_{kA} \\ 0 & \rho \end{bmatrix},$$

where $s_t = [\hat{k}_t\ \hat{A}_t]'$ and $\epsilon_t = [0 \quad \epsilon_{A,t+1}]$. Coefficients in the matrices Π and M—implied by the Table III-1 deep parameters—are given in Table III-2. Stationarity of deviations from trend ($\mu_1 < 1$) is assured by diminishing

[9] Our approximation strategy—which works off the first order conditions to the representative agent's dynamic optimization problem—is detailed in King [1987] and King, Plosser and Rebelo [1987]. In the present context, it is equivalent to the log linear approximation strategy of Christiano [1988], which uses quadratic approximation to the objective function.

returns to capital (holding fixed labor input). Thus, s_t is stationary so long as A_t is stationary ($\rho < 1$).

Thus, it is relatively easy to compute the population moments of the economic variables, z_t and s_t, by the two step procedure common in state space systems. First, one computes the moments of the states and then one exploits the simple relations that are readily shown to exist for moments of the z variables. For example, if $V=E(s_t s_t')$ is the variance-covariance matrix of the states, then $E(z_t z_t') = \Pi V \Pi'$ is the variance of the z variables. Results reported in subsequent tables involve application of these ideas in a straightforward manner.

Filtering the System

Table III-3 reports the consequences of application of the Hodrick and Prescott [1980] cyclical filter for the population moments of the model. Panel A of the table reports the moments of the original series (i.e., linearly detrended); panel B reports moments of the filtered series.

Looking first at the unfiltered moments in panel A, a researcher would draw one set of conclusions about the relative volatility of different series: labor input is about half as volatile as output; consumption is about two-thirds as variable; and investment is about twice as variable. The real wage is less volatile than output (about two thirds) but more variable than labor input. Further, one would conclude that labor input is at best only slightly more procyclical than capital input, on the basis that each has a contemporaneous correlation with output of about three quarters. Finally, one would view the stochastic components of output as relatively persistent given that the correlation of output with its fourth lag is about .75 and the correlation with its twelfth lag is about .4.

Turning now to panel B, one finds the population moments for the components of time series isolated by the HP cyclical filter, with the smoothing parameter λ set to 1600. Through this filter, a very different picture of business cycles emerges. Consumption is now only about a quarter as variable as output, labor input is two thirds as variable and investment is now nearly three times as variable. The volatility of the real wage is only one sixth that of output. Further, with an application of the HP filter, the real wage it is sharply less volatile than labor input (only about one half as volatile).

One now also has a very different picture of cyclical movements in inputs: labor input is very highly correlated with output (.98) and capital is unrelated to cyclical activity (its correlation is .07). Finally, autocorrelation in output is close to zero at a lag of one year (four quarters) and negative at a lag of three years (twelve quarters).

Considering the state space system, it is not too hard to see what is happening to produce these results. The evolution of all variables depends on their weights placed on the state variables, the capital stock and the technology shock. The technology shock is given by $\hat{A}_t = \rho \hat{A}_{t-1} + \epsilon_{At}$, which implies that it is representable as $\hat{A}_t = \sum_{s=0}^{\infty} \rho^s \epsilon_{A,t-s}$. Given the law of motion for capital, $\hat{k}_{t+1} = \mu_1 \hat{k}_t + \pi_{kA} \hat{A}_t$ with $\mu_1 \simeq .95$, the capital stock is a moving average of technology shocks, with weights that die out very slowly. Relative to the technology shock, then, the capital stock is very slow moving and application of the low frequency filter down plays its influence and raises that of the technology shock. Notice that this occurs despite the fact that both capital and technology are driven by ϵ_{At}, since they are related to it by different (one sided) linear filters.

Figure III–1 shows the impact of HP filtering on the spectral densities of these two variables (the dashed line is the spectral density of the unfiltered series and the solid line is the spectral density of the filtered series). Despite the fact that both variables display Granger's [1961] typical spectral shape, the power of the capital stock is more concentrated at low frequencies and, consequently, the HP filter downplays its relative influence.

Random Walk Technology Shocks

It is possible to solve this model under the alternative assumption that technology shocks are integrated processes (see Christiano [1988] or King, Plosser and Rebelo [1988b, section 2]). In view of the Nelson and Plosser [1982] results and given the intuitive idea that technology shocks are well modeled as a random walk (with positive drift), we present some final results based on that alternative specification in Table III–2. Since the levels of variables are not stationary, population moments are not finite. Thus, we present results for the first difference filter and for the HP cyclical filter. In the presence of this nonstationarity, the HP filter produces results that broadly resemble those of Table III–3, although the shift to a random walk technology shock does reduce the extent of labor volatility, as stressed by Hansen [1988].

Does Filtering Affect Moments that Are Very Important?

In viewing the foregoing results, one is naturally led to ask whether the practice of filtering affects moments that are very important from the standpoint of real business cycle research. In this research area, it is established practice to focus on a subset of moments—typically contemporaneous correlations and selected autocorrelations or cross correlations—in evaluating whether an alteration in a model's physical environment is quantitatively important. For example, Hansen's [1985] analysis of the influence of indivisibilities in labor supply on a real business cycle model concentrates on its implications for the contemporaneous covariance matrix of the model's variables, notably the relative volatility of hours and productivity. Clearly, given the foregoing, HP filtering will alter the moments studied by Hansen. However, no major alteration in one's views of the importance of this structural change is indicated by a careful comparison of Hansen's [1985] analysis (which uses HP filtering) and King, Plosser and Rebelo's [1988a] analysis of a similar economy (which does not employ HP filtering).

By contrast, with complicated model elements that are capitalistic in nature—that is, those which alter intertemporal substitution opportunities—HP filtering is likely to be far more important. To take one example, Rouwenhorst [1988] studies the influence of the "time to build" technology of Kydland and Prescott [1982]. He concludes that the major differences between models with and without time to build lie in the autocovariances—with jumps in otherwise smooth generating functions occurring at the lags that are integer multiples of the delay between the initiation and fruition of an investment project. The application of a smoothing procedure—such as the HP cyclical filter—would likely mask this key implication of the model. To take another example, there has been recent interest in the cyclical implications of models with endogenous long run growth (King and Rebelo [1986], King, Plosser and Rebelo [1988b] and Christiano and Eichenbaum [1988b]). A major feature of these models is the endogenous generation of a stochastic growth component of the form that is eliminated by the HP cyclical filter. We conclude that there are important and numerous extensions of real business cycle models in which essential information will be lost if the HP cyclical filter is the unique mechanism for viewing model implications.

IV. Implications of Our Analysis for Practice

To this point, we have provided an exposition and critique of an established practice in real business cycle research, the low frequency filtering of model and actual time series with a method due to Hodrick and Prescott [1980]. In our view, this procedure has gained widespread acceptance for three reasons, which are important background to our recommendations for alterations of research practice. First, as stressed by Hodrick and Prescott [1980, page 1], their method is a simple procedure that can be mechanically applied to economic time series. This characteristic reduces the judgmental decisions by a researcher and thus makes easier the process of cross–investigation comparison which is essential to scientific progress. Second, we have seen that HP cyclical filtering renders stationary series that have persistent changes in the underlying growth rate. Thus, as stressed by Hodrick and Prescott [1980, pp. 4–5], it is capable of accommodating phenomena such as "the productivity slowdown" in underlying time series. Third, the procedure implements a traditional view that economic growth and business cycles are phenomena that are to be studied separately. Further, application of the HP procedure generates summary statistics for real U.S. data that correspond to many economists' prior notions of "business cycle facts."

We now provide some suggestions for how researchers should modify practice based on the results of our investigation, so as to maximize scientific communication. These suggestions are based on three ideas: (i) it is desirable on statistical grounds to report sample moments only when time series have finite population counterparts; (ii) economic models generally contain explicit instructions about how to transform the data so that it will be stationary; and (iii) since the traditional separation of growth and business cycles is not an attribute of modern dynamic equilibrium theories, which embody concrete and extensive cross frequency restrictions, economists pursuing the real business cycle research program cannot have sharp priors about the decomposition of macroeconomic time series along these lines.

Reporting Attributes of Dynamic Macroeconomic Models: The moment implications of a dynamic equilibrium model are governed by its reduced form, e.g., the linear dynamic system summarized by the Π and M matrices. Investigators should always report sufficient information for calculation of alternative moment implications to be undertaken by another researcher without solution of the model.[10]

Reporting HP Growth Components: Researchers utilizing the HP filtering procedure should report moments of the actual and model generated "stochastic growth" components so that comparisons between models can be made on the basis of this information.

First, since the "prior" under the HP filtering procedure is that the actual data contain a stochastic trend, a transformation to achieve stationarity is necessary. Below, we report results for the HP stochastic

[10]Although not present in such important contributions as Kydland and Prescott [1982], Hansen [1985], and Prescott [1986], this information is provided in other early work by Kydland and Prescott [1979] and Long and Plosser [1983]. More recently, this practice is followed by Christiano [1988], Hansen and Sargent [1988], King, Plosser and Rebelo [1988a,b], and Kydland and Prescott [1988] in the recent *Journal of Monetary Economics* special issue on Real Business Cycles.

growth component extracted from a model filtered with $(1-B)(1-B^{-1})/2$ for this purpose. We have experimented with this centered second difference for two reasons: (i) it induces no phase shift; and, (ii) it renders stationary a time series with a random walk growth rate.

Second, the researcher should report statistics on this stochastic growth component under the transformation implied by the specified dynamic economic model that naturally achieves stationarity: we give two examples of this transformation below.

Reporting Moments for Model Based Data Transformations: Dynamic stochastic economic models generally suggest ways of treating nonstationarity in economic time series. Any investigation should at minimum report the direct transformations suggested by the model, since evidence against this transformation is useful in judging the adequacy of the model.

<u>Trend Stationary Models</u>: For example, one common model building strategy is to view economic time series as stationary relative to a common deterministic trend, as is implicit in Hansen [1985] and explicit in King, Plosser and Rebelo [1988a]. Under this scenario, our results suggest that HP filtering can dramatically alter how a researcher views a model economy as working, for example in terms the relative importance of variation in capital and labor in response to persistent but stationary technology shocks. Major components of time series on output, consumption etc. are treated by the filter as stochastic growth, when the posited model involves none. We recommend two alterations in practice for this case. First, researchers should report unfiltered moment implications as well as HP filtered moment implications. Researchers using the HP filter should also report attributes of the isolated stochastic growth components, under the model's assumption that these are stationary and the alternative assumption that there is a

random walk growth rate, which is implicit in analysis underlying HP filtering. In table IV.1, we provide an application of these methods to the population moments of the model economy described in section III above. These tables make clear that there is a substantial component of the time series removed by the HP filter and that this component in many ways resemble those of the unfiltered series. Thus, in such a trend stationary model, a much clearer picture of the operation of the theoretical model is provided by table III–3, panels A&B, and table IV–1, panels A&B, than by individual components.[11]

Presumably, it is not feasible to report all of the information in the tables we have presented here, given constraints on journal space. But it would be very easy to add information on HP growth components to our table I–1, which is a standard device for reporting implications of business cycle models.

Stochastic Trend Models: Frequently, low frequency filtering is motivated by concern over potential nonstationarity of macroeconomic time series as suggested, for example, by Nelson and Plosser's [1982] investigation of individual series and King, Plosser, Stock and Watson's [1987] investigation of common stochastic trends. For models with explicit stochastic growth elements—as in, for example, Christiano [1988], Hansen [1988] and King, Plosser and Rebelo [1988b]—it will generally not be meaningful to produce simulated moments for the levels of economic variables, since the population counterparts are not finite. Some transformation of

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[11]Further, in such trend stationary environments, we caution that research which focuses on dynamic elements of model construction—like that of Rouwenhorst (1988) discussed earlier—should be wary of interpreting HP filtered moments as providing much information about the importance of structural changes.

actual and model generated data will be necessary: two natural transformations that are consistent with the economic structure of stochastic steady state models are first differencing and construction of ratios of variables possessing common stochastic trends. Motivated by concern over nonstationarity, some recent investigations do undertake exploration of model sensitivity to filtering and data transformation in the way that our investigation suggests. Examples are provided in Christiano and Eichenbaum [1988a], who explore HP filtering and first differencing, and King, Plosser and Rebelo [1988b], who use first difference filtering and a ratio form that involves imposition of a common stochastic trend. Again, for researchers using HP filtering, our recommended practice requires reporting of statistics on stochastic growth components under (i) the model based assumption that the first difference is stationary and (ii) using the second difference filter discussed earlier.

No real business cycle research to this point has explicitly incorporated the persistent changes in productivity growth originally cited by Hodrick and Prescott [1980] as a major motivation for application of their filter to post war U.S. data. This feasible investigation could well shed further light on the interaction of growth and business cycles.

V. Summary and Conclusions

This paper has reported on implications of low frequency filtering, focusing on the HP filter—due to Hodrick and Prescott [1980]—which is commonly used in investigations of the stochastic properties of real business cycle models.

First, application of the filter to U.S. real gross national product and a measure of labor input illustrates the impact of HP filtering on the character of cyclical components. Second, we derive convenient expressions for the HP filter and the closely related exponential smoothing (ES) filter in forms appropriate for both the time domain and frequency domain. These results are used (i) to discuss the influence of smoothing parameters and (ii) to demonstrate that the cyclical components which these filters generate are stationary, when the underlying time series are differenced stationary stochastic processes in the sense of Nelson and Plosser [1982]. Third, we consider the conditions under which the HP filter is the optimal linear filter in the sense of Wiener [1949] and Whittle [1963]. These conditions are unlikely to be even approximately true in practice. Fourth, application of the HP filter to a basic real business cycle model demonstrates that this filter substantially influences the perception of the operation of the model economy, as viewed by researchers studying its moment implications. Fifth, based on the results of our investigations, we recommend some new practices designed to facilitate scientific communication between researchers.

At the end of our investigation, however, we remain struck by the Figures presented in section 1: macroeconomic research focusing on the component of the time series that is isolated by the HP cyclical filter—in terms of either devising stylized facts or evaluating dynamic economic models—is likely to capture only a subset of the time series variation that most economists associate with cyclical fluctuations. A major facet of our ongoing research is the construction of dynamic models that more completely integrate the explanation of these components.

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Appendix A

Analysis of the HP Filter in Time Domain

The starting point for our analysis is the (first order condition) requirement that

$$y_t = F(B)y_t^g$$

where $F(B) = \lambda \left[B^{-2} - 4B^{-1} + (6 + \frac{1}{\lambda}) - 4B + B^2\right]$.

$$= \lambda \left[(1 - B)^2 (1 - B^{-1})^2 + \frac{1}{\lambda}\right] .$$

A.1. <u>Zeros of F Polynomial</u>

We develop properties of the polynomial F(z), especially the location of its zeros, establishing the claims made in the main text.

(a) *Reciprocal Character of Roots* — since the polynomial F(z) is symmetric if z^* is a root then $1/z^*$ is also a root. To see this, note that F(z) = $\lambda\left[(1 - z)^2 (1 - \frac{1}{z})^2 + \frac{1}{\lambda}\right]$ for arbitrary z. Thus, if z^* implies $F(z^*) = \lambda\left[(1 - z^*)^2 (1 - \frac{1}{z^*})^2 + \frac{1}{\lambda}\right] = 0$ then $F(1/z^*) =$ $\lambda \left[(1 - 1/z^*)^2 (1 - z^*)^2 + \frac{1}{\lambda}\right] = F(z^*) = 0.$

(b) *Complex Character of Roots* — For any real number z, F(z) > 0. Thus, the roots must be complex. Further, it follows that z^* and $1/z^*$ are complex conjugates.

A.2. Inverting F(B) and Related Matters

The previous results imply that we can express F(B) as:

$$F(B) = (\lambda/\theta_1\theta_2)\ (1 - \theta_1 B)\ (1 - \theta_2 B)\ (1 - \theta_1 B^{-1})\ (1 - \theta_2 B^{-1}),$$

where $|\theta_i| < 1$, for i=1,2.

Thus to determine a useful form for $[F(B)]^{-1} = G(B)$, it is necessary to decompose

$$\frac{1}{1-\theta_1 z}\ \frac{1}{1-\theta_2 z}\ \frac{1}{1-\theta_1 z^{-1}}\ \frac{1}{1-\theta_2 z^{-1}}$$

$$\text{into}\quad A_0 + \frac{A_1}{1-\theta_1 z} + \frac{A_2}{1-\theta_2 z} + \frac{A_3}{1-\theta_1 z^{-1}} + \frac{A_4}{1-\theta_2 z^{-1}}\ .$$

To determine A_0, A_1, A_2, A_3 and A_4 we require that

$$1 = A_0(1-\theta_1 z)\ (1-\theta_2 z)\ (1-\theta_1 z^{-1})\ (1-\theta_2 z^{-1})$$

$$+ A_1(1-\theta_2 z)(1-\theta_1 z^{-1})(1-\theta_2 z^{-1}) + A_2(1-\theta_1 z)\ (1-\theta_1 z^{-1})\ (1-\theta_2 z^{-1})$$

$$+ A_3(1-\theta_1 z)\ (1-\theta_2 z)\ (1-\theta_2 z^{-1}) + A_4(1-\theta_1 z)\ (1-\theta_2 z)\ (1-\theta_1 z^{-1})\ .$$

Evaluating this expression at z=1 yields

$$\frac{1}{1-\theta_1}\ \frac{1}{1-\theta_2}\ \frac{1}{1-\theta_1}\ \frac{1}{1-\theta_2} = A + \frac{A_1}{1-\theta_1} + \frac{A_2}{1-\theta_2} + \frac{A_3}{1-\theta_1} + \frac{A_4}{1-\theta_2}$$

Evaluating this expression at $z = 1/\theta_1$ yields

$$A_1 = \left[(1 - \theta_2/\theta_1)\ (1 - \theta_1^2)\ (1 - \theta_2\theta_1)\right]^{-1}.$$

and evaluating at the other roots yields

$$A_2 = \left[(1 - \theta_1/\theta_2)\ (1 - \theta_1\theta_2)\ (1 - \theta_2^2)\right]^{-1}$$

$$A_3 = \left[(1 - \theta_1^2)\ (1 - \theta_1\theta_2)\ (1 - \theta_2/\theta_1)\right]^{-1}$$

$$A_4 = \left[(1 - \theta_1\theta_2)\ (1 - \theta_2^2)\ (1 - \theta_1/\theta_2)\right]^{-1}.$$

Some useful properties of these expressions are as follows. First, $A_1 = A_3$ and $A_2 = A_4$. Second, A_1 and A_2 are complex conjugates, as is most readily evident if we move to the (polar form) representation θ_1 = rexp (im) and θ_2 = rexp (−im). Then, when we substitute these expressions for θ_1 and θ_2 into the preceding expressions for A_1 and A_2, we find that:

$$A_1 = \left[(1 - \exp(-2im))\ (1 - r^2\exp(2im))\ (1 - r^2)\right]^{-1}$$

$$A_2 = \left[(1 - \exp(2im))\ (1 - r^2\exp(-2im))\ (1 - r^2)\right]^{-1},$$

so that the conjugate status of these coefficients becomes clear. Hence, combining the results of the foregoing, we can express the growth filter as

$$G(B) = [F(B)]^{-1}$$

$$= \left[\frac{\theta_1\theta_2}{\lambda}\right]\left\{A_o + \left[\frac{A_1}{1-\theta_1 B} + \frac{A_2}{1-\theta_2 B}\right] + \left[\frac{A_1}{1-\theta_1 B^{-1}} + \frac{A_2}{1-\theta_2 B^{-1}}\right]\right\} .$$

A.3 Coefficients in the Growth Filter

To establish that the coefficients in the growth filter — which depend $A_1\theta_1^j + A_2\theta_2^j$ for $j \geq 0$ — are real, it is again convenient to adopt the polar form representation

$$\theta_1 = r \exp (im) \qquad \theta_2 = r \exp (-im)$$

$$A_1 = R \exp (iM) \qquad A_2 = R \exp (-iM).$$

Then, it follows that

$$\left[A_1\theta_1^j + A_2\theta_2^j\right] = Rr^j \exp (i(M+jm)) + Rr^j \exp (-i(M+jm))$$

$$= 2Rr^j \cos (M+jm).$$

Thus, we can write $G(B) = \sum_{j=-\infty}^{\infty} g_j B^j$ as

$$= \left[\frac{r^2}{\lambda}\right] \left\{ A_o + 2R \sum_{j=0}^{\infty} r^j \cos(M+jm) B^j \right.$$

$$\left. + 2R \sum_{j=0}^{\infty} r^j \cos(M+jm) B^{-j} \right\}$$

which indicates that the roots are real. Further, using $\cos(jm + M) = [\cos(mj) \cos(M) - \sin(mj) \sin(M)]$ it is direct to establish the form of the filter provided by Hodrick and Prescott [1980] and Singleton [1988]. For this purpose, we note that A_o turns out to be $-2R \cos(M)$. Then, the previous expression for G(B) may be written as:

$$G(B) = \sum_{j=-\infty}^{\infty} g_j B^j$$

where

$$g_j = r^j [a_1 \cos(bj) + a_2 \sin(bj)] \qquad \text{for } j \geq 0$$

$$g_j = g_{-j} \qquad \text{for } j \leq 0$$

with the constants $a_1 = [r^2/\lambda]\ 2R \cos(M)$, $a_2 = [r^2/\lambda]\ 2R \sin(M)$, $b = |m|$.

Appendix B

Inverse Optimal Linear Filtering

Taking as given a specific filter, the Hodrick and Prescott [1980] filter in our context—one can ask what the implicit model for the underlying time series must be for this filter to be optimal in the sense of minimizing the mean square error as in Wiener [1949] and Whittle [1963]. In order to be possible for the HP filter to be optional we start with a statistical representation of the underlying time series which is linear and in which growth and cycles are separate phenomena.

Suppose that we view the growth and cyclical components as being generated by ARMA models,

$$A^g(B)\ y_t^g = M^g(B)\ \epsilon_t^g$$
$$A^c(B)\ y_t^c = M^c(B)\ \epsilon_t^c$$

where ϵ_t^g and ϵ_t^c are white noise processes whose variances are $s^2(\epsilon^c)$ and $s^2(\epsilon^g)$. By assumption, the roots of the autoregressive polynomials lie outside the unit circle (stationarity) and the roots of the moving average polynomial lie outside the unit circle (invertibility). The innovations ϵ_t^g and ϵ_t^c are serially uncorrelated and, for simplicity, we assume that $E[\epsilon_t^g \epsilon_t^c] = 0$. Further, for convenience, we define the ratio of variances $\psi = s^2(\epsilon^c)/[s^2(\epsilon^c) + s(\epsilon^g)]$.

Whittle [1963, chapter V] shows that the optimal (two sided) signal extraction filter for the cyclical component is:

$$C^*(B) = \frac{\Gamma_{cc}(B)}{\Gamma_{cc}(B) + \Gamma_{gg}(B)}$$

where $\Gamma_{cc}(B)$ is the autocovariance generating function of the cyclical component and $\Gamma_{gg}(B)$ is the autocovariance generating function of the growth component. From the ARMA structure it follows directly that

$$\Gamma_{cc}(z) = \frac{M^c(z)\ M^c(z^{-1})}{A^c(z)\ A^c(z^{-1})}\ s^2(\epsilon_t^c)$$

$$\Gamma_{gg}(z) = \frac{M^g(z)\ M^g(z^{-1})}{A^g(z)\ A^g(z^{-1})}\ s^2(\epsilon_t^g)$$

Hence, it follows that the optimal filter may be expressed as:

$$C^*(B) = \frac{\psi\ A^g(B)\quad A^g(B^{-1})}{\psi A^g(B)A^g(B^{-1})+(1-\psi)Q(B)}$$

where $Q(B) = [A^c(B)A^c(B^{-1})][M^g(B)M^g(B^{-1})]/[M^c(B)M^c(B^{-1})]$.

Whittle's analysis [1963] is limited to stationary ARMA processes. However, recent work extends these formulas to cases with unit roots (Watson [1986] provides a brief summary of Bell's [1984] work on these cases).

<u>Matching the HP Cyclical Filter</u>

The HP cyclical filter may be written as

$$C(B) = [F(B)-1][F(B)^{-1}] = \frac{\lambda\ [1-B]^2\ [1-B^{-1}]^2}{1 + \lambda[1-B]^2\ [1-B^{-1}]^2}$$

The problem is to find AR and MA polynomials ($A^g(B)$, $A^c(B)$, $M^g(B)$, and $M^c(B)$) such that $C(B)$ and $C^*(B)$ coincide.

One example of such an inverse optimal filtering rule is discussed by Hodrick and Prescott [1980, page 5] and involves assuming that

$$A^g(B) = (1-B)^2$$

$$A^c(B) = M^g(B) = M^c(B) = 1$$

That its, under this specification, the change in the growth rate is a white noise as is the cyclical component. Further, the parameter λ corresponds to $\psi/(1-\psi)$ which is equal to the ratio of variances $\lambda = s^2(\epsilon^c)/s^2(\epsilon^g)$ or $\lambda^{(1/2)} = s(\epsilon^c)/s(\epsilon^g)$. Hodrick and Prescott [1980] use a "prior view that a five percent cyclical component is moderately large as is a one-eighth of one percent change in the rate of growth in a quarter. This led us to select or $\lambda^{(1/2)} = 5/(1/8)$ or $\lambda = 1600$ as a value for the smoothing parameter."

Pursuing this line further, suppose that we require that $A^g(B) = (1-B)^2$ so as to accommodate nonstationarity in the growth rate. Then, it follows that $C(B) = C^*(B)$ requires that

$$\frac{1}{\lambda} = \left(\frac{1-\psi}{\psi}\right) Q(B).$$

Thus, the optimality of the HP filter requires—apart from the constant terms—implies restrictions across the $A^c(B)$, $M^c(B)$, and $M^g(B)$ polynomials. In particular, it requires that

$$M^c(B) = \left[\frac{\lambda(1-\psi)}{\psi}\right]^{(1/2)} A^c(B) M^g(B).$$

In our view, these sorts of restrictions are unlikely to arise directly from the structure of dynamic economic models since in these models growth and cycles do not tend to arise as separate phenomena.

Table I-1

EFFECTS OF FILTERING ON SELECTED POPULATION MOMENTS:
BASIC NEOCLASSICAL MODEL

	Deviations from Linear Trend			Deviations from HP Trend		
Variable	Std. Dev.	Relative Std. Dev.	Correlation w/ Cyclical Output	Std. Dev.	Relative Std. Dev.	Correlation w/ Cyclical Output
Output	4.26	1.00	1.00	2.07	1.00	1.00
Consumption	2.73	0.64	0.82	0.53	0.26	0.78
Investment	9.82	2.31	0.92	6.08	2.94	0.72
Hours	2.04	0.48	0.79	1.35	0.65	0.98
Real Wage	2.92	0.69	0.90	0.79	0.17	0.59
Capital Stock	3.74	0.88	0.73	0.58	0.28	0.07

The relative standard deviation of a variable (z) denotes the ratio std(z)/std(y), where y is output. It is thus each entry in the standard deviation column divided by the first entry.

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Table I-2
Output and Hours, 1948.1-1987.4

Panel A: Autocorrelations

Lag:	1	2	3	4	8	12	16	24
$HP^c(y_t)$	.85	.62	.34	.09	−.33	−.32	−.07	−.05
y_t^r	.96	.89	.81	.72	.41	.20	.13	−.02
$HP^c(N_t)$	.83	.61	.37	.14	−.40	−.34	−.02	.04
$N_t-\bar{N}$	.93	.85	.75	.65	.37	.24	.19	.11

Panel B: Correlation Matrix of Variables

	y_t^r	$HP(y_t)$	$HP(y_t)$	N_t-N	$HP(N_t)$	$HP(N_t)$
y_t^r	1	0.60	0.89	0.06	0.48	−.24
$HP^c(y_t)$		1	0.17	0.56	0.86	0.15
$HP^g(y_t)$			1	−.24	0.11	−.37
$N_t-\bar{N}$				1	0.62	0.86
$HP^c(N_t)$					1	0.14
$HP^g(N_t)$						1

Table I–2(cont'd)

Panel C: Standard Deviations of Variables					
y_t^r	$HP^c(y_t)$	$HP^g(y_t)$	$N_t-\bar{N}$	$HP^c(N_t)$	$HP^g(N_t)$
.043	.020	.035	.030	.016	.024

Notes:

#1: The data employed in this table and Figures 1–5 are quarterly U.S. time series over 1948.1–1987.4 constructed from entries in CITIBASE. GNP refers to U.S. real gross national product in 19982 base (CITIBASE code GNP82). The hours percapita series is constructed as follows: First, monthly series on civilian noninstitutional population 16 years and older (CITIBASE code D16); total workers (CITIBASE code LHEM); and average weekly hours (CITIBASE code LHCH) were obtained on a monthly basis. Second, percapita hours at the monthly frequency (N) was formed as N = LHCH*LHEM/D16. Third, the monthly entries were averaged to form quarterly average hours per capita.

#2: Although we report moments for the growth component of the series, this information has to be interpreted with caution, since despite the fact that a linear trend has been removed prior to filtering, the growth components may be nonstationary.

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Table II–1:

Impact of Smoothing Parameter (λ) on Exponential Smoothing Parameter (θ)	
λ	θ
7.5	0.6955
15.0	0.7730
30.0	0.8333
60.0	0.8790
120.0	0.9128
240.0	0.9375
480.0	0.9554
960.0	0.9682

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Table II-2:

Impact of Smoothing Parameter (λ) on Parameters of the HP Filter				
λ	re(θ_1)	im(θ_1)	re(A_1)	im(A_1)
100	0.7792	0.1764	0.0566	0.0552
400	0.8429	0.1341	0.0398	0.0393
1600	0.8885	0.0997	0.0280	0.0279
6400	0.9211	0.0729	0.0198	0.0197

notes:

(i) re(θ_1) is the real part of θ_1 and im(θ_1) is the imaginary part of θ_1.

(ii) since each pair θ_1, θ_2 and A_1, A_2 are complex conjugates, it suffices to report the real and imaginary parts of each since, for example, θ_2 = re(θ_1) − im(θ_1).

man8901:tII2

Table III-1:

Economic Parameter Values	
Depreciation rate:	$\delta = .025$
Labor's share	$\alpha = .58$
Gross growth rate:	$\gamma_x = 1.004$
Discount factor:	$\beta = 1/(1+.016)$
Steady state fraction of time spent working*:	$N = .20$
Technology persistence parameter	$\rho = .9$
Standard Deviation of Technology Innovation	$s^2(\epsilon) = 1.00$

* It is equivalent for us to specify the steady state fraction or the utility parameter η, since there is a simple relation that links these two parameters.

man8901:tIII1

Table III-2:

Parameters of the Log-linear System

	State Variable:	
	Capital	Technology Shock
Consumption	π_{ck} = .617	π_{cA} = .298
Labor Input	π_{Nk} = −.294	π_{NA} = 1.048
Investment	π_{ik} = −.629	π_{iA} = 4.733
Output	π_{yk} = .249	π_{yA} = 1.608
Real Wage	π_{wk} = .544	π_{wA} = .560
Capital Stock	μ_1 = .953	π_{kA} = .137

note: For details on derivations of these coefficients, see King, Plosser and Rebelo (1988a).

man8901:tIII2

Table III–3

EFFECTS OF FILTERING ON POPULATION MOMENTS

Variable	Std. Deviation	Std. Dev. Relative to $\hat{y}$	Autocorrelations 1	2	3	Cross-correlations with $\hat{y}_{t-j}$ 12	8	4	2	1	0	–1	–2	–4	–8	–12
Panel A: Moments of Original Series																
$\hat{y}$	4.26	1.0	.93	.86	.80	.42	.55	.74	.86	.93	1.0	.93	.86	.74	.55	.42
$\hat{c}$	2.73	.64	.99	.98	.97	.76	.82	.86	.85	.84	.82	.76	.71	.61	.47	.36
$\hat{i}$	9.82	2.31	.88	.77	.67	.11	.26	.52	.70	.80	.92	.85	.79	.68	.50	.38
$\hat{N}$	2.04	.48	.86	.73	.62	–.11	.04	.32	.52	.65	.79	.73	.67	.57	.42	.31
$\hat{w}$	2.92	.69	.98	.96	.94	.69	.78	.85	.88	.90	.90	.84	.78	.67	.51	.39
$\hat{A}$	2.29	.54	.90	.81	.73	.28	.42	.64	.80	.88	.98	.91	.84	.72	.54	.40
$\hat{k}$	3.74	.88	1.00	.99	.98	.81	.85	.82	.77	.73	.68	.63	.59	.51	.39	.30
$\hat{r}$	0.11	.03	.87	.76	.66	–.47	–.34	–.07	.14	.28	.43	.40	.36	.30	.22	.16

Table III-3 (Continued)

EFFECTS OF FILTERING ON POPULATION MOMENTS

Variable	Std. Deviation	Std. Dev. Relative to $\hat{y}$	Autocorrelations			Cross-correlations with $\hat{y}_{t-j}$										
			1	2	3	12	8	4	2	1	0	−1	−2	−4	−8	−12
Panel B: Moments of Filtered Series (HP filter)																
$\hat{y}$	2.07	1.0	.70	.45	.23	−.23	−.23	.08	.45	.70	1.0	.70	.45	.08	−.23	−.23
$\hat{c}$	.53	.26	.86	.69	.52	−.13	.10	.49	.68	.74	.78	.42	.13	−.23	−.43	−.29
$\hat{i}$	6.08	2.94	.69	.43	.23	−.24	−.29	.00	.38	.65	.99	.72	.49	−.15	−.18	−.21
$\hat{N}$	1.35	.65	.69	.43	.22	−.24	−.31	−.05	.34	.62	.98	.72	.51	.18	−.15	−.20
$\hat{w}$	.79	.17	.77	.55	.36	−.19	−.07	.31	.60	.77	.94	.59	.31	−.08	−.35	−.28
$\hat{A}$	1.28	.62	.69	.44	.23	−.24	−.26	.04	.41	.67	1.00	.71	.47	.12	−.20	−.22
$\hat{k}$	.58	.28	.95	.84	.69	.06	.43	.68	.55	.37	.07	−.15	−.30	−.46	−.41	−.19
$\hat{r}$	.07	.03	.69	.43	.22	−.24	−.36	−.12	.27	.57	.95	.73	.53	.22	−.10	−.17

Table III-3 (Continued)

EFFECTS OF FILTERING ON POPULATION MOMENTS

Variable	Std. Deviation	Std. Dev. Relative to $\hat{y}$	Autocorrelations			Cross-correlations with $\hat{y}_{t-j}$										
			1	2	3	12	8	4	2	1	0	−1	−2	−4	−8	−12
Panel C: Moments of Filtered Series (first-difference filter)																
$\hat{y}$	1.64	1.00	−.04	−.04	−.03	−.02	−.02	−.03	−.04	−.04	1.00	−.04	−.04	−.03	−.02	−.02
$\hat{c}$	0.32	.19	.27	.23	.19	.02	.05	.09	.13	.14	.88	−.10	−.09	−.08	−.05	−.04
$\hat{i}$	4.89	2.98	−.05	−.05	−.04	−.02	−.03	−.05	−.06	−.07	1.00	−.03	−.03	−.02	−.02	−.01
$\hat{N}$	1.09	.67	−.06	−.05	−.05	−.02	−.04	−.06	−.07	−.08	.99	−.03	−.02	−.02	−.01	−.01
$\hat{w}$	0.57	.35	.05	.04	.03	.00	.01	.02	.03	.04	.98	−.07	−.06	−.05	−.04	−.03

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Table III-4

Effects of Filtering on Population Moments: Unit Root Model

Variable	Std. Deviation	Relative Std. Dev.*	Autocorrelations 1	2	3	Cross-correlations with $\Delta\log(Y_{t-j})$ 12	8	4	2	1	0	−1	−2	−4	−8	−12
Panel A: Moments of First Differences of Original Series																
$\Delta\log(Y_t)$	.75	1.00	.02	.02	.02	.01	.01	.02	.02	.02	1.0	.02	.02	.02	.01	.01
$\Delta\log(C_t)$	.39	.52	.13	.12	.12	.05	.06	.07	.08	.09	.98	.03	.03	.02	.02	.02
$\Delta\log(I_t)$	1.63	2.17	−.01	−.01	−.01	−.01	−.02	−.02	−.02	−.02	.99	.01	.01	.01	.01	.01
$\Delta\hat{N}$	.30	.40	−.02	−.02	−.02	−.03	−.04	−.05	−.05	−.05	.98	.01	.01	.01	.01	.00
$\hat{N}$	.97	1.29	.95	.91	.87	.20	.24	.29	.32	.34	.35	.05	.05	.04	.04	.03

* denotes standard deviation of x relative to standard deviation of growth rate of output.

Table III–4 (Continued)

Effects of Filtering on Population Moments: Unit Root Model

Variable	Std. Deviation	Relative Std. Dev.*	Autocorrelations 1	2	3	Cross-correlations with $\log(Y_{t-j})$ 12	8	4	2	1	0	−1	−2	−4	−8	−12
						Panel B: Moments of Filtered Series (HP Filter)										
Y_t	.99	1	.68	.47	.28	−.24	−.20	.13	.47	.68	1	.68	.47	.13	−.20	
C_t	.53	.53	.72	.53	.35	−.21	−.10	.26	.56	.73	.98	.63	.40	.04	−.29	
I_t	2.15	2.16	.67	.46	.27	−.26	−.26	.05	.41	.64	.99	.70	.51	.18	−.15	
$\hat{N}$	.39	.40	.67	.45	.26	−.27	−.31	−.02	.35	.60	.98	.71	.53	.23	−.11	

* denotes standard deviation of HP filtered x relative to standard deviation of filtered output.

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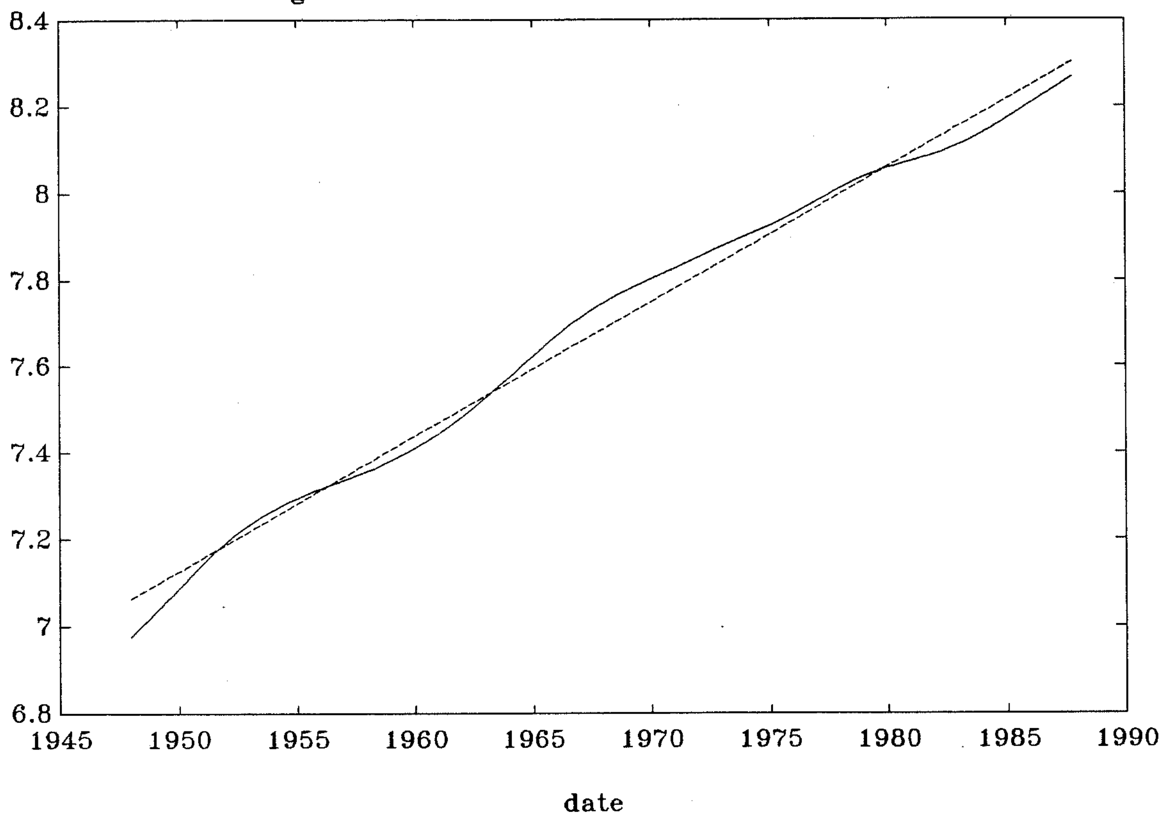
Table IV-1

Moments of Alternative HP Growth Components for Trend Stationary Neoclassical Model

			Autocorrelations			Cross Correlations with $\hat{y}_{t-j}$										
Variable	sd	sd(z)/sd(y)	1	2	3	−12	−8	−4	−2	−1	0	1	2	4	8	12
Panel A: Standard HP Growth Component																
$\hat{y}$	3.43	1.00	1.0	.99	.98	.76	.88	.97	.99	1.0	1.0	1.0	.99	.97	.88	.76
$\hat{c}$	2.60	.76	1.0	1.00	.99	.95	.98	.96	.93	.92	.90	.88	.85	.80	.68	.56
$\hat{i}$	6.63	1.93	1.0	.99	.97	.44	.63	.80	.86	.89	.91	.93	.94	.95	.91	.81
$\hat{N}$	1.27	.37	1.0	.98	.96	.08	.31	.52	.61	.65	.69	.72	.75	.78	.79	.73
$\hat{w}$	2.71	.79	1.0	1.0	.99	.92	.97	.98	.97	.95	.94	.92	.90	.86	.75	.62
$\hat{r}$	.08	.02	1.0	.99	.97	−.56	−.36	−.14	−.03	.02	.06	.11	.15	.22	.31	.33
$\hat{k}$	1.70	.50	1.0	.99	.98	.63	.79	.92	.96	.97	.98	.99	.99	.98	.91	.80
$\hat{A}$	3.59	1.05	1.0	1.0	.99	.97	.96	.91	.88	.85	.83	.80	.77	.72	.59	.47

Figure I.1: Linear and HP Trends in Real GNP



- - - - - - Linear Trend Component of Logarithm of Real GNP

——— HP Trend Component of Logarithm of Real GNP

Figure I.2: HP Growth Component v. Linear Trend Residual

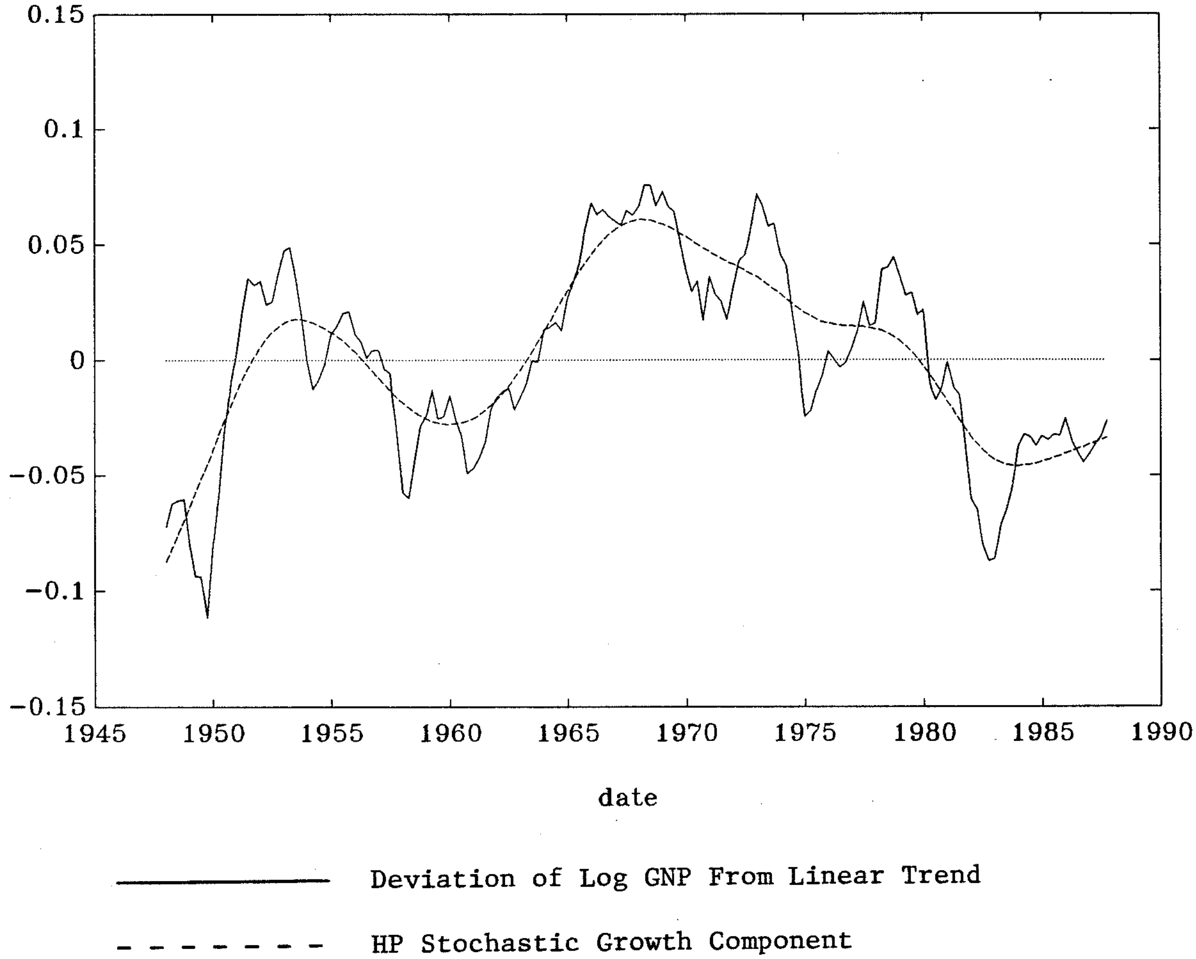


Figure I.3: HP Cyclical Component of Real GNP

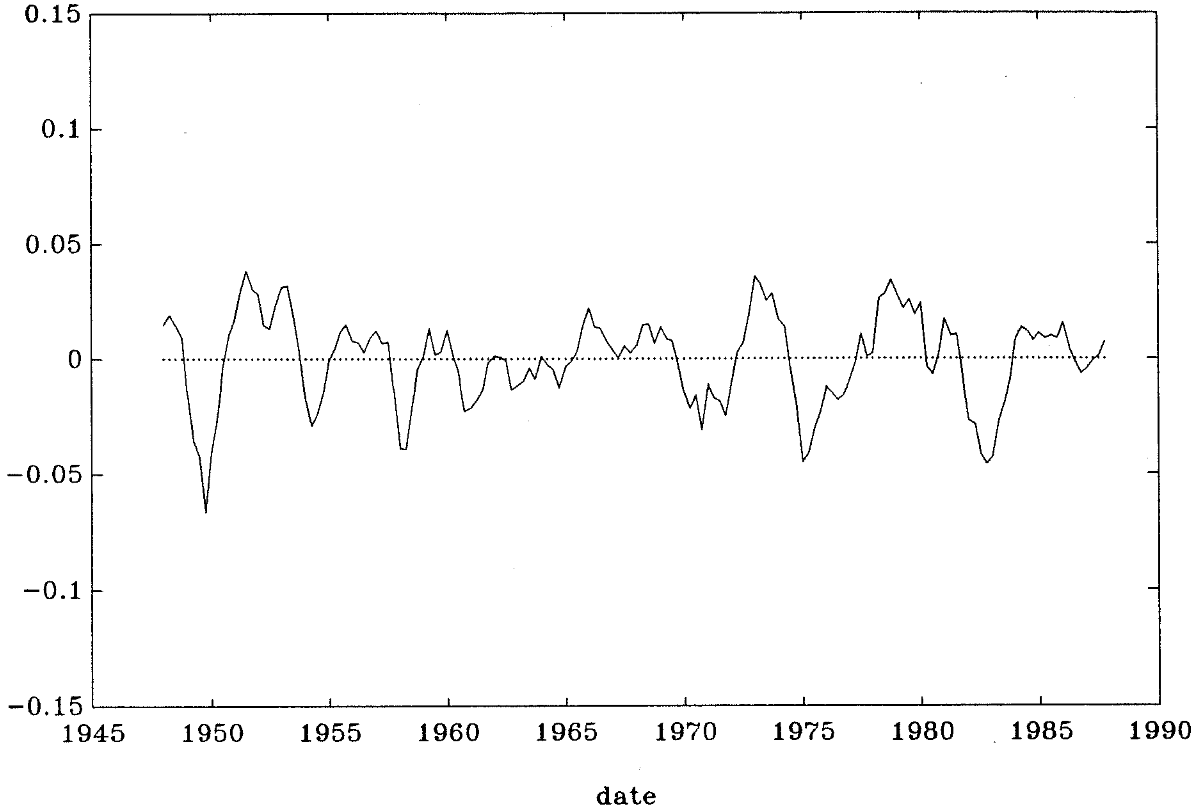


Figure I-4: logHours - mean v. logGNP-linear trend

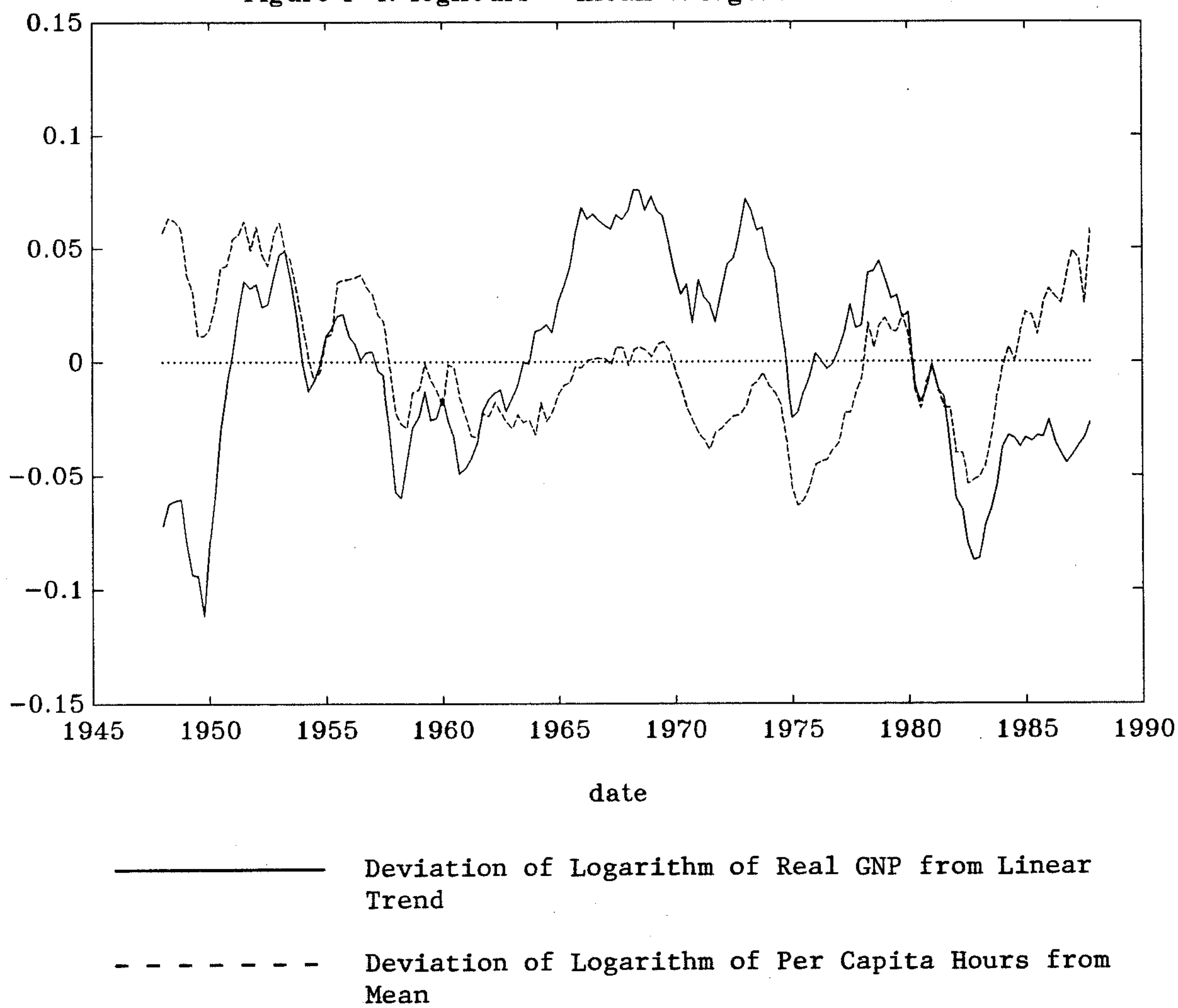


Figure I–5: HP cyclical components of output and hours

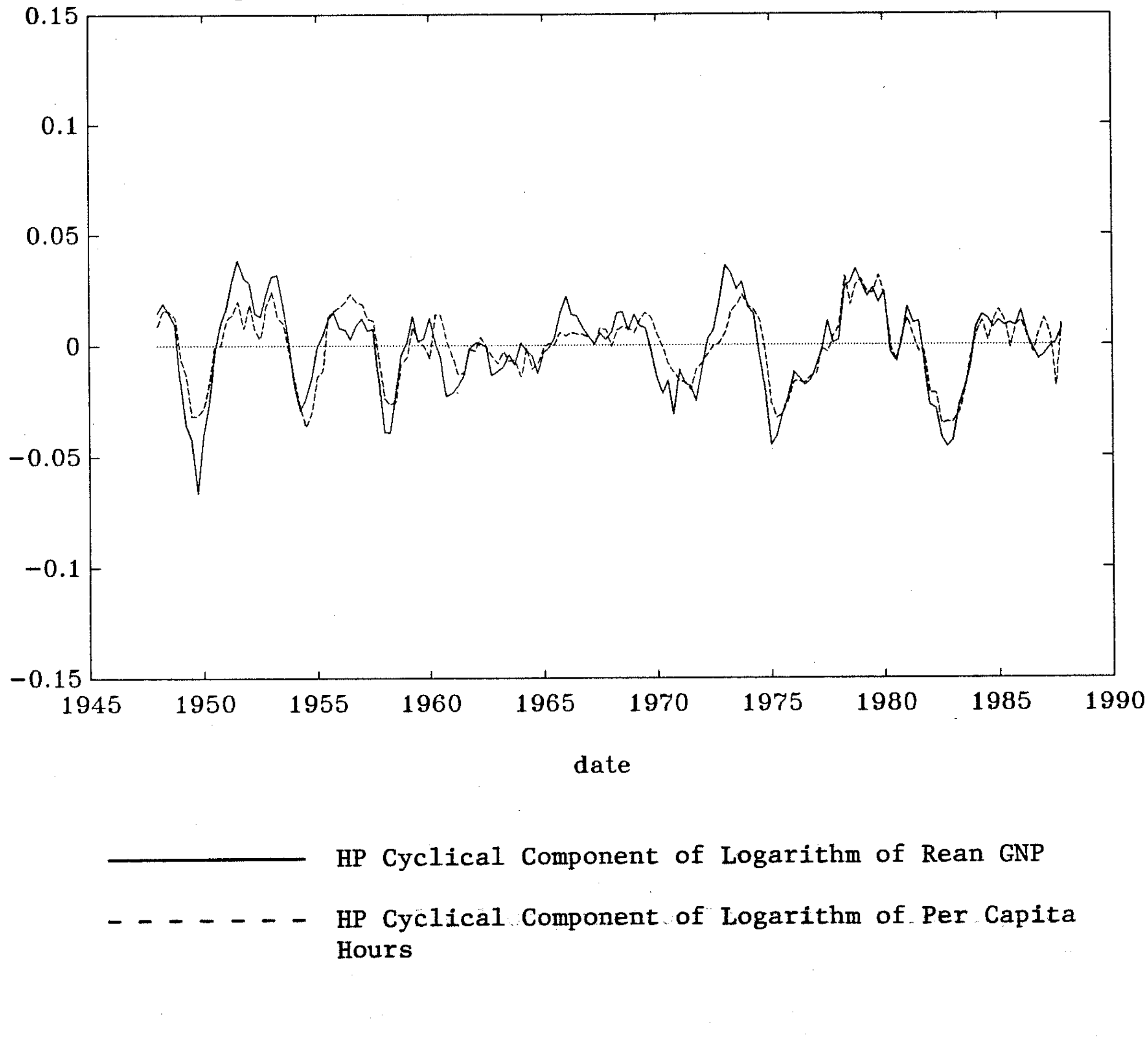


Figure II-1a: Impact of Filtering--Increase in Gain

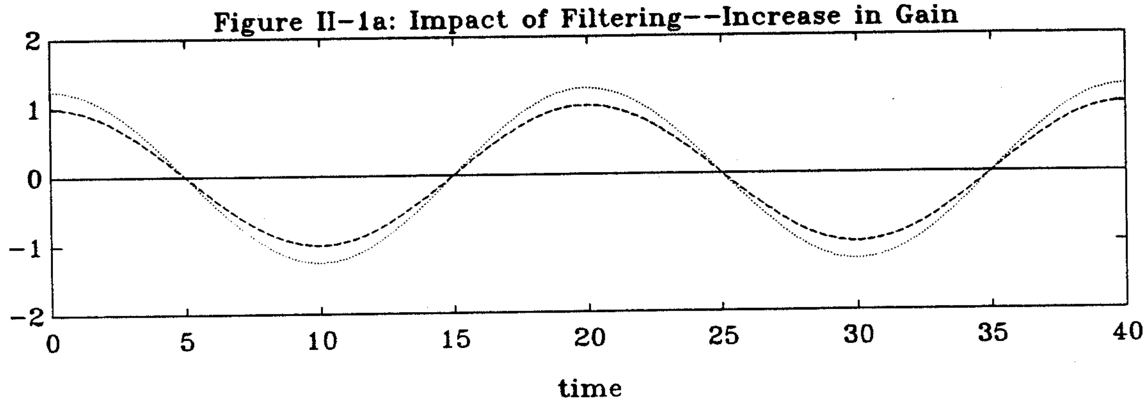
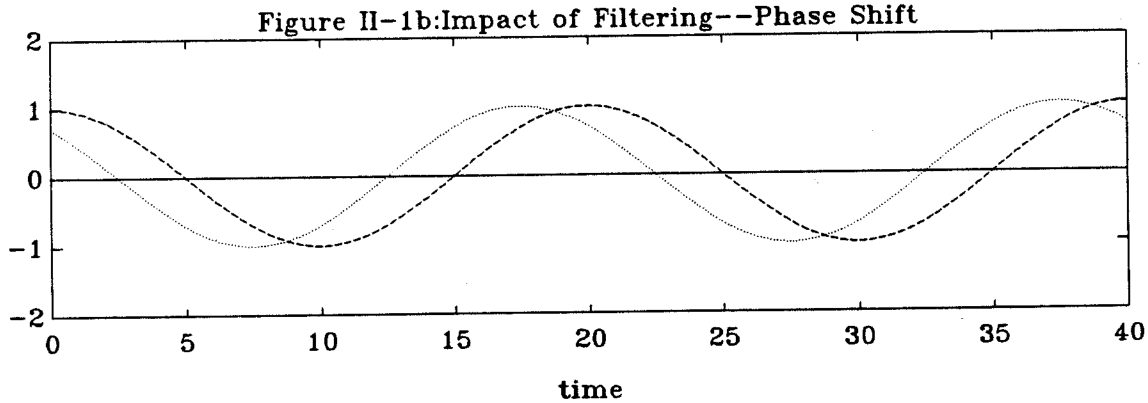


Figure II-1b:Impact of Filtering--Phase Shift



- - - - - - - Original Series

. . . . Filtered Series

Figure II-2a: idealized low frequency filter

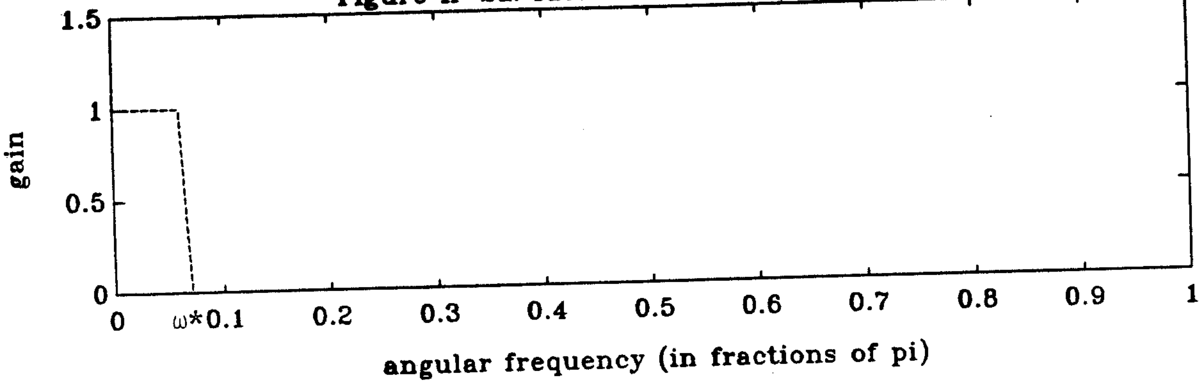


Figure II-2b: implied cyclical filter

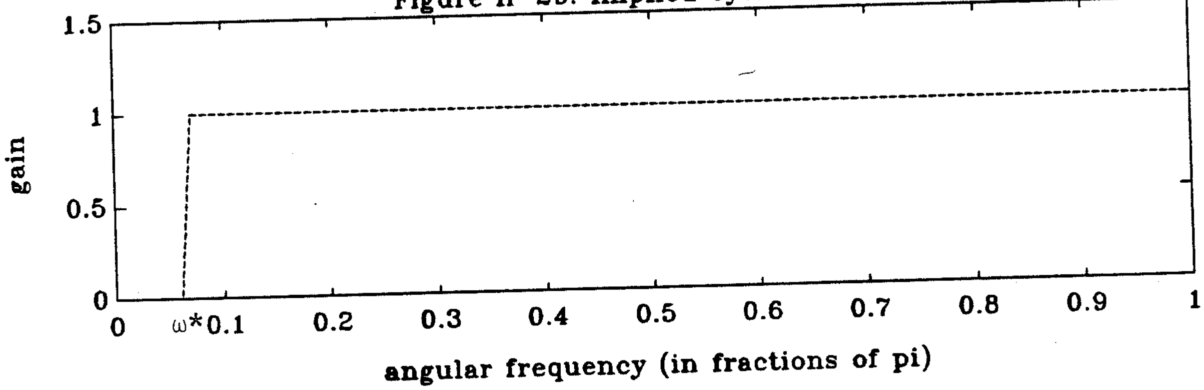
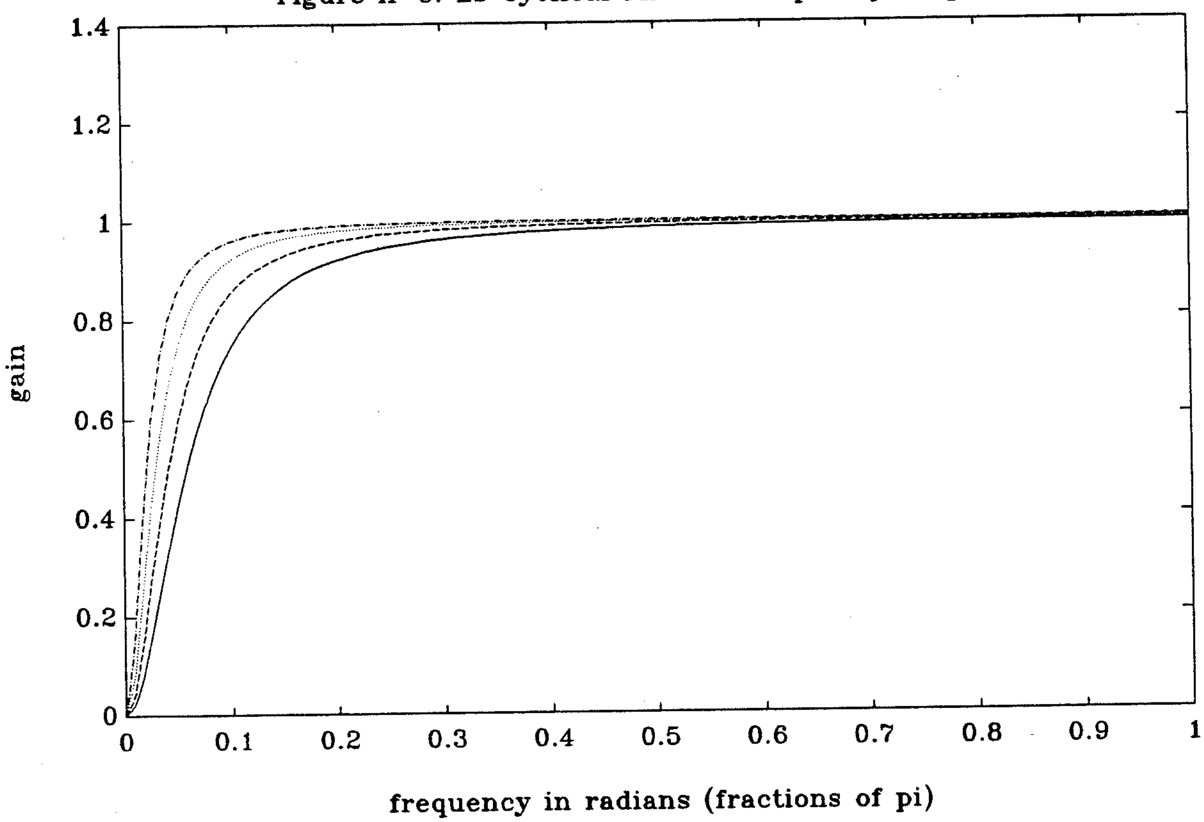
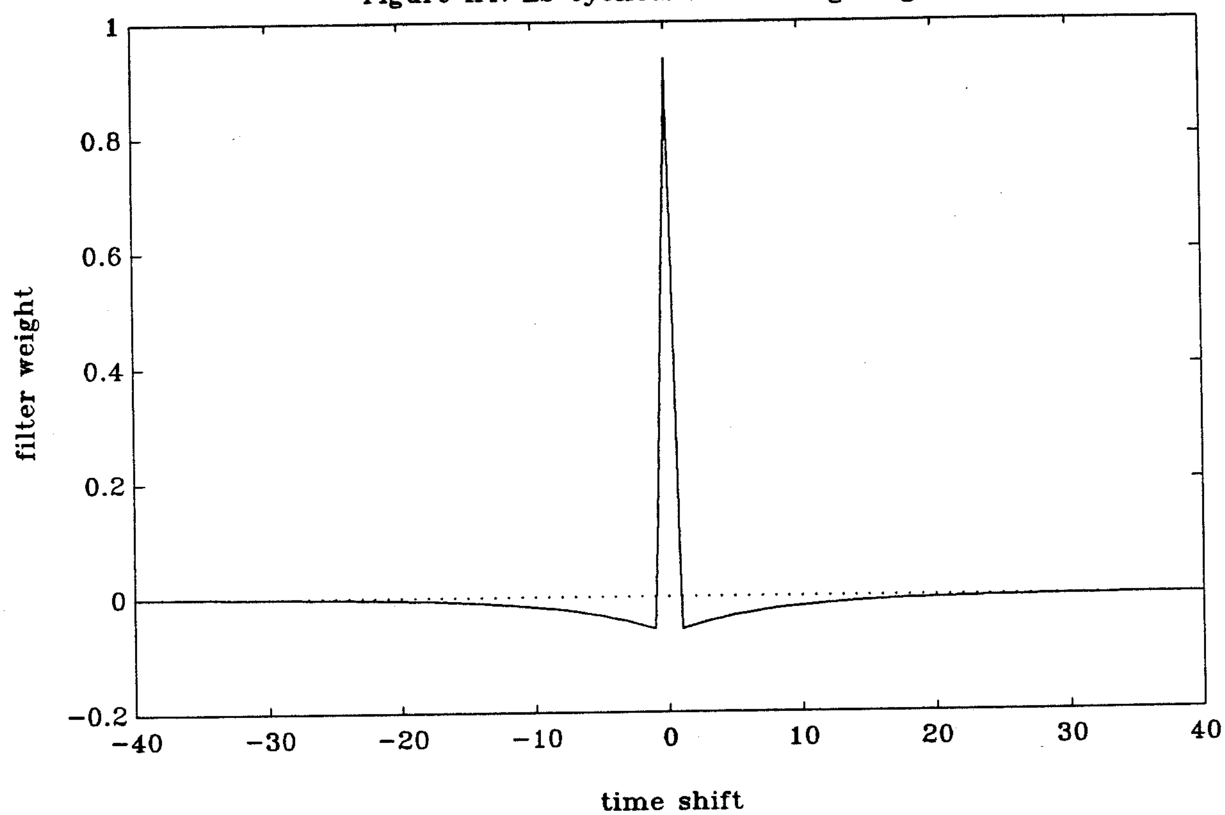


Figure II-3: ES Cyclical Filter--Frequency Response



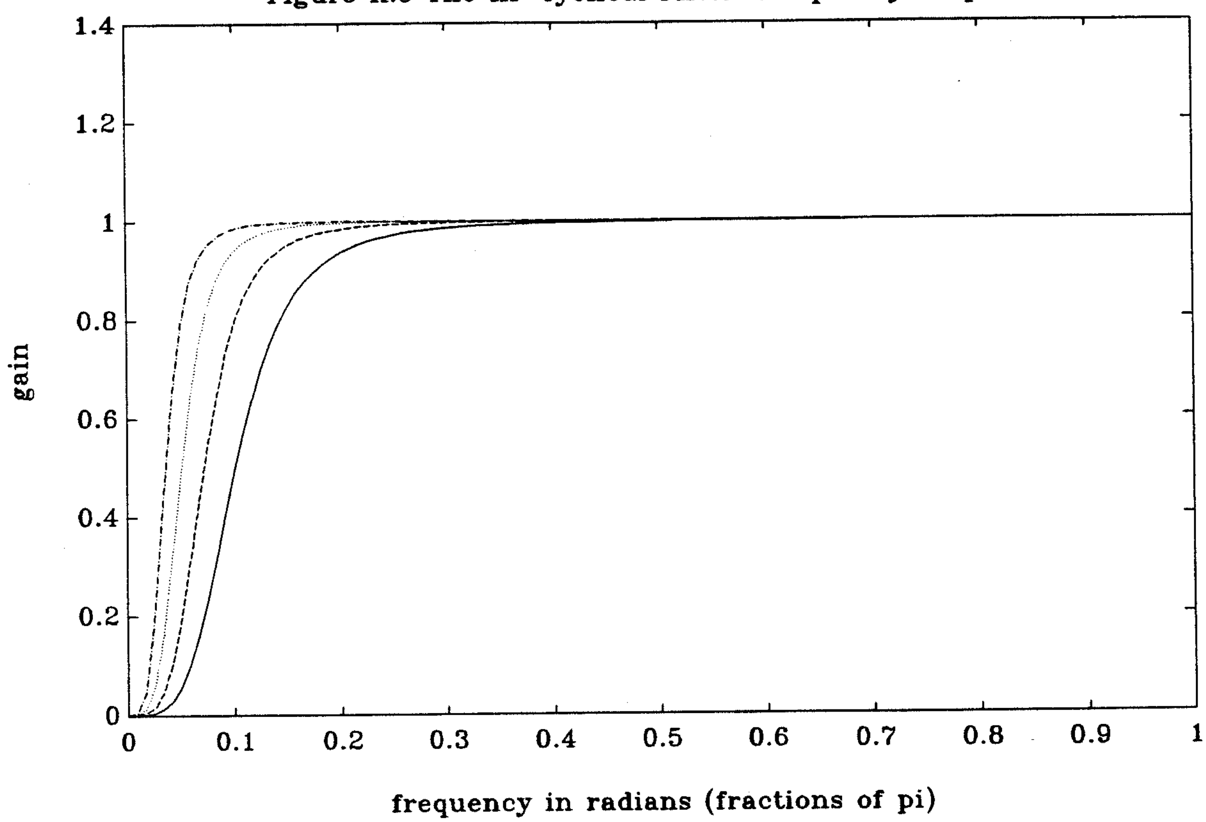
Smoothing Parameter Values = [30 60 120 240]

Figure II4: ES cyclical filter--lag weights



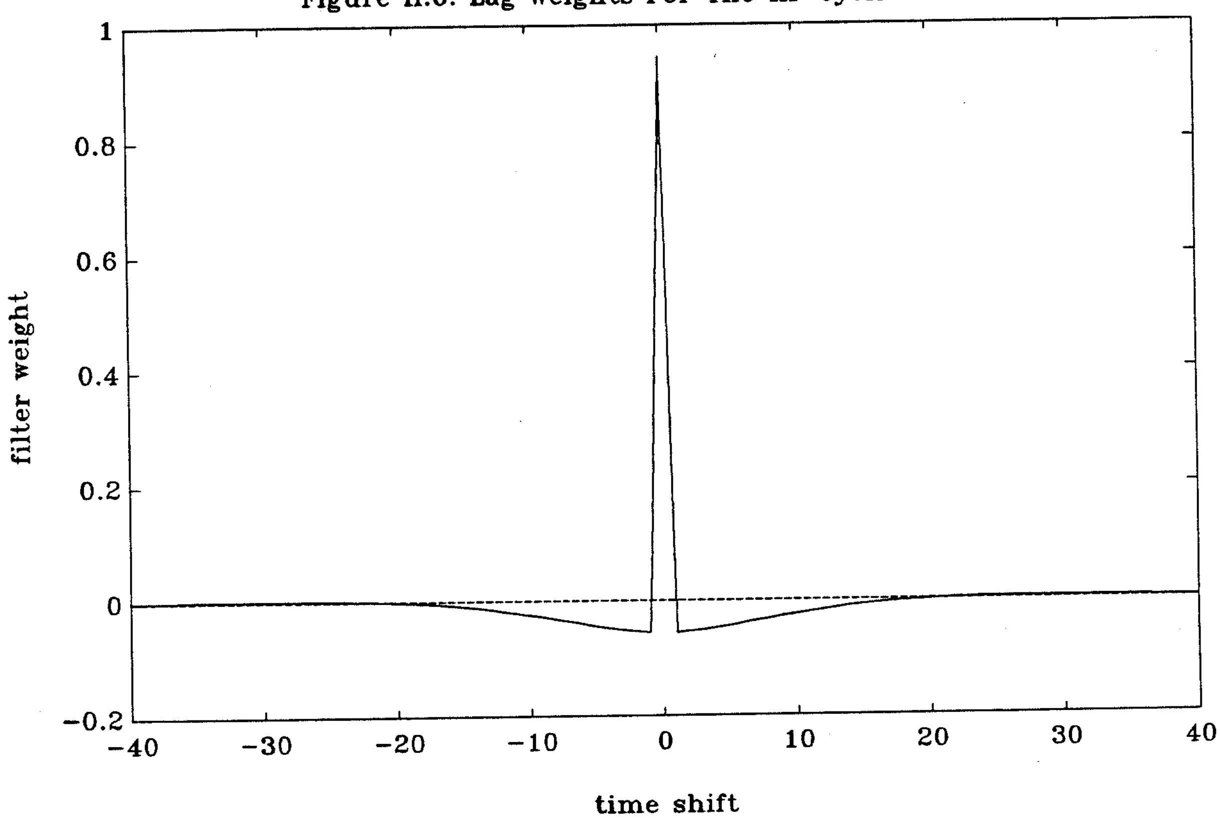
Smoothing Parameter Value = 60

Figure II.5 The HP cyclical Filter: Frequency Response



Smoothing Parameter Values = [100 400 1600 6400]

Figure II.6: Lag Weights For The HP cyclical filter



Smoothing Parameter Value = 1600

Figure II.7: The HP & ES Cyclical Filters

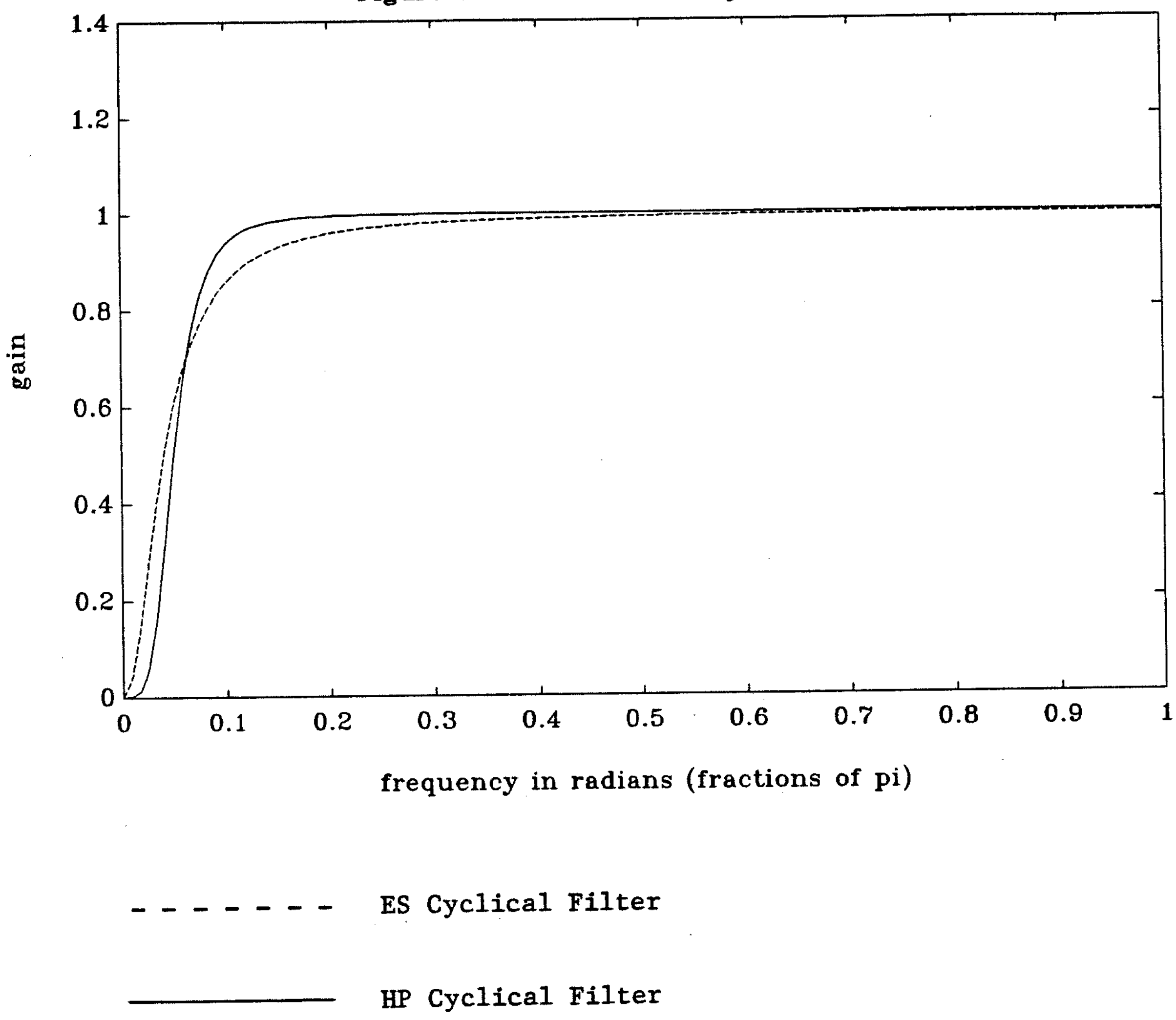
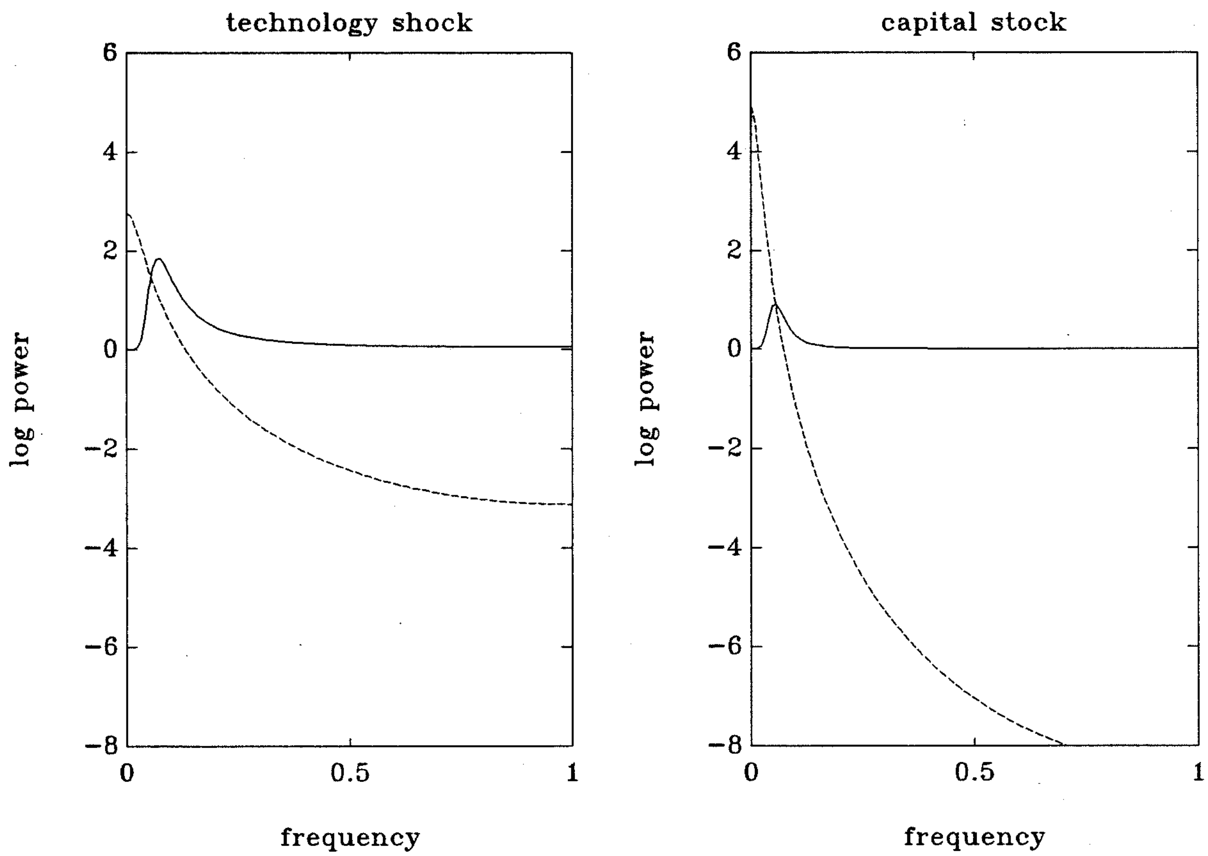


Figure III.1: Implications of HP Filtering



------- Unfiltered Spectral Density (Gain Function)

_______ Filtered Spectral Density (Gain Function)