Equal Sacrifice and Incentive Compatible Income Taxation

Berliant, Marcus and Miguel Gouveia

Working Paper No. 219
February 1990

**Equal Sacrifice and Incentive Compatible Income Taxation**

Marcus Berliant and Miguel Gouveia

Rochester Center for Economic Research
Working Paper No. 219

# EQUAL SACRIFICE AND INCENTIVE COMPATIBLE INCOME TAXATION*

Marcus Berliant
University of Rochester

and

Miguel Gouveia
University of Rochester
and
Portuguese Catholic University

Working Paper No. 219

February 1990

*The authors would like to thank John Ledyard, James Kahn, Christophe Faugere and Patrick Legros for helpful discussions but retain responsibility for any remaining errors.

## Abstract

The concepts of equal sacrifice and incentive compatibility in income taxation have been of interest to economists for some time. Here we attempt to integrate the two literatures in a model that offers a labor-leisure choice to consumers while imposing an equal relative or absolute sacrifice tax system. The main result is that equal sacrifice incentive compatible income tax systems generally exist and are unique. Examples are presented where this tax system is easy to compute. Comparative statics are calculated. Finally, some remaks about the welfare properties of these tax systems are made.



Key words : Income taxation, Equal Sacrifice, Incentive Compatibility. *Journal of Economic Literature* classification : 323.

## 1. INTRODUCTION

The search for a just and efficient system of taxation has always been a major concern in economics. Income taxation in particular was a subject of much attention of the classical economists, even though their efforts to deal formally with the conflict between equity and efficiency were not very successful. Some of the most interesting thoughts the classical economists had on the subject are related to the principle of equal sacrifice. This paper is an attempt to integrate this time-honored principle and the more modern concept of incentive compatibility in a model that offers a labor-leisure choice with income taxation. As some historical perspective shows, this integration is but a logical development of the normative theory of income taxation.

Following closely Musgrave's " Theory of Public Finance", we can distinguish two basic approaches to the theory of taxation : the benefit approach, which puts taxes and public expenditures in an efficiency context, and the ability to pay approach, which puts taxation and public expenditures in an equity context.

The latter approach emphasized that the state should impose taxes in an equitable fashion, according to one's ability to pay them. The ability to pay approach led to the principle of equal sacrifice. An important early advocate of equal sacrifice was J. S. Mill. According to Mill (1848), the tax version of the principle that everyone should have equal treatment under the law is that everyone should be taxed so as to incur equal sacrifice. The questions are then how to define an (observable) index of ability to pay (property, income, etc... ), how to precisely define equal sacrifice ( absolute, relative, marginal ) and what the shape of the reference utility function should be.

The proponents of equal sacrifice taxes relied on the assumption of cardinal preferences, with diminishing marginal utility of income. For a while, these assumptions were thought to imply progressive taxes, but in fact progression as defined by an increasing average tax depends on the shape of the marginal utility of income schedule, not just on its slope.

A general condition for progressivity can be found in Samuelson's Foundations (1983). Define y as gross income, t(y) as the taxes paid by an individual with gross income y and u( ) as the reference utility function. Then an equal absolute sacrifice tax satisfies

$$u(y) - u(y - t(y)) = s, \forall \quad y \in Y$$

where s is a constant, the level of sacrifice, and $Y \subset R_{++}$ is the set of all possible gross incomes. Differentiating the equation above with respect to y and rearranging we have

$$\frac{u'(y)y}{u'(y - t(y))(y - t(y))} = \frac{1 - t'(y)}{1 - t(y)/y}.$$

If the right hand side of the last equation is less than one, then t(y) is progressive. This inequality holds when the absolute value of the elasticity of the marginal utility of income, or the coefficient of relative risk aversion as it is usually known, is greater than one.

Using this result, equal relative sacrifice can be handled easily. The equal absolute sacrifice tax for the utility function $u(y)$ is the equal relative sacrifice tax for utility function $v(y) = e^{u(y)}$.

Finally, equal marginal sacrifice taxes ( a misleading name ) are taxes that equate the marginal utilities of all taxpayers. This rule can result from the maximization of an utilitarian Social Welfare Function subject to a constraint given by the total endowment of the consumption good in the economy.

Equal sacrifice theories were criticised on several grounds, mainly on the need to rely on interpersonal comparisons of utility and the implicit knowlege of each individual's utility function. However, it is hard to dismiss completely the equal sacrifice approach on those grounds, because any normative theory of taxation will have to rely on a similar set of assumptions or value judgements. As Musgrave puts it " for purposes of policy formation in a democracy -or, for that matter in a nondiscriminating dictatorship - the best solution may be to proceed as if individuals were alike. (...) we may then postulate a marginal utility schedule that seems proper as a matter of social policy (...) so as to derive a tax formula from this schedule."

Another criticism of equal sacrifice has to do with the relationship between sacrifice and labor. Some proponents of equal sacrifice taxation advocated discrimination in favor of wage income based on the idea that sacrifice involves the loss of utility from consumption and the disutility of earning labor income. Typically, some sort of additive preferences were assumed. Again quoting Musgrave: " Pigou defined net satisfaction as satisfaction from income minus dissatisfaction from work." We shall see later that Pigou's assumption indeed justifies much of the results of early theories of equal sacrifice, even when the incentive problems they ignored are taken into account.

The approach used by the classical economists was too simplistic, in that they did not recognize formally the efficiency and incentive effects any tax system causes. These effects should be taken into account when designing such a system. We think that the failure of this literature to take incentives into account cannot be charged to the equal sacrifice principle "per se", but rather to the state of economics "technology" by the time this development unfolded.

Whatever the reasons, the normative theory of income taxation did not suffer major alterations until the late 1960's when Sheshinsky and, especially, Mirrlees initiated the literature on optimal income taxation. This literature adopts a typical planning approach: the planner maximizes a Social Welfare Function subject to the technical constraints of the economy and the incentives constraint. This last aspect of the economy is the crucial development brought by this approach, because it introduces, in a formal way, the optimizing behavior of the economic agents when confronted with the tax function.

In this way, equity and efficiency were embedded into the solution of the problem. Equity concerns were thought to be behind the structure of the Social Welfare Function to be used, since the weights given to each individual expressed the redistributive concerns of the planner. We can also adopt Seade's (1977) terminology and think of a utilitarian objective where the planner is very careful when picking a particular cardinalization of preferences. Among other things this also means that there is a cardinal aspect to the optimal income tax approach, much along the same lines seen in equal sacrifice theories. In fact, one can view this approach as a development of the old theories of equal marginal sacrifice, when a utilitarian Social Welfare Function is used. However, it is intriguing that no modern developments appeared for the other concepts of equal sacrifice.

The optimal income tax approach has been fruitful and an interesting result appeared: the need for a zero marginal tax rate at the top of the income distribution as a condition for Pareto efficiency. However, there are a few weak links in this approach, which we will enumerate next.

a) The planning problem is a complex one. We have no guarantees it will always be a well-behaved problem. In some cases there may not exist solutions or we may not have uniqueness. There is a proof of existence in the literature, Kaneko (1981), but even a superficial reading of this proof will reveal that the model being discussed is not really the standard one. Instead, Kaneko discusses a different model where there is no explicit treatment of the incentive compatibility problem. In fact, one can reasonably suspect some interesting cases of the standard formulation will not be well behaved ( see Mirrlees (1986)).

b) Since there might be too many solutions, or they may be too broadly defined, some research has been directed to the consequences of imposing further restrictions on the class of admissible functions. Berliant (1988) followed this strategy. In order to find further qualitative necessary conditions, Berliant tried to find the core of an income tax schedule. The core would be composed of the Pareto optimal tax schedules that survive the threat to secede by every coalition. There are informational asymmetries , since everyone knows the global distribution of the agents' types but not each agent's identity. These asymmetries can lead to two distinct core concepts. In the first one, a coalition can block a tax function if all its members are not worse and some are strictly better of under an alternative feasible tax function. In this case, Berliant shows that the core consists only of head taxes : it is optimal for each coalition to have a zero marginal tax rate at the top, so the tax rate must be zero everywhere. The second core concept has an additional incentive compatibility constraint : agents outside the coalition should not wish to lie about their type and join the coalition. The core is often empty in this case. Thus in both cases the results undermine our confidence in the optimal income tax approach.

c) The lack of robustness of the results also extends to the characteristics of the environment where economic agents operate. Minor changes in the assumptions about the economy lead to important changes in the qualitative results ( see Journal of Public Economics,1982, Special Issue on Optimal Taxation).

d) The way the optimal income tax problem is specified in the literature makes it almost a pure redistributional problem. This is a very delicate issue, one where we think different ideas might be mixed. One might be interested in apportioning taxes in a just and efficient way without actually wanting to redistribute income, say for financing some types of public expenditures or some other purpose that we may assume to be approximately redistributive neutral, for practical purposes. The separation of these two problems seems to be worthwhile, but unfortunately is not a feature of the research in this area.

f)There is an ethically undesirable property of the solutions implied by the optimal income tax formulations: the marginal tax faced by a taxpayer depends, other things equal, in the density of the population on the domain of taxpayer characteristics. This externality is a violation of the standards for horizontal equity. One would want the tax function to depend on the general characteristics of the taxpayer population and not to be overly sensitive to the specific properties of the neighbourhood where a taxpayer is located.

At the same time these problems were recognized, there has been a revival of the old principle of equal sacrifice.

Young (1988) showed that the equal sacrifice approach can be justified without having to support interpersonal utility comparisons, but can rely instead on an axiomatic approach. For him, equal sacrifice taxes are a consequence of basic notions of distributive justice. Along with a regularity assumption of continuity, Young used the axioms of symmetry ( equal gross incomes pay equal taxes ), strict monotonicity ( when the total tax revenue increases everyone pays more ), composition ( the increase in each individual's tax depends on previous ability to pay, i.e. previous net income ), strict order preservation ( the ranking of after tax incomes is the same as the ranking of gross incomes ) and consistency ( an allocation of taxes that is fair to a group is fair to all its subgroups ). Young proves that a tax system satisfying these properties is equal sacrifice.

The reappearance of a criterion as simple and transparent as the equal sacifice principle is something we should salute, but again, we must remark that none of the applications of the principle known to us includes incentives in the problem. The reason for this seems to be that the modern settings where equal sacrifice has appeared have a cooperative nature. In fact cooperative games leave little room for incentives to play a major role. By its own nature, the participants in a cooperative game are commited to certain actions, specified in the solution adopted, so there is no relevant extra individual optimizing behavior.

We think it is possible to give a non-cooperative justification for the equal sacrifice principle (something our future research will address), but we persist in stressing the importance of including incentives in the equal sacrifice inspired solutions, in particular for the income taxation problem. This is exactly what we are trying to accomplish in this paper: to derive an incentive compatible and equal sacrifice tax function, prove its existence and uniqueness and study its qualitative characteristics, namely progressivity and efficiency.

Section 2 lays down the basic labor-leisure choice model, makes explicit our assumptions on technology and preferences, and formalizes the concept of sacrifice. Section 3 will examine briefly the first best case. In section 4 we will study the second best case, establishing the existence and uniqueness of the equal sacrifice and incentive compatible income tax under our assumptions, and determining some of its properties. Section 5 concludes.

## 2. BASIC MODEL

### *2.1 Assumptions*

In the set-up of the standard optimal income tax model, consumers differ by an ability parameter,w,strictly positive, which can be interpreted as a wage rate or productivity. The support of w is an interval on the real line:

$$\mathrm{w} \in [\underline{w},\overline{w}] \subseteq R_{++}$$

and w has a population density function f(w).

We make the following assumptions on preferences over consumption c and labor $l$, as given by a twice differentiable utility function $u(c, l)$. These assumptions are used in the standard optimal tax model, and the propositions below will employ various subsets of these assumptions.

A1 Standard assumptions on preferences

$$u_1 > 0 \; ,u_2 < 0 \; ,u_{22} < 0 \; ,u_{11} < 0$$

A2- The utility function is quasi-concave:

$$u_{11}{u_2}^2 - 2u_{12}u_1u_2 + u_{22}{u_1}^2 \equiv D' < 0$$

A3- Leisure is normal:

$$u_{11}u_2 - u_{21}u_1 > 0$$

A4- Consumption is normal:

$$u_{21}u_2 - u_{22}u_1 > 0$$

A5- Interior solutions:

$$u(.) \to \inf \quad u(.) \quad if \quad c \to 0 \text{ or } l \to 1,$$

$$\lim_{l \to 0} u_2(c, l) = 0$$

A6- Unboundedness of utility:

$$\inf \quad u(c, l) = -\infty$$

Note : Assumption 6 is used in the study of the general case of the incentive compatible equal sacrifice tax. We will dispense with it in some particular cases, to be used as examples, at the cost of having a restricted choice for the sacrifice level yielded by the tax function derived.

A7- The production technology is CRS in labor, with coefficient w for type w workers.

A8- Monotonicity under equal sacrifice :

$$(u_{222}u_{21}w + u_{221}u_{22})l + 2(u_{22}u_{21} - u_{221}u_2) \geq 0$$

for all admissible $c, l$ and $w$.

This assumption and the reason for its use will be discussed in the appendix. For now, let us just remark that A8 holds trivially when preferences are additively separable.

We define gross income as y and, when there are no taxes, we have that

$y \equiv wl$ and $y = c$.

The consumer's problem for type w is

$$\max_y u(y, y/w)$$

We assume interior solutions for this problem, satisfying the first order condition

$$u_1(y, y/w) + u_2(y, y/w)/w = 0 \tag{1}$$

and the second order condition

$$u_{11}(y, y/w) + 2u_{12}(y, y/w)/w + u_{22}/w^2 = D' u_2{}^{-2} \equiv D < 0 \tag{2}$$

By assumption A2 the second order condition is always met. Solving the consumer's problem, we find

$$\overline{y}(w) \equiv arg \max_y \quad u(y, y/w)$$

and the indirect utility function

$$g(w) \equiv u(\overline{y}(w), \overline{y}(w)/w).$$

The following results will prove to be useful:

$$\frac{d\overline{y}}{dw} = \frac{u_2/w^2 + \overline{y}/w^2[u_{12} + u_{22}/w]}{D} > 0 \tag{3}$$

$$\frac{dg}{dw} = [u_1 + u_2/w]\frac{d\overline{y}}{dw} - u_2\overline{y}(w)/w^2 = -u_2\overline{y}(w)/w^2 > 0. \tag{4}$$

These results prove that gross income and individual welfare are increasing functions of the ability parameter.

### *2.2 The model with taxes*

We now look at the effects of an arbitrary income tax. Define the tax function $\tau$(y) as the tax paid by an individual with gross income y. The net income function, $\theta$(y), is defined as:

$$\theta(y) \equiv y - \tau(y)$$

Given that income will be taxed, the consumer's problem is now

$$\max_y u(\theta(y), y/w).$$

Assuming we have a smooth tax function and interior solutions ( for a discussion of why, in general, this assumption is not strong, see Berliant(1989,p.9)), we have the first order condition

$$u_1 \frac{d\theta}{dy} + \frac{u_2}{w} = 0 \tag{5}$$

and the second order condition

$$\Delta \equiv u_{11}(\frac{d\theta}{dy})^2 + 2u_{12}(\frac{d\theta}{dy})/w + u_{22}/w^2 + u_1 \frac{d^2\theta}{dy^2} < 0. \tag{6}$$

The solution to the problem is denoted as y(w). Alternatively, we can think of (5) and (6) as being restrictions we can use to check the tax functions we will derive using the equal sacrifice principle and incentive compatibility. From this perspective, equation (5) gives us the marginal tax rate for any chosen $y$. Less direct, but also very interesting, is the fact that equation (6) provides an upper bound for the "local "progressivity:

$$\frac{d^2\theta}{dy^2} < -\frac{1}{{u_1}^2 w^2}\{u_{11}{u_2}^2 - 2u_{12}u_1u_2 + u_{22}{u_1}^2\}.$$

A few comparative statics results can be obtained. Consider the existence of a given exogenous income I, which will be set at zero unless otherwise stated. Utility is

$$u(\theta(y) + I, y/w).$$

The following results are straightforward.

$$\frac{dy}{dI} = -\frac{u_{11}\frac{d\theta}{dy} + u_{21}/w}{\Delta} \tag{7}$$

This will be negative, by our assumption of leisure normality.

For total consumption,we have

$$\frac{dc}{dI} = \frac{u_{12}\frac{d\theta}{dy}/w + u_{22}/w^2 + u_1\frac{d^2\theta}{dy^2}}{\Delta} \tag{8}$$

which will be greater than zero under consumption normality if the net income function is not too convex.

The gross income response to a change in w is

$$\frac{dy}{dw} = \frac{u_{12}\frac{y}{w^2}\frac{d\theta}{dy} + u_{22}\frac{y}{w^3} + u_2/w^2}{\Delta}. \tag{9}$$

This will be positive, by our previous assumptions.

Finally, a couple of results we will use subsquently are:

$$\lambda \equiv \frac{du}{dI} = u_1 = -u_2/(w\frac{d\theta}{dy}) \tag{10}$$

by the envelope theorem, and

$$\frac{du}{dw} = u_1\frac{d\theta}{dy}\frac{y}{w} = -u_2 y/w^2 > 0. \tag{11}$$

*2.3 Definition of sacrifice*

We now face the question of how to define sacrifice. A trivial definition is that sacrifice is the difference between the pre-tax and the post-tax levels of utility. There is no ambiguity in what the post tax level is. However, the definition of the pre-tax level of utility seems to be more problematic.

The first possible answer is that it corresponds to the no tax case, i.e. it is g(w). For this definition we have that, in general, the pre-tax and the post tax levels of labor supply will differ. But we may wish to define sacrifice holding labor and gross income fixed. This second possibility seems to agree with the definition of sacrifice in the classical case of exogenous income, where only consumption changes due to tax payment. Again we face a choice: we may fix gross (pre tax) income at the post tax level or at the no tax level.

The first concept of equal absolute sacrifice can be stated as

$$G(w, y, \theta; s) \equiv g(w) - u(\theta(y), y/w) - s = 0 \quad \forall\, w, w \in [\underline{w}, \bar{w}]. \tag{12}$$

This concept of equal sacrifice allows different levels of labor, before and after the introduction of income taxation.

The second concept of equal absolute sacrifice fixes labor at the post tax level. Formally we can state this second concept as

$$G'(w, y, \theta; s) \equiv u(\bar{y}(w), y(w)/w) - u(\theta(y(w)), y(w)/w) - s = 0, \; \forall\, w \in [\underline{w}, \overline{w}]. \tag{13}$$

We will generally work with the first concept (given in (12)). On the one hand it is a more "natural" concept, and on the other hand it is a more tractable formalization.

If we are interested in relative sacrifice we can use the transformation previously discussed: equal relative sacrifice for $v(c, l)$ is obtained by having equal absolute sacrifice for $ln\, v(c, l)$.

Once the gross and net income functions are defined, we can determine the revenue produced by an equal sacrifice tax function with sacrifice level s:

$$R(s) = \int_{\underline{w}}^{\overline{w}} [y(w; s) - \theta(y(w; s); s)] f(w) dw. \tag{14}$$

We will omit (.;s) from our notation for convenience, except when strictly necessary.

Under certain conditions, we will find a one-to-one relationship between the level of sacrifice s and the revenue collected. This relationship will allows us to consider the sacrifice level as the single index relating the aggregate budgetary equilibrium to the taxes each individual pays.

## 3. The First Best Case.

### *3.1 Derivation of the tax function*

The first best case is interesting because it is a case closer to what the classics had in mind, as there are no distortions in the labor-leisure choice.

Also important is the following intuition: a first best or lump sum tax that yields a given level of revenue can be considered as a lower bound for the distortionary tax achieving the same level of revenue.The reason is the existence of a deadweight loss: for the same utility loss the lump-sum tax must yield a higher revenue than any distortionary tax system.

The first best case uses the assumption that the government or social planner knows the type of each particular person (for otherwise lump-sum taxes might not be feasible). Hence, in reality, a worker/consumer does not face a tax schedule $\tau(y)$ or net income schedule $\theta(y)$ common to all, but rather a *personal* tax schedule, $\tau(w) = y(w) - \theta(w)$, which is trivial and specific to type w.

For the time being we will use the first definition of sacrifice ( given in (12)).

*Proposition* 1- Under assumptions A1,A2 and A5-A7 the first best equal sacrifice tax exists and is unique.

Proof/

The function $\theta$(w) is completely determined by the equal sacrifice equation and the incentive constraint. The second equation is just the first order condition for the consumer when taxation is perceived to be of the lump-sum kind.

$$g(w) - u(\theta, y/w) - s = 0 \tag{15}$$

$$u_1(\theta, y/w) + u_2(\theta, y/w)/w = 0 \qquad \forall\, w \in [\underline{w}, \overline{w}] \tag{16}$$

The Jacobian of the system of equations above is $-D'u_1$ . Under our assumptions on preferences it is strictly positive, so we can use the implicit function theorem to establish the existence and uniqueness of a pair of functions y(w) and $\theta$(w). Also, by A5 and A6 we have that $\theta(w)$ and $y(w)$ are strictly positive, so corner solutions are ruled out.//

Differentiating the system given in (15) and (16), we find

$$\frac{dy}{dw} = (Du_1)^{-1}[u_1/w^2(u_2 + u_{12}y + u_{22}y/w) - (u_{11} + u_{21}/w)(g'(w) + u_2y/w)] \tag{17}$$

and

$$\frac{d\theta}{dw} = (Du_1)^{-1}[(g'(w) + u_2y/w^2)(u_{12}/w + u_2/w^2) - u_2/w^3(u_2 + u_{12}y + u_{22}y/w)] \tag{18}$$

and thus

$$\frac{d\tau}{dw} = \frac{dy}{dw} - \frac{d\theta}{dw} = -\frac{[g'(w) + u_2y/w^2]}{u_1}. \tag{19}$$

The assumptions made do not allow us to sign unequivocally the expressions above. In particular we need to sign

$$g'(w) + u_2y/w^2 \tag{20}$$

If taxes are to increase with ability this expression must be negative; however that need not always be the case. To see under what conditions this relationship holds, we state the following.

<u>*Proposition* 2</u> - The first best incentive compatible equal sacrifice tax increases with ability if the sum of the elasticity of the marginal utility of income and the income elasticity of labor supply is negative.

<u>Proof/</u>

Using (4) , (20) can be rewritten as $\overline{\lambda}\overline{y} - \lambda y$, where we follow the convention that overlined variables represent levels in the no tax case. The sign of the $\overline{\lambda}\overline{y}$ depends on the sign of

$$\frac{\partial\lambda(I)y(I)}{\partial I}. \tag{21}$$

where I is exogenous income. Because we are considering w as given, a full income interpretation is possible. Specifically we have that

$$\text{Full income } = w + I.$$

The condition for (21) to be negative can thus be written as

$$\frac{\partial\lambda}{\partial FI}\frac{FI}{y} + \frac{\partial y}{\partial FI}\frac{FI}{y} < 0. \tag{22}$$

Recall that in the usual case, taxes would rise with ability/income if the elasticity of the marginal utility of income were negative, i.e. if the utility function were concave. Our result says that once we introduce endogenous labor supply, the requirement for taxes to increase with ability is weaker: some convex utility functions may still have increasing taxes. //

We will call (22) the sum of elasticities assumption and we will return to its use in the study of the second best case, so we now include it formally in our list of assumptions.

A9–Negative sum of elasticities. See (22) above.

Assuming (22) is verified we can establish the signs of expressions (17) and (18). Under our assumptions A1-A7, $\frac{dy}{dw}$ will always be positive. However, the sign of $\frac{d\theta}{dw}$ is ambiguous. The fact that people with higher ability end up receiving less consumption may seem strange, but this is in fact a well known result in first best analysis of income taxation (e.g. Stiglitz (1982)).

In order to gain some intuition, we now present two examples for which we were able to obtain a simple closed form for the equal sacrifice tax function.

EXAMPLE 1 – $u(c,l) = 2\sqrt{c(1-l)}$

$=2\sqrt{y(1-y/w)}$.

Then we have that $\theta = y + s^2/2 - \frac{\sqrt{8ys^2+s^4}}{2}$.

In this case (22) is obeyed. It is trivial to check that

$$\tau' > 0 \quad , \quad \tau'' < 0 \quad , \quad \text{and} \quad \lim_{y\to\infty} \tau' = 0.$$

EXAMPLE 2 – u(c,l) = 4c(1-l)

=4y(1-y/w).

In this case we have $T = 2/3y + s/6 - 3/3\sqrt{y^2 - ys + s^2/16}$,

$$\tau' < 0 \quad , \quad \tau'' > 0 \quad , \quad \text{and} \quad \lim_{y\to\infty} \tau' = 0.$$

The fact that the utility function in example 2 is just a increasing monotonic transformation of the utility function in example 1 highlights the cardinality assumption already discussed.

### *3.2 Progressivity and the effects of changes in s.*

The consequences of increasing the level of sacrifice are fairly obvious

$$\frac{dy}{ds} = \frac{u_{11} + u_{21}/w}{Du_1} > 0$$

$$\frac{d\theta}{ds} = -\frac{u_{12}w + u_{22}}{w^2 Du_1} < 0$$

where the signs can be determined by assumptions A2 and A3, respectively. In fact this is the same result as the one we get for the change in the level of taxes:

$$\frac{d\tau}{ds} = \frac{dy}{ds} - \frac{d\theta}{ds} = \frac{u_{11}w^2 + 2u_{21}w + u_{22}}{w^2 Du_1} = \frac{1}{u_1} > 0,$$

which simply means we have a direct conversion of utility sacrifice into taxes, where the rate of conversion is just the marginal utility of income $\lambda$, equal to $u_1$ in the consumer's equilibrium.

A direct attempt to study progressivity in this framework is not an easy task. A strategy that eventually proved to be more successful was to reformulate equations (15) and (16) so as to make the average tax rate (actually the average retention rate ) explicit. The transformed equations, with $\rho = \frac{\theta}{y}$ being this average retention rate, are

$$u(\rho y, y/w) = g(w) - s \tag{23}$$

and

$$u_1(\rho y, y/w) + u_2(\rho y, y/w) = 0 \tag{24}$$

It is easy to check that, for finite s $\geq 0$ , we have $0 < \rho \leq 1$. The system of equations above defines the average tax function that yields equal sacrifice. The Jacobian of the system is

$$J = u_1 y[(u_{11} + u_{21}/w)\rho + (u_{21}/w + u_{22}/w^2)] - y(u_{11} + u_{21}/w)(u_1\rho + u_2/w)$$

and it is always negative by our assumptions on preferences. We can now state a surprising result:

*Proposition* 3 - Under assumptions A1-A4 and A5-A7, if the equal sacrifice first best tax increases with ability, then it is regressive.

Proof/

Differentiating the system above we have that

$$\frac{d\rho}{dw} = \frac{g'(w)[(u_{11} + u_{21}/w)\rho + (u_{21}/w + u_{22}/w^2)] - (u_{21}y/w^2 + u_{22}y/w^3 + u_2/w^2)(u_1\rho + u_2/w)}{J}$$

With our assumptions and for admissible $\rho$'s the expression above is always negative. So as long as $y'(w)$ and $\theta'(w)$ are positive the first best equal sacrifice tax will always be regressive.//

The reader may wish to go back to our first example and check that, indeed, the equal sacrifice tax system is regressive.

An attempt was made to look at the first best case using the alternate concept of sacrifice. Unfortunately, the problem becomes more complex as no use of the indirect utility function can be made, so that no interesting results were obtained.

## 4. The Second Best Case

### *4.1 Derivation of the tax function using the revelation principle.*

The first best case studied in the previous section is just an abstraction. Economic agents do not perceive income taxation as lump-sum but as something their behavior can change. The substitution effects in the labor-leisure choice induced by taxation will create some deadweight loss.

In the case of non-linear taxation, the choice consumers make is dependent not only on the fact that there is a distortion on the margin but also on the changes in that distortion as the marginal tax rate changes with gross income. In other words, under a non-linear tax structure the ( possibly non-constant ) derivative of the net income function will be a part of the first order condition of the consumer, making the task of determining $\theta(y)$ more complex.

To solve this problem we will make use of the results in the literature on incentive compatibility, as exposited in modern texbooks such as Laffont (1988).

As in the implementation literature we can consider that the y's are the messages the agents send to the government. The government then allocates consumption levels $\theta$(y), selecting one of the feasible social states. The incentive problem arises from the fact that the economic agents will choose y strategically, so as to lead to their preferred social state. In our particular case this means that we are looking for a function $\theta$(y) such that, in equilibrium, all agents have made an equal sacrifice.

What we have described in the previous paragraph is an indirect mechanism. A direct mechanism is one where the message space coincides with the space of characteristics. This concept is useful because, instead of trying to find $\theta(y)$ directly, we can follow an easier path by using the Revelation Principle.

Again turning to our particular problem, we can transform the problem of finding $\theta(y)$ into one of finding two functions $\theta(w)$ and $y(w)$. The implementation process can be roughly described in the following way:

Each agent announces his w. He is then required to to produce $y(w)$ but is allowed to retain $\theta(w)$ for his consumption. Our assumptions on the utility function make it possible for us to use the differential method to find the $y(w)$ and the $\theta(w)$ that implement the equal sacrifice rule. Call $w'$ the value of $w$ announced by an agent. Then the final utility level reached by the agent is $u(\theta(w'), y(w')/w)$. The agent chooses to announce the w' that maximizes his utility. We now establish the properties any incentive compatible income tax function must have, so that we can later use them in the derivation of the equal sacrifice tax.

The first order condition for incentive compatibility is

$$u_1 \frac{d\theta}{dw'} + \frac{u_2}{w} \frac{dy}{dw'} = 0. \tag{25}$$

To be able to implement the Social Choice function, we must have that all types of agents choose to reveal their true characteristics, i.e. the following identity must hold

$$u_1(\theta(w), y(w)/w) \frac{d\theta}{dw} + u_2(\theta(w), y(w)/w) \frac{1}{w} \frac{dy}{dw} \equiv 0. \tag{26}$$

The second order condition for incentive compatibility is

$$D^2 \equiv u_{11}(\frac{d\theta}{dw})^2 + 2\frac{u_{21}}{w} \frac{d\theta}{dw} \frac{dy}{dw} + u_{22}(\frac{dy}{dw})^2 + u_1 \frac{d^2\theta}{dw^2} + \frac{u_2}{w} \frac{d^2y}{dw^2} < 0. \tag{27}$$

On the other hand, we can differentiate (26) with respect to $w$ and obtain

$$D^2 - u_{21} \frac{y}{w^2} \frac{d\theta}{dw} - \frac{u_2}{w^2} \frac{dy}{dw} - u_{22} \frac{y}{w^3} \frac{dy}{dw} \equiv 0. \tag{28}$$

Combining the last three equations we can write the second order condition for incentive compatibility as

$$[\frac{u_{21}u_2 - u_{22}u_1}{u_2} \frac{y}{w^2} - \frac{u_1}{w}] \frac{d\theta}{dw} < 0. \tag{29}$$

Using assumptions A1 and A4 the second order condition is reduced to $\frac{d\theta}{dw} > 0$. Hence, under assumptions A1,A2,A4 and A5 an incentive compatible tax function exists if and only if $d\theta/dw > 0$. This is a well known result, the so called monotonicity condition. With this preliminary work done, we can now state our main result.

*Proposition 4* - If assumptions A1,A2,A4-A8 hold, the second best equal sacrifice and incentive compatible tax function exists and is unique.

Proof/

By definition of equal sacrifice we must have

$$g(w) - u(\theta(w), y(w)/w) \equiv s. \tag{30}$$

Differentiating (30) and using the identity (26) we obtain

$$g'(w) + u_2(\theta(w), y(w)/w)\frac{y(w)}{w^2} \equiv 0. \tag{31}$$

Thus the functions $\theta(w)$ and $y(w)$ are the solution to the system of parametric equations:

$$u(\theta, y/w) = g(w) - s \tag{32}$$

$$u_2(\theta, y/w)y/w^2 = -g'(w). \tag{33}$$

The Jacobian of the system is $J = (u_{22}u_1 - u_{21}u_2)y/w^3 + u_1u_2/w^2$. It is always negative, so we can use the implicit function theorem to establish the existence, uniqueness and differentiability of $\theta(w)$ and $y(w)$. Again by assumption A5, $\theta(w)$ and $y(w)$ will be strictly positive, so that we are always dealing with interior solutions. We now need to prove that $\theta(w)$ satisfies the second order condition for incentive compatibility. Since $\theta(w)$ is differentiable, this amounts to proving $\theta'(w) > 0$. This inequality can be proved by differentiating (32) and (33) and using A8 ( see the appendix for the details of this part of the proof).

From (25) we have

$$\frac{dy}{dw} = -\frac{u_1}{u_2 w}\frac{d\theta}{dw}.$$

By A1 and monotonicity $dy/dw > 0$, so y(w) is invertible. Hence $\theta(y)$ is well defined.//

Notice that the ambiguity we saw in the first best case cannot arise here: both gross income and consumption must increase with ability if an equal sacrifice tax function is to exist.

Another issue of interest relates to the assumption of monotonicity, A8. This assumption might seem rather arbitrary. However, as far as we are able to tell, the earlier literature on optimal income taxation does not even address the verification of second order conditions for incentive compatibility. More recent literature (since L'Ollivier and Rochet(1983)) focuses on the case of simple functional forms with additive separable preferences, as in Weymark(1987). These preferences obey A8 trivially, so in fact our results seem to be a generalization of the literature on the subject rather than the study of a limited case. This issue is further discussed in the appendix to the paper.

Finally, in order to relate our existence result to existing literature, we stress that J not equal to zero is the usually called "single crossing" property in signaling problems (see Mailath (1987)). In the optimal income tax literature, the negativity of J is known as "agent monotonicity", as in Seade (1982).

### 4.2 *The additively separable case and some examples*

If preferences are additively separable we can obtain further results, one of them very interesting.

*Proposition* 5 - If preferences are additively separable, an incentive compatible equal sacrifice tax function will induce the same level of labor supply ( and gross income ) as in the no tax case.

Proof/

Take a utility function of the form $p(y) + v(y/w)$. Then $\bar{y}(w)$ is implicitly defined by the first order condition

$$p'(\bar{y}(w)) + v'(\bar{y}/w)\frac{1}{w} \equiv 0.$$

On the other hand $y(w)$ is defined implicitly by

$$g'(w) + v'(y(w)/w)y(w)/w^2 \equiv 0.$$

Since $g'(w) = -v'(\bar{y}(w)/w)\bar{y}(w)/w^2$ we can write the second of these identities as

$$v'(\bar{y}(w)/w)\frac{\bar{y}(w)}{w^2} - v'(y(w)/w)\frac{y(w)}{w^2} \equiv 0.$$

This is true only when $\bar{y}(w) \equiv y(w)$.//

Another way to put it is that, for additively separable preferences, an equal sacrifice and incentive compatible tax has an income effect that exactly offsets its substitution effect.

The result in Proposition 5 has several implications. To begin with, we obviously have that the two possible concepts of sacrifice discussed in section 2.3 yield exactly the same results, since there is no difference between pre-tax and post-tax levels of labor supply.

Another and far more interesting implication relates to the treatment of equal sacrifice presented by the "classics" and by Peyton Young. In fact, if preferences are additively separable, we can ignore the incentives problem and proceed as if gross income were exogenous. Did they realize that separability would justify their results ? Whatever the answer to this question may be, Pigou seemed to have additivity in mind, as seen by the quote in the introduction to this paper. It might be the case that most normative analyses of income taxes where no attention is paid to the endogeneity of income, such as the analysis of equity or the analysis of inequality indexes, can actually be compatible with the existence of incentives in labor supply with this or some other notion of separability in preferences.

A third implication of our result concerns the interpretation of empirical studies on the effects of taxation on labor supply. Typically these studies report small, sometimes negligible, effects of taxation on labor supply. But, if preferences are of the sort described here we should expect to find no effects at all, provided that the tax structure is equal sacrifice.

We look next at a few simple examples, where it is easy to obtain closed form solutions for the tax function.

EXAMPLE 3 - $u(c,l) = p(c) - \alpha l$ with $p' > 0$ and $p'' < 0$.

In this case we have that $y(w) = {p'}^{-1}(\alpha/w)$, $g(w) = p({p'}^{-1}(\alpha/w) - \frac{\alpha}{w}{p'}^{-1}(\alpha/w)$ and $g'(w) = \frac{\alpha}{w^2}{p'}^{-1}(\alpha/w)$. Then we have $\theta(w) = p^{-1}[g(w)+g'(w)w-s]$ and $\tau(w) = y(w)-\theta(w)$. Since $y(w)$ is monotonic, it can be inverted to obtain $w = h(y)$. The tax function is then $t(y) = g'(h(y))\frac{h(y)^2}{\alpha} - u^{-1}[g(h(y)) + g'(h(y))h(y) - s]$.

Let us look now at two particular specifications.

- $u(c,l) = ln(c) - \alpha l$. Then $y(w) = w/\alpha$, $\theta(w) = w/\alpha e^{-s}$ and $t(y) = y(1 - e^{-s})$ . This is a very simple case where proportional taxes are equal sacrifice.

- $u(c,l) = c^{.5} - \alpha l$. Then $y(w) = \frac{w^2}{4\alpha^2}$, $\theta(w) = (\frac{w}{2\alpha} - s)^2$ and $t(y) = y - (y^{.5} - s)^2$.

EXAMPLE 4 - $u(c,l) = \beta c + v(l)$, with $v' < 0$ and $v'' < 0$.

Here, we have that $y(w) = v'^{-1}(-\beta w)w$ and $g(w) = \beta v'^{-1}(-\beta w)w + v(v'^{-1}(-\beta w))$.

After a few steps we have $\theta(w) = \frac{1}{\beta}[\beta v'^{-1}(-\beta w)w - s]$. So the tax function is $t(y) = s/\beta$, a direct head tax !

EXAMPLE 5 - $u(c,l) = p(c) + v(l)$, with $p' > 0$, $p'' < 0$, $v' < 0$, $v'' < 0$. Then we have $y(w)$ defined as in Proposition 5, and invertible. Thus we obtain $w = h(y)$. The net income function is $\theta(w) = p^{-1}[p(y(w)) - s]$ so, the tax function is $t(y) = y - p^{-1}[p(y) - s]$, exactly the usual formula for equal sacrifice.

Again we present two particular specifications that allow closed form solutions.

- $u(c,l) = \alpha ln(c) + \beta ln(1-l)$. Then $y(w) = \frac{\alpha}{\alpha+\beta}w$ and $\theta(w) = \frac{\alpha w}{\alpha+\beta}e^{-\frac{s}{\alpha}}$.

So, $t(y) = y(1 - e^{-\frac{s}{\alpha}})$. Again we find proportional taxes to be equal sacrifice.

- $u(c,l) = \alpha c^{.5} + (1-l)^{.5}$ Then $y(w) = \frac{\alpha^2 w}{\alpha^2 w+1}$ and $\theta(w) = \frac{1}{\alpha^2}[(\alpha^2 w + 1)^{.5} - (\alpha^2 w + 1)^{-.5} - s]^2$. The tax function is $t(y) = y - \frac{1}{\alpha^2}[(1-y)^{-.5} - (1-y)^{.5} - s]^2$

EXAMPLE 6 – Our last example is a specific non-additive utility function, $u(c,l) = c^{.5}(1-l)^{.5}$.

We have that $\bar{y}(w) = w/2$ and that $g(w) = \frac{w^{.5}}{2}$. Solving the system of equations (32) and (33) we obtain $y(w) = \frac{w}{2-sw^{-.5}}$ The net income function is $\theta(w)\frac{w}{4}(2 - 3sw^{-.5} + s^2 w)$. These equations define the tax function parametrically. Unfortunately, a closed form solution for $t(y)$ does not seem to be possible.

### *4.3 Comparative statics, the sacrifice level, tax revenue and progressivity.*

Differentiating the system of equations (32) and (33), we can determine the effects of a change in the level of sacrifice on the tax function.

$$\frac{d\theta}{ds} = -\frac{u_{22}y/w^2 + u_2/w}{J} < 0 \tag{34}$$

$$\frac{dy}{ds} = \frac{u_{21}y/w}{J} \tag{35}$$

$$\frac{d\tau}{ds} = \frac{u_{21}y/w + u_{22}y/w^2 + u_2/w}{J} > 0 \tag{36}$$

An increase in sacrifice reduces consumption and increases taxes, even though the effects on labor supply are indeterminate. Again, for the case of additively separable preferences, we have that a change in the tax function does not affect labor supply and gross income. That taxes increase with sacrifice proves that an equal sacrifice and incentive compatible tax function will never be on the "wrong" side of the Laffer curve. This result makes the derivation of the relationship between sacrifice and revenue easy to obtain. We just need to differentiate (14) and use (36) in order to obtain revenue as a monotonic increasing function of sacrifice:

$$R'(s) = \int_{\underline{w}}^{\overline{w}} [\frac{dy(w;s)}{ds} - \frac{d\theta(w;s)}{ds}] f(w)dw = \int_{\underline{w}}^{\overline{w}} \frac{d\tau(w;s)}{ds} f(w)dw > 0$$

The existence of a one-to-one relationship between the level of sacrifice and the aggregate revenue collected has two implications. The first implication is that the social planner is indifferent between fixing either sacrifice or revenue as the main exogenous parameter of the tax function, as long as the revenue required is feasible in the sense of not violating the conditions for existence of the tax. The second implication is that equal sacrifice tax systems are completely specified once the exogenous parameter (revenue or sacrifice) is chosen. Unlike optimal tax schedules, where the value of the population density function at each w determines the marginal tax rate faced by consumers of that type, in the equal sacrifice tax model a single parameter/index determines the shape of the entire tax schedule. This feature of equal sacrifice taxes, intuitively, seems to be more in line with horizontal equity requirements and, at the same time, makes the taxes more economical in terms of the information required for implementation.

Another aspect that may be of some interest is how the shape of the tax schedule is affected by a change in the sacrifice level.

The marginal retention rate is defined as MRR$= 1 - t'(y) = \theta'(w)/y'(w)$. In order to gain some intuition, differentiate the marginal retention rate with respect to sacrifice

$$\frac{dMRR}{ds} = -\frac{1}{wu_1^2}[(u_{21}u_1 - u_2u_{11})\frac{d\theta}{ds} + (u_{22}u_1 - u_2u_{21})\frac{1}{w}\frac{dy}{ds}]. \tag{37}$$

In general, no statement can be made on how a change in the level of sacrifice affects the marginal tax rates. However, in the cases of additive preferences or where labor supply increases with sacrifice ($u_{21} \leq 0$ ) we have that an increase in sacrifice brings about an increase in the marginal tax rates.

To study progressivity, we use the concept of average retention rate, ARR$= 1 - t(y)/y$. A progressive tax function is defined here to be one where the average tax rate increases with gross income. This condition is equivalent to a tax function having a MRR less than the ARR. We can now state the following proposition.

*Proposition* 6- If the reference utility function has an elasticity of marginal utility of consumption greater than one, then the equal sacrifice and incentive compatible income tax is progressive.

Proof/

Use equation (25) to determine the ARR

$$\frac{\theta(w)}{y(w)} = -\frac{\theta(w)u_2(\theta(w), y(w))}{g'(w)w^2}.$$

The ratio of the MRR to the ARR is then

$$MRR/ARR = -\frac{u_2(\bar{y}(w), \bar{y}(w)/w)\bar{y}(w)/w}{u_1(\theta(w), y(w)/w)\theta(w)}.$$

We now use equation (5) to write the ratio above as

$$= \frac{u_1(\bar{y}(w), \bar{y}(w)/w)\bar{y}(w)}{u_1(\theta(w), y(w)/w)\theta(w)}. \tag{38}$$

Thus if the elasticity of the marginal utility of income is greater than one, in absolute value, the function is progressive.//

Note that this condition is partially related to the condition for the first best tax to increase with gross income. The latter condition was that the sum of the income elasticity of labor supply and the elasticity of the marginal utility of income was negative. Here, we require that the algebraic value of only the elasticity of the marginal utility of income be less than negative one, a much stronger condition.

By now the reader will have already recognized Samuelson's rule ( Samuelson (1983)) for progressivity of the equal sacrifice income tax. This result, along with Proposition 5, seems to indicate that the old normative theory of income taxation is a great deal more robust than one might think at first glance.

Finally, the question of the existence of absolute bounds for marginal tax rates can be addressed. We already saw that marginal tax rates can never be greater than 100 percent when the second order conditions for incentive compatibility were examined. Is there a lower bound ? The answer is yes, under reasonable conditions. The reader will recall (from the proof of Proposition 2) the "sum of elasticities" assumption, A9.

*Proposition7* - Under assumptions A1-A9, marginal tax rates are strictly greater than zero.

Proof/

The proof depends crucially on assumptions A3 (normality of leisure) and A9 (sum of elasticities). Rewrite (40) as

$$\frac{\theta'(w)}{y'(w)} = \frac{u_1(\bar{y}(w), \bar{y}(w)/w)\bar{y}(w)}{u_1(\theta(w), y(w)/w)y(w)}$$

Thus, the marginal retention rate can also be expressed as the ratio of the marginal utility of consumption times gross income without taxes to the same product after taxes. If the level of sacrifice is null the ratio is obviously one. It is sufficient to establish that $u_1 y$ is monotonically increasing in the level of sacrifice. Differentiate $u_1 y$ with respect to sacrifice, to obtain

$$\frac{du_1 y(w)}{ds} = -\frac{(u_{11}u_2 - u_{21}u_1)y/w + (u_{11}u_{22} - u_{21}^2)(y/w)^2}{J}.$$

The conditions for this expression to be positive involve a term related to the effects of income on labor supply and the concavity of the utility function; it is the same condition we used in the first best analysis to prove that taxes increase with income. This condition, A9, is satisfied for a wide range of preferences, including some convex utility functions.//

As a curiosity the reader may recall example 4. In that example the marginal tax rates were zero. However, the example does not use one of our assumptions on preferences, namely normality of leisure. Nevertheless, it is interesting to check that the expression just above still applies: just take $u_{11} = u_{21} = 0$.

### *4.4 Some thoughts on the Welfare properties of Equal Sacrifice taxes.*

The result in Proposition 7 is similar to one in Seade (1982), but it has a completely different nature. In Seade (1977) the marginal tax rate is strictly positive in the interior of $[\underline{w}, \overline{w}]$, but it is zero at the endpoints.

In the derivation of the equal sacrifice incentive compatible tax no information about the distribution of abilities is used, so even at endpoints we do not have zero tax rates. Since a zero marginal tax rate at the top is a necessary condition for Pareto optimality in our framework, Proposition 7 rules out Pareto optimality of Equal Sacrifice taxes. This alone might seem reason enough to reject the Equal Sacrifice principle as a good normative concept, but we should not be hasty. One must wonder if the losses in efficiency may not be compensated by gains in the simplicity of the tax, equity of the tax and in the relative ease of implementation. There is no obvious way to proceed in the cost-benefit analisys required for such a problem, but an important step would be to find a measure of the welfare costs ( or to derive bounds for the losses ) implied by Equal Sacrifice taxes.

Unfortunately, the determination of these losses in a meaningful and general way is not an easy task. We could proceed along the lines of the appendix in Seade (1977) and derive the differential equation defining a Pareto improvement for a given Equal Sacrifice tax schedule, while keeping total tax payments for every agent fixed. However, it is not clear this is the "closest" Pareto optimal tax schedule we can find, meaning that there might be other Pareto optimal tax schedules relative to which the measured efficiency loss would be smaller. Even if the " Seade improvement" is accepted as a benchmark, it is of little help. We were able to calculate this improvement for simple examples but could not obtain general qualitative results.

Another problem relates to the loss measure iself. A candidate would be to use a Social Welfare Function as a metric and compute the decline in social welfare from using an Equal Sacrifice tax instead of an optimal tax. Again this is something that can be done numerically but seems less promising when trying to arrive at general conclusions, due to the complexity of optimal tax schedules. A more subtle approach is to introduce an equal sacrifice constraint in the optimal income tax problem and use the corresponding multiplier as a measure of the social loss. There are some technical problems with this approach, but it nevertheless allowed us to derive some fairly intuitive, albeit weak, results. These basically say that the cost of equal sacrifice taxes relative to Pareto optimal taxes, at the extrema of the ability distribution, increase with the local density of the population and the level of sacrifice ( or revenue ).

We regard this as an area requiring more research. In fact, the traditional approaches to deadweight loss measurement imply perfect information and the possibility of lump-sum taxes. Extending this analisys to nonlinear income taxes while taking into account informational imperfections when considering feasible alternatives might solve our problem as well as advance our knowledge in the general theory of tax reform.

## 5. Conclusion

Our results confirm that equal sacrifice theories of income taxation appear to have interesting implications, and deserve further study using the tools of modern economic analysis.

For the model studied here we found that the comparative statics are easy to perform, in contrast with the rather complex case of the optimal income tax model, studied by Weymark (1987), and that the results for the second best case are rather intuitive.

We now have a fairly good knowledge of the structure of the model, something that will enable us to understand what we can and cannot ask it to accomplish. For example, Young (1990) suggests that a possible line of research is to find an "optimal equal sacrifice tax". But if one regards this tax as the result of the optimization of a Social Welfare Function with respect to the parameters of the functional form proposed by Young ( based on classic equal sacrifice with constant relative risk aversion utility ) only two situations can arise:

- The reference utility function is additively separable. In this case the functional form achieves equal sacrifice indeed, but the function is unique for each level of revenue. Since the specification of a required revenue level is part of an optimal tax problem, there are no degrees of freedom left and there is only one equal sacrifice tax, with no particular claims to optimality.

-For other preferences structures the functional form proposed by Young does not achieve equal sacrifice. It is just an arbitrary functional form with no particular claims either to efficiency or equity.

One of the aspects of the model we still need to clarify are its welfare properties. The condition of zero marginal tax rates at the top ability level, emphasized in Sadka (1976) and Seade (1977), is not generally satisfied. Our examples where equal sacrifice generates proportional taxes demonstrate this fact. On the one hand, it seems interesting to derive bounds for the welfare losses implied by an equal sacrifice relative to an optimal tax. On the other hand, the optimal tax model requires the government to know exactly the densities of the distribution of abilities everywhere in the economy. This seems to be a strong informational assumption. In contrast, the only time the distribution of abilities enters in the equal sacrifice tax derivation is in establishing the relationship between aggregate revenue and the sacrifice level. Giving that revenue is a monotonic function of sacrifice, a trivial iterative procedure finds the tax function that raises the desired revenue, without using any information on the distribution of abilities. Further exploration of the trade-off between information requirements and efficiency of allocations in income taxation may prove to be interesting but falls outside the scope of this paper.

Finally, the remarkable empirical results in Young (1990) give us reasons to engage in the study of income taxes in a positive perspective. Young found that income taxes in several countries, including the U.S., do look like classical equal sacrifice taxes. This finding has at least two interpretations. First, equal sacrifice is seen as the equity yardstick the community agrees upon, and the political system enforces. Alternatively, we can use the paradigm of the rational preference maximizer economic agent and look at actual tax functions as politico-economic equlibria. We think this second approach is promising and we are now devoting our research to a public choice model of the income tax shape determination.

## Appendix

In this appendix we discuss the monotonicity assumption used in Proposition 4 and we prove that A8 is a sufficient condition for monotonicity in an equal sacrifice model of income taxation.

Differentiating (33) and using (25) we find that

$$\frac{d\theta}{dw} = -\frac{u_2/w(u_{22}y^2/w^4 + 2u_2y/w^3 - g''(w))}{J}. \qquad \text{(ap.1)}$$

The monotonicity assumption is then equivalent to assuming (ap.1) is positive. The main conclusion to extract from (ap.1) is that the existence of the equal sacrifice tax is going to depend on higher order derivatives of the utility function, i.e. existence depends on the negativity of the expression $u_{22}y^2/w^4 + 2u_2y/w^3 - g''(w)$.

Microeconomic theory only tells us that g(w) must be non-decreasing. If the indirect utility function is convex in w then existence is assured. We must investigate the negativity requirement above.

The second derivative of the indirect utility function is

$$g''(w) = -D(\frac{d\bar{y}}{dw})^2 + u_{22}\frac{\bar{y}^2}{w^4} + 2u_2\frac{\bar{y}}{w^3}$$

where D is defined in (2). Using (4) and (33), the numerator of $\frac{d\theta}{dw}$ becomes

$$\frac{1}{w^4}[u_{22}(\theta, y/w)y^2 - u_{22}(\bar{y}, \bar{y}/w)\bar{y}^2] + D(\frac{d\bar{y}}{dw})^2 \qquad \text{(ap.2)}$$

Since the last term is negative, there is a presumption that the monotonicity condition will be verified for most preference structures. Unfortunately, the first term does not seem to be easy to sign. Moreover there are problems with existence of third derivatives in continuous representations of preferences. Nevertheless we can say that monotonicity will hold for quadratic approximations to arbitrary utility functions that satisfy our requirements of normality and have interior solutions.

If we want to be more rigorous, and unless some more fundamental characteristic underlying (ap.2) is found, we will have to specify its negativity as an explicit condition for the existence of an incentive compatible equal sacrifice tax function. A sufficient condition for this to hold is

$$u_{22}(\theta, y/w)y^2 - u_{22}(\bar{y}, \bar{y}/w)\bar{y}^2 \leq 0$$

which can be rewritten as a line integral on sacrifice z:

$$\int_0^s [u_{221}y^2\frac{d\theta}{dz} + (u_{222}y^2 + 2u_{22}y)\frac{dy}{dz}]dz$$

Since this integral is independent of the integration path chosen, we can choose the actual paths followed when sacrifice changes with no loss of generality. This will be helpful because y may increase or decrease with sacrifice. Using equations (34) and (36), the integrand becomes

$$\frac{y^2}{w^2 J}[(u_{222}u_{21}w - u_{221}u_{22})y + 2[u_{22}u_{21} - u_{221}u_2)] \leq 0 \quad \text{(ap.3)}$$

by using A8 and where J is defined after (33).

If $u_{21} = 0$, this last condition holds trivially as an equality. This is the reason why the literature on incentive compatibility relies so heavily on the use of additive utility functions.

As a final note, we should be explicit that (ap.3) is a "strongly" sufficient condition: even if it does not hold in a particular case we may still have existence of an equal sacrifice income tax. In most cases a more informal approach is probably easier to follow : just compute $\frac{d\theta}{dw}$ from (32) and (33) and check whether it is positive. If positivity holds we can be sure the tax function exists and is well behaved.

Clearly this area needs additional research to arrive at necessary and sufficient conditions for incentive compatibility.

## References


[1] **Atkinson, A. and J. Stiglitz (1980),** *Lectures on Public Economics* (Berkshire, England : McGraw-Hill) .

[2] **Atkinson, A. and J. Stiglitz (1976),** "The design of tax structure : direct versus indirect taxation", *Journal of Public Economics,* **6**, 55-75 .

[3] **Berliant, M. (1989),** "On Income Taxation and the Core", *University of Rochester* , mimeo .

[4] **Kaneko, M. (1981) ,** "On the Existence of an Optimal Income Tax Schedule", *Review of Economic Studies,* **48**, 633-642 .

[5] **Laffont, J.J. (1988),** *Fundamentals of Public Economics* ( Cambridge, MA : M.I.T. Press) .

[6] **Lollivier, S. and J-C Rochet (1983),** "Bunching and Second Order Conditions: A Note on Optimal Tax Theory", *Journal of Economic Theory,* **31**, 392-400 .

[7] **Mailath, G. (1987),** "Incentive Compatibility in Signaling Games with a Continuum of Types", *Econometrica,* **55**, 1349-1365 .

[8] **Mill, J. S. (1848),** *Principles of Political Economy* (London : Longmans Green) .

[9] **Mirrlees, J. (1971),** "An Exploration in the Theory of Optimum Income Taxation", *Review of Economic Studies,* **38**, 175-208 .

[10] **Mirrlees, J. (1986),** "The Theory of Optimal Taxation", in Arrow and Intriligator (eds.), *Handbook of Mathematical Economics, Volume III* (Amsterdam : North-Holland) .

[11] **Musgrave, R. A. (1959),** *The Theory of Public Finance* ( New York : McGraw Hill) .

[12] **Phelps, E. (1973)** , "The Taxation of Wages for Economic Justice ", *Quaterly Journal of Economics,* **87**, 331-354 .

[13] **Sadka, E. (1976),** "On Income Distribution Incentive Effects and Optimal Income Taxation", *Review of Economic Studies ,* **43**, 261-268 .

[14] **Samuelson, P. A. (1983)** , *Foundations of Economic Analysis (enlarged edition)* (Cambridge, MA : Harvard University Press) .

[15] **Seade, J. (1977)** , "On the Shape of Optimal Tax Schedules ", *Journal of Public Economics,* **7**, 203-236 .

[16] **Seade, J. (1982)** , "On the Sign of the Optimuum Marginal Income Tax ", *Review of Economic Studies,* **49**, 637-643 .

[17] **Weymark, J. P. (1987),** "Comparative Static Properties of Optimal Nonlinear Income Taxes", *Econometrica ,* **55**, 1165-1185 .

[18] **Young, H.P. (1987)** , "Progressive Taxation and the Equal Sacrifice Principle ", *Journal of Public Economics ,* **32**, 203-214 .

[19] **Young, H.P. (1988),** "Distributive Justice in Taxation", *Journal of Economic Theory,* **44**, 321-335 .

[20] **Young, H.P.(1990)** , "Equal Sacrifice and Progressive Taxation ", *American Economic Review ,* (forthcoming) .