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Working Paper No. 217
January 1990

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ALONSO'S DISCRETE POPULATION MODEL OF LAND USE: EFFICIENT ALLOCATIONS AND COMPETITIVE EQUILIBRIA*

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*The authors thank Tony Smith for useful conversations, but retain full responsibility for any remaining errors.

Proposed Running Head:

Alonso's Model of Land Use

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ABSTRACT

In his 1964 book, Alonso proposed several concepts of land use equilibrium in terms of discrete population. However, the so–called new urban economics (developed after Alonso) has been concerned exclusively with continuous population models, in which household distributions are represented in terms of densities. It is our belief that in order to develop logical foundations for continuous population models, it would be useful to go back to the original models of Alonso. In this paper, we focus on the additive land price model of Alonso, and examine its solution characteristics. In particular, we demonstrate the existence and Pareto efficiency of equilibria, and the supportability of efficient allocations through additive land price systems. Journal of Economic Literature Classification Numbers: 930, 021.

1. Introduction

The standard land–use model of the urban economics and regional science literature employs a continuum of consumers and locations. For classical examples, see Beckmann [4], Muth [10], Mills [8] and Solow [12]. The class of models has proved to be useful and empirically rich. However, the theoretical foundations of these class of models have not been sufficiently explored.[1]

In order to develop the foundations for this model, it seems useful to consider a finite model that was proposed by Alonso [1]. Here we focus on the additive land price model of Alonso, and examine its general solution characteristics. The model is a discrete or finite household version of the monocentric city model. It is a very natural model of an urban economy that Alonso takes pains to connect to standard microeconomics in his book. Surprisingly, to our knowledge this model has never been analyzed with the tools of modern economic theory. In particular, existence of equilibrium and the welfare theorems have never been explored formally for this model.

The purpose of the present work is to conduct this exploration. Unfortunately, many of the tools of mathematical economics, such as fixed point or separation theorems, are not applicable due to important nonconvexities in the model[2]. Instead we construct supporting prices and equilibria *explicitly*. Thus, it is our hope that such solutions have empirical relevance. In fact, this model can be viewed as a tractable example of the finite or discrete household approach to urban economics.[3]

The model detailed below is a one–dimensional model of location with a central business district. As in the standard land–use model, the utilities of households are restricted to be location–independent and the same for all households, but the households can have different endowments. Under standard assumptions, the welfare theorems and existence of a competitive equilibrium are demonstrated.

The paper is organized as follows. Section 2 contains a description of the model and the first welfare theorem. Section 3 contains an example of the techniques and concepts using Edgeworth box diagrams that illustrates how the model differs from the standard model of microeconomics. Section 4 contains the statement and proof of the second welfare theorem, while section 5 contains the statement and proof of existence of a competitive equilibrium. Finally, Section 6 contains our conclusions and suggestions for future research.

2. The Economy and Efficient Allocations

Imagine a long narrow area of length l and width 1. Since the width of the area is sufficiently small, the area is treated as one–dimensional and is represented by the interval $X = [0, l]$ of the real line. Location and distance from the origin are denoted by $x \in X$ (or by $y \in X$). The density of land at x is equal to 1 for all $x \in X$.

In this area, n households are to be accommodated, where n is a positive integer. Each household is assumed to occupy a lot in X and consume an amount of the composite (consumer) good. Each lot is represented by a half open interval, $[x, x + s) \subset X$, where s represents the size of the lot. All households have the same utility function U(s,z), where s and z represent respectively the lot size and the amount of the composite good. [This implies that the utility function is location–independent.] The composite good is chosen as the numeraire, so its price is unity. Each household commutes (for working and shopping) to the central business district (CBD) located at the origin of X. If a household occupies a lot $[x, x + s)$, the associated transport cost (per unit of time) is assumed to be equal to tx (measured in terms of the numeraire good z), where t is a positive constant.[4] Absentee landlords own all land initially and only want consumption good. As this sector of the model is inessential (as well as irrelevant), we suppress it throughout the remainder. The utility function is assumed to be well behaved in the following sense:[5]

<u>Assumption 1:</u> The utility function, $U:\mathbb{R}^2_{++} \to \mathbb{R}$, satisfies the following conditions:

(i) On $\mathbb{R}^2_{++}$, U is twice continuously differentiable, strictly quasi-concave, and increasing in both components s and z.

(ii) No indifference curve cuts an axis, and every indifference curve has the z-axis as an asymptote.

(iii) Lot size (or land) s is a normal good.

To explain Assumption 1(iii), let us consider the following standard utility maximization problem: Given a "land price" $p > 0$ and income $Y > 0$,

$$\max_{s,z} U(s, z), \text{ s.t. } z + ps = Y \text{ and } s > 0, z > 0. \tag{2.1}$$

Under Assumptions 1(i) and (ii), the optimal lot size [for problem (2.1)] exists uniquely for each $(p, Y) \in \mathbb{R}_{++}$, which is denoted by $s(p, Y)$ and called the <u>Marshallian demand for land</u>. The normality of land means that $s(p, Y)$ is increasing in income Y. For example, if $U(s,z) = \alpha \log s + \beta \log z$ ($\alpha > 0$ and $\beta > 0$), all three conditions above are satisfied.

Since U is increasing in each component, we can assume without loss of generality that

$$\inf\{U(s, z): s > 0, z > 0\} = -\infty, \sup\{U(s, z): s > 0, z > 0\} = \infty. \tag{2.2}$$

For each $u \in \mathbb{R}$ and $s > 0$, the equation

$$u = U(s, z) \tag{2.3}$$

can be solved for z (by Assumption 1) and we denote the solution by $Z(s, u)$. For each u, $z = Z(s, u)$ represents the equation of the indifference curve associated with utility level u. We can readily see that Assumption 1 implies

<u>Lemma 2.1:</u> Z is a function such that

(i) $Z:\mathbb{R}_{++} \times \mathbb{R} \to \mathbb{R}_{++}$.

(ii) Z is twice continuously differentiable on $(0, \infty)\times\mathbb{R}$, where for each given $u \in \mathbb{R}$

$$\lim_{s\downarrow 0} Z(s, u) = \infty, \tag{2.4}$$

$$Z_s(s,u) \equiv \partial Z(s,u)/\partial s < 0, \; Z_u(s,u) \equiv \partial Z(s,u)/\partial u > 0, \tag{2.5}$$

$$\lim_{s\downarrow 0} Z_s(s,u) = -\infty, \; \lim_{s\uparrow\infty} Z_s(s,u) = 0, \tag{2.6}$$

(iii) At each $(s,u) \in \mathbb{R}_{++}\times\mathbb{R}$ [6]

$$Z_{ss}(s,u) \equiv \partial Z_s(s,u)/\partial s > 0 \text{ and } Z_{su}(s,u) \equiv \partial Z_s(s,u)/\partial u < 0. \tag{2.7}$$

Next, given $p > 0$ and $u \in \mathbb{R}$, let us consider the following expenditure minimization problem:

$$\min_{s,z} \; z + ps, \text{ s.t. } U(s,z) = u \text{ and } s > 0, \; z > 0. \tag{2.8a}$$

Using function Z above, this problem can be restated as

$$\min_{s>0} \; Z(s, u) + ps. \tag{2.8b}$$

Since $Z(\cdot, u)$ represents a smooth, strictly convex curve in the consumption space $\mathbb{R}^2_{++}$, the following first-order condition,

$$- Z_s(s, u) = p, \tag{2.9}$$

defines uniquely the optimal lot size, which is denoted by $S(p, u)$ and called the <u>Hicksian demand for land</u>. By definition

$$-Z_s(S(p, u), u) = p \quad \text{for all } p > 0, \; u \in \mathbb{R}. \tag{2.10}$$

Using Lemma 2.1, it can be readily shown that:

<u>Lemma 2.2:</u> Relation (2.10) defines a unique function, $S:\mathbb{R}_{++}\times\mathbb{R}\to(0, \infty)$, which is continuously differentiable and

$$\partial S(p, u)/\partial p < 0, \; \partial S(p, u)/\partial u > 0, \tag{2.11}$$

at each $(p, u)\in\mathbb{R}_{++}\times\mathbb{R}$. Furthermore, for each $u \in \mathbb{R}$,

$$\lim_{p\downarrow 0} S(p, u) = \infty, \; \lim_{p\uparrow\infty} S(p, u) = 0. \tag{2.12}$$

Next, to study the problem of optimal resource allocations for the area under consideration, we introduce several definitions. There are n (integer and finite) consumers in the economy. Let N = {1, 2, ..., n}. Each allocation in X is represented by $\{(s_i, z_i, x_i)\}_{i=1}^{n}$ such that $s_i > 0$, $z_i > 0$, $x_i \geq 0$ and $x_i + s_i \leq l$. Here, s_i represents the size of the lot occupied by household i, z_i the amount of the composite good consumption by i, and x_i the front location of i's lot. This allocation implies that each household i occupies the lot, $[x_i, x_i + s)$. An allocation $\{(s_i, z_i, x_i)\}_{i=1}^{n}$ is feasible iff

$$\bigcup_{i=1}^{n} [x_i, x_i + s_i) \subset X \text{ and } [x_i, x_i + s_i) \cap [x_j, x_j + s_j) = \emptyset \text{ for all consumers } i,j \in N \text{ with } i \neq j. \quad (2.13)$$

That is, the lot of each household must be inside the area X, and no pair of lots be overlapping. Given a feasible allocation $\{(s_i, z_i, x_i)\}_{i=1}^{n}$, the associated total cost C is defined as

$$C = \sum_{i=1}^{n} (z_i + tx_i), \quad (2.14)$$

which is the sum of the composite–good costs and transport costs for the n households.[7] Now, given any pair of feasible allocations, $\{(s_i, z_i, x_i)\}_{i=1}^{n}$ and $\{(s_i', z_i', x_i')\}_{i=1}^{n}$, we say that the latter dominates (or, Pareto–dominates) the former if

$$U(s_i, z_i) \leq U(s_i', z_i') \quad \text{for all } i \in N, \text{ and} \quad (2.15a)$$

$$\sum_{i=1}^{n} (z_i + tx_i) \geq \sum_{i=1}^{n} (z_i' + tx_i'), \quad (2.15b)$$

with a strictly inequality for at least one relation. An allocation is said to be efficient (or, Pareto optimal) if no feasible allocation dominates it.

The main task of the rest of this section is to examine the characteristics of efficient allocations. One systematic way for identifying the set of all efficient allocations may be as follows. Suppose we choose arbitrarily a target utility level,

$u_i \in \mathbb{R}$, for each $i \in N$. Given the target utility vector, $(u_1, u_2, ..., u_n)$, the problem of finding a feasible allocation that achieves this target utility vector with the least cost can be formulated as follows:

Problem A:
$$\min_{\{(s_i, z_i, x_i)\}_{i=1}^n} \quad C = \sum_{i=1}^n (z_i + tx_i)$$
s.t
$$U(s_i, z_i) \geq u_i \quad \forall\, i \in N,$$
$$\bigcup_{i=1}^n [x_i, x_i + s_i) \subset X$$
$$[x_i, x_i + s_i] \cap [x_j, x_j + s_j) = \emptyset \;\; \forall\, i,j \in N \text{ with } i \neq j$$
and
$$s_i > 0,\; z_i > 0,\; x_i \geq 0 \quad \forall\, i \in N.$$

Since the utility function U is assumed to be continuous and increasing in z, it can be readily seen that each solution to Problem A under any target utility vector $(u_1, u_2, ..., u_n) \in \mathbb{R}^n$ is efficient; conversely, any efficient allocation must be a solution to Problem A under an appropriate target utility vector.

For each $(u_1, u_2, ..., u_n) \in \mathbb{R}^n$, we denote by Problem $A(u_1, u_2, ..., u_n)$ the Problem A under the target utility vector $(u_1, u_2, ..., u_n)$.

Again, since U is assumed to be increasing in both s and z, if $\{(s_i, z_i, x_i)\}_{i=1}$ is a solution to problem A, then it follows immediately that

$$U(s_i, z_i) = u_i \qquad \forall\, i \in N, \text{ and} \tag{2.16}$$

$$\sum_{i=1}^n s_i = l \tag{2.17}$$

That is, in any efficient allocation, each target utility constraint is satisfied by equality, and no vacant land is left in the area X. The following result is less obvious.

Lemma 2.3: Let $\{(s_i, z_i, x_i)\}_{i=1}^n$ is a solution to the Problem $A(u_1, u_2, ..., u_n)$. [Hence, it is efficient]. Then, for any pair i and $j \in N$,

$$x_i < x_j \Leftrightarrow s_i < s_j, \tag{2.18}$$

and

$$x_i < x_j \Rightarrow u_i \leq u_j, \qquad \text{and } u_i < u_j \Rightarrow x_i < x_j. \tag{2.19}$$

Proof. If $\{(s_i, z_i, x_i)\}_{i=1}^n$ is a solution to Problem A$(u_1, u_2, ..., u_n)$, then by (2.17) no vacant land is left in X. Hence, if we show relations (2.18) and (2.19) for each pair of adjacent households, then by transitivity, these relations hold for any pair of households. Therefore, in the following analyses, we consider an arbitrary pair of adjacent households, i, j $\in$ N.

(i) First,

$$x_i < x_j \Longrightarrow s_i \leq s_j \tag{2.20}$$

must hold. For if $x_i < x_j$ and $s_i > s_j$, then we can give the two households the same consumption bundles, (s_i, z_i) and (s_j, z_j), but by switching their positions (i.e., household i now occupies the lot $[x_j, x_j + s_i)$, and household j the lot $[x_i, x_i + s_j)$) $(s_i - s_j)t$ can be saved in transport cost (without affecting the rest of the allocation). This contradicts that the original allocation was efficient, so (2.20) must hold.

(ii) Next, given that $x_i < x_j$, considering (2.20) we have the following four possible situations in the relationship between (u_i, s_i) and (u_j, s_j): (a) $u_i > u_j$ and $s_i = s_j$, (b) $u_i > u_j$ and $s_i < s_j$, (c) $u_i \leq u_j$ and $s_i = s_j$, (d) $u_i \leq u_j$ and $s_i < s_j$. We show in turn that each of (a), (b) and (c) leads to a contradiction; hence (d) must hold.

(a) Suppose $x_i < x_j$, $u_i > u_j$, and $s_i = s_j \equiv s'$. Then, since $Z_{su}(s, u) < 0$ [by (2.7)], we have that $Z_s(s', u_i) < Z_s(s', u_j)$. Since $Z_s(\cdot, u)$ is continuous on $(0, \infty)$ [by Lemma 2.1(ii)], and since $s' > 0$ by definition, this implies that there exists a positive Δs such that $s' - \Delta s > 0$ and

$$Z_s(s' + \epsilon, u_i) < Z_s(s' - \epsilon, u_j) \qquad \forall\ \epsilon \in [0, \Delta s],$$

from which it follows that $\int_0^{\Delta s} Z_s(s' + \epsilon, u_i)\, d\epsilon < - \int_0^{\Delta s} Z_s(s' - \epsilon, u_j) d\epsilon$ and hence $Z(s' + \Delta s, u_i) - Z(s', u_i) < Z(s', u_j) - Z(s' - \Delta s, u_j)$, or

$$Z(s' + \Delta s, u_i) + Z(s' - \Delta s, u_j) < Z(s', u_i) + Z(s', u_j).$$

Suppose we switch the positions of i and j, and increase the lot size of i (now located further away from the CBD) from $s'(\equiv s_i)$ to $s' + \Delta s$ and reduce that of j (now located closer to the CBD) from $s'(\equiv s_j)$ to $s' - \Delta s$. This switching reduces the total transport cost of the two households by $t\Delta s$, and also reduces the total cost of z – consumption (for maintaining the same utility levels u_i and u_j) as is shown by the last expression. This implies that the original allocation was not efficient, a contradiction.

(b) Suppose $x_i < x_j$, $u_i > u_j$, and $s_i < s_j$. Then, since $Z_{su}(s, u) < 0$ by (2.7), $\int_{s_i}^{s_j} Z_s(s, u_i)ds < \int_{s_i}^{s_j} Z_s(s, u_j)ds$, which leads to $Z(s_j, u_i) - Z(s_i, u_i) < Z(s_j, u_j) - Z(s_i, u_j)$, or

$$Z(s_j, u_i) + Z(s_i, u_j) < Z(s_i, u_i) + Z(s_j, u_j).$$

Now consider switching the land allotments of i and j (while maintaining the rest of the allocations). That is, i gets j's previous lot, $[x_j, x_j + s_j)$, and j gets i's previous lot, $[x_i, x_i + s_i)$. The total transport cost of the two households is the same as the lots are the same, while the total cost of z–consumption is reduced as is shown by the last expression. Thus, the original allocation was not efficient, a contradiction.

(c) Finally, suppose $x_i < x_j$, $u_i \leq u_j$, and $s_i = s_j \equiv s'$. Then, since $Z_s(s', u_i) \geq Z_s(s', u_j)$ [by (2.7)] and $s' > 0$ by definition, and since $Z_s(\cdot, u)$ is continuous and increasing on $(0, \infty)$, there exists a positive Δs such that $s' - \Delta s > 0$ and

$$Z_s(s' - \epsilon, u_i) > Z_s(s' + \epsilon, u_j) - t \qquad \forall\ \epsilon \in [0, \Delta s],$$

which implies $-\int_0^{\Delta s} Z_s(s' - \epsilon, u_i)d\epsilon > \int_0^{\Delta s}[Z_s(s' + \epsilon, u_j) - t]d\epsilon$, and hence $Z(s', u_i) - Z(s' - \Delta s, u_i) > Z(s' + \Delta s, u_j) - Z(s', u_j) - t\Delta s$, or

$$Z(s', u_i) + Z(s', u_j) > Z(s' - \Delta s, u_i) + Z(s' + \Delta s, u_j) - t\Delta s.$$

Now consider reducing the land consumption of i from $s'(\equiv s_i)$ to $s' - \Delta s$ and increasing that of j from $s'(\equiv s_j)$ to $s' + \Delta s$. By the last expression, this reallocation of land reduces the total cost (of z–consumption and transportation) by the two households (while maintaining the same utility levels). This implies that the original allocation was not efficient, a contradiction. Therefore, we can conclude that (d) must be true.

(iii) Namely, $x_i < x_j \Rightarrow u_i \leq u_j$ and $s_i < s_j$. Then, since $x_i \neq x_j$ by definition, we can immediately conclude that (2.18) and (2.19) hold. Q.E.D.

Relation (2.18) implies that at any efficient allocation, households with larger lots locate farther from the CBD than households with smaller lots, while (2.19) means that households with higher utilities never locate closer to the CBD than households with lower utilities.

Given any Problem $A(u_1, u_2, ..., u_n)$, without loss of generality we can assume that

$$u_1 \leq u_2 \leq ... \leq u_n. \tag{2.21}$$

Furthermore, if $u_i = u_j$, then the two households i and j are identical for the purpose of our analysis. In this case we can switch indices i and j with each other whenever necessary. Therefore, if $\{(s_i, z_i, x_i)\}_{i=1}^{n}$ is a solution the Problem $A(u_1, u_2, ..., u_n)$, then considering Lemma 2.3, without loss of generality we can assume that

$$x_1 < x_2 < ... < x_n, \tag{2.22}$$

$$s_1 < s_2 < ... < s_n. \tag{2.23}$$

That is, in any efficient allocation, if we number households in order ascending with utility level, the order is also ascending with distance (from the CBD) as well as with land consumption.

Conditions (2.17) and (2.22) together imply that

$$x_1 = 0,\ x_i = \sum_{j=1}^{i-1} s_j \text{ for } i = 2, ..., n.$$

Therefore, given any target utility vector $(u_1, u_2, ..., u_n)$ that satisfies condition (2.21), considering equalities (2.16) we can restate the corresponding Problem A as follows:

Problem $A'(u_1, u_2, ..., u_n)$: $$\min_{s_1,...,s_n} \sum_{i=1}^{n} Z(s_i, u_i) + \sum_{i=2}^{n} t \sum_{j=1}^{i-1} s_j, \tag{2.24}$$

$$\text{s.t.} \qquad \sum_{i=1}^{n} s_i \le l, \tag{2.25}$$

$$s_i > 0 \qquad \forall\ i \in N. \tag{2.26}$$

[Here, for mathematical convenience, we use a land constraint (2.25) instead of (2.17).] This problem has a solution, and its solution is unique.

To see this, let $F = \{(s_1, s_2, ..., s_n) \in \mathbb{R}^n : \Sigma_{i=1}^n s_i \le l,\ s_i > 0\ \forall\ i \in N\}$ be the feasible set of the problem. Since each function $Z(\cdot, u_i)$ is strictly convex on $(0, \infty)$ and $\lim_{s \downarrow 0} Z(s, u_i) = \infty$, it can be readily seen that the objective function (2.24) attains a minimum in the relative interior of F and is strictly convex and continuous on F. Therefore, there exists a unique solution to Problem $A'(u_1, ..., u_n)$.

Notice also that the feasible set F is convex, and it has interior points. Hence, if we define the Lagrangian function as

$$\mathscr{L} = \sum_{i=1}^{n} Z(s_i, u_i) + \sum_{i=2}^{n} t \sum_{j=1}^{i-1} s_j + p_n\left(\sum_{i=1}^{n} s_i - l\right)$$

then the following Kuhn–Tucker conditions [together with (2.25) and (2.26)] represent the necessary and sufficient conditions for optimality:

$$\partial \mathscr{L} / \partial s_i = Z_s(s_i, u_i) + t(n - i) + p_n = 0,\ i = 1, ..., n,$$

$$p_n \ge 0 \text{ and } \left(\sum_{i=1}^{n} s_i - l\right) p_n = 0,$$

which implies $p_n = - Z_s(s_n, u_n) > 0$ [by (2.5)], and hence $\sum_{i=1}^{n} s_i = l$ as was expected.

Therefore, these conditions can be restated equivalently as

$$p_i = - Z_s(s_i, u_i), \text{ i.e., } s_i = S(p_i, u_i),\ \forall\ i \in N,$$

$$p_i = p_n + t(n - i) \text{ for } i = 1, ..., n-1, \text{ i.e., } p_{i-1} = p_i + t \text{ for } i = 2,...,n.$$

Summarizing the results, we can conclude as follows.

Proposition 1: Given any target utility vector $(u_1, u_2, ..., u_n) \in \mathbb{R}^n$, the unique solution to Problem $A(u_1, u_2, ..., u_n)$ provides the (unique) efficient allocation attaining this target utility vector. If we adopt convention (2.21) and let $\{(s_i, z_i, x_i)\}_{i=1}^n$ represent this efficient allocation, then the following conditions hold:

$$u(s_i, z_i) = u_i, \text{ i.e. } z_i = Z(s_i, u_i), \ \forall\, i \in N, \tag{2.27}$$

$$s_1 < s_2 < \dots < s_n, \tag{2.28}$$

$$x_1 = 0,\ x_i = \sum_{j=1}^{i-1} s_j \text{ for } i = 2, \dots, n,\ \sum_{i=1}^{n} s_i = x_n + s_n = l, \tag{2.29}$$

and thus

$$x_1 < x_2 < \dots < x_n. \tag{2.30}$$

Conversely, let $(u_1, u_2, \dots, u_n) \in \mathbb{R}^n$ be any target utility vector that satisfies condition (2.21). Then, $\{(s_i, z_i, x_i)\}_{i=1}^n$ is the unique efficient allocation satisfying this target utility vector iff conditions (2.28) and (2.30) hold and there exists a vector $(p_1, p_2, \dots, p_n) \in \mathbb{R}^n$ such that

$$p_i = - Z_s(s_i, u_i), \text{ i.e. } s_i = S(p_i, u_i), \ \forall\, i \in N, \tag{2.31}$$

$$p_i = p_n + t(n-i) \text{ for } i=1, \dots, n-1, \text{ i.e., } p_{i-1} = p_i + t \quad \text{for } i=2, \dots, n \tag{2.32}$$

Before closing this section, we introduce the additive land price model of competitive land markets due to Alonso [1], and demonstrate the efficiency of its equilibrium solutions. Suppose that the given n households, $i = 1, 2, \dots, n$, are to choose their residential locations inside the area, $X \equiv [0, l]$, under a competitive land market. The entire land of this area is owned by absentee landlords. All n households have the same utility function, $U(s, z)$, that was introduced previously. Each household $i \in N \equiv \{1, 2, \dots, n\}$ commutes to the CBD [located at the origin of X], and earns a fixed income $Y_i > 0$.

A function, $P{:}X \to \mathbb{R}_+$, is called a <u>land price (density) function</u> if it is integrable on X. Given such a function P, if a household chooses a lot, $[x, x + s) \subset X$, then the household is assumed to pay the (total) land rent, $\int_x^{x+s} P(y)dy$, and to pay the transport cost tx (where t is the same constant as before). In this context, it is postulated that the residential choice behavior of each household $i \in N$ can be described by the following utility maximization problem:

$$\max_{s,z,x} U(s, z), \text{ s.t. } z + \int_x^{x+s} P(y)dy + tx = Y_i, \tag{2.33}$$

where $s > 0$, $z > 0$, $x \geq 0$, and $x + s \leq l$. The utility function U is assumed, as before, to satisfy Assumption 1. We call this competitive land market model for n households the <u>additive land price model (the ALP–model)</u> of residential land market. When we emphasize that incomes of n households are fixed at $Y_1, Y_2, ..., Y_n$ respectively, we call it the ALP($Y_1, Y_2, ..., Y_n$)–model.

Given a land price function P^* and an allocation $\{(s_i^*, z_i^*, x_i^*)\}_{i=1}^n$, we say that $[P^*, \{(s_i^*, z_i^*, x_i^*)\}_{i=1}^n]$ represents a competitive equilibrium for the ALP($Y_1, Y_2, ..., Y_n$)–model if the following four conditions are satisfied:

$$U(s_i^*, z_i^*) = \max_{s,z,x} \{U(s, z): z + \int_x^{x+s} P^*(y)dy + tx = Y_i, s > 0, z > 0, x \geq 0, x + s \leq l\}, \forall i \in N, \tag{2.34}$$

$$\bigcup_{i=1}^n [x_i^*, x_i^* + s_i^*) \subset X, \tag{2.35}$$

$$[x_i^*, x_i^* + s_i^*) \cap [x_j^*, x_j^* + s_j^*) = \emptyset, \forall i,j \in N \text{ with } i \neq j, \text{ and} \tag{2.36}$$

$$P^*(x) = 0 \text{ at each } x \in (X \setminus \bigcup_{i=1}^n [x_i^*, x_i^* + s_i^*]). \tag{2.38}$$

In this event, we call $\{(s_i^*, z_i^*, x_i^*)\}_{i=1}^n$ an <u>equilibrium allocation</u>. The last condition (2.38) means that on the vacant area, land price must be zero.

<u>Proposition 2 (The First Welfare Theorem):</u> Every equilibrium allocation is efficient.

<u>Proof:</u> Let $[P^*, \{(s_i^*, z_i^*, x_i^*)\}_{i=1}^n]$ be a competitive equilibrium for the ALP($Y_1, Y_2, ..., Y_n$)–model. Suppose the allocation, $\{(s_i^*, z_i^*, x_i^*)\}_{i=1}^n$, is not efficient. Then, there must exist a feasible allocation $\{(s_i', z_i', x_i')\}_{i=1}^n$ that Pareto–dominates $\{(s_i^*, z_i^*, x_i^*)\}_{i=1}^n$, i.e., such that

$$U(s_i', z_i') \geq U(s_i^*, z_i^*), \forall i \in N, \text{ and}$$

$$\sum_{i=1}^n (z_i' + tx_i') \leq \sum_{i=1}^n (z_i^* + tx_i^*)$$

with a strict inequality for at least one relation. By continuity and nonsatiation of the utility function U, then it must be the case that

$$z_i' + \int_{x_i'}^{x_i'+s_i'} P^*(y)dy + tx_i' \geq z_i^* + \int_{x_i^*}^{x_i^*+s_i^*} P^*(y)dy + tx_i^*, \ \forall\ i \in N, \text{ and}$$

$$\sum_{i=1}^{n} (z_i' + tx_i') \leq \sum_{i=1}^{n} (z_i^* + tx_i^*)$$

with a strict inequality for at least one relation. Thus,

$$\sum_{i=1}^{n} \int_{x_i'}^{x_i'+s_i'} P^*(y)dy + \sum_{i=1}^{n} (z_i' + tx_i') \geq \sum_{i=1}^{n} \int_{x_i^*}^{x_i^*+s_i^*} P^*(y)dy + \sum_{i=1}^{n} (z_i^* + tx_i^*),$$

and

$$\sum_{i=1}^{n} (z_i' + tx_i') \leq \sum_{i=1}^{n} (z_i^* + tx_i^*),$$

with a strict inequality for at least one relation. This implies

$$\sum_{i=1}^{n} \int_{x_i'}^{x_i'+s_i'} P^*(y)dy > \sum_{i=1}^{n} \int_{x_i^*}^{x_i^*+s_i^*} P^*(y)dy = \int_0^l P^*(y)dy \text{ [by (2.38)]},$$

which is impossible. Therefore, it must be true that the equilibrium allocation $\{(s_i^*, z_i^*, x_i^*)\}_{i=1}^n$ is efficient. Q.E.D.

Proposition 2 holds under any continuous transport cost function $T_i(x)$ and any continuous and locally non-satiated (even location dependent) utility function $U_i(s, z, x)$ where U_i can differ for each $i \in N$.

3. An Example with Two Households

In this section, we consider a simple example with two households. Efficient allocations, along with the first-order conditions that characterize them, are illustrated using a modified Edgeworth box diagram. Using the competitive land market of the ALP – model, a land price function that supports a given efficient allocation is constructed. The basic ideas and techniques of proof used in the next section are explained within this simplified framework.

Suppose that the two households, i = 1,2, are to be accommodated in the area X. To illustrate efficient allocations for the two households graphically, it is convenient to consider the following maximization problem, which is dual to problem A:

<u>Problem A</u>*:

$$\max_{\{s_i, z_i, x_i\}} U(s_1,z_1)$$

$$\text{s.t.} \quad U(s_2,z_2) \geq u_2, \tag{3.1}$$

$$z_1 + z_2 + tx_1 + tx_2 \leq C, \tag{3.2}$$

$$\text{and} \quad [x_1,x_1+s_1) \cup [x_2,x_2+s_2) \subset X, \quad [x_1,x_1+s_1) \cap [x_2,x_2+s_2) = \emptyset, \tag{3.3}$$

where u_2 and C (>0) are given constants. The problem is to find a feasible allocation that achieves the highest utility for household 1 while satisfying the given target utility constraint (3.1) for household 2 and the material balance constraint (3.2).

If we know the highest value of the objective that can be obtained in Problem A^*, then using lemma 2.3 we can determine which household should locate closer to the CBD at the optimal allocation. However, in the absence of such information, we can determine the efficient spatial order of the two households only by guessing. Hence, let us first assume that household 1 locates closer to the CBD than household 2. Call this locational configuration <u>Pattern α</u>. Since no vacant land should be left in X at the optimum, it must be the case that $x_1 = 0$, $x_2 = s_1$, and $s_1 + s_2 = l$. Furthermore, at the optimum the material balance constraint holds with equality. Thus, Problem A^* can be restated as follows:

Problem $A^*-\alpha$:

$$\max_{\{s_i, z_i\}} U(s_1,z_1),$$

$$\text{s.t.} \quad U(s_2,z_2) \geq u_2,$$

$$s_1 + s_2 = l \text{ and } z_1 + z_2 = C - ts_1.$$

If the transport cost ts_1 were absent, this would be a typical example of a resource allocation problem in a standard Edgeworth box. However, in the present context, the net amount of composite commodity left for consumption, $C - ts_1$, depends on the land allocation (s_1,s_2). Hence, it is necessary to modify the standard Edgeworth box diagram as in Figure 1. The entire figure represents the feasible allocations for the two households in

Figure 1

the absence of transportation cost ts_1, where O_i represents the origin of the consumption space for household i. With transport cost ts_1, the net amount of numeraire available to the economy decreases in proportion to s_1. This is represented by the slanted line ending at O_2'. With transport costs, the origin O_2 is moved to O_2', and an indifference curve u_2 of household 2 shifts downward by ts_1 for each s_1. This shifted indifference curve is represented by the broken curve $\mathbf{u}_2$ in Figure 1. [The equation of the new indifference curve $\mathbf{u}_2$ under the initial origin O_2 is given by $z_2 = Z(s_2,u_2) + t(l - s_2)$.] Given this utility constraint for household 2, the highest utility attainable for household 1 is determined by the indifference curve u_1^* (of household 1 with origin O_1) that is tangent to the $\mathbf{u}_2$ curve at point G in Figure 1. Since $-Z_s(s_1^*,u_1^*) = -Z_s(s_2^*,u_2) + t$ at point G, if we define

$$p_1 = -Z_s(s_1^*,u_1^*),\ p_2 = -Z_s(s_2^*,u_2), \tag{3.4}$$

then

$$p_1 = p_2 + t, \tag{3.5}$$

as required by (2.32). Relation (3.5) reflects the following fact: A unit increase in the land consumption of household 1 pushes outward the front location of household 2's lot by a unit distance, which in turn increases the transport cost of household 2 by t. Therefore, the social "price" p_1 of the last unit of land allocated to household 1 (which is directly in front of household 2's lot) must be t dollars higher than the social "price" of the last unit of land allocated to household 2 (which is located at the end of X).

At the solution to Problem A*–α, if $u_2 \geq u_1^*$ (the highest attainable utility level of household 1), then from Lemma 2.3, our use of Pattern α is vindicated. If this inequality does not hold, then the reverse spatial ordering (called Pattern β) is the optimal one. In this case, $x_2 = 0$ and $x_1 = s_2$ imply that the optimal allocation can be found by solving the following problem:

Problem A^*–β: $\max_{\{s_i, z_i\}} U(s_1,z_1)$,

s.t. $U(s_2,z_2) \geq u_2$,

$s_1 + s_2 = l$ and $z_1 + z_2 = C - ts_2$.

As a consequence, the contract curve (the locus of efficient allocations) is not connected (see Figure 2).

Figure 2

The curve O_1AO_2 (curve O_2BO_1) represents the locus of optimal allocations under the spatial ordering of Pattern α (Pattern β). The households achieve the highest common utility level, say $\tilde{u}$, at points A and B. Each allocation on the broken curve AO_2 (BO_1) is dominated by some allocation on the solid curve BO_2 (AO_1). The contract curve consists of two segments, O_1A and O_2B, such that for each point on O_1A (O_2B), household 1 (household 2) occupies the lot closer to the CBD, and we have $u_2 > \tilde{u} > u_1^*$ ($u_1^* > \tilde{u} > u_2$)[8].

The question of main interest is as follows. Given any efficient point on the contract curve in Figure 2, can we find an appropriate combination of a land price function (P) and a pair of incomes (Y_1,Y_2) such that the selected allocation is realized as a solution to the corresponding ALP – model? To investigate this question in our modified Edgeworth box, let us focus on the efficient allocation G in Figure 1, where

$$(s_1^*,z_1^*,x_1^*) = (s_1^*,Z(s_1^*,u_1^*),0),\ (s_2^*,z_2^*,x_2^*) = (s_2^*,Z(s_2^*,u_2),s_1^*) \tag{3.6}$$

We also assume without loss of generality that the point G belongs to the segment O_1A in Figure 2, and hence $u_1^* < \tilde{u} < u_2$ and $s_1^* < s_2^*$.

Given any land price function P and a utility level $u \in \mathbb{R}$, for each (s,x) such that $s > 0$, $x \geq 0$ and $x + s \leq l$, let us define

$$E(s,x;P,u) = Z(s,u) + \int_x^{x+s} P(y)\,dy + tx \tag{3.7}$$

which represents the total expenditure necessary for a household to achieve the given utility u (when the household chooses the front location x and lot size s). Define

$$\hat{E}(P,u) = \min \{E(s,x;P,u) : s > 0,\ x \geq 0,\ x + s \leq l\}. \tag{3.8}$$

$\hat{E}(P,u)$ represents the minimum expenditure necessary to attain the given utility level u under the land price function P.

It is readily apparent that a land price function P is a supporting land price function of allocation (3.6) if and only if

$$E(s_1^*,0;P,u_1^*) = \hat{E}(P,u_1^*),\ E(s_2^*,s_1^*;P,u_2) = \hat{E}(P,u_2). \tag{3.9}$$

Setting $Y_1 = \hat{E}(P,u_1^*)$ and $Y_2 = \hat{E}(P,u_2)$, the allocation (3.6) is clearly an equilibrium allocation for the ALP(Y_1,Y_2) – model under the land price function P.

Given the conditions of optimality (3.4) and (3.5), consider the following function as a candidate for a supporting price:

$$P^{\circ}(x) = \begin{cases} p_1 & \text{for } 0 \leq x \leq s_1^* \\ p_2 & \text{for } s_1^* < x \leq l, \end{cases}$$

where p_1 and p_2 are defined by (3.4). This land price function is depicted by the step function HIJM in Figure 3.

Figure 3

It clearly cannot support the allocation (3.6) due to its discontinuity at $x = s_1^*$. Since $-Z_s(s_1^*,u_1^*) = p_1 > p_2$, if the allocated lot size of household 1 is increased by one marginal unit, then $-Z_s(s_1^*,u_1^*)$ units of expenditure on numeraire are saved, while expenditure on land is increased only by p_2.

The price function must be modified to eliminate this key discontinuity and provide price support. There are many ways to accomplish this, which we will return to later. For now, let us propose a second natural candidate for price support (recall that S is the Hicksian demand function).

$$P(x) = \begin{cases} p_1 & \text{for } 0 \leq x \leq S(p_1,u_1^*) \equiv s_1^* \\ -Z_s(x,u_1^*) & \text{for } s_1^* < x \leq S(p_2,u_1^*) \\ p_2 & \text{for } x > S(p_2,u_1^*) \end{cases} \tag{3.10}$$

This price function is depicted in Figure 3 by the curve HIKLM. Next we give a sketch of the reasoning behind why this is a supporting price function. Recall that expenditure minimization for each household is necessary and sufficient for a price function to support in our framework. Consider first household 1. If household 1 changes its allocation of land so that its front location is still zero but the quantity of land s_1 is expanded, then at each $s_1 \in [s_1^*, S(p_2,u_1^*)]$ the marginal cost ($= P(s_1)$) is equal to the marginal willingness to pay ($= -Z_s(s_1,u_1^*)$) of household 1 for additional land; the utility level and total expenditure of household 1 stays constant. If s_1 is expanded beyond the point K in figure 3, the marginal cost exceeds the marginal willingness to pay. If the front location x_1 for household 1 is moved away from zero while keeping lot size at $s < S(p_2, u_1^*)$, then the transportation cost increases faster than the price of land decreases. If the front location x_1 is moved away from zero while keeping lot size at $S(p_2,u_1^*)$, land price savings exactly offset transport cost increases. When the front location x_1 reaches point I in Figure 3, a further increase in x_1 causes a greater increase in transport cost than land cost savings. Hence, household 1 attains the minimum expenditure $\hat{E}(P,u_1^*)$ if the front location is kept at zero and the end location is chosen at any point between I and K in Figure 3, or if its lot size is kept at $S(p_2,u_1^*)$ while the front location is chosen at any point between H and I in Figure 3.

Now consider household 2. If household 2 moves its front location x_2 closer to the CBD without changing $s_2 (= s_2^*)$, then as long as its end location $s_2 + x_2$ is beyond point L in Figure 3, land price increases exactly offset transport cost savings. When the front location x_2 reaches zero, the end location $x_2 + s_2$ is at point L. In this situation, if household 2 reduces its lot size s_2, then at each $s_2 < s_2^*$ the increase in the cost (= $-Z_s(s_2,u_2)$) of numeraire to maintain the utility level exceeds savings (= p_2) in the cost of land. Hence, household 2 can attain the minimum expenditure $\hat{E}(P,u_2)$ iff its lot size is kept at s_2^* and the end location is chosen at any point between L and M in Figure 3. Therefore, we can conclude that the land price function (3.10) supports the efficient allocation G.

The reader might already have noticed that the supporting price function proposed above is not the only supporting price function for the efficient allocation G. In fact, there are many supporting price functions that differ in shape. For example, if we define a land price function $\tilde{P}$ by

$$\tilde{P}(x) = \begin{cases} p_1 & \text{for } 0 \le x \le S(p_1,u_2) \\ -Z_s(x,u_2) & \text{for } S(p_1,u_2) < x \le S(p_2,u_2) \equiv s_2^* \\ p_2 & \text{for } x > s_2^*, \end{cases} \tag{3.11}$$

then the same arguments used for P show that $\tilde{P}$ also supports the efficient allocation G. This land price function is depicted by the curve HINLM is Figure 3. It is also not difficult to see that any land price function P^* such that $\tilde{P}(x) \ge P^*(x) \ge P(x)$ for all $x \in X$ [where P is given by (3.10)] also supports the allocation G.[9] In the next section, we generalize these land price functions to the case of an arbitrary number of households and make the argument given above formal.

4. Price-Supportability of Optimal Allocations

A land price function P is said to <u>support</u> a feasible allocation $\{(s_i,z_i,x_i)\}_{i=1}^n$ if for each i and for any bundle $(\hat{s}_i,\hat{z}_i,\hat{x}_i)$ with $U_i(\hat{s}_i,\hat{z}_i) > U_i(s_i,z_i)$, $\hat{z}_i + \int_{\hat{x}_i}^{\hat{x}_i+\hat{s}_i} P(y)dy + t\hat{x}_i > z_i + \int_{x_i}^{x_i+s_i} P(y)dy + tx_i$. The objective of this section is to show that:

Proposition 3 (The Second Welfare Theorem): Every efficient allocation has a supporting price.

In order to prove this proposition, choose an efficient allocation $\{(s_i,z_i,x_i)\}_{i=1}^{n}$. Let $U(s_i,z_i) \equiv u_i \in \mathbb{R}\ \forall\ i \in N$. Without loss of generality, condition (2.21) is assumed to be satisfied. Recall that relations (2.28) – (2.30) are satisfied for this allocation. Furthermore, if p_i is defined as in (2.31), then (2.32) follows.

We propose below a land price function that generalizes the function P defined by (3.10) in the previous section. However, before introducing the formal definition, it is useful to explain the basic idea behind our generalized land price function. To this end, recall the following prominent feature of the land price function P depicted by the curve HIKLM in Figure 3. If household 1 sets the end location, $x_1+s_1 \equiv k$, of its lot at any point between I and K in Figure 3 so that $p_1 > P(k) > p_2$, and if the optimal lot size $s_1 = S(P(k),u_1^*)$ is chosen, then the expenditure function $E(s,x;P,u_1^*)$ is minimized: i.e.,

$$\hat{E}(P,u_1^*) = E[S(P(k),u_1^*),\ k - S(P(k),u_1^*);P,u_1^*]$$
$$\text{for all } k \in X \text{ such that } p_1 > P(k) > p_2.$$

We wish to develop a land price function that assures that the same condition holds for each household i=1, 2, ..., n–1: i.e., for each $i < n$,

$$\hat{E}(P,u_i) = E[S(P(k),u_i),\ k - S(P(k),u_i);P,u_i]$$
$$\text{for all } k \in X \text{ such that } p_i > P(k) > p_{i+1}. \tag{4.1}$$

In other words, if household i ($< n$) sets its end location at any point $k \in X$ such that $p_i > P(k) > p_{i+1}$ [and chooses the optimal lot size $S(P(k),u_i)$], then expenditure is minimized over all bundles generating utility u_i.[10] To find such a land price function, let us observe that if we set $u = u_i$, then the first-order (Kuhn–Tucker) conditions for (s, x) to be a solution to the expenditure–minimization problem on the right–hand side of (3.8) are given as follows (provided that $x + s < l$):[11]

$$P(x) \leq P(x + s) + t, \tag{4.2}$$

$$P(x) = P(x + s) + t \text{ if } x > 0, \tag{4.3}$$

$$P(x + s) = -Z_s(s,u_i), \text{ i.e., } s = S(P(x + s),u_i). \tag{4.4}$$

Setting $x + s \equiv k$, these conditions can be restated as

$$P(k - S[P(k),u_i]) \leq P(k) + t, \tag{4.5}$$

$$P(k - S[P(k),u_i]) = P(k) + t \text{ if } k - S(P(k),u_i) > 0. \tag{4.6}$$

Therefore, to assure that relation (4.1) holds for each $i < n$, it is necessary to design a land price function P so that for each $i < n$, conditions (4.5) and (4.6) hold for all $k \in \{k \in X;\ p_i > P(k) > p_{i+1}\}$. Even though condition (4.5) or (4.6) is only necessary for expenditure minimization, in order to construct our price function we require that they hold on these intervals. Based on this consideration, we ask that our land price function P satisfy the following set of conditions:

(i) P is continuous and nonincreasing.

(ii) P(x) is decreasing on each interval $X_i \equiv \{x \in X:\ p_i > P(x) > p_{i+1}\}$, where i=1,2, ...,n–1.

(iii) $P(0) = p_1$, $P(l) = p_n$.

(iv) $S(P(x),u_1) = x$ for all $x \in X_1$.

(v) For each i=2,3, ...,n–1, $P(x - S[P(x), u_i]) = P(x) + t$ for all $x \in X_i$.

Notice that (iv) implies that for i=1, condition (4.5) holds at each $k \in X_1$. [To see this, set $x = k$ in (iv), and use (ii) and (iii).] Similarly, (v) implies that for each i=2,...,n–1, condition (4.6) holds at each $k \in X_i$. the remainder of the conditions are natural extensions of the properties of the function defined in (3.10).

It turns out that these five conditions, (i) to (v), uniquely determine a land price function. To show this, first observe that for each r=2,3,...,n–1, by (2.32),

$$x \in X_r \Leftrightarrow p_r > P(x) > p_{r+1} \Leftrightarrow p_j > P(x) + (r - j)t > p_{j+1} \text{ for all } j < r. \quad (4.7)$$

Next, (ii) implies that for each $i < n$, the inverse of P exists on the interval X_i. We call this inverse f_i. Now, let us fix $i \in \{2, ..., n-1\}$ and $x \in X_i$. Since $P(x - S[P(x),u_i]) = P(x) + t$ by (v) and $p_{i-1} > P(x) + t > p_i$ by (4.7), we have $p_{i-1} > P(x - S[P(x),u_i]) > p_i$, which implies $x - S[P(x),u_i] \in X_{i-1}$. Thus, $f_{i-1}(P(x) + t) = f_{i-1}(P(x - S[P(x),u_i])) = x - S[P(x),u_i]$. Setting $x_{i-1} \equiv f_{i-1}(P(x) + t)$, we have that

$$x - S(P(x),u_i) = x_{i-1} \in X_{i-1} \text{ and } P(x_{i-1}) = P(x) + t. \quad (4.8)$$

In turn, $x_{i-1} \in X_{i-1}$ implies that $P(x_{i-1} - S[P(x_{i-1}),u_{i-1}]) = P(x_{i-1}) + t$ by (v) and $p_{i-2} > P(x_{i-1}) + t > p_{i-1}$ by (4.7), provided $i > 2$. Thus, $p_{i-2} > P(x_{i-1} - S[P(x_{i-1}),u_{i-1}]) > p_{i-1}$, which implies $x_{i-1} - S[P(x_{i-1}),u_{i-1}] \in X_{i-2}$. Hence, as in (4.7) and (4.8), by setting $x_{i-2} \equiv f_{i-2}(P(x_{i-1}) + t)$, we have that

$$x_{i-1} - S(P(x_{i-1}),u_{i-1}) = x_{i-2} \in X_{i-2} \text{ and } P(x_{i-2}) = P(x) + 2t.$$

Continuing the process, it follows that

$$x_j - S(P(x_j),u_j) = x_{j-1} \in X_{j-1} \text{ and } P(x_{j-1}) = P(x) + (i-j-1)t \text{ for } 2 \le j < i. \quad (4.9)$$

Finally, for $j = 2$, (4.9) implies $x_1 \in X_1$, so we have by (iv) that

$$S(P(x_1),u_1) = x_1. \quad (4.10)$$

From (4.8) through (4.10), it follows that for each $i < n$,

$$x = \sum_{j=1}^{i} S(P(x) + (i - j)t, u_j) \text{ for } p_i > P(x) > p_{i+1}. \quad (4.11)$$

Notice that relation (4.11) defines the inverse f_i of the land price function P on each interval (p_{i+1}, p_i). Namely, for each $i < n$,

$$f_i(p) = \sum_{j=1}^{i} S(p + (i-j)t, u_j), \ \forall\, p \in (p_{i+1}, p_i). \quad (4.12)$$

Furthermore, to satisfy conditions (i) and (iii), these (n–1) segments of the land price curve [defined by (4.12)] must be connected by horizontal lines at each p_i $(i < n)$; thus, a unique land price function has been obtained. This observation leads us to the formal definition of our land price function. Define

$$s_i' = S_i(p_{i+1}, u_i), \text{ so } -Z_s(s_i', u_i) = p_{i+1}, \; i = 1, 2, ..., n-1. \tag{4.13}$$

Since $p_i > p_{i+1} = p_i - t$, (2.11) and (2.31) imply that

$$s_i < s_i', \; i = 1, 2, ..., n-1. \tag{4.14}$$

Next, for each $i = 1, 2, ..., n-1$, define the function f_i as follows:

$$f_i(p) = \sum_{j=1}^{i} S(p + (i-j)t, u_j) \text{ for } p \in (p_{i+1}, p_i). \tag{4.15}$$

Then, it is clear from Lemma 2.2 that f_i is continuous and decreasing on (p_{i+1}, p_i). Moreover,

$$\lim_{p\uparrow p_i} f_i(p) = \sum_{j=1}^{i} s_j < \sum_{j=1}^{i} s_j' = \lim_{p\downarrow p_{i+1}} f_i(p). \tag{4.16}$$

Hence, the inverse f_i^{-1} exists for $x \in (\Sigma_{j=1}^{i} s_j, \Sigma_{j=1}^{i} s_j')$. Now we propose our candidate P for a supporting price function:

$$P(x) = \begin{cases} p_1 & \text{for } x \in [0, s_1] \\ p_i & \text{for } x \in [\Sigma_{j=1}^{i-1} s_j', \Sigma_{j=1}^{i} s_j] \text{ where } i > 1 \\ f_i^{-1}(x) & \text{for } x \in (\Sigma_{j=1}^{i} s_j, \Sigma_{j=1}^{i} s_j') \text{ where } i < n. \end{cases} \tag{4.17}$$

See Figure 4 for an illustration of the price function when $n = 3$.

Figure 4

P is a well-defined function on X with the following properties:

<u>Lemma 4.1:</u> P is a positive, continuous, nonincreasing function on X such that:

(i) $P(x) = p_1 \Leftrightarrow x \in [0,s_1]$, (4.18a)

$P(x) = p_i \Leftrightarrow x \in [\Sigma_{j=1}^{i-1} s_j', \Sigma_{j=1}^{i} s_j]$, $i = 2,...,n$, (4.18b)

$p_i > P(x) > p_{i+1} = p_i - t \Leftrightarrow x \in (\Sigma_{j=1}^{i} s_j, \Sigma_{j=1}^{i} s_j')$, $i = 1, ..., n-1$. (4.18c)

(ii) For each $i = 1,2, ...,n-1$,

$dP(x)/dx < 0$ for $x \in (\Sigma_{j=1}^{i} s_j, \Sigma_{j=1}^{i} s_j')$. (4.19)

(iii) For each $i = 1,2,...,n-1$ and $x \in X$,

$x \in [\Sigma_{j=1}^{i} s_j, \Sigma_{j=1}^{i} s_j'] \Leftrightarrow \Sigma_{j=1}^{i} S[P(x) + (i-j)t, u_j] = x$. (4.20)

(iv) $x \in [s_1, s_1'] \Rightarrow P(x - S[P(x),u_1]) < P(x) + t$, (4.21)

$x \in [\Sigma_{j=1}^{i} s_j, \Sigma_{j=1}^{i} s_j'] \Rightarrow P(x - S[P(x),u_1]) = P(x) + t$, $i=2,...,n-1$. (4.22)

Proof: By construction, P is a positive, continuous, nonincreasing function (on X) having properties (i) and (ii). In (4.20), the left–to–right relationship ($\Rightarrow$) follows by construction and continuity of P. Using (2.11), (2.32) and (4.18), it is easy to prove the converse relationship ($\Leftarrow$) in (4.20). Next, by (4.20), $x \in [s_1, s_1']$ $\Rightarrow$ $S(P(x),u_1) = x$ $\Rightarrow$ $P(x - S[P(x),u_1]) = P(0) = p_1 < p_1 + t = P(x) + t$ for $x \in [s_1,s_1']$, which proves (4.21). Finally, to show (4.22), fix $i \in \{2,3,\ldots,n-1\}$. Then, by (2.32), (4.18c) and (4.20)

$$x \in (\Sigma_{j=1}^{i} s_j, \Sigma_{j=1}^{i} s_j') \quad \Rightarrow \begin{cases} p_i > P(x) > p_{i+1} \\ \Sigma_{j=1}^{i} S(P(x) + (i-j)t, u_j) = x \end{cases}$$

$$\Rightarrow \begin{cases} p_i > P(x) + t > p_i \\ x - S(P(x), u_i) = \Sigma_{j=1}^{i-1} S[P(x) + (i-j)t, u_j]. \end{cases}$$

Furthermore, since S(p,u) is decreasing in p, $p_i > P(x) > p_{i+1}$ implies that

$$\Sigma_{j=1}^{i-1} S[p_i + (i-j)t, u_j] < \Sigma_{j=1}^{i-1} S[P(x) + (i-j)t, u_j] < \Sigma_{j=1}^{i-1} S[p_{i+1} + (i-j)t, u_j]$$

$$\Rightarrow \Sigma_{j=1}^{i-1} S(p_j, u_j) < \Sigma_{j=1}^{i-1} S[P(x) + (i-j)t, u_j] < \Sigma_{j=1}^{i-1} S(p_{j+1}, u_j) \text{ by (2.32)}$$

$$\Rightarrow \Sigma_{j=1}^{i-1} s_j < \Sigma_{j=1}^{i-1} S[P(x) + (i-j)t, u_j] < \Sigma_{j=1}^{i-1} s_j'$$

$$\begin{aligned} \Rightarrow P(\Sigma_{j=1}^{i-1} S[P(x) + (i-j)t, u_j]) &= f_{i-1}^{-1}(\Sigma_{j=1}^{i-1} S[P(x) + (i-j)t, u_j]) \text{ by (4.17)} \\ &= f_{i-1}^{-1}(\Sigma_{j=1}^{i-1} S[P(x) + t + (i-1-j)t, u_j]) \\ &= f_{i-1}^{-1}(f_{i-1}(P(x) + t)) \text{ by (4.15)} \\ &= P(x) + t \end{aligned}$$

Therefore, $x \in (\Sigma_{j=1}^{i} s_j, \Sigma_{j=1}^{i} s_j') \Rightarrow P(x - S[P(x), u_i]) = P(\Sigma_{j=1}^{i-1} S[P(x) + (i-j)t, u_j])$ $= P(x) + t$. By continuity of P and S, we can conclude that (4.22) holds. Q.E.D.

Showing that the land price function P supports the efficient allocation is equivalent to showing that

$$E(s_i, x_i; P, u_i) = \hat{E}(P, u_i), \ \forall \ i \in N, \tag{4.23}$$

where functions E and $\hat{E}$ are defined by (3.7) and (3.8) respectively. Note that for each $i = 2,3, \ldots,n$, $s_i + x_i = s_i + \Sigma_{j=1}^{i-1} s_j$ [by (2.29)] $= \Sigma_{j=1}^{i} s_j$, and $s_i \equiv S(p_i,u_i) = S(P(\Sigma_{j=1}^{i} s_j),u_i)$ [by (4.18b)] $= S(P(s_i + x_i),u_i)$. Hence, setting $x = s_i + x_i$ ($= \Sigma_{j=1}^{i} s_j$) in (4.22), for each $i = 2,3,\ldots,n$ we have that

$$P(x_i) = P(x_i + s_i) + t \text{ and } s_i = S[P(x_i + s_i),u_i]. \tag{4.24}$$

Similarly, since $x_1 = 0$ and $s_1 \equiv S(p_1,u_1) = S(P(s_1),u_1)$ [by (4.18a)] $= S(P(s_1 + x_1),u_1)$, using (4.21) we can see that condition (4.24) also holds for $i = 1$. Therefore, for each $i \in N$, the first-order conditions (4.2) to (4.4) for expenditure minimization [implied by the right-hand side of (3.8)] are satisfied at (s_i,x_i). Hence, if these first-order (necessary) conditions were also sufficient for the expenditure minimization, we could conclude that (4.23) holds. Unfortunately, it turns out that for each $i \in N$, these first-order conditions are satisfied at (uncountably) many combinations of (s,x) that are not minimizing $E(\cdot, \cdot;P,u_i)$. Therefore, we prove (4.23) using an alternative approach involving several steps.

Define

$$\begin{aligned} E_i^*(k) &= \min_{s,x} \{E(s,x;P,u_i) : s > 0,\ x \geq 0,\ x + s = k\} \\ &= \min_{s} \{E(s,k-s;P,u_i) : 0 < s \leq k\}, \end{aligned} \tag{4.25}$$

which represents the minimum expenditure of household i necessary to achieve utility level u_i when household i is constrained to have the end of its lot, $x + s$, at k.

Notice that for $0 < s < k$,

$$\partial E(s,k-s;P,u_i)/\partial s = Z_s(s,u_i) + P(k-s) - t. \tag{4.26}$$

If $P(y)$ is differentiable at $y = k - s$, then

$$\partial^2 E(s,k-s;P,u_i)/\partial s^2 = Z_{ss}(s,u_i) - dP(y)/dy|_{y=k-s}. \tag{4.27}$$

Since $Z_{ss}(s,u_i) > 0$ by (2.7) and $dP(y)/dy|_{y=k-s} \leq 0$ wherever P is differentiable. Thus, $\partial^2 E_i(s,k-s;P,u_i)/\partial s^2 > 0$ provided that $P(y)$ is differentiable at $y = k - s$. Since $P(y)$ is not differentiable at a finite number of points only (by definition), we can conclude that $E_i(s,k-s;P,u_i)$ is strictly convex in s on $(0,k]$. By (2.4), $\lim_{s\downarrow 0}$

$E(s,k-s;P,u_i) = \infty$. Therefore, given any $k \in (0,l]$, $E(s,k-s;P,u_i)$ achieves its minimum at a unique point of $(0,k]$. We denote this point by $s_i^*(k)$. Then by definition

$$E_i^*(k) = E(s_i^*(k),k-s_i^*(k);P,u_i). \tag{4.28}$$

Since $E(s,k-s;P,u_i)$ is differentiable (but not necessarily twice differentiable), the necessary and sufficient conditions for $s_i^*(k)$ to be the unique minimizer of $E(s,k-s;P,u_i)$ on $(0,k]$ are:

If $s_i^*(k) < k$,

$$\partial E(s,k-s;P,u_i)/\partial s|_{s=s_i^*(k)} = Z_s(s_i^*(k),u_i) + P(k-s_i^*(k)) - t = 0 \tag{4.29a}$$

If $s_i^*(k) = k$,

$$\lim_{s\uparrow k} \partial E(s,k-s;P,u_i)/\partial s|_{s=s_i^*(k)} = Z_s(s_i^*(k),u_i) + P(k-s_i^*(k)) - t \leq 0 \tag{4.29b}$$

Since $E(s,k-s;P,u_i)$ is strictly convex in s, it follows that

$$s_i^*(k) = k \Leftrightarrow Z_s(k,u_i) + P(0) - t \leq 0. \tag{4.30}$$

Since $P(0) - t = p_1 - t = p_2$, $Z_s(k,u_i) + P(0) - t \leq 0 \Leftrightarrow -Z_s(k,u_i) \geq p_2 \Leftrightarrow k \leq S(p_2,u_i)$. The last step follows using (2.7) and (2.10). Finally,

$$s_i^*(k) < k \Leftrightarrow k > S(p_2,u_i) \tag{4.31a}$$

$$s_i^*(k) = k \Leftrightarrow k \leq S(p_2,u_i) \tag{4.31b}$$

Using conditions (4.31a) and (4.31b), the following characteristics of function E_i^* can be derived (see Appendix 1 for a proof):

Lemma 4.2: For each $i = 1,2,\ldots,n$, E_i^* is continuous on $(0,l]$. Furthermore,

(i) for each $i = 1,2,\ldots,n$, $E_i^*(k)$ is differentiable at any $k \in (0,S(p_2,u_i))$, where

$$dE_1^*(k)/dk \begin{cases} < 0 \text{ for } k < s_1 \\ = 0 \text{ for } s_1 \leq k < S(p_2,u_1) \equiv s_1' \end{cases} \tag{4.32}$$

$$dE_i^*(k)/dk \leq 0 \text{ for all } k < S(p_2,u_i),\ i = 1,2,\ldots,n. \tag{4.33}$$

(ii) For each $i = 1,2,\ldots,n$, $E_i^*(k)$ is differentiable at $k \in (S(p_2,u_i),l)$ iff $k \neq \sum_{j=1}^{r} s_j + S(P(\sum_{j=1}^{r} s_j)-t,u_i)$ and $k \neq \sum_{j=1}^{r} s_j' + S(P(\sum_{j=1}^{r} s_j')-t,u_i)$ for any $r = 1,2,\ldots,n-1$.

When $E_i^*(k)$ is differentiable at $k \in (S(p_2,u_i),l)$,

$$dE_i^*(k)/dk \lesseqgtr 0 \text{ as } P(k-S(P(k),u_i)) \gtreqless P(k) + t, \tag{4.34}$$

where $P(k - S(P(k),u_i))$ is defined to be ∞ if $k - S(P(k),u_i) < 0$.

For the relation (4.34) to be useful, it is necessary to know whether $P(k-S(P(k),u_i))$ is greater or smaller than $P(k) + t$ at each $k \in (S(p_2,u_i),l]$. The next lemma provides this information (see Appendix 2 for a proof).

Lemma 4.3: Let $P(k-S(P(k),u_i)) \equiv \infty$ if $k < S(P(k),u_i)$. Then, we have:

(i) $P(k-S(P(k),u_1)) \leq P(k) + t$ for all $k \in (S(p_2,u_1),l]$. (4.35)

(ii) For $i = 2,3,\ldots,n-1$,

$$P(k-S(P(k),u_i)) \begin{cases} \geq P(k) + t \text{ for } k \in (S(p_2,u_i),\Sigma_{j=2}^{i} s_j] & (4.36a) \\ = P(k) + t \text{ for } k \in [\Sigma_{j=1}^{i} s_j, \Sigma_{j=1}^{i} s_j'] & (4.36b) \\ \leq P(k) + t \text{ for } k \in (\Sigma_{j=1}^{i} s_j', l] & (4.36c) \end{cases}$$

(iii) $P(k-S(P(k),u_n)) \geq P(k) + t$ for all $k \in (S(p_2,u_n),l]$. (4.37)

It follows from (4.34) and Lemma 4.3 that for each $i \in N$, wherever $E_i^*(k)$ is differentiable,

$$dE_1^*(k)/dk \geq 0 \text{ for } k \in (S(p_2,u_1),l), \tag{4.38}$$

$$dE_i^*(k)/dk \begin{cases} \leq 0 & \text{for } k \in (S(p_2,u_i), \Sigma_{j=1}^{i} s_j] \\ = 0 & \text{for } k \in (\Sigma_{j=1}^{i} s_j, \Sigma_{j=1}^{i} s_j'] \\ \geq 0 & \text{for } k \in (\Sigma_{j=1}^{i} s_j', l) \end{cases} \text{ where } 1 < i < n, \tag{4.39}$$

$$dE_n^*(k)/dk \leq 0 \text{ for } k \in (S(p_2,u_n), l). \tag{4.40}$$

Now, for each $i \in N$, define

$$\hat{E}_i = \min\{E_i^*(k) : 0 < k \leq l\}, \tag{4.41}$$

$$\hat{K}_i = \{k \in (0,l] : E_i^*(k) = \hat{E}_i\}. \tag{4.42}$$

By definition, $\hat{K}_i$ represents the set of lot endpoints at which expenditure is minimized. Recall that E_i^* is differentiable on $(S(p_2,u_i),l)$ except at a finite number of points.

Using (4.32), (4.33), (4.38) through (4.40), and recalling the continuity of E_i^* on $(0, l]$, we can conclude that

$$[\Sigma_{j=1}^{i} s_j, \Sigma_{j=1}^{i} s_j'] \subset \hat{K}_i \text{ for each } i = 1,2,\ldots,n-1, \tag{4.43}$$

$$\Sigma_{j=1}^{n} s_j = l \in \hat{K}_n. \tag{4.44}$$

Next, we claim that

$$\hat{E}(P,u_1) = E(s_1,x_1;P,u_1) = E(k,0;P,u_1) \text{ for all } k \in [s_1,s_1']. \tag{4.45a}$$

$$\hat{E}(P,u_i) = E(s_i,x_i;P,u_i) = E[S(P(k),u_i),k-S(P(k),u_i);P,u_i]$$

$$\text{for all } k \in [\Sigma_{j=1}^{i} s_i, \Sigma_{j=1}^{i} s_j'], \text{ where } 1 < i < n, \tag{4.45b}$$

$$\hat{E}(P,u_n) = E(s_n,x_n;P,u_n). \tag{4.45c}$$

To see (4.45a), notice that $k \in [s_1, s_1'] \Rightarrow k \le s_1' \equiv S(p_2,u_1) \Rightarrow s_1^*(k) = k$ [by (4.31b)] $\Rightarrow E_1^*(k) = E_1(k,0;P,u_1)$ [by (4.28)]. Furthermore, by definition $\hat{E}(P,u_1) = \min_{0<k\le l} E_1^*(k)$. Hence, using (4.43) and recalling that $x_1 = 0$ we can conclude that (4.45a) holds. To show (4.45b), let $1 < i < n$. Then $k \in [\Sigma_{j=1}^{i} s_j, \Sigma_{j=1}^{i} s_j'] \Rightarrow k \ge s_1 + s_2 > s_2 = S(p_2,u_2) \ge S(p_2,u_i) \Rightarrow k > S(p_2, u_i) \Rightarrow s_i^*(k) < k$ [by (4.31a)]. Using (4.29a), for each $k \in [\Sigma_{j=1}^{i} s_j, \Sigma_{j=1}^{i} s_j']$, $s^* \equiv s_i^*(k)$ is uniquely determined by the following equation in s^*:

$$Z_s(s^*,u_i) + P(k-s^*) - t = 0.$$

Notice that for each $k \in [\Sigma_{j=1}^{i} s_i, \Sigma_{j=1}^{i} s_i']$, $Z_s(S(P(k),u_i),u_i) + P(k-S(P(k),u_i)) - t = -P(k) + P(k-S(P(k),u_i)) - t = 0$. The last step uses (2.10) and (4.22). Hence, $s_i^*(k) = S(P(k),u_i)$. In particular, when $k = \Sigma_{j=1}^{i} s_j$, $s_i^*(\Sigma_{j=1}^{i} s_j) = S(P(\Sigma_{j=1}^{i} s_j),u_i) = S(p_i,u_i)$ [by (4.18b)] $\equiv s_i$, and $\Sigma_{j=1}^{i} s_j - S(P(\Sigma_{j=1}^{i} s_j),u_i) = \Sigma_{j=1}^{i} s_j - S(p_i,u_i) = \Sigma_{j=1}^{i-1} s_j = x_i$. Therefore, since $\hat{E}(P,u_i) = \min_{0<k\le l} E_i^*(k)$, using (4.43) we can conclude that (4.45b) holds. Similarly, using (4.29a), (4.31a) and (4.44), we can prove (4.45c). Since (4.45a) through (4.45c) imply (4.23), the proof of Proposition 3 is now complete.

Next, let numbers ϵ_i ($i = 2,3,\ldots,n$) be defined as follows:

$$\epsilon_i = \int_{p_i}^{p_{i-1}} (S(p,u_{i-1}) - s_{i-1})dp \equiv \int_{s_{i-1}}^{s_{i-1}'} (-Z_s(s,u_{i-1}) - p_i)dp, \tag{4.46}$$

where the last identity follows from (2.10), (2.31) and (4.13). Define also

$$Y_i \equiv E(s_i,x_i;P,u_i),\ i \in N. \tag{4.47}$$

Recalling (3.7) and using (4.17) and (4.46),

$$Y_1 = Z(s_1,u_1) + p_1 s_1, \tag{4.48a}$$

$$Y_i = Z(s_i,u_i) + \{p_i s_i + \Sigma_{j=2}^{i} \epsilon_j\} + t\cdot\Sigma_{j=1}^{i-1} s_j,\ i = 2, \ldots, n. \tag{4.48b}$$

Now for each $i \in N$, define $z_i \equiv Z(s_i,u_i)$ and income Y_i as (4.48). Then, since $\hat{E}(P,u_i) = E(s_i,x_i;P,u_i) = Y_i$, we can conclude that $[p,\{s_i,z_i,x_i\}_{i=1}^{n}]$ is a competitive equilibrium for ALP$(Y_1, \ldots, Y_n)$. This observation leads us to a constructive proof of equilibrium existence in the next section.

Finally, as in the two–household case in section 3, the reader might have noticed that P [defined by (4.17)] is not the only supporting land price function for the efficient allocation $\{s_i,z_i,x_i\}_{i=1}^{n}$. For example, let

$$s_i'' \equiv S_i(p_{i-1},u_i),\ i = 2,3,\ldots,n, \tag{4.49}$$

$$\tilde{f}_i(p) \equiv \Sigma_{j=2}^{i} S(p + (i-j)t,\ u_j) \text{ for } p \in (p_i,p_{i-1}),\ i = 2,3,\ldots,n \tag{4.50}$$

and define a new land price function $\tilde{P}$ by[12]

$$\tilde{P}(x) = \begin{cases} p_1 & \text{for } x \in [0,\ s_2''] \\ p_i & \text{for } x \in [\Sigma_{j=2}^{i} s_j,\ \Sigma_{j=2}^{i+1} s_j''],\ i = 2,\ldots,n-1 \\ p_n & \text{for } x \in [\Sigma_{j=2}^{n} s_j,\ l] \\ \tilde{f}_i^{-1}(x) & \text{for } x \in (\Sigma_{j=2}^{i} s_j'',\ \Sigma_{j=2}^{i} s_j),\ i = 2,\ldots,n. \end{cases} \tag{4.51}$$

Then we can show that $\tilde{P}$ also supports the efficient allocation using the same arguments as used for P. Furthermore, it can also be shown that any land price function P* such that $\tilde{P}(x) \geq P^*(x) \geq P(x)$ for all $x \in X$ supports the efficient allocation. As will be shown in the next section, these price functions can generate equilibria that differ in the rent collections of landlords as well as the utility levels of households.

5. Existence of a Competitive Equilibrium

In this section we demonstrate that an equilibrium exists for any set of positive endowments of numeraire.

Proposition 4: Given any $(Y_1, \ldots, Y_n) \in \mathbb{R}^n_{++}$, there exists a competitive equilibrium for the ALP$(Y_1, \ldots, Y_n)$ – model.

In the following we present a constructive proof, which is based on the land price function P in the previous section. Beginning with an arbitrarily chosen p_1, by inductive steps we construct a land price function $P(\cdot;p_1)$ which has essentially the same shape as P. By demonstrating that there exists a number p_1^* such that the land consumption of the n households under $P(\cdot;p_1^*)$ sums to l (≡ the size of the land area), the proof is completed.

Before starting the proof, it is necessary to define the indirect utility function. For each p>0, Y>0, define

$$V(p,Y) \equiv \max\{U(s,z) : s>0,\ z>0,\ z + ps = Y\}$$

Without loss of generality, order households so that $Y_1 \leq Y_2 \leq \ldots \leq Y_n$. The proof now proceeds by inductive steps.

Step 0: Arbitrarily fix $p_1 > (n-1)t$.

Step 1: Define

$$u_1(p_1) = V(p_1, Y_1),$$

$$s_1(p_1) = S(p_1, u_1(p_1)).$$

Then it follows by definition that

$$Y_1 = Z(s_1(p_1), u_1(p_1)) + p_1 s_1 \tag{5.1}$$

Define

$$s_1'(p_1) = S(p_1 - t, u_1(p_1)), \tag{5.2}$$

$$\epsilon_2(p_1) = \int_{s_1(p_1)}^{s_1'(p_1)} [-Z_s(s, u_1(p_1)) - (p_1 - t)]dp. \tag{5.3}$$

Step i ($i > 1$): From step i–1 we have $s_j(p_1)$, $s'_j(p_1)$ and $u_j(p_1)$ for $j = 1,2,\dots,i-1$. We also have $\epsilon_j(p_1)$ for $j = 2,\dots,i$. Define

$$I_i(p_1) = Y_i - t\cdot\sum_{j=1}^{i-1} s_j(p_1) - \sum_{j=2}^{i} \epsilon_j(p_1), \tag{5.4}$$

$$u_i(p_1) = V(p_1-(i-1)t, I_i(p_1)),$$

$$s_i(p_1) = S(p_1-(i-1)t, u_i(p_1)).$$

Using these three definitions,

$$Y_i = Z(s_i(p_1),u_i(p_1)) + \{(p_1-(i-1)t)s_i(p_1) + \sum_{j=2}^{i} \epsilon_j(p_1) + t\cdot\sum_{j=2}^{i-1} s_j(p_1). \tag{5.5}$$

Also define

$$s'_i(p_1) = S(p_1 - it, u_i(p_1)), \tag{5.6}$$

$$\epsilon_{i+1}(p_1) = \int_{s_i(p_1)}^{s'_i(p_1)} [-Z_s(s,u_i(p_1)) - (p_i-it)]dp. \tag{5.7}$$

Continue the induction until step n is complete [where the definition of $\epsilon_{n+1}(p_1)$ is omitted in step n]. Finally, define

$$x_{n+1}(p_1) = \sum_{j=1}^{n} s_j(p_1) \text{ for each } p_1 \in ((n-1)t,\infty).$$

Lemma 5.1:

(i) x_{n+1} is continuous and decreasing on $((n-1)t,\infty)$.

(ii) $\lim_{p_1\downarrow(n-1)t} x_{n+1}(p_1) = \infty$, $\lim_{p_1\uparrow\infty} x_{n+1}(p_1) = 0$.

Proof: For (i), continuity is obvious. That x_{n+1} is decreasing follows from the fact that land is a normal good.

For (ii), first we show that $u_i(p_1) \leq u_{i+1}(p_1)$ for $i = 1,2,\dots,n-1$. Recalling (5.1) and (5.5), and using (5.7), we can readily see that for each $i = 1,2,\dots,n-1$,

$$Y_i = Z(s'_i(p_1),u_i(p_1)) + \{(p_1-it)s'_i(p_1) + \Sigma_{j=2}^{i+1}\epsilon_j(p_1)\} + t\cdot\Sigma_{j=1}^{i}s_j(p_1). \tag{5.8}$$

For each $i = 1,2,\dots,n-1$, from (5.1), (5.4), (5.5), and (5.8),

$$Z(s'_i(p_1),u_i(p_1)) + (p_1-it)s'_i(p_1)$$

$$= Y_i - \Sigma_{j=2}^{i+1}\epsilon_j(p_1) - t\cdot\Sigma_{j=1}^{i} s_j(p_1)$$

$$= \quad I_{i+1}(p_1) - (Y_{i+1} - Y_i). \tag{5.9}$$

From (5.6) and (5.9),

$$\begin{aligned} u_i(p_1) &= V(p_1-it, I_{i+1}(p_1) - (Y_{i+1}-Y_i)) \\ &\le V(p_1-it, I_{i+1}(p_1)) \text{ [since } Y_{i+1} \ge Y_i] \\ &= u_{i+1}(p_1) \text{ [by definition].} \end{aligned}$$

Hence, $u_i(p_1) \le u_{i+1}(p_1)$ for $i = 1,2,\ldots,n-1$, as was to be shown. Since land is a normal good, it follows $s_n(p_1) \equiv S(p_1 - t(n-1), u_n(p_1)) \ge S(p_1 - t(n-1), u_1(p_1))$. Hence,

$$\begin{aligned} \lim_{p_1 \downarrow t(n-1)} x_{n+1}(p_1) &\ge \lim_{p_1 \downarrow t(n-1)} S(p_1 - t(n-1), u_1(p_1)) \\ &\equiv \lim_{p_1 \downarrow t(n-1)} S(p_1 - t(n-1), V(p_1, Y_1)) \\ &= \lim_{p_1 \downarrow t(n-1)} S(p_1 - t(n-1), \lim_{p_1 \downarrow t(n-1)} V(p_1, v_1)) \\ &= \lim_{p \downarrow 0} S(p, V(t(n-1), Y_1)) \\ &= \infty. \end{aligned}$$

Next, for $p_1 > t(n-1)$, define

$$\bar{s}(p_1) = Y_n/[p_1 - t(n-1)].$$

Then

$$s_i(p_1) < \bar{s}(p_1) \text{ for all } i = 1,2,\ldots,n,$$

and

$$\lim_{p_1 \uparrow \infty} \bar{s}(p_1) = 0.$$

Therefore $\lim_{p_1 \uparrow \infty} x_{n+1}(p_1) = 0.$ Q.E.D.

From Lemma 5.1, there exists a unique p_1^* such that

$$x_{n+1}(p_1^*) \equiv \sum_{j=1}^{n} s_j(p_1^*) = l.$$

Next a number of definitions are proposed:

$$s_i = s_i(p_1^*)$$
$$s_i' = s_i'(p_1^*)$$

$$u_i = u_i(p_1^*)$$
$$p_i = p_i^* \equiv p_1^* - (i-1)t.$$

Define f_i as in (4.15) using the definitions of the parameters given just above. Now define a land price function

$$P(x;p_1^*) = \begin{cases} p_1^* & \text{for } x \in [0, s_1(p_1^*)] \\ p_i^* & \text{for } x \in [\Sigma_{j=1}^{i-1} s_j'(p_1^*), \Sigma_{j=1}^{i} s_j(p_1^*)], i > 1 \\ f_i^{-1}(x) & \text{for } x \in (\Sigma_{j=1}^{i} s_j(p_1^*), \Sigma_{j=1}^{i} s_j'(p_1^*)), i < n. \end{cases}$$

Finally, for i = 2,3,...,n, define

$$x_1^*(p_1^*) = 0, \; x_i^*(p_1^*) = \sum_{j=1}^{i-1} s_j(p_1^*)$$

and for i = 1,2,...,n, define

$$z_i(p_1^*) = Z(s_i(p_1^*), u_i(p_1^*)).$$

Employing these definitions in the arguments in section 4 (which we will not repeat here for the sake of brevity), it is clear that $P(\cdot;p_1^*)$ is a supporting land price function for the feasible allocation $\{(s_i(p_1^*), z_i(p_1^*), x_i(p_1^*))\}_{i=1}^n$. Therefore, $[P(\cdot;p_1^*), \{(s_i(p_1^*), z_i(p_1^*), x_i(p_1^*))\}_{i=1}^n]$ is a competitive equilibrium for the $ALP(Y_1,...,Y_n)$ – model. This completes the proof of Proposition 4.

As the reader may have already noticed, the competitive equilibrium obtained above is not the only competitive equilibrium for $ALP(Y_1, ..., Y_n)$. For example, using the shape of the land price function $\tilde{P}$ given by (4.51), another competitive equilibrium can be found. It is easy to see that this new competitive equilibrium yields a larger total land rent collection by absentee landlords and lower equilibrium utility levels for all n households compared to those implied by the competitive equilibrium constructed above. Similarly, employing different shapes of land price functions, we can generate a continuum of competitive equilibria that differ in the rent collections of landlords and the utility levels of households. All of the equilibria are Pareto optimal.

6. Conclusions

In this paper, we have attempted to apply the tools of modern economic analysis to Alonso's classical model. We have shown that the welfare theorems hold and that equilibrium exists provided that the standard assumptions are satisfied *and all households have the same (well–behaved) utility function.* It is clearly desirable to allow households to employ different utility functions, and this should be a priority for future work. It appears that techniques similar to those used here might be used for this extension. In particular, the same techniques appear to be applicable if incomes and steepness of bid rent order the consumers and yield identical rankings.

We have also shown that there is a *continuum* of supporting prices for *each* efficient allocation. In addition, there is a *continuum* of equilibria for any distribution of endowments. This contrasts with both the canonical general equilibrium model as well as the standard monocentric city model. All of these equilibrium allocations are efficient (even if passive landlords are included in the model). It would be interesting to investigate this phenomenon further. We conjecture that this unusual result is a consequence of the discreteness involved in ordering the location of the consumers.

It is interesting to ask how the indeterminacy of equilibrium changes as the parameters of the economy change. The answer to this question might provide a clue to the connection between the discrete model and the standard density model discussed in the introduction.

In this paper we have considered a bounded (one–dimensional) land area because we intended to make our problems as similar as possible to those of traditional microeconomics. In contrast, the standard density model often employs the assumption of an unbounded land area together with a given opportunity cost of land. We can, however, readily see that with appropriate definitions of efficient allocation and competitive equilibrium, all the results of this paper (mutatis mutandis) hold for a model with an unbounded land area.[13]

Footnotes

[1]Some recent research has addressed this problem. See, for example, Papageorgiou and Pines [11]; Asami, Fujita, and Smith [3]; Kamecke [7]; and Berliant and ten Raa [6]. However, it seems that much is left to be done in this area.

[2]Namely, the demand correspondence is not convex valued. For discussion, see Berliant [5, chapter III].

[3]Given the form of transportation cost, it seems unlikely to us that the finite model used in this paper can be derived from a continuous density model using the approach given in [11].

[4]More generally, we may assume that the transport cost associated with a lot, $[x, x + s)$, equals $t[\alpha x + (1 - \alpha)(x + s)]$, whre α is a given constant such that $0 \leq \alpha \leq 1$. It would not be difficult to show that the main results of the paper hold under this generalization (mutatis mutandis).

[5]Throughout this paper, we let $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_{++}$ denote the real numbers, nonnegative reals, and positive reals, respectively. The terms "increasing" and "decreasing" are taken to mean "strictly increasing " and "strictly decreasing" respectively.

[6]Strictly speaking, Assumption 1 alone cannot prevent the possibility that each Z_u, Z_{ss} and Z_{su} becomes zero on a set of points with measure zero. However, this minor difference does not affect our results in any essential way, and hence we neglect it in the following discussion.

[7]For simplicity, the opportunity cost of land is assumed to be zero. For a reformulation of the problems arising from a positive opportunity cost of land, see footnote 13.

[8]This result is closely related to Mirrlees' "unequal treatment of equals." Maximization of a Benthamite social welfare function (where households are equally weighted) can result in higher utility levels for households located farther from the CBD compared with households located closer to the CBD. There are many articles that discuss this phenomenon; see, for example, Mirrlees [9], Arnott and Riley [2], and Wildasin [13].

[9]It is not difficult to see that the land price function P [given by (3.10)] represents the lowest land price curve among the family of supporting land price curves (of allocation G) that are nonincreasing on X. The question of the existence of a broader class of supporting land price functions is deferred to future research.

[10]In (4.1), if equality holds for all k such that $p_i > P(k) > p_{i+1}$, then it also holds at both ends of the interval $\{k \in X : p_i > P(k) > p_{i+1}\}$. However, we cannot replace this interval by the set $\{k \in X : p_i \geq P(k) \geq p_{i+1}\}$. For example, if n=2 and i=1, then we can see from Figure 3 that the set $\{k \in X : p_1 \geq P(k) \geq p_2\}$ coincides with the entire location space, X.

[11]If $x + s = l$, then the first-order conditions are: $P(x) \leq -Z_s(s,u) + t$, $P(x) = -Z_s(s,u) + t$ if $x > 0$, and $P(l) \leq -Z_s(s,u)$.

[12]The land price function $\tilde{P}$ is a generalization of the function (3.11) in Section 3.

[13]Specifically, set $X = [0, \infty)$ and rewrite (2.14) as

$$C = \Sigma_{i=1}^{n}(Z_i + x_i + R_A s_i),$$

where R_A represents the agricultural land rent (or the net revenue from each unit of land used for agricultural production). A new cost term, $\Sigma_{i=1}^{n} R_A s_i$, must be added where appropriate after (2.14). In the definition of a competitive equilibrium in section 2, (2.38) must be rewritten as

$P^*(x) \geq R_A$ for all $x \in X$, and

$P^*(x) = R_A$ at each $x \in (X \setminus \bigcup_{i=1}^{n} [x_i^*, x_i^* + s_i^*])$.

Appendix 1: Proof of Lemma 4.2

Proof of Lemma 4.2(ii):

Recall from the discussion above equation (4.28) that given any $k \in (0,l]$, $E(s,k-s;P,u_i)$ achieves its (finite) minimum at a unique point $s_i^*(k)$ in $(0,k]$. Suppose $S(p_2,u_i) < k \leq l$. Then using (4.29a) and (4.31a), $s_i^*(k)$ is the unique solution of the following equation in s^*:

$$Z_s(s^*,u_i) + P(k-s^*) - t = 0. \tag{A1.1}$$

Provided that $P(y)$ is differentiable at $y = k - s^*$, it follows that

$$Z_{ss}(s^*,u_i)ds^* + P'(k-s^*)(dk - ds^*) = 0,$$

where $P'(y) \equiv dP(y)/dy$. Then

$$\frac{ds^*}{dk} = \frac{-P'(k - s^*)}{Z_{ss}(s^*, u_i) - P'(k - s^*)}. \tag{A1.2}$$

Hence $s_i^*(k)$ is <u>differentiable at</u> $k \in (S(p_2,u_i),l)$ iff $P(k-s^*)$ is <u>differentiable at</u> $y = k - s^*$. In this event, since $Z_{ss} > 0$ and $P' \leq 0$, we have that

$$0 \leq \frac{ds_i^*(k)}{dk} < 1. \tag{A1.3}$$

Now suppose $k \in (S(p_2,u_i),l]$ and $k - s_i^*(k) = \bar{y}$ for some $\bar{y} \in \mathbb{R}$. [Due to (4.31a), this implies $l > k - s_i^*(k) = \bar{y} > 0$.] It follows from (A1.1) that $s_i^*(k) = S[P(\bar{y})-t,u_i]$ and hence

$$k = \bar{y} + S[P(\bar{y})-t,u_i].$$

By construction, $P(y)$ is differentiable at $\bar{y} \in (0, l)$ iff $\bar{y} \neq \sum_{j=1}^{r} s_j$ and $\bar{y} \neq \sum_{j=1}^{r} s_j'$ for any $r \in \{1,\ldots,n-1\}$. Since S is differentiable, $s_i^*(k)$ is differentiable at $k \in (S(p_2,u_i),l)$ iff $k \neq \sum_{j=1}^{r} s_j + S[P(\sum_{j=1}^{r} s_j)-t,u_i]$ and $k \neq \sum_{j=1}^{r} s_j' + S_i[P(\sum_{j=1}^{r} s_j')-t,u_i]$ for any $r \in \{1,\ldots,n-1\}$.

By definition,

$$E_i^*(k) = Z(s_i^*(k),u_i) + \int_{k-s_i^*(k)}^{k} P(y)dy + t(k - s_i^*(k)). \tag{A1.4}$$

$E_i^*(k)$ is differentiable iff s_i^* is differentiable at k. Therefore, the first half of Lemma 4.2(ii) is proved. Next, notice that condition (A1.1) can be restated as

$$s^* = S[P(k-s^*)-t,u_i]. \tag{A1.5}$$

Hence, when s_i^* is differentiable at $k \in (S(p_2,u_i),l)$, it follows from (A1.4) that

$$\begin{aligned}\frac{dE_i^*(k)}{dk} &= Z_s(s_i^*(k),u_i)\cdot\frac{ds_i^*}{dk} + P(k) - P[k-s_i^*(k)](1 - \frac{ds_i^*}{dk}) + t(1 - \frac{ds_i^*}{dk}) \\ &= P(k) + t - P[k - s_i^*(k)].\end{aligned} \tag{A1.6}$$

The last step follows using $Z_s(s_i^*(k),u_i) = Z_s(S[P(k-s_i^*(k))-t,u_i],u_i)$ [by (A1.5)] $= -\{P(k-s_i^*(k))-t\}$ by (2.10). Hence

$$\frac{dE_i^*(k)}{dk} \lesseqgtr 0 \text{ as } P[k-s_i^*(k)] \gtreqless P(k) + t. \tag{A1.7}$$

For notational simplicity set $s^* \equiv s_i^*(k)$. Then

$$\begin{aligned}P(k-s^*) = P(k) + t &\Rightarrow s^* = S[P(k),u_i] \text{ by (A1.5)} \\ &\Rightarrow P(k-S[P(k),u_i]) = P(k) + t,\end{aligned}$$

and

$$\begin{aligned}P(k-s^*) < P(k) + t &\Rightarrow P(k-s^*) - t < P(k) \\ &\Rightarrow s^* = S[P(k-s^*)-t,u_i] > S[P(k),u_i] \text{ by (A1.5) and} \\ &\qquad \text{since } S(p,u_i) \text{ is decreasing in p} \\ &\Rightarrow k - S[P(k),u_i] > k - s^* \\ &\Rightarrow P(k-S[P(k),u_i]) \leq P(k-s^*) \text{ since P is nonincreasing} \\ &\Rightarrow P(k-S[P(k),u_i]) < P(k) + t.\end{aligned}$$

Recalling that $P(y) \equiv \infty$ if $y < 0$, similar reasoning yields

$$\begin{aligned}P(k-s^*) > P(k) + t &\Rightarrow P(k - s^*) - t > P(k) \\ &\Rightarrow s^* = S[P(k-s^*)-t,u_i] < S[P(k),u_i] \\ &\Rightarrow k - S[P(k),u_i] < k - s^* \\ &\Rightarrow P(k-S[P(k),u_i]) \geq P(k-s^*) \text{ since P is nonincreasing} \\ &\qquad \text{on } (-\infty, l] \\ &\Rightarrow P(k-S[P(k),u_i]) > P(k) + t.\end{aligned}$$

Setting $s^* \equiv s_i(k)$,

$$P[k-s_i^*(k)] \gtrless P(k) + t \text{ as } P(k-S[P(k),u_i]) \gtrless P(k) + t. \tag{A1.8}$$

(A1.7) and (A1.8) together imply (4.34).

Proof of Lemma 4.2(i):

Suppose that $0 < k \leq S(p_2,u_i)$. Then, $s_i^*(k) = k$ by (4.31b).

Hence,

$$E_i^*(k) = E(s_i^*(k),k-s_i^*(k);P,u_i) = Z(k,u_i) + \int_0^k P(y)dy \tag{A1.9}$$

which is differentiable at any $k \in (0,S(p_2,u_i))$. Also

$$\frac{dE_i^*(k)}{dk} = Z_s(k, u_i) + P(k). \tag{A1.10}$$

By construction of P,

$$P(x) \begin{cases} = p_1 < -Z_s(x,u_1) \text{ for } x < s_1 \\ = -Z_s(x,u_1) \text{ for } s_1 \leq x \leq s_1' \\ > -Z_s(x,u_1) \text{ for } x > s_1' \end{cases} \tag{A1.11}$$

(A1.10) and (A1.11) together imply (4.32). Then, for i=1, (4.32) implies (4.33). To show (4.33) for each $i > 1$, observe that $Z_{su}(s,u) < 0$ by (2.7), and $u_i \geq u_1$ by assumption. Hence, at each $k > 0$,

$$Z_s(k,u_1) \geq Z_s(k,u_i). \tag{A1.12}$$

Using (A1.10) through (A1.12),

$$\frac{dE_i^*(k)}{dk} = Z_s(k,u_i) + P(k) \leq Z_s(k,u_1) + P(k) \leq 0 \text{ for } k < s_1'. \tag{A1.13}$$

Furthermore, since $u_i \geq u_1$, by (2.11) $s' \equiv S(p_2,u_1) \leq S(p_2,u_i)$; since $P(s_1') = p_2$ and P is nonincreasing, $P(k) \leq p_2$ for $k \geq s_1'$; by (2.10) and (2.11), $k < S(p_2, u_i) \Leftrightarrow Z_s(k,u_i) < -p_2$. Therefore

$$\frac{dE_i^*(k)}{dk} = Z_s(k,u_i) + P(k) < -p_2 + P(k) \leq 0 \text{ for } s_1' \leq k < S(p_2,u_i). \tag{A1.14}$$

(A1.13) and (A1.14) together imply (4.33) for $i > 1$.

Finally, the continuity of the function E_i on $(0,S(p_2u_i))$ follows from (A1.9) [because $Z(\cdot,u_i)$ and P are continuous]. Recall also that on $(S(p_2,u_i)l)$, E_i is differentiable except at (N–2)x2 points. The continuity of E_i at these (N–2)x2 points, at $k \equiv S(p_2,u_i)$, and at $k \equiv l$ can be confirmed by using the continuity of $Z(\cdot,u_i)$ and P. Therefore, E_i is continuous on $(0,l]$. Q.E.D.

Appendix 2: Proof of Lemma 4.3

Since relations (4.35) and (4.37) are special uses of (4.36), (4.36) is proved first. Since (4.22) implies (4.36b), only (4.36a) and (4.36c) need to be proved.

Proof of (4.36a): Fix any $i \in \{2,\dots,n-1\}$. Then since $i \geq 2$ implies that $S(p_2,u_i) \geq S(p_2,u_1) \equiv s_1'$ it follows that

$$(S(p_2,u_i), \sum_{j=1}^{i} s_j] = (\{\bigcup_{r=1}^{i-1}(\sum_{j=1}^{r} s_j', \sum_{j=1}^{r+1} s_j)\}\cup\{\bigcup_{r=2}^{i}[\sum_{j=1}^{r} s_j, \sum_{j=1}^{r} s_j']\})\backslash[0,S(p_2,u_i)] \tag{A2.1}$$

(a–1) Suppose that

$$k \in (\sum_{j=1}^{r} s_j', \sum_{j=1}^{r+1} s_j) \text{ for some } r \in \{1,\dots,i-1\}. \tag{A2.2}$$

Then, $P(k) = p_{r+1}$ by (4.18b). Hence,

$$P(k-S[P(k),u_i]) < P(k) + t$$

$$\Rightarrow P(k-S(p_{r+1},u_1)) < p_{r+1} + t = p_r$$

$$\Rightarrow k - S(p_{r+1},u_i) > \sum_{j=1}^{r} s_j \text{ by (4.18)}$$

$$\Rightarrow k > \sum_{j=1}^{r} s_j + S(p_{r+1},u_i)$$

$$\Rightarrow k > \sum_{j=1}^{r+1} s_j \tag{A2.3}$$

The last step follows because $r \leq i-1 \Rightarrow i \geq r+1 \Rightarrow S(p_{r+1},u_i) \geq S(p_{r+1},u_{r+1}) \equiv s_{r+1}$. However, (A2.3) contradicts (A2.2). Hence, it must be true that

$$k \in (\sum_{j=1}^{r} s_j', \sum_{j=1}^{r+1} s_j) \text{ and } r \in \{1,\dots,i-1\} \Longrightarrow P(k-S[P(k),u_i]) \geq P(k) + t. \tag{A2.4}$$

(a–2) Next, observe that

$$k \in [\sum_{j=1}^{r} s_j, \sum_{j=1}^{r} s_j'] \text{ for some } r \in \{2,\dots,i\}$$

$\Rightarrow P(k-S[P(k),u_r]) = P(k) + t$ by setting $i = r$ in (4.22)

$\Rightarrow P(k-S[P(k),u_i]) \geq P(k) + t.$ (A2.5)

The last step follows because $r \leq i \Rightarrow S[P(k),u_r] \leq S[P(k),u_i] \Rightarrow k - S[P(k),u_r] \geq k - S[P(k),u_i] \Rightarrow P(k-S[P(k),u_i]) \leq P(k-S[P(k),u_i])$. (4.36a) follows from (A2.3), (A2.4) and (A2.5).

Proof of (4.36c): Notice that for each $i \in \{2,\ldots,n-1\}$,

$$(\sum_{j=1}^{i} s_j', \ell] = \{\bigcup_{j=1}^{n-1} (\sum_{j=1}^{r} s_j', \sum_{j=1}^{r+1} s_j)\} \cup \{\bigcup_{r=i+1}^{n-1} [\sum_{j=1}^{r} s_j, \sum_{j=1}^{r} s_j']\}. \quad \text{(A2.6)}$$

(c–1) Suppose that

$$k \in (\sum_{j=1}^{r} s_j', \sum_{j=1}^{r+1} s_j) \text{ for some } r \in \{i,\ldots,n-1\}. \quad \text{(A2.7)}$$

Then $P(k) = p_{r+1}$ by (4.18b). Hence,

$P(k-S[P(k),u_i]) > P(k) + t$

$\Rightarrow P(k-S(p_{r+1},u_i)) > p_{r+1} + t = p_r$

$\Rightarrow k - S(p_{r+1},u_i) < \sum_{i=1}^{r-1} s_j'$ by (4.18)

$\Rightarrow k < \sum_{j=1}^{r-1} s_j' + S(p_{r+1},u_i)$

$$\Rightarrow k < \sum_{j=2}^{r} s_j'. \quad \text{(A2.8)}$$

The last step follows because $r \geq i$ implies that $S(p_{r+1},u_i) \leq S(p_{r+1},u_r) \equiv s_r'$.

However, (A2.8) contradicts (A2.7). Hence, we must have that

$$k \in (\sum_{j=1}^{r} s_j', \sum_{j=1}^{r+1} s_j) \text{ and } r \in \{i,\ldots,n-1\} \Rightarrow P(k-S[P(k),u_i]) \leq P(k) + t. \quad \text{(A2.9)}$$

(c–2) Next, observe that

$k \in [\sum_{j=1}^{r} s_j, \sum_{j=1}^{r} s_j']$ for some $r \in \{i+1,\ldots,n-1\}$

$\Rightarrow P(k-S[P(k),u_r]) = P(k) + t$ by setting $i = r$ in (4.22)

$\Rightarrow P(k-S[P(k),u_i]) \leq P(k) + t.$ (A2.10)

The last step follows because $r \geq i + 1 \Rightarrow S[P(k),u_r] \geq S[P(k),u_{i+1}] \geq S[P(k),u_i] \Rightarrow k - S[P(k),u_r] \leq k - S[P(k),u_i] \Rightarrow P(k-S[P(k),u_r]) \geq P(k-S[P(k),u_i])$. (4.36c) follows from (A2.6), (A2.9), and (A2.10). This concludes the proof of (4.36).

Proof of (4.35): The method of proof used for (4.36c) is completely valid even when i = 1. Therefore (4.35) holds.

Proof of (4.37): The method of proof used for (4.36a) is valid for i=n if we replace $\bigcup_{r=2}^{i}$ with $\bigcup_{r=2}^{n-1}$ in the middle term, $\{\bigcup_{r=2}^{i} [\sum_{j=1}^{r} s_j, \sum_{j=1}^{r} s_j']\}$, of the right hand side of (A2.1). Therefore (4.37) holds.

Q.E.D.

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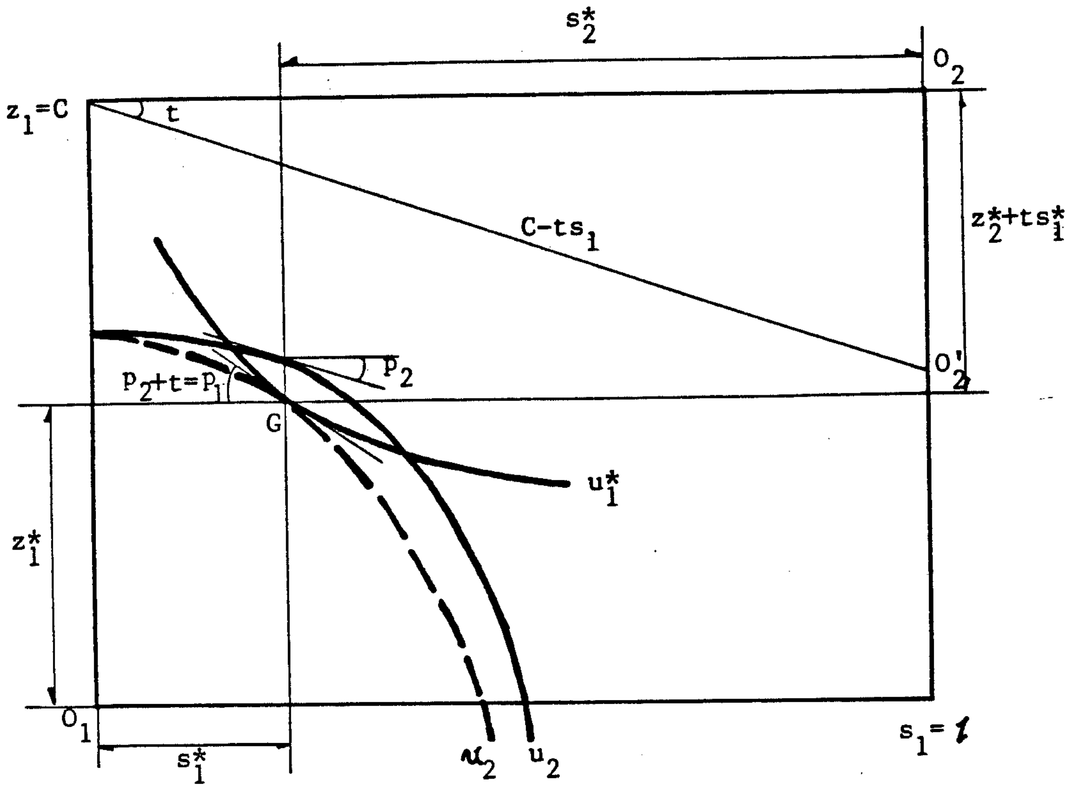

Figure 1. An Efficient allocation.

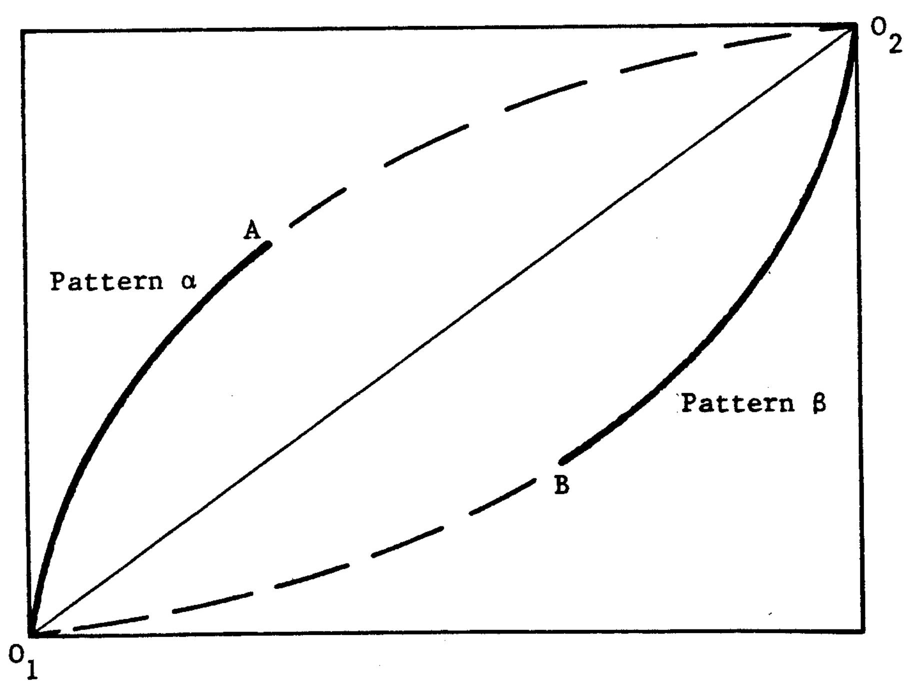

Figure 2. Contract curve O_1A-BO_2

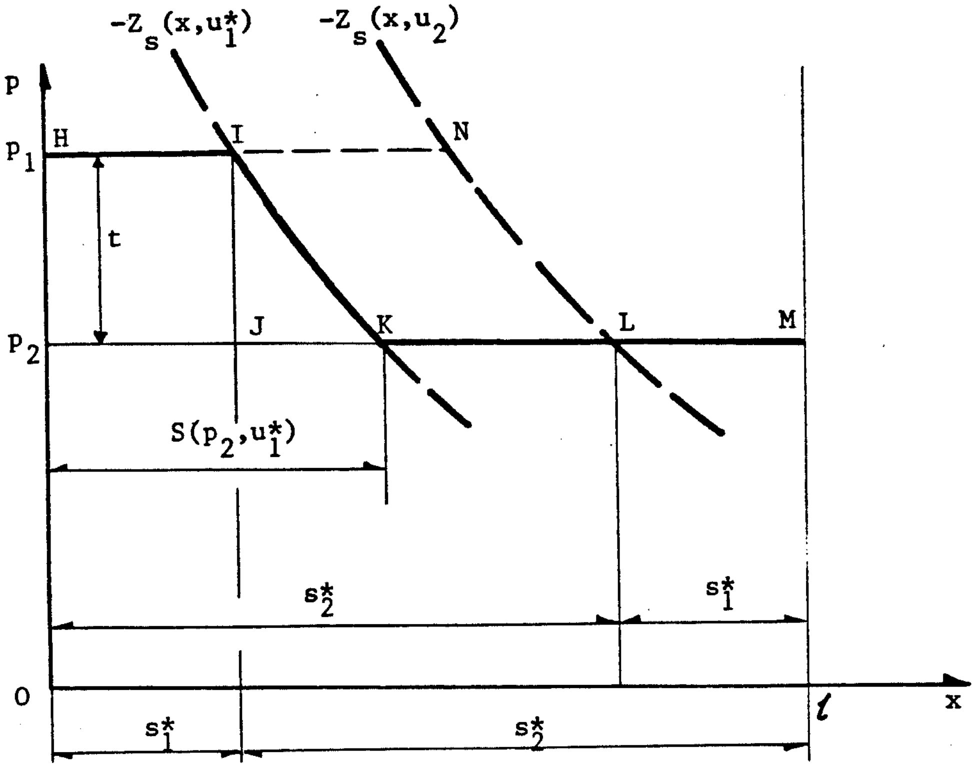

Figure 3. A supporting land price function (n=2).

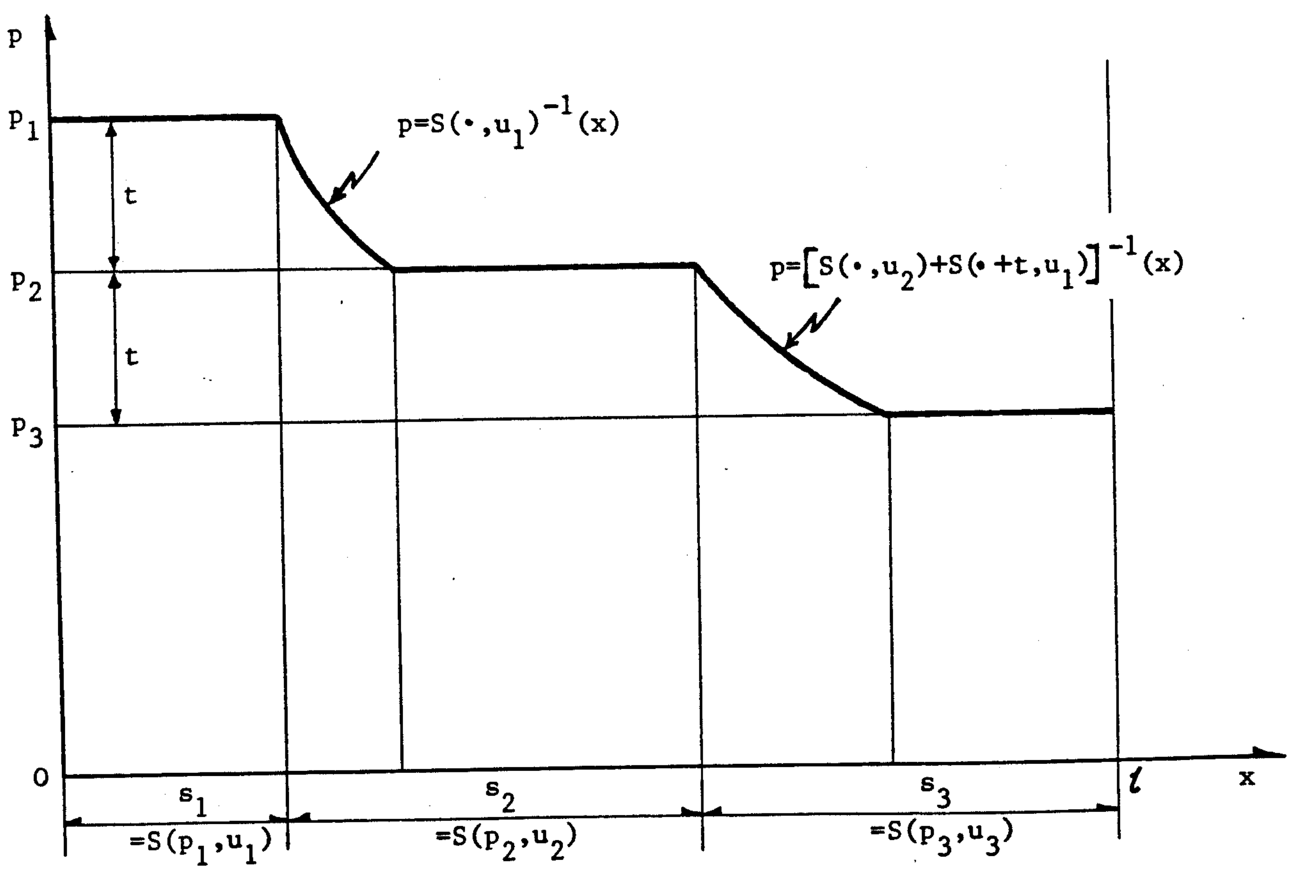

Figure 4. A supporting land price function (n=3).