Rochester Center for
Economic Research

The Stolper-Samuelson Theorem: Links to Dominant Diagonals

Jones, Ronald W. and Sugata Marjit

Working Paper No. 221
February 1990

University of
Rochester

The Stolper-Samuelson Theorem: Links to Dominant Diagonals

Ronald W. Jones, Sugata Marjit and Tapan Mitra

Rochester Center for Economic Research
Working Paper No. 221

THE STOLPER–SAMUELSON THEOREM: LINKS TO DOMINANT DIAGONALS

Ronald W. Jones
University of Rochester

Sugata Marjit
Jadavpur University

Tapan Mitra
Cornell University

Working Paper No. 221

Revised Draft
February 1990

The Stolper–Samuelson Theorem: Links to Dominant Diagonals

Ronald W. Jones
University of Rochester

Sugata Marjit
Jadavpur University

Tapan Mitra
Cornell University

Some thirty years ago Lionel McKenzie (1960) prepared an eloquent statement exposing the important role of matrices with dominant diagonals in economic theory. These matrices were shown to be directly relevant to issues such as the existence of positive solutions in a Leontief system, the stability of a competitive equilibrium in the case of gross–substitutes, and the unique determination of factor prices by goods prices. Regarding this latter issue, it is, of course, well known in the field of international trade that the Stolper–Samuelson result, whereby a rise in any commodity's price lowers all factor returns save that of the factor "intensively" used in producing that commodity, requires much stronger restrictions on the production matrix than does the factor–price equalization theorem.[1] Nonetheless, we intend to show in this article that there exists a link between matrices which exhibit the dominant diagonal property and the Stolper–Samuelson theorem. Such a link emerges naturally from a new set of conditions on production technology which are shown to be sufficient to establish the Stolper–Samuelson results. These new conditions are, in turn, related to the stronger set of sufficient conditions exemplified by the Produced Mobile Factor (PMF) structure laid out in Jones and Marjit (1985, 1988), the older set of necessary conditions

[1]For earlier contributions exploring this issue see Minabe (1967), Kemp and Wegge (1969), Chipman (1969) and Uekawa (1971). Of special interest is Chipman's demonstration that diagonal dominance of the share matrix is neither necessary nor sufficient for the strong Stolper–Samuelson results.

discussed by Kemp and Wegge (1969), and a theorem on sufficiency established in the mathematics literature by Willoughby (1977).

1. The Strong Factor–Intensity Condition

The standard setting in which the Stolper–Samuelson theorem is discussed involves the active production in competitive markets of n commodities, each produced non–jointly by the use of n distinct factors in processes that are linearly independent of each other at prevailing factor prices. The existence of any "holes" in the input–output technology matrix, whereby there exists one or more factors not used in one or more industries, plays havoc with the strong Stolper–Samuelson results.[2] Therefore we assume that the distributive share of factor i in industry j, θ_{ij}, is strictly positive for all i, j.

To proceed with our investigation we assume that a single commodity price, p_s, rises, with all other commodity prices held fixed. Using "hat" terminology to indicate relative changes in variables, we are assuming

$$\hat{p}_s > 0, \text{ with } \hat{p}_i = 0 \quad \forall i \neq s.$$

In such a case we know that some factor return must rise (and by relatively more than $\hat{p}_s$) and, since some sectors have experienced no price change, at least one other factor must have its return lowered. But this could be the return to factor s, instead of some off–diagonal term, $\hat{w}_k$, $k \neq s$.[3] The following Factor Intensity (FI) condition,

[2]That is, if some $a_{ik} = 0$ for an input–output coefficient, the matrix A^{-1} cannot exhibit strictly positive diagonal elements and negative off–diagonal elements. For explanation, see Jones (1987).

[3]Inada (1971) explores in some detail the case in which all diagonal elements of the inverse of the share matrix are negative and all off–diagonal elements are positive. Our Factor Intensity condition rules out this kind of behavior.

due to Kemp and Wegge (1969), we now impose on the technology, and then use to show that when p_s rises, at least one off–diagonal w_k, $k \neq s$, must fall.

<u>*Definition:*</u> *The matrix of distributive factor shares, [θ], satisfies the following <u>Factor Intensity</u> (FI) condition if for all s, i, j ≠ s,*

$$\frac{\theta_{ss}}{\theta_{is}} > \frac{\theta_{sj}}{\theta_{ij}}$$

Factor s is said to be used intensively in industry s if its share there, relative to that of any other factor, i, is greater than the corresponding ratio of s to i distributive shares in any other industry. With this restriction in mind we establish a weak–sounding lemma which, nonetheless, is of value in establishing our principal theorem:

<u>*Lemma:*</u> *If the [θ] matrix of distributive factor shares satisfies the Factor–Intensity (FI) condition and one price, p_s, rises with all other commodity prices constant, at least one factor return, w_k, must fall for k ≠ s.*

This lemma is easily established by considering the competitive profit equations of change for commodity s and any other commodity, j:

$$(1) \qquad \theta_{ss}\hat{w}_s + \sum_{i \neq s} \theta_{is}\hat{w}_i = \hat{p}_s$$

$$(2) \qquad \theta_{sj}\hat{w}_s + \sum_{i \neq s} \theta_{ij}\hat{w}_i = 0$$

Solve for $\hat{w}_s$ in each equation and equate to obtain:

$$(3) \qquad \frac{\hat{p}_s}{\theta_{ss}} = \sum_{i \neq s} \left(\frac{\theta_{is}}{\theta_{ss}} - \frac{\theta_{ij}}{\theta_{sj}}\right)\hat{w}_i$$

Since the matrix of factor shares satisfies the FI conditions, each bracketed term in the summation must be negative. Therefore a rise in p_s requires at least one $\hat{w}_i$, $i \neq s$, to be negative.

The Factor Intensity condition posits the importance of diagonal terms in the $[\theta]$ matrix relative to off–diagonal terms. Indeed, as demonstrated in Kemp and Wegge, the Factor Intensity condition is always necessary for the Stolper–Samuelson theorem to hold, but is not sufficient if $n \geq 4$. As discussed in Jones and Marjit (1985, 1988) more structure is generally required in order to guarantee strong Stolper–Samuelson results. For example, the Produced Mobile Factor (PMF) structure, so–called because it can be derived from the (n+1) x n specific factors model (Jones, 1975) by letting the single mobile input itself be produced by all the "specific" factors, implies that all the ratios of off–diagonal terms in any pair of rows, $\frac{\theta_{sj}}{\theta_{ij}}$, be equal for all industries, $j \neq s, i$. And this condition suffices to establish the Stolper–Samuelson result. The idea behind the Strong Factor Intensity (SFI) condition, to be defined below, is to allow these off–diagonal ratios to differ from each other, but to set an aggregate limit on the extent of these discrepancies.[4] Thus:

> *Definition:* A matrix of distributive factor shares satisfies the *Strong Factor–Intensity* condition (SFI) if for any pair of productive factors, *s* and *r*, and industries *s* and *t* $(t \neq r)$,

$$\left(\frac{\theta_{ss}}{\theta_{rs}} - \frac{\theta_{st}}{\theta_{rt}}\right) > \sum_{i \neq r, s, t} \left|\frac{\theta_{is}}{\theta_{rs}} - \frac{\theta_{it}}{\theta_{rt}}\right| \qquad \forall s;\ i,t \neq s.$$

Consider two industries, *s* and *t*, and focus on some factor, $r \neq s, t$. By the (FI) condition the left–hand side of the (SFI) criterion is positive, and features the excess of

[4]In section 2 an alternative characterization of the Strong Factor Intensity condition will be provided, one which fits more closely the concept of diagonal dominance used by McKenzie (1960).

the $\underline{s}$ to $\underline{r}$ input share ratio in industry $\underline{s}$ (where factor $\underline{s}$ is the intensive factor used in industry $\underline{s}$) over that for industry $\underline{t}$. The right–hand side collects the absolute values of the discrepancy between the ratio of other factor inputs, $i \neq s$, to $\underline{r}$ in industry $\underline{s}$ compared to industry $\underline{t}$. The SFI criterion posits that the sum of such absolute values still falls short of the excess of θ_{ss}/θ_{rs} over the comparable ratio in the $\underline{t}^{th}$ sector, θ_{st}/θ_{rt}.

Armed with the lemma previously established we now prove the basic theorem:

<u>Theorem</u>: *If the n x n productive structure satisfies the Strong Factor–Intensity condition, the Stolper–Samuelson results hold.*

Let p_s rise, keeping all other commodity prices constant. This leads to an array of factor price changes, $\hat{w}_i$, and let k,j be such that:

$$\hat{w}_k = \min_i \hat{w}_i; \quad \hat{w}_j = \max_i \hat{w}_i \qquad \text{over all } i \neq s.$$

The lemma ensures that $\hat{w}_k$ is negative. Denote by set Q the values for $i \neq s,k$ or j.

The competitive profit conditions for commodities $\underline{j}$ and $\underline{k}$ are shown in equations (4) and (5):

$$(4) \qquad \theta_{sj}\hat{w}_s + \theta_{kj}\hat{w}_k + \theta_{jj}\hat{w}_j + \sum_{i \epsilon Q} \theta_{ij}\hat{w}_i = 0$$

$$(5) \qquad \theta_{sk}\hat{w}_s + \theta_{kk}\hat{w}_k + \theta_{jk}\hat{w}_j + \sum_{i \epsilon Q} \theta_{ik}\hat{w}_i = 0$$

Again, solve each equation for $\hat{w}_s$ and equate to obtain:

$$(6) \qquad \sum_{i \epsilon Q} \frac{\theta_{ij}}{\theta_{sj}} \hat{w}_i + \frac{\theta_{kj}}{\theta_{sj}} \hat{w}_k + \frac{\theta_{jj}}{\theta_{sj}} \hat{w}_j = \sum_{i \epsilon Q} \frac{\theta_{ik}}{\theta_{sk}} \hat{w}_i + \frac{\theta_{kk}}{\theta_{sk}} \hat{w}_k + \frac{\theta_{jk}}{\theta_{sk}} \hat{w}_j.$$

Re–write this equation so as to isolate the terms in $\hat{w}_j$ and $\hat{w}_k$, noting that $\hat{w}_k$ is negative:

$$(7) \qquad \left[\frac{\theta_{jj}}{\theta_{sj}} - \frac{\theta_{jk}}{\theta_{sk}}\right] \hat{w}_j + \left[\frac{\theta_{kk}}{\theta_{sk}} - \frac{\theta_{kj}}{\theta_{sj}}\right] |\hat{w}_k| = \sum_{i \epsilon Q} \left[\frac{\theta_{ik}}{\theta_{sk}} - \frac{\theta_{ij}}{\theta_{sj}}\right] \hat{w}_i$$

By the FI condition note that the coefficients of $\hat{w}_j$ and $|\hat{w}_k|$ on the left–hand side are both positive.

Now consider possible values for $\hat{w}_j$. We prove that $\hat{w}_j$ must be negative, and we do this by showing, first, that $\hat{w}_j$ cannot exceed the positive value, $|\hat{w}_k|$, and, secondly, that $\hat{w}_j$ cannot lie in the range $0 \leq \hat{w}_j \leq |\hat{w}_k|$. If $\hat{w}_j$ were to exceed $|\hat{w}_k|$, the right–hand side of (7) would be less than

$$\hat{w}_j \sum_{i \epsilon Q} \left|\frac{\theta_{ij}}{\theta_{sj}} - \frac{\theta_{ik}}{\theta_{sk}}\right|,$$

while in any case the left–hand side exceeds

$$\hat{w}_j \left[\frac{\theta_{jj}}{\theta_{sj}} - \frac{\theta_{jk}}{\theta_{sk}}\right].$$

If $\hat{w}_j$ were positive, this inequality would violate SFI. If $\hat{w}_j$ were non–negative but less than or equal to $|\hat{w}_k|$, the right–hand side of (7) would be less than or equal to

$$|\hat{w}_k| \sum_{i \epsilon Q} \left|\frac{\theta_{ik}}{\theta_{sk}} - \frac{\theta_{ij}}{\theta_{sj}}\right|,$$

and the left–hand side of (7) would exceed or equal

$$|\hat{w}_k| \left[\frac{\theta_{kk}}{\theta_{sk}} - \frac{\theta_{kj}}{\theta_{sj}}\right].$$

Once again a contradiction with SFI is involved. Therefore $\hat{w}_j$ must be negative. A rise in p_s lowers all w_i for $i \neq s$. ($\hat{w}_s$ must therefore exceed unity). This is the Stolper–Samuelson result.

Remark:

While our strong factor–intensity condition and our generalized Stolper–Samuelson theorem are phrased in terms of economic magnitudes (the factor shares), we have actually established the following purely mathematical result: Suppose an n x n positive non–singular matrix, $\alpha = (\alpha_{ij})$ satisfies the following condition:
For every triple s, t, r of distinct indices,

$$\left(\frac{\alpha_{ss}}{\alpha_{rs}} - \frac{\alpha_{st}}{\alpha_{rt}}\right) > \sum_{i \neq r,s,t} \left|\frac{\alpha_{is}}{\alpha_{rs}} - \frac{\alpha_{it}}{\alpha_{rt}}\right|$$

Then, the diagonal terms of α^{-1} are positive while the off–diagonal terms of α^{-1} are negative.

2. A Dominant Diagonal Matrix

The Strong Factor Intensity condition, if satisfied by the matrix of distributive factor shares, allows the construction of a new matrix of order (n – 2) x (n – 2) which

(a) Has a positive diagonal if the Factor Intensity (FI) condition is satisfied, and

(b) Has a dominant diagonal if the Strong Factor Intensity (SFI) condition is satisfied.

From the original $[\theta]$ matrix of distributive shares we select an arbitrary r^{th} row and t^{th} column ($t \neq r$) and construct a new matrix, B(r,t), which will be (n – 2) x (n – 2). The procedure involves several steps:

(i) For each column, s, divide all elements by θ_{rs}. (Thus the r^{th} row becomes the unit vector).

(ii) In this new matrix subtract the t^{th}column (consisting of elements such as θ_{jt}/θ_{rt}) from all columns. (This subtraction makes both the t^{th} column and the r^{th} row consist entirely of zeroes).

(iii) Delete the r^{th} row <u>and</u> column and the t^{th} row <u>and</u> column to obtain B(r,t). Retain the original numbering of rows and columns so that, for example, the $(n - 2)^{nd}$ row index of B(r,t) is <u>n</u>.

The typical diagonal element of B(r,t) is

$$b_{ss} \equiv \frac{\theta_{ss}}{\theta_{rs}} - \frac{\theta_{st}}{\theta_{rt}}$$

If (FI) is satisfied, all the diagonal elements are positive. The typical off–diagonal element in the $\underline{s}^{th}$ column is

$$b_{is} \equiv \frac{\theta_{is}}{\theta_{rs}} - \frac{\theta_{it}}{\theta_{rt}}.$$

The usual definition of diagonal dominance requires

$$b_{ss} > \sum_{i \neq r, s, t} |b_{is}|,$$

which is precisely the condition for SFI. That is, diagonal dominance of all the B(r,t) matrices implies that the underlying production structure satisfies the Stolper–Samuelson conditions.

In the subsequent section we find it convenient to use an alternative characterization of diagonal dominance, the one which McKenzie employed (1960). Applied to the the Strong Factor Intensity (SFI) condition this leads to:

<u>Definition</u>: *A matrix of distributive shares satisfies the Generalized Strong Factor Intensity condition if there exists a set of positive numbers, $d_1,\ldots,d_n$, such that for any pair of productive factors, s and r, and industries s and t ($t \neq r$),*

$$d_s\left(\frac{\theta_{ss}}{\theta_{rs}} - \frac{\theta_{st}}{\theta_{rt}}\right) > \sum_{i \neq r,\, s,t} d_i \left|\frac{\theta_{is}}{\theta_{rs}} - \frac{\theta_{it}}{\theta_{rt}}\right| \qquad \forall s;\ i,t \neq s$$

Suppose the non–singular matrix of distributive shares, θ, satisfies the generalized SFI condition. We can then define a diagonal matrix, μ satisfying $\mu_{ii} = d_i$ for i = 1,..., n, and a matrix, $\nu = \mu\theta$. Then, $\nu_{ij} = d_i\theta_{ij}$ for all i,j, and so we have for any pair of distinct productive factors, s and r, and distinct industries s and t ($t \neq r$),

$$d_s\left[\frac{(\nu_{ss}/d_s)}{(\nu_{rs}/d_r)} - \frac{(\nu_{st}/d_s)}{(\nu_{rt}/d_r)}\right] > \sum_{i \neq r,s,t} d_i \left|\frac{(\nu_{is}/d_i)}{(\nu_{rs}/d_r)} - \frac{(\nu_{it}/d_i)}{(\nu_{rt}/d_r)}\right|$$

which simplifies to

$$\left[\frac{\nu_{ss}}{\nu_{rs}} - \frac{\nu_{st}}{\nu_{rt}}\right] > \sum_{i \neq r,s,t} \left|\frac{\nu_{is}}{\nu_{rs}} - \frac{\nu_{it}}{\nu_{rt}}\right|$$

This means that the matrix, ν, is non–singular, and (by the remark following our theorem) ν^{-1} has positive diagonal and negative off–diagonal elements. Since $\nu^{-1} = \theta^{-1}\mu^{-1}$ and μ^{-1} is a positive diagonal matrix, clearly θ^{-1} has positive diagonal and negative off–diagonal elements. This is the Stolper–Samuelson result.

3. <u>The Willoughby Theorem</u>

In Willoughby (1977) there is a theorem which states sufficient conditions which, in effect, guarantee the Stolper–Samuelson result. Here we present a version of his result phrased in terms of the distributive share matrix.

There are two strands to Willoughby's result. First is the dominance of any diagonal element over all off–diagonal elements (assumed positive) in the same row:

$$(8) \qquad 0 < m \equiv \underset{i \neq j}{\mathrm{Min}} \frac{\theta_{ij}}{\theta_{ii}} \leq \underset{i \neq j}{\mathrm{Max}} \frac{\theta_{ij}}{\theta_{ii}} \equiv M < 1$$

Such dominance was shown by Kemp and Wegge (1969) to be a necessary consequence of a share matrix exhibiting the Stolper–Samuelson property. More generally, it follows from the Factor Intensity condition, and thus may hold in larger dimensional cases even when Stolper–Samuelson does not.[5]

The second strand of Willoughby's result, as reflected in the following statement of his theorem, is that if <u>m</u> and <u>M</u> are sufficiently close together, the Stolper–Samuelson result holds:

<u>*Theorem*</u>: *(Willoughby). Suppose (8) holds for an (n x n) distributive share matrix and, furthermore,*

$$(9) \qquad (n-2)\,\frac{(M^2 - m^2)}{(m - m^2)} < 1.$$

[5]Rewrite the FI condition as $\theta_{ss}\theta_{ij} > \theta_{sj}\theta_{is}$. For given s there are (n – 1) inequalities of this type, for i≠s. Adding them yields:

$$\theta_{ss}\Big(\sum_{i \neq s} \theta_{ij}\Big) > \theta_{sj}\Big(\sum_{i \neq s} \theta_{is}\Big),$$

Since $\sum_{i \neq s} \theta_{ij} = (1 - \theta_{sj})$ and $\sum_{i \neq s} \theta_{is} = (1 - \theta_{ss})$, substitution reveals

$$\theta_{ss} > \theta_{sj} \quad \forall j \neq s.$$

Thus even though a share matrix which satisfies FI may not have a dominant diagonal, it does have a "dominating" diagonal over other row elements.

Then the matrix of distributive shares must be such that θ^{-1} *has negative off-diagonal elements (and diagonal elements all exceeding unity).*

The proof involves showing that the generalized characterization of the SFI condition is satisfied with positive weights $d_i = \frac{1}{\theta_{ii}}, i = 1,\ldots,n$. We begin by observing that

$$(10) \qquad \frac{m}{M} \le \frac{\theta_{is}/\theta_{ii}}{\theta_{rs}/\theta_{rr}} \le \frac{M}{m} \quad \text{and} \quad \frac{m}{M} \le \frac{\theta_{it}/\theta_{ii}}{\theta_{rt}/\theta_{rr}} \le \frac{M}{m},$$

for i,r≠s,t since m is less than or equal to the smallest θ_{ij}/θ_{ii} and M exceeds or equals the largest θ_{ij}/θ_{ii}, i≠j. From this follow the bounds set on the difference between ratios of distributive shares:

$$(11) \qquad \frac{\theta_{rr}}{\theta_{ii}}\left[\frac{\theta_{is}}{\theta_{rs}} - \frac{\theta_{it}}{\theta_{rt}}\right] \le \frac{M}{m} - \frac{m}{M} = \frac{M^2 - m^2}{mM}$$

$$(12) \qquad \frac{\theta_{rr}}{\theta_{ii}}\left[\frac{\theta_{it}}{\theta_{rt}} - \frac{\theta_{is}}{\theta_{rs}}\right] \le \frac{M}{m} - \frac{m}{M} = \frac{M^2 - m^2}{mM}$$

Since there are (n − 3) values for i≠r,s,t, addition over factors i (≠r,s,t) yields:

$$(13) \qquad \theta_{rr} \sum_{i\neq r,s,t} d_i \left|\frac{\theta_{is}}{\theta_{rs}} - \frac{\theta_{it}}{\theta_{rt}}\right| \le (n-3)\left[\frac{M^2 - m^2}{mM}\right]$$

Turning to the discrepancy between the ratio of the diagonal share (θ_{ss}) to an off-diagonal share (θ_{rs}) in an industry and the comparable ratio in another industry, obtain:

$$(14) \qquad \theta_{rr} d_s \left[\frac{\theta_{ss}}{\theta_{rs}} - \frac{\theta_{st}}{\theta_{rt}} \right] \geq \frac{1}{M} - \frac{M}{m}$$

$$= \frac{m - M^2}{mM} = \frac{(m - m^2)}{mM} - \frac{(M^2 - m^2)}{mM}.$$

Finally, Willoughby's condition (9) can be restated as

$$(15) \qquad (n-2) \frac{(M^2 - m^2)}{mM} < \frac{(m - m^2)}{mM}$$

Now, subtract $\frac{(M^2 - m^2)}{mM}$ from each side of (15) to obtain:

$$(16) \qquad (n-3) \frac{(M^2 - m^2)}{mM} < \frac{(m - m^2)}{mM} - \frac{(M^2 - m^2)}{mM}$$

The right–hand sides of (14) and (16) are equivalent, so that from (13), (14) and (16),

$$(17) \qquad d_s \left[\frac{\theta_{ss}}{\theta_{rs}} - \frac{\theta_{st}}{\theta_{rt}} \right] > \sum_{i \neq r,s,t} d_i \left| \frac{\theta_{is}}{\theta_{rs}} - \frac{\theta_{it}}{\theta_{rt}} \right|$$

That is, Willoughby's restriction (9) implies that the matrix of distributive factor shares satisfies the generalized SFI condition and thus leads to Stolper–Samuelson results.

The preceding proof utilized the fact that a matrix satisfying the Willoughby conditions must also satisfy SFI. But we now provide an example to show that the converse does not hold: Consider the following A matrix, which is the product of a share matrix and the diagonal matrix D where $d_{ii} = \frac{1}{\theta_{ii}}$:

$$A = \begin{bmatrix} 1 & 0.1 & 0.1 & 0.1 \\ 0.3 & 1 & 0.3 & 0.3 \\ 0.3 & 0.3 & 1 & 0.3 \\ 0.1 & 0.3 & 0.1 & 1 \end{bmatrix}$$

The inverse of A has a positive diagonal and strictly negative off–diagonal elements. Furthermore, the Factor Intensity condition is satisfied, as is (SFI). But the Willoughby condition is not: $M = 0.3$, $m = 0.1$, $n = 4$ and $(n - 2)\frac{(M^2 - m^2)}{(m - m^2)} = 16/9 > 1$.

4. <u>The Produced Mobile Factor (PMF) Structure</u>

Jones and Marjit (1985, 1988) develop a simple structure based on the (n + 1)–factor, n–sector specific factor model, modified so that the single "mobile" factor in such a setting is itself the output of a linear homogeneous production function with all other factors as inputs. In effect each productive activity is a positive convex combination of an activity using only the factor "intensively" used and a common activity used in all sectors. In each industry there is one (intensive) factor which is used in two ways – in helping to produce an intermediate good used in all sectors and in combining with this intermediate good (the "produced mobile factor") to help produce the final output in that industry. All other factors enter only indirectly through their employment in producing the intermediate good. This leads to a natural additive decomposition of the share matrix into the direct factor use in the final stage (a positive diagonal matrix) and the indirect use (matrix of rank one since all sectors use varying amounts of the same intermediate good).

Such a structure must satisfy the SFI conditions. To see this consider the ratio of a pair of off–diagonal shares. As shown by (18), this ratio must be the same for all industries:

$$(18) \qquad \frac{\theta_{is}}{\theta_{rs}} = \frac{w_i a_{is}}{w_r a_{rs}} = \frac{w_i a_{im} a_{ms}}{w_r a_{rm} a_{ms}} = \frac{w_i a_{im}}{w_r a_{rm}} \qquad \forall s.$$

The term a_{ms} refers to the input of the produced mobile factor required per unit of commodity s, whereas a_{im} and a_{rm} denote input requirements to produce a unit of the "mobile" factor. Thus the right–hand side of the SFI criterion vanishes, and the left–hand side is positive by construction.

In Jones and Marjit (1988) use is made of the geometrical apparatus employed recently by Leamer (1987) and earlier by McKenzie (1955) to depict the composition of factor input requirements in a triangle (for the case of three inputs) or an $(n - 1)$ dimensional tetrahedron (for the case of n inputs), with the use of barycentric co–ordinates. A feature of the PMF structure is that rays from each factor origin passing through the point representing the activity level for the industry intensive in its use of that factor all pass through a common point. That point represents the composition of inputs for the produced "mobile" factor. The weaker conditions provided by the SFI criterion provide a measure of how far apart such rays can be without sacrificing the Stolper–Samuelson result. It is as if the PMF structure combines in each activity a single factor and a common "off–the–rack" input (the common mobile factor), whereas the more general structure exhibiting SFI combines each factor with a "tailor–made" input which may differ somewhat from the custom–made input used in any other sector.

Reconsider condition (9) required for the Willoughby result. As the number of distinct productive sectors, n, expands, the discrepancy between minimal and maximal relative off–diagonal elements, m and M, must get smaller and smaller. Thus the PMF structure represents the limiting case of the Willoughby condition.

The PMF structure yields a particularly simple form of the dominant diagonal B(r,t) matrices; these matrices reduce to pure diagonal matrices since ratios of off–diagonal shares are equal. The most simple case of the PMF structure is the one in which the share of the factor used intensively in any industry is the same, d, over all industries and all unintensive factor shares have a common value s, where $s < d$. Such a matrix clearly has only two types of factors and supports Stolper–Samuelson conclusions since when p_j alone rises, at least one factor return must rise by a magnified amount ($\hat{w}_j$) and one (or more) factor return must fall ($\hat{w}_i$, the same value for all $i \neq j$). This special matrix will exhibit a "dominating" diagonal as long as d exceeds s, but not a dominant diagonal unless $d > (n-1)s$. Conversely, suppose d does exceed $(n-1)s$ in all industries save one, and that in that special sector the diagonal term is $(d + s)$ and one of the off-diagonal terms is zero. Such a matrix has a (weakly) larger diagonal than the straightforward case with all diagonal elements possessing value d and off-diagonal elements, s. Nonetheless, it cannot exhibit Stolper–Samuelson properties. The latter characteristic of a production structure depends not only on the degree of intensity in the use of a single factor in each sector but also on a limitation in the divergence of factor shares of the non-intensive factors.

5. Concluding Remarks

Relatively few production structures have been developed which are sufficient to lead to Stolper–Samuelson results. The Produced Mobile Factor (PMF) structure is an example which does support Stolper–Samuelson. The Factor Intensity (FI) condition by itself has been shown by Kemp and Wegge (1969) not to be sufficient to yield Stolper–Samuelson results in dimensions greater than three, although it is a necessary condition. The Strong Factor Intensity (SFI) condition developed in this paper, which imposes more constraints on factor intensities than simple (FI) alone, has been shown to be sufficient for Stolper–Samuelson. Indeed, it encompasses the mathematical theorem due to Willoughby as well as the Produced Mobile Factor Structure. In imposing extra constraints on the degree of asymmetry in off–diagonal terms it provides yet another example of the concept of dominant diagonals, whose applications in economic theory were pioneered thirty years ago by Lionel McKenzie.

It is important to emphasize that although some forms of strong symmetry are required to obtain Stolper–Samuelson results, the direction taken in the Strong Factor Intensity Condition is not the only route that is possible. An alternative path was suggested by Willoughby in his example of a circulant matrix. Before appropriate scaling to render it a share matrix, the circulant structure has value unity along the diagonal, values equal to some fraction a to the immediate right of the diagonal (with a_{n1} also equal to a), the smaller fraction a^2 to the right of that, and so on until the term a^{n-1} lies in the position immediately to the left of the diagonal (with a_{1n} equal to a^{n-1}). Such a matrix does not satisfy the Willoughby conditions discussed earlier nor, indeed, the SFI conditions, yet the inverse matrix is "borderline" Stolper–Samuelson in that although no positive off–diagonal elements appear, there are many zeroes. Thus the kind of symmetry imposed is genuinely different from the SFI structure in that every sector looks like every other sector except for a renumbering of most intensively used factor, etc.[6] It is the former (SFI) structure that makes use of the bounds on asymmetry provided by the concept of diagonal dominance of the differences between ratios of factor shares.

[6]In future work we propose to investigate the significance of the shape of such "share ribs" which are assumed to be similar in structure from sector to sector. For example, Stolper–Samuelson results emerge only if the typical schedule of shares from highest to lowest is sufficiently bowed in. A linear "rib" will not do.

References

Chipman, J. (1969): "Factor Price Equalization and the Stolper–Samuelson Theorem", International Economic Review, 10, pp 399–406.

Inada, K. (1971): "The Production Coefficient Matrix and the Stolper–Samuelson Condition", Econometrica, Vol 39: 219–40.

Jones, R. W. (1987): "The Heckscher–Ohlin Trade Theory", in Eatwell et al. eds. The New Palgrave.

Jones, R. W. and S Marjit (1985): "A Simple Production Model with Stolper–Samuelson Properties", International Economic Review, 26, pp. 565–567.

Jones, R. W. and S. Marjit (1988): "The Stolper–Samuelson Theorem, the Leamer Triangle and the Produced Mobile Factor Structure", unpublished.

Kemp, M. C. and L. Wegge (1969): "On the Relation Between Commodity Prices and Factor Rewards", International Economic Review, 10, 407–413.

McKenzie, L. (1955): "Equality of Factor Prices in World Trade", Econometrica, 23, 239–257.

McKenzie, L. (1960): "Matrices with Dominant Diagonals and Economic Theory", in Mathematical Methods in the Social Sciences, edited by K. Arrow, S. Karlin and P. Suppes, Stanford University Press, pp. 47–60.

Minabe, N. (1967): "The Stolper–Samuelson Theorem, the Rybczynski Effect, and the Heckscher–Ohlin Theory of Trade Pattern and Factor Price Equalization: The Case of Many–Commodity, Many–Factor Country", Canadian Journal of Economics and Political Science, Aug., 33;401–19.

Uekawa, Y. (1971): "Generalization of the Stolper–Samuelson Theorem", Econometrica, 39, 197–213.

Willoughby, R. A. (1977): "The Inverse M–Matrix Problem", Linear Algebra and its Applications, 18, 75–94.